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Phys. Rev. D **94**, 125026 — Published 28 December 2016

DOI: 10.1103/PhysRevD.94.125026

Study of the Six-Loop Beta Function of the $\lambda\phi^4_4$ Theory

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We investigate whether the six-loop beta function of the $\lambda\phi^4_4$ theory exhibits evidence for an ultraviolet zero. As part of our analysis, we calculate and analyze Padé approximants to this beta function. Extending our earlier results at the five-loop level, we find that in the range of λ where the perturbative calculation of the six-loop beta function is reliable, the theory does not exhibit robust evidence for an ultraviolet zero.

PACS numbers: 11.10.-z,11.10.Hi

I. INTRODUCTION

There has long been interest in the renormalization-group (RG) behavior of the $\lambda\phi^4$ field theory in $d = 4$ spacetime dimensions, where ϕ is a real scalar field. This theory is described by the Lagrangian [1]

$$\mathcal{L} = \frac{1}{2}(\partial_\nu\phi)(\partial^\nu\phi) - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \ . \qquad (1.1)$$

The coupling λ in $\mathcal{L}$ is taken to be positive for the stability of the theory. The Lagrangian (1.1) is invariant under the global discrete $\mathbb{Z}_2$ symmetry $\phi \to -\phi$. This theory is sometimes denoted $\lambda\phi^4_4$, with the subscript 4 indicating the spacetime dimensionality; henceforth, this value of d will be understood implicitly. The sign of m^2 will not be important for our analysis of the ultraviolet behavior of the theory; for definiteness, we assume that $m^2 > 0$.

The dependence of the running coupling $\lambda(\mu)$ on the Euclidean energy/momentum scale, μ, where it is measured, is described by the beta function of the theory [2], $\beta_\lambda = d\lambda/dt$, where $dt = d\ln\mu$. (The argument μ will often be suppressed in the notation.) The one-loop term in this beta function has a positive coefficient, so that for small λ, $\beta_\lambda > 0$ and hence as $\mu \to 0$, the coupling $\lambda(\mu) \to 0$, i.e., the theory is infrared-free. This perturbative result is in agreement with nonperturbative approaches [4] and is sometimes described as the "triviality" property of the theory. One then interprets the theory as an effective one that is applicable only over a limited range of scales μ (e.g., [5, 6]). In this theory, as μ increases from small values in the infrared (IR) to larger values toward the ultraviolet (UV), the running coupling $\lambda(\mu)$ increases. If one were to retain only the one-loop term in the beta function, then this would lead to an apparent pole in $\lambda(\mu)$ at a finite value of μ, denoted μ_p. As is well known, it would not be valid to infer the existence of a pole in $\lambda(\mu)$ at $\mu = \mu_p$, since $\lambda(\mu)$ would become too large for the perturbative calculation to be reliable before μ reached μ_p. Nevertheless, this provides a motivation to calculate and analyze higher-loop terms in the beta function for this theory.

An important question is whether this beta function has a UV zero, which could thus constitute an ultraviolet fixed point (UVFP) of the renormalization group (RG), so that as μ increases from the infrared (IR) limit $\mu = 0$ to the UV limit $\mu \to \infty$, $\lambda(\mu)$ would increase, but would approach a finite value, $\lambda_{_{UV}}$. In [3] we investigated this question for the general O(N) $\lambda|\vec{\phi}|^4$ theory with a real N-component scalar field $\vec{\phi} = (\phi_1, ..., \phi_N)$, using the beta function calculated to the highest loop order available, namely five loops. Our conclusion from that analysis was that the beta function for the O(N) model $\lambda|\vec{\phi}|^4$ theory does not exhibit evidence for such a UVFP. This finding is consistent with the view of this theory as an effective field theory, to be applied only over a restricted range of momentum scales μ. The $\lambda\phi^4$ theory of Eq. (1.1) is the special case of the O(N) $\lambda|\vec{\phi}|^4$ theory with $N = 1$, where the continuous global O(N) symmetry is reduced to a discrete $\mathbb{Z}_2$ symmetry.

