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Gauging Unbroken Symmetries in **F**-theory

Chia-Yi Ju* and Warren Siegel†

C. N. Yang Institute for Theoretical Physics
State University of New York, Stony Brook, NY 11794-3840

Abstract

F-theory attempts to include all **U**-dualities manifestly. Unlike its **T**-dual manifest partner, which is based on string current algebra, **F**-theory is based on higher dimensional brane current algebra. Like the **T**-dual manifest theory, which has $O(D-1,1)^2$ unbroken symmetry, the **F**-theory vacuum also enjoys certain symmetries ("H"). One of its important and exotic properties is that worldvolume indices are also spacetime indices. This makes the global brane current algebra incompatible with H symmetry currents. The solution is to introduce worldvolume covariant derivatives, which depend on the H coordinates even in a "flat" background. We will also give as an explicit example the 5-brane case.

1 Introduction

It is well known that many string theories ("**S**-theories") are related by different dualities [1–3]. On the one hand, this led to the idea of **M**-theory [4,5] (a theory that is one dimension higher than **S**-theory) that links all different types of **S**-theories to each other by dualities. It's also known that **M**-theory (or **S**-theory) would further imply **U**-duality [3, 6, 7], conjectured to be a discrete subgroup of $E_{n(n)}$ (n is the dimension of **M**-theory), which is the most general duality (including **S**-duality and **T**-duality) of **S**-theory. However, **M**-theory cannot give us every type of **S**-theory

*cju@insti.physics.sunysb.edu
†siegel@insti.physics.sunysb.edu

without using duality [8], i.e. the theory does not manifest all dualities. On the other hand, **T**-theory [9–11] (sometimes called "Double Field Theory" [12]) has **T**-duality manifest by including the "**T**-dual" coordinates (coordinates are doubled).

F-theory [13–18] is meant to take advantage of both **M**-theory and **T**-theory – having **U**-dualities manifest by including all the **U**-dual coordinates. To accomplish that, we have to use branes rather than strings. The low energy limit of that is a theory with $G = E_{n(n)}$ symmetry. However, some of the symmetries are also symmetries of the vacuum, which form the group H. Therefore, we can treat H as a gauge group. Since **F**-theory includes both **M**-theory and **T**-theory, H should at least include both the symmetries of the vacuum in **M**-theory and **T**-theory.

Although the currents of the theory have been found [17], it is usually easier to work with by having H symmetry manifest instead of gauging them to zero directly. We therefore use the group element g to make H symmetry local even in "flat" spacetime. The group coordinates are then included with the other "spacetime" coordinates as worldvolume fields. However, we found that the global current algebra is not compatible with H symmetry currents. This requires that the derivatives of δ-functions that appear in the Schwinger terms be covariantized with g. These derivatives were found previously to need covariantization in nontrivial backgrounds, but when gauging H even "flat" spacetime has a vielbein that is not constant.

In this paper, we start by reviewing **F**-theory using worldvolume approach. Then we show the global brane current algebra does not go along with H symmetry currents. Then we modify the theory and give a very general construction for arbitrary finite dimensional current algebras and check its consistency with Jacobi identities. We will give an explicit example for the 5-brane case, where the H group is $Spin(3, 2)$.

2 Brief review of worldvolume approach to F-theory

The idea of **F**-theory is that it should be able to include all types of **S**-theories in one single theory. Therefore, the **F**-theory should also contain **T**-theory and **M**-theory for the reasons stated in the Introduction. Therefore, **M**-theory, **T**-theory, and **S**-theories should be a limit of **F**-theory (in fact, they are different solutions to the constraints that will be mentioned later in the section).