In this paper we use the recently calculated six-loop term in the ($\mathbb{Z}_2$-invariant) $\lambda\phi^4$ theory [13] to extend our investigation of the question of a possible UV zero in the beta function to the six-loop level. For perspective, one might ask whether a (Lorentz-invariant) infrared-free quantum field theory is known whose beta function does exhibit a UV zero. The answer is yes, and an example is provided by the nonlinear O(N) σ model in $d = 2 + \epsilon$ spacetime dimensions. In this theory, an exact solution was obtained in the limit $N \to \infty$ with $\lambda(\mu)N = x(\mu)$ a fixed function of μ and yielded the beta function

$$\beta_x = \frac{dx}{dt} = \epsilon x\Big(1 - \frac{x}{x_{_{UV}}}\Big) \qquad (1.2)$$

for small ϵ, where $x_{_{UV}} = 2\pi\epsilon$ is a UV fixed point of the renormalization group [7]. Thus, in this nonlinear O(N) σ model in $d = 2 + \epsilon$ dimensions, the coupling $x(\mu)$ flows (monotonically) from $x = 0$ at $\mu = 0$ to $x = x_{_{UV}}$ as $\mu \to \infty$. The question that we investigate here is whether there is evidence for a similar type of behavior in the $\lambda\phi^4$ theory in $d = 4$ dimensions at the six-loop level.

This paper is organized as follows. In Section II we discuss some background and list the coefficients in the beta function that we will use for our study. In Section III we investigate the question of the presence or absence of a UV zero of the beta function up to six-loop order. Section IV contains a further analysis of this question of a UV zero using Padé approximants. Our conclusions are summarized in Section V.

II. BETA FUNCTION AND PROPERTIES OF COEFFICIENTS UP TO FIVE LOOPS

The beta function $\beta_\lambda = d\lambda/dt$ has the series expansion $\beta_\lambda = \lambda \sum_{\ell=1}^{\infty} b_\ell \, a^\ell$, where

$$a \equiv \frac{\lambda}{16\pi^2} \ . \tag{2.1}$$

The corresponding beta function $\beta_a = da/dt$ has the series expansion

$$\beta_a = a \sum_{\ell=1}^{\infty} b_\ell \, a^\ell \ . \tag{2.2}$$

The n-loop ($n\ell$) beta function, denoted $\beta_{a,n\ell}$, is given by Eq. (2.2) with the upper limit of the loop summation index $\ell = n$ instead of $\ell = \infty$. Thus, $\beta_{a,n\ell}$ is a polynomial in a of degree $n+1$ having an overall factor of a^2. It is convenient to extract this factor and define a reduced beta function

$$\beta_{a,r} = \frac{\beta_a}{b_1 a^2} = 1 + \frac{1}{b_1} \sum_{\ell=2}^{\infty} b_\ell a^{\ell-1} \ . \tag{2.3}$$

We denote $\beta_{a,r,n\ell}$ as the n-loop truncation of this series. Thus, $\beta_{a,r,n\ell}$ is a polynomial of degree $n-1$ in a. For a table of coefficients to be presented below it will also be convenient to define the rescaled coefficients

$$\bar{b}_\ell \equiv \frac{b_\ell}{(4\pi)^\ell} \ . \tag{2.4}$$

The one-loop and two-loop coefficients in the beta function, b_1 and b_2, are independent of the scheme used for regularization and renormalization, while the coefficients at loop order three and higher, b_ℓ for $\ell \ge 3$, are scheme-dependent. The first two coefficients are [9]

$$b_1 = 3 \tag{2.5}$$

and

$$b_2 = -\frac{17}{3} \ . \tag{2.6}$$

As noted above, since $b_1 > 0$, it follows that for small a, β_a is positive, so that as $\mu \to 0$, $a(\mu) \to 0$, i.e., the theory is IR-free. As μ increases, $a(\mu)$ increases. The question to be investigated here is whether this increase in $a(\mu)$ stops, i.e., whether $a(\mu)$ approaches a finite value $a_{UV} = \lambda_{UV}/(16\pi^2)$ as $\mu \to \infty$, with $\beta(a) \to 0$ as $a \nearrow a_{UV}$, or whether, instead, β_a has no (reliably perturbatively calculable) UV zero, so that $a(\mu)$ continues to grow with μ until it passes out of the region in which β_a can be reliably calculated perturbatively. Here we extend our earlier five-loop analysis of this question in [3] to the six-loop level.