Different from most literature that try to approach **F**-theory using supergravity limit of string theory, in this paper, we approach **F**-theory using string/brane theories. M-theory tells us that the most general duality in string theory is U-duality, which is conjectured to be the discrete subgroup of $E_{11(11)}$. Since there are some technical issue that we couldn't attack the **F**-theory with $E_{11(11)}$ symmetry directly, we start with lower dimensional **S**-theories as toy models and find their general properties. The corresponding **F**-theory should have $E_{d+1(d+1)}$ symmetry, where d is the dimension of **S**-theory.

Although the theory has $E_{11(11)}$ symmetry, the vacuum does not have to recognize the symmetry. For example, the linear part of low energy **S**-theory has $GL(d)$ symmetry, but the vacuum is only invariant under $SO(d-1,1)$ symmetry. Also, the vacuum of **M**-theory is only invariant under $SO(d,1)$ symmetry. Last example is **T**-theory, the theory has $SO(d,d)$ symmetry, but its vacuum is only invariant under $SO(d-1,1)^2$ symmetry. We call the symmetry obeyed by the vacuum "unbroken symmetry". To find the unbroken symmetry of **F**-theory, we use the fact that **F**-theory contains both **M**-theory and **T**-theory, hence, the unbroken symmetry of **F**-theory should contain the unbroken symmetry of **M**-theory and **T**-theory as well. Therefore, the unbroken symmetry group, H, is the group that can double cover $SO(d,1)$ and $SO(d-1,1)^2$.

For example, for $d=3$ **S**-theory, the corresponding **F**-theory should have $E_{4(4)}$ symmetry and the unbroken symmetry should be $Sp(4)$ since $Sp(4) = Spin(3,2)$ which double covers $SO(3,2)$ and we know that $SO(3,2) \supset SO(2,2) = SO(2,1)^2$ and also $SO(3,2) \supset SO(3,1)$.

The followings are some reviews on **T**-theory and **F**-theory that will be useful for the later sections.

2.1 T-theory review

To understand worldvolume/brane approach to **F**-theory, it is useful to review worldsheet/string approach to **T**-theory. We review **T**-theory using worldsheet approach in this subsection. Later we'll generalize the techniques used in **T**-theory to **F**-theory.

It is well known that **T**-theory is a theory with $SO(d,d)$ symmetry and requires twice coordinates than string theory. It includes the coordinates in string theory and their **T**-dual coordinates. We start with bosonic string algebra for both canonical momenta and their dual momenta in flat background. Later we'll treat them on equal footing and form "doubled coordinates".

We know that string coordinate can be decomposed into left-moving and right-moving modes:

$$x(\tau,\sigma) = \frac{1}{2}\left(x_{_L}(\tau+\sigma) + x_{_R}(\tau-\sigma)\right).$$

T-duality can be defined as flipping the sign of the right moving-mode, i.e.

$$\begin{aligned} x_{_L} &\to x_{_L}, \quad x_{_R} \to -x_{_R} \\ \Rightarrow \tilde{x}(\tau,\sigma) &\equiv \frac{1}{2}\left(x_{_L}(\tau+\sigma) - x_{_R}(\tau-\sigma)\right), \end{aligned}$$

where $\tilde{x}$ is the dual coordinate of x.

Let p and $\tilde{p}$ be the canonical momentum of x and $\tilde{x}$ respectively. Then the canonical commutation relations are

$$\begin{cases} [p_m(1), x_n(2)] = -i\mathring{\eta}_{mn}\delta(1-2) \\ [\tilde{p}_m(1), \tilde{x}_n(2)] = -i\mathring{\eta}_{mn}\delta(1-2) \end{cases},$$

where $\mathring{\eta}_{mn}$ is the usual Minkowski metric. Like x and $\tilde{x}$, p and $\tilde{p}$ can also be decomposed into left- and right-moving modes as well. The left- and right-moving modes can be written in terms of x' and its canonical momentum p as

$$\begin{cases} p_{_L} = p + x' = x'_{_L} \\ p_{_R} = p - x' = -x'_{_R} \end{cases},$$

where $x' \equiv \dfrac{\partial}{\partial\sigma}x$. By direct calculation, we found the following commutation relations:

$$\begin{cases} \left[p_{_L m}(1), p_{_L n}(2)\right] = -i2\mathring{\eta}_{mn}\delta'(1-2) \\ \left[p_{_R m}(1), p_{_R n}(2)\right] = i2\mathring{\eta}_{mn}\delta'(1-2) \\ \left[p_{_L m}(1), p_{_R n}(2)\right] = 0 \end{cases}.$$