The n-loop coefficients b_n have been calculated for the general $O(N)$ $\lambda|\vec{\phi}|^4$ theory up to $n = 5$ loop order in the $\overline{\rm MS}$ scheme [8]. For our present purposes, we only need the values of these b_n for the theory with $N = 1$. These coefficients at the three-, four-, and five-loop level, as calculated in the $\overline{\rm MS}$ scheme, are [9–12]

$$b_3 = \frac{145}{8} + 12\zeta_3 \ , \tag{2.7}$$

$$b_4 = -\frac{3499}{48} - 78\zeta_3 + 18\zeta_4 - 120\zeta_5 \ , \tag{2.8}$$

and

$$b_5 = \frac{764621}{2304} + \frac{7965}{16}\zeta_3 - \frac{1189}{8}\zeta_4 + 987\zeta_5 + 45\zeta_3^2 - \frac{675}{2}\zeta_6 + 1323\zeta_7 \ , \tag{2.9}$$

where

$$\zeta_s = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2.10}$$

is the Riemann zeta function. If $s = 2r$ is even, then ζ_s can be expressed as a rational number times π^{2r}, namely $\zeta_{2r} = (-1)^{r+1} B_{2r} (2\pi)^{2r}/[2(2r)!]$, where B_n are the Bernoulli numbers; however, we leave these ζ_{2r} in their generic form here and below. Recently, the six-loop coefficient has been calculated (in the $\overline{\rm MS}$ scheme) [13] and is

$$b_6 = -\frac{18841427}{11520} - \frac{779603}{240}\zeta_3 + \frac{16989}{16}\zeta_4 - \frac{63723}{10}\zeta_5 - \frac{8678}{5}\zeta_3^2 + \frac{6691}{2}\zeta_6 + 162\zeta_3\zeta_4 - \frac{63627}{5}\zeta_7 - 4704\zeta_3\zeta_5 + \frac{264543}{25}\zeta_8 - \frac{51984}{25}\zeta_{3,5} - 768\zeta_3^3 - \frac{46112}{3}\zeta_9 \ , \tag{2.11}$$

where

$$\zeta_{s_1,s_2} = \sum_{1 \le n_1 < n_2} \frac{1}{n_1^{s_1} \, n_2^{s_2}} \tag{2.12}$$

(with n_1 and n_2 positive integers) is the double zeta value

TABLE I: The first and second columns of this table list the loop order n and the numerical values of the n-loop coefficients $\bar{b}_n = b_n/(4\pi)^n$ in the beta function for $1 \le n \le 6$. The coefficients $\bar{b}_n$ with $n \ge 3$ are calculated in the $\overline{\rm MS}$ scheme. The third column lists values of the UV zero $a_{_{UV,n\ell}}$ of the n-loop beta function, $\beta_{a,n\ell}$ for $n = 2, ..., 6$ (with b_n calculated in the $\overline{\rm MS}$ scheme). The dash notation − means that $\beta_{a,n\ell}$ has no physical UV zero.

n	$\bar{b}_n$	$a_{_{UV,n\ell}}$
1	0.2387	−
2	−0.035885	0.5294
3	0.01640	−
4	−0.01089	0.2333
5	0.009090	−
6	−0.008831	0.1604

[14]. In Table I we list the values of the $\bar{b}_n = b_n/(2\pi)^n$ for $1 \le n \le 6$. The values for $1 \le n \le 5$ were used in our previous work, Ref. [3].

III. ZEROS OF THE n-LOOP BETA FUNCTION UP TO LOOP ORDER $n = 6$

Here we investigate a possible UV zero, denoted $a_{_{UV,n\ell}}$, of the n-loop beta function, $\beta_{a,n\ell}$. The double zero of $\beta_{a,n\ell}$ at $a = 0$ is always present (independent of n); this is an infrared zero and hence will not be of interest here. We denote a UV zero of the n-loop beta function (or equivalently, the reduced beta function $\beta_{a,r,n\ell}$) as $a_{_{UV,n\ell}}$, if such a zero exists. As background for our new six-loop study, we first review the results from our earlier five-loop analysis [3] on the question of a possible UV zero in the beta function. The two-loop beta function has a UV zero at $a = a_{_{UV,2\ell}}$, where

$$a_{_{UV,2\ell}} = \frac{9}{17} = 0.5294 \ , \tag{3.1}$$

where here and below, floating-point values are given to the indicated accuracy.