As stated before, we put coordinates and dual coordinates into doubled coordinate multiplet, $X \equiv (x, \tilde{x})$. We can do the same with canonical momentum:

$$P_M \equiv (p_m, \tilde{p}^m) = \left(\frac{1}{2}\left(p_{_L m} + p_{_R m}\right), \frac{1}{2}\left(p_{_L}^m - p_{_R}^m\right)\right).$$

The commutation relation between doubled momentum is

$$[P_M(1), P_N(2)] = i2\eta_{MN}\delta'(1-2),$$

where

$$\eta_{MN} = \begin{pmatrix} 0 & \delta_m^n \\ \delta_n^m & 0 \end{pmatrix}.$$

It is obvious that η is an $SO(d,d)$ metric. We can, therefore, conclude that the metric of the theory can be read off from the commutation relations between the momenta. A more general case has been worked out in reference [19].

We have doubled the number of coordinates (and the momenta or symmetry currents in general), however, our goal is not to write down a new theory but to find a theory with all the symmetry in **S**-theory. Therefore, we need a constraint to reduce the number coordinates in **T**-theory by a half. This constraint has to be covariant under $SO(d,d)$ transformation and should come from **S**-theory. One such constraint can be found in **S**-theory, namely, Virasoro constraint, i.e. all physical states are killed by

$$T_{\tau\sigma} = p_m \tilde{p}^m = \frac{1}{2} P_M \eta^{MN} P_N.$$

It is worth noting that the particle limit (zero mode and integrating over the worldsheet coordinates) of the above constraint is equivalent to $\frac{1}{2}\partial_M \eta^{MN} \partial_N$ in coordinate bases. We have found the weak/strong constraint in Double Field Theory (particle limit of **T**-theory) and is known to kill half of the coordinates.

2.2 F-theory review

We begin this subsection with an example – bosonic 5-brane current algebra [16]. The corresponding **F**-theory is a 10 dimensional theory with $E_{4(4)} = SO(3,2)$ symmetry, and should reduce to 3 dimensional **S**-theory after solving all the constraints (Other different dimensional **F**-theories examples can be found in [17, 18]). The 5-brane action (after choosing conformal gauge) is

$$S = \frac{1}{12} \int d^6\sigma F_{MNO} F^{MNO}, \tag{2.1}$$

where $M, N, O = -2 \cdots 3$ and $F_{MNO} = \frac{1}{2}\partial_{[M} X_{NO]}$ (the reason why the indices runs from -2 to 3 rather than 0 to 5 because the action is $SO(3,3)$ invariant rather than $SO(5,1)$ invariant). X_{NO} is both a spacetime coordinate and a worldvolume gauge field. However, we know that the gauge field that carries worldvolume time index is nondynamical, it is a Lagrangian multiplier. To single them out, we rewrite the action in terms of Hamiltonian:

$$S = -\int d\tau d^5\sigma \frac{1}{2} P_{mn} \partial_\tau X^{mn} + \int d\tau H,$$

where

$$H = \int d^5\sigma \left(\frac{1}{4} P_{mn} P^{mn} + \frac{1}{12} F_{mno} F^{mno} + X^{\tau m} \partial^n P_{mn} \right),$$

$m, n, o = -1 \cdots 3$ and τ is the "-2" coordinate. The last term is the constraint term and the $\partial^n P_{mn}$ is the corresponding constraint, called U-constraint. Since the physical fields are X_{mn} where m, n runs over 5 indices, there are 10 independent coordinates. Moreover, spacetime coordinate, X_{mn}, carries worldvolume indices when it should be spacetime index. This is a general property of **F**-theory – there is no difference between spacetime indices and worldvolume indices. In other word, spacetime coordinate transforms the same way as worldvolume coordinates and vise versa.

Using Hamiltonian formalism, we found U-constraint and reduced some nonphysical components. However, we know that in 6 dimensions a 3-form field strength is reducible. We, therefore, should work with (anti)self-dual field strength rather than full field strength. We can define self-dual momentum as

$$\mathring{\triangleright}_{mn} \equiv P_{mn} + \frac{1}{2} \epsilon_{mnopq} \partial^q X^{op}.$$

Using canonical commutation relation

$$[P_{mn}(1), X^{op}(2)] = -i \delta^{op}_{mn} \delta(1-2),$$

we get

$$\left[\mathring{\triangleright}_{mn}(1), \mathring{\triangleright}_{op}(2) \right] = 2i \epsilon_{mnopq} \partial^q \delta(1-2).$$

The ϵ sits in the same place as η in **T**-theory, which can be treated as the "metric" of the corresponding **F**-theory. We have seen that two antisymmetric worldvolume indices count as one spacetime index, therefore, the metric on the right hand side has one worldvolume index more than usual metric which contracts with the partial derivative. This is also a general property of **F**-theory – the metric in **F**-theory contains two symmetric spacetime indices and one additional worldvolume index.

As in **T**-theory, there are too many coordinates than **S**-theory. To kill the extra coordinates, we have to impose more constraints. It can be seen that U-constraints doesn't give us a 3 dimensional **S**-theory, however, it gives us a 6 dimensional **T**-theory. To find other constraints of the theory, we go back to the energy-momentum tensor give by action (2.1). We found that some components of the energy-momentum tensor can be written as

$$S^q \equiv \overset{\circ}{\triangleright}_{mn}\epsilon^{mnopq}\overset{\circ}{\triangleright}_{op},$$

which is covariant under $SO(3,2)$ transformation. And, again, ϵ sits in the same place as η in **T**-theory. S^q can serve as constraints by applying the concept of Virasoro constraint – all the physical fields are killed by S^q's. Solving the Virasoro constraints does not give us a 3 dimensional **S**-theory, instead, it gives us a 4 dimensional **M**-theory. However, together with U-constraint, it gives us a proper 3 dimensional **S**-theory (for higher than 3 dimension **S**-theory, more constraints will be needed).

Many properties in the above example holds in every known dimensions, such as

1. $\left[\overset{\circ}{\triangleright}_M(1), \overset{\circ}{\triangleright}_N(2)\right] = 2i\eta_{MNr}\partial^r\delta(1-2)$, where M, N are spacetime indices and r is worldvolume index.
2. $S^r \equiv \overset{\circ}{\triangleright}_M\eta^{MNr}\overset{\circ}{\triangleright}_N$ acting on any fields gives zero.

We can even generalize this to include supersymmetry and other symmetry currents [17]. For example, we can define supersymmetry currents $\overset{\circ}{\triangleright}_\alpha$'s such that

$$\left\{\overset{\circ}{\triangleright}_{\alpha_1}(1), \overset{\circ}{\triangleright}_{\alpha_2}(2)\right\} = i\left(\gamma^M\right)_{\alpha_1\alpha_2}\overset{\circ}{\triangleright}_M\delta(1-2).$$

We can right down the most general commutation relation between currents

$$\left[\overset{\circ}{\triangleright}_M(1), \overset{\circ}{\triangleright}_N(2)\right\} = if_{MN}{}^O\overset{\circ}{\triangleright}_O\delta(1-2) + 2i\eta_{MNr}\partial_1^r\delta(1-2).$$

The Jacobi identity not only gives us the usual $f_{[MN|}{}^O f_{O|P)}{}^Q = 0$ but also a "new" identity

$$f_{M(N|}{}^O\eta_{O|P]r} = 0.$$

For the rest of the paper, the notations follows the rules in Appendix A.