In order to determine whether this two-loop UV zero is a reliable perturbative result, one must calculate higher-loop contributions to the beta function and ascertain if this zero is reproduced in a stable manner in these higher-loop calculations. This program was carried out to the five-loop level in [3] and the answer was negative. Since $\beta_{a,n\ell}$ at loop order $n \ge 3$ is scheme-dependent and hence so are the zeros, it is incumbent upon one to study the effect of a scheme transformation on this answer, and this was done in [3], with the result that the evidence against a UV zero in the beta function was robust under such scheme transformations. We recall the results obtained in the $\overline{\rm MS}$ scheme. At the three-loop level, $\beta_{a,3\ell}$ has no IR zero; its zeros away from the origin consist of the complex-conjugate pair $a = 0.087046 \pm 0.29084i$). At the four-loop level, $\beta_{a,4\ell}$ has three zeros away from the origin, namely $a = -0.056739 \pm 0.21005i$ and $a = a_{_{UV,\ell}} = 0.23332$. As indicated, one of these is physical and may be denoted $a_{_{UV,\ell}}$, as listed in Table I, but its value is more than a factor of two smaller than the two-loop value $a_{_{UV,2\ell}}$. At the five-loop level, $\beta_{a,5\ell}$ does not have any physical zeros away from the origin; instead, its four such zeros consist of the two complex-conjugate pairs $-0.094402 \pm 0.14585i$ and $0.14208 \pm 0.12127i$. The physical zeros of these n-loop beta functions up to loop order $n = 5$ were given in Table II of Ref. [3].

With the recent calculation of the six-loop coefficient, b_6 in [13], we can analyze the zeros of the resultant six-loop beta function. This function is a polynomial of degree 7 in a and has the numerical form

$$\beta_{a,6\ell} \ = \ a^2\Big(3 - \frac{17}{3}a + 32.5497a^2 - 271.6058a^3 + \ 2848.568a^4 - 34776.131a^5\Big) \ . \tag{3.2}$$

The lower-loop functions $\beta_{a,n\ell}$ with $1 \le n \le 5$ are the corresponding truncations of this function with degree $n+1$. Aside from the double IR zero at $a = 0$, this six-loop beta function has the zero $a_{_{UV,6\ell}} = 0.16041$ as well as two complex-conjugate pairs $a = -0.10272 \pm 0.10558i$ and $a = 0.063473 \pm 0.14406i$. We list this six-loop UV zero, together with the lower-loop results, in Table I.

A necessary condition for a perturbative computation of the beta function β_a at a given a to be reliable is that, for this value of a, the fractional difference

$$\left| \frac{\beta_{a,n+1} - \beta_{a,n}}{(1/2)(\beta_{a,n+1} + \beta_{a,n})} \right| \tag{3.3}$$

should tend to decrease as the loop order n increases. A related necessary condition for the reliability of a perturbative calculation of a zero of the beta function is that if one calculates the value of a that yields this zero at two successive loop orders, then (i) if this zero is present at one order, it should also be present at the successive order, and (ii) the magnitude of the fractional difference between successive loop orders, $\Delta_{n,n+1}$, where

$$\Delta_{n,n+k} = \frac{|a_{_{UV,(n+k)\ell}} - a_{_{UV,n\ell}}|}{(1/2)(a_{_{UV,(n+k)\ell}} + a_{_{UV,n\ell}})} \ , \tag{3.4}$$

should be reasonably small and should tend to decrease as the loop order n increases. Specifically, one would expect that $\Delta_{n,n+1}/a_{_{UV,n\ell}}$ and $\Delta_{n,n+1}/a_{_{UV,(n+1)\ell}}$ should be small compared with unity and should tend toward zero with increasing loop order n. Our analysis up to the five-loop level in [3] showed that neither of these two requirements is met for this theory. Indeed, the fractional differences between successive-loop orders, $\Delta_{n,n+1}$, are not usable for $2 \le n \le 4$, since $\beta_{a,n\ell}$ has no UV zero for $n = 3$ and $n = 5$.