3 General Construction

3.1 Problem With Naive Approach

We start this subsection with an observation, and then work the way to the general case, showing why the naive current algebra is not compatible with global H symmetry.

As described in [16], we know that in general the worldvolume current $\overset{\circ}{\triangleright}_M$ obeys the following algebra:

$$\left[\overset{\circ}{\triangleright}_N(1), \overset{\circ}{\triangleright}_N(2)\right] = i f_{MN}{}^O \overset{\circ}{\triangleright}_O \delta(1-2) + 2i\eta_{MNr}\partial_1^r \delta(1-2).$$

We could naively introduce additional H-group worldvolume currents $\overset{\circ}{\triangleright}_S$ and force

$$\left[\overset{\circ}{\triangleright}_S(1), \overset{\circ}{\triangleright}_M(2)\right] = i f_{SM}{}^{M'} \overset{\circ}{\triangleright}_{M'} \delta(1-2).$$

Then a symmetric part of Jacobi identities gives

$$f_{SM}{}^{M'}\eta_{M'Nr} + f_{SN}{}^{N'}\eta_{MN'r} = 0. \tag{3.1}$$

However, this doesn't work because spacetime indices in F-theory are also worldvolume indices. If we want spacetime indices to transform under H group, then worldvolume should transform as well. Since η is a "constant" under group G-transformations, it should also be invariant under H-transformation. And we're led to the following identity:

$$f_{SM}{}^{M'}\eta_{M'Nr} + f_{SN}{}^{N'}\eta_{MN'r} + f_{Sr}{}^{q}\eta_{MNq} = 0,$$

which contradicts equation (3.1).

To find a solution, it is useful to go back to the general construction of symmetry generators of H group. The generalized symmetry generator method from particle to brane is listed in the table below:

Particle	Brane
$(dg)g^{-1} = dx^i (e^{-1})_i{}^s G_s$	$(\delta g)g^{-1} = \delta\alpha^I (e^{-1})_I{}^S G_S$
$\triangleright_s = i e_s{}^i \frac{\partial}{\partial x^i}$	$\overset{\circ}{\triangleright}_S = i e_S{}^I \frac{\delta}{\delta \alpha^I}$
	$\overset{\circ}{\triangleright}{}^{Sr} = \left(\partial^r \alpha^I\right)\left(e^{-1}\right)_I{}^S$

Here G is the symmetry generator of the group. The last brane current is the "dual" current of $\triangleright_S$, which does not have a particle analog. The relations between structure constants and generalized metric are listed as follows:

$$\begin{aligned}
\left[\mathring{\triangleright}_{S_1}(1), \mathring{\triangleright}_{S_2}(2)\right] &= -e_{[S_1|}{}^I \left(\frac{\delta}{\delta\alpha^I} e_{|S_2)}{}^J\right) (e^{-1})_J{}^O e_O{}^L \frac{\delta}{\delta\alpha^L}\delta(1-2) \\
&= ie_{[S_1|}{}^I \left(\frac{\delta}{\delta\alpha^I} e_{|S_2)}{}^J\right) (e^{-1})_J{}^O \mathring{\triangleright}_O \delta(1-2) \\
&= if_{S_1 S_2}{}^O \mathring{\triangleright}_O \delta(1-2), \\
\left[\mathring{\triangleright}_{S_1}(1), \mathring{\triangleright}^{S_2 r}(2)\right] &= ie_P{}^I \left(\frac{\delta}{\delta\alpha^I} e_{S_1}{}^J\right) (e^{-1})_J{}^O e_O{}^{S_2} \left(\partial^r \alpha^L\right) (e^{-1})_L{}^P \delta(1-2) \\
&+ i\delta_{S_1}^{S_2} \partial_2^r \delta(1-2) \\
&= if_{S' S_1}{}^{S_2} \mathring{\triangleright}^{S' r} \delta(1-2) + i\eta_{S_1}{}^{S_2 r}{}_q \partial_2^q \delta(1-2), \\
\left[\mathring{\triangleright}^{S_1 r}(1), \mathring{\triangleright}^{S_2 q}(2)\right] &= 0.
\end{aligned}$$