Here we extend this analysis to the next higher-loop order, namely $n = 6$ loops. Our six-loop results confirm and extend our previous conclusion in [3]. Although $\beta_{a,n\ell}$ has UV zeros at loop order $n = 2$, $n = 4$, and $n = 6$, they are absent at loop orders $n = 3$ and $n = 5$, so the first condition above is not satisfied. Second, even if one changes the fractional difference test to relate not successive-loop values of $a_{UV,n\ell}$ but values separated by two loop orders, i.e., $\Delta_{n,n+2}$, these fractional differences are substantial:

$$\Delta_{2,4} = 0.776 \tag{3.5}$$

and

$$\Delta_{4,6} = 0.370 \ . \tag{3.6}$$

Furthermore, none of the quantities

$$\frac{\Delta_{2,4}}{a_{UV,2\ell}} = 1.467 \ , \tag{3.7}$$

$$\frac{\Delta_{2,4}}{a_{UV,4\ell}} = 3.328 \ , \tag{3.8}$$

$$\frac{\Delta_{4,6}}{a_{UV,4\ell}} = 1.587 \ , \tag{3.9}$$

and

$$\frac{\Delta_{4,6}}{a_{UV,6\ell}} = 2.309 \tag{3.10}$$

is small compared to unity.

In Fig. 1 we plot the respective n-loop beta functions $\beta_{a,n\ell}$ for $2 \le n \le 6$ loops. This plot shows the intervals in a over which the calculations of $\beta_{a,n\ell}$ to the respective n-loop orders are in mutual agreement. An alternative way to investigate this is to plot the reduced beta function (2.3). We have

$$\beta_{a,r,n\ell} = \frac{\beta_{a,n\ell}}{\beta_{a,1\ell}} \equiv R_n \ . \tag{3.11}$$

We plot R_n in Fig. 2.

As one can see from Fig. 2, the n-loop beta functions $\beta_{a,n\ell}$ with $2 \le n \le 6$ only agree with each other well over the small interval of couplings $0 \le a \lesssim 0.05$; as a increases beyond the upper part of this interval, they deviate from each other. As is shown in Fig. 1, the beta functions $\beta_{a,n\ell}$ with even $n = 2,\ 4,\ 6$ reach maxima and then decrease, crossing the (positive) real axis at respective values $a_{UV,2} = 0.529$, $a_{UV,4} = 0.233$ and $a_{UV,6} = 0.160$ that decrease strongly with increasing n, while the $\beta_{a,n\ell}$ with odd n increase monotonically as a increases from zero. The corresponding behaviors are evident for the ratios R_n shown in Fig. 2. These results extend to the six-loop level our previous five-loop results reported in [3] and continue the same trends observed there. Particularly noteworthy is our present finding that even using very high-order calculations up to six-loop order does not

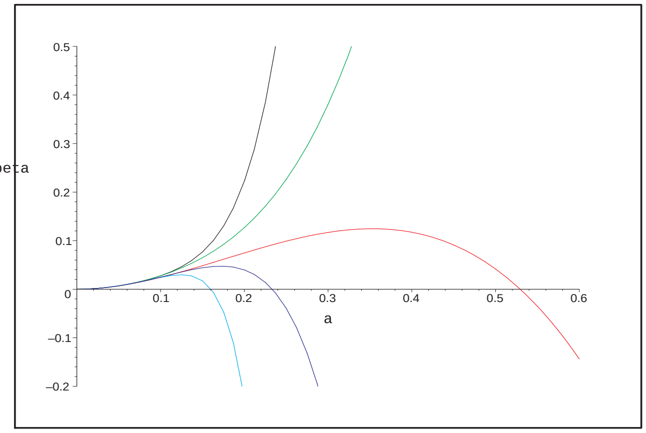

FIG. 1: Plot of the n-loop β function $\beta_{a,n\ell}$ as a function of a for (i) $n = 2$ (red), (ii) $n = 3$ (green), (iii) $n = 4$ (blue), (iv) $n = 5$ (black), and (v) $n = 6$ (cyan) (colors in online version). At $a = 0.16$, going from bottom to top, the curves are for $n = 6$, $n = 4$, $n = 2$, $n = 3$, and $n = 5$.

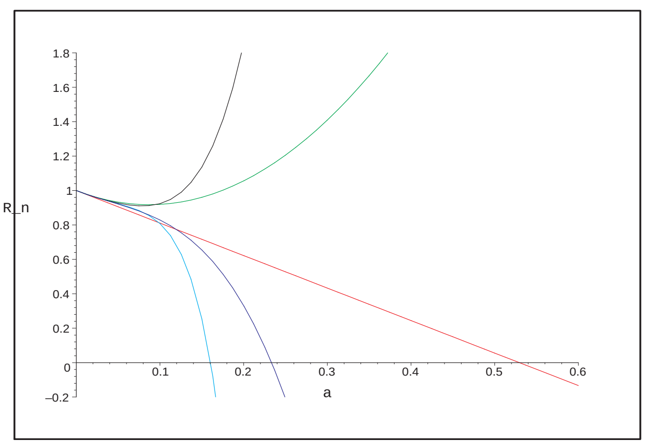

FIG. 2: Plot of the ratio R_n of $\beta_{a,n\ell}$ divided by $\beta_{a,1\ell}$, as a function of a for (i) $n = 2$ (red), (ii) $n = 3$ (green), (iii) $n = 4$ (blue), (iv) $n = 5$ (black), and (v) $n = 6$ (cyan) (colors in online version). At $a = 0.16$, going from bottom to top, the curves are for $n = 6$, $n = 4$, $n = 2$, $n = 3$, and $n = 5$.

significantly increase the range in a in which the beta functions calculated to adjacent-loop orders $(n, n+1)$ agree with each other. With this six-loop analysis, we thus confirm and strengthen our conclusion in [3] that the zero in the two-loop beta function of the $\lambda\phi^4$ theory occurs at too large a value of a for the perturbative calculation to be reliable.

IV. ANALYSIS WITH PADÉ APPROXIMANTS

In this section we analyze the six-loop beta function for the $\lambda\phi^4$ theory using Padé approximants (PAs). Since we are not interested in the double zero in $\beta_{a,n\ell}$ at the origin, it is convenient to utilize the reduced beta function $\beta_{a,r,n\ell}$ for this Padé analysis. The $[p,q]$ Padé approximant to $\beta_{a,r,n\ell}$ is the rational function

$$[p,q]_{\beta_{a,r,n\ell}} = \frac{1+\sum_{j=1}^{p} r_j a^j}{1+\sum_{k=1}^{q} s_k\, a^k} \tag{4.1}$$

with

$$p+q = n-1 \ , \tag{4.2}$$

where the coefficients r_j and s_j are independent of a. As is well-known, the series expansion for β_a is not a Taylor series with a finite radius of convergence, but instead only an asymptotic expansion; hence, in using the Padé approximants, we will only require the absence of any pole on the positive real a axis at a value less than a possible zero. At loop order n, we can calculate the $[p,q]_{\beta_{a,r,n\ell}}$ Padé approximants with $p+q=n-1$. There are thus n Padé approximants to the n-loop reduced beta function $\beta_{a,r,n\ell}$, viz., the set $\{\ [n-k,k-1]_{\beta_{a,r,n\ell}}\ \}$ with $1 \le k \le n$. Because the value of loop order n is obvious for a given Padé approximant $[p,q]_{\beta_{a,r,n\ell}}$ from Eq.(4.2), one may omit the subscript and write this approximant simply as $[p,q]$, and we shall do so below.

There are several necessary requirements for a zero of a $[p,q]$ Padé approximant to be physically relevant. These include the requirement that this zero should occur on the positive real axis in the complex a plane at a value that is not too different from $a_{_{UV,2\ell}}$ and the requirement that this zero of the PA should be closer to the origin $a=0$ than any pole on the real positive a-axis, since otherwise the pole would dominate the IR to UV flow starting at the origin. The second requirement is clearly satisfied if a Padé approximant has a denominator polynomial in which all of the coefficients are positive. As will be evident from the PAs to be displayed, this positivity condition on the coefficients in the denominators is met for all of the PAs except for the [1,3] PA at the five-loop level and the [1,4] PA at the six-loop level, so we only need to check the poles explicitly for these two approximants (and neither has any relevant physical pole).

An analysis of these Padé approximants up to the five-loop level in [3] confirmed the conclusions reached by analysis of the zeros of $\beta_{a,r,n\ell}$ for $2 \le n \le 5$, namely evidence against a stable, reliably calculable UV zero in the beta function. Here we display the actual Padé approximants up to the five-loop level (which were not given explicitly in [3]) for reference, and, furthermore, we analyze the Padé approximants to the reduced six-loop beta function, $\beta_{a,r,6\ell}$. For comparison with our new six-loop results, we recall the values of the zeros of the PAs up to the five-loop level from [3]. A general result that we established up to the five-loop level in [3] was that none of the PAs has any physical pole, i.e. a pole occurring at a real positive value. We extend this result to the six-loop level here. This is clear from an inspection of the coefficients of the denominator polynomials; these are all positive, which immediately proves that the denominators never vanish for any real positive value of a.

At the three-loop level, one can calculate the following set of Padé approximants to the reduced beta function $\beta_{a,r,3\ell}$: $\{[2,0],\ [1,1],\ [0,2]\}$. The $[2,0]$ PA is $\beta_{a,r,3\ell}$ itself, and the $[0,2]$ approximant has no zeros and is therefore not useful for the analysis of a possible UV zero. This leaves the [1,1] PA to be examined. In [3] we gave the zeros and poles of this PA and mentioned it has no physical UV zero. Here we list it for reference:

$$[1,1] = \frac{1+3.85517a}{1+5.74406a} \ . \tag{4.3}$$

This [1,1] PA has an unphysical zero at $a=-0.2594$.

At the four-loop level, one can calculate the following set of Padé approximants to the reduced beta function $\beta_{a,r,4\ell}$: $\{[3,0],\ [2,1],\ [1,2],\ [0,3]\}$. The $[3,0]$ PA is $\beta_{a,r,4\ell}$ itself, and the $[0,3]$ approximant has no zeros and is therefore not useful here. In [3] we gave the zeros and poles of these approximants and noted that none of these was a physical UV zero. Here we present the actual Padé approximants for reference. They are

$$[2,1] = \frac{1+6.45546a-4.91165a^2}{1+8.344345a} \tag{4.4}$$

and

$$[1,2] = \frac{1+7.72950a}{1+9.61839a+7.31817a^2} \ . \tag{4.5}$$

The [2,1] PA has an unphysical zero at $a=-0.1400$ and a UV zero at $a=1.4543$, which is much larger than the four-loop zero of $\beta_{a,4\ell}$ at $a_{_{UV,4\ell}}=0.2333$ and hence is not of physical relevance. The [1,2] PA has an unphysical zero at $a=-0.1294$.

At the five-loop level, one can calculate the following set of Padé approximants to the reduced beta function $\beta_{a,r,5\ell}$: $\{[4,0],\ [3,1],\ [2,2],\ [1,3],\ [0,4]\}$. The $[4,0]$ PA is $\beta_{a,r,5\ell}$ itself, and the $[0,4]$ approximant has no zeros and is therefore not useful for our analysis. In [3] we gave the zeros and poles of these approximants and noted that none of these was a physical UV zero. Here we present the actual Padé approximants. They are

$$[3,1] = \frac{1+8.5989a-8.9605a^2+23.2571a^3}{1+10.4879a} \ , \tag{4.6}$$

$$[2,2] = \frac{1 + 13.3341a + 21.6066a^2}{1 + 15.2230a + 39.51125a^2} , \qquad (4.7)$$

and

$$[1,3] = \frac{1 + 10.5387a}{1 + 12.4276a + 12.6245a^2 - 20.4568a^3} . \qquad (4.8)$$

As noted in [3], none of these PAs has any physical zeros. The [3,1] PA has unphysical zeros at $a = -0.10245$ and $a = 0.2439 \pm 0.6002i$; the [2,2] PA has unphysical zeros at $a = -0.5298$ and $a = -0.08736$; and the [1,3] PA has an unphysical zero at $a = -0.9489$. As a special case of the general discussion above, since the coefficients of all terms in the denominators of the [3,1] and [2,2] PAs are positive, it follows that neither one has any physical pole. As was remarked in [3], the [1,3] PA has unphysical poles at $a = -0.46439$ and $a = -0.08986$ and a third pole at $a = 1.1714$. Since this third pole lies farther from the origin than $a_{_{UV,n\ell}}$ with $n = 2,\ 4$, one may infer that it does not affect the RG flow from the origin in the IR to the UV and hence is not physically relevant.

At the level of $n = 6$ loops, we can calculate the following set of Padé approximants to $\beta_{a,r,6\ell}$ (a polynomial of degree 5 in a): $\{[5,0],\ [4,1],\ [3,2],\ [2,3],\ [1,4],\ [0,5]\}$. The $[5,0]$ PA is $\beta_{a,r,6\ell}$ itself, which we have already analyzed, and the $[0,5]$ approximant has no zeros and hence is not useful for our analysis. This leaves us with the other four PAs in the set above. We calculate the following Padé approximants to the six-loop reduced beta function $\beta_{a,r,6\ell}$:

$$[4,1] = \frac{1 + 10.3193a - 12.2102a^2 + 41.9233a^3 - 155.757a^4}{1 + 12.2083a} , \qquad (4.9)$$

$$[3,2] = \frac{1 + 17.0166a + 45.3789a^2 - 18.0872a^3}{1 + 18.9055a + 70.2394a^2} , \qquad (4.10)$$

$$[2,3] = \frac{1 + 17.8537a + 56.5411a^2}{1 + 19.7426a + 82.9828a^2 + 33.0754a^3} , \qquad (4.11)$$

and

$$[1,4] = \frac{1 + 12.48863a}{1 + 14.3775a + 16.3076a^2 - 34.6560a^3 + 109.7524a^4} . \qquad (4.12)$$

We find that none of these Padé approximants has a physical UV zero at a value near to $a_{_{UV,6\ell}} = 0.1604$. The [4,1] PA has only one physical UV zero, at $a = 0.4675$, about 3 times larger than $a_{_{UV,6\ell}}$, as well as unphysical zeros at $a = -0.085055$ and $a = -0.05663 \pm 0.3978i$. The [3,2] PA has a UV zero at the value $a = 2.846$, which is too large to be trustworthy and, moreover, is much larger than $a_{_{UV,6\ell}}$. It also has two unphysical zeros at $a = -0.2637$ and $a = -0.07366$. The [2,3] PA has unphysical zeros at $a = -0.2430$ and $a = -0.07279$. Finally, the [1,4] PA has a zero at the unphysical value $a = -0.08007$. Since the [4,1], [3,2], and [2,3] PAs have denominators with completely positive coefficients, it is clear that they do not have any poles on the positive real a axis. The [1,4] PA also has only unphysical poles, which occur at $a = -0.301245$, $a = -0.07784$, and $a = 0.3474 \pm 0.5175i$.

Thus, our analysis with Padé approximants of the six-loop beta function yields the same conclusion as our analysis of the beta function itself, namely that there is no evidence for a stable, reliably perturbatively calculable UV zero up to this six-loop level.

V. CONCLUSIONS

In this paper we have investigated whether the real scalar field theory with a $\lambda\phi^4$ interaction (in four spacetime dimensions) exhibits evidence of an ultraviolet zero in the beta function. Using the recently calculated six-loop term b_6 from [13], our present study extends our previous five-loop study in [3] to the six-loop level. From a study of the six-loop beta function itself together with an analysis of Padé approximants, we conclude that this beta function does not exhibit evidence for a stable, reliably perturbatively calculable UV zero to the highest loop order, namely six loops, to which it has been computed.

Acknowledgments

This research was partly supported by the U.S. National Science Foundation Grant NSF-PHY-16-1620628.

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[14] More generally, the multiple zeta value $\zeta_{s_1,\ldots,s_p}$ is defined as $\zeta_{s_1,\ldots,s_p} = \sum_{1\leq n_1<n_2<\cdots n_p}(n_1^{s_1}\cdots n_p^{s_p})^{-1}$; see, e.g., J. Blumlein, D. J. Broadhurst, and J. A. M. Vermaseren, Comput. Phys. Commun. **181**, 582 (2010) and references therein.

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