It's useful to inspect the simplest case with metric only (we neglect f term):

$$\left[\mathring{\triangleright}_{S_1}(1), \mathring{\triangleright}^{S_2 r}(2)\right] \sim i\eta_{S_1}{}^{S_2 r}{}_q \partial_2^q \delta(1-2).$$

In **F**-theory, again, worldvolume indices are also spacetime indices. They both have to transform the same way under H group. However, by construction, ∂^r is not a function of α, therefore, both r and q don't transform under $\mathring{\triangleright}_S$. We are led to an impasse.

For the rest of the paper we will denote $\mathring{\triangleright}_\Sigma$ instead of $\triangleright^{Sr}$ for simplicity.

3.2 Solution

To include the group H in the theory, we introduce a set of H group coordinates as worldvolume fields $\alpha^I(\sigma)$, and the corresponding group elements $g(\alpha(\sigma))) \in H$, and their inverses. By definition, they obey the following commutation relation:

$$\begin{aligned}
\left[\triangleright_S(1), g_A{}^M(2)\right] &= if_{SA}{}^B g_B{}^M \delta(1-2), \\
\left[\triangleright_S(1), (g^{-1})_M{}^A(2)\right] &= i(g^{-1})_M{}^B f_{BS}{}^A \delta(1-2),
\end{aligned}$$

where $\triangleright_S$ is the symmetry generator of group H. We also define new sets of currents by multiplying the old ones with $g's$, i.e.

$$\triangleright_A \equiv g_A{}^M \mathring{\triangleright}_M,$$

so that all the indices transform under H-group as well.

We should point out that since we have introduced a current $\triangleright_S$, we should also introduce its dual current $\overset{\circ}{\triangleright}_\Sigma$:

$$\overset{\circ}{\triangleright}_\Sigma \equiv \overset{\circ}{\triangleright}^{Sr} = \left(\partial^r \alpha^I\right)\left(e^{-1}\right)_I{}^S.$$

It can be shown that the original metrics are unaffected if the worldvolume derivative is also multiplied by g:

$$\begin{aligned}
&\left[\overset{\circ}{\triangleright}_M(1), \overset{\circ}{\triangleright}_N(2)\right\} = i f_{MN}{}^O \overset{\circ}{\triangleright}_O \delta(1-2) + 2i\eta_{MNr}\partial_1^r \delta(1-2) && (3.2)\\
\Rightarrow &\left[\triangleright_A(1), \triangleright_B(2)\right\} = \left[g_A{}^M \overset{\circ}{\triangleright}_M(1), g_B{}^N \overset{\circ}{\triangleright}_N(2)\right\}\\
&= i g_A{}^M g_B{}^N f_{MN}{}^O \overset{\circ}{\triangleright}_O \delta(1-2) + 2i g_A{}^M(1) g_B{}^N(2)\eta_{MNr}\partial_1^r \delta(1-2)\\
&= i g_A{}^M g_B{}^N f_{MN}{}^O (g^{-1})_O{}^C \triangleright_C \delta(1-2) + 2i g_A{}^M(1) g_B{}^N(2)\eta_{MNr}\partial_1^r \delta(1-2)\\
&= i g_A{}^M g_B{}^N f_{MN}{}^O (g^{-1})_O{}^C \triangleright_C \delta(1-2) + i g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-2)\\
&\quad + i\left(g_A{}^M g_B{}^N \eta_{MNr}\right)((1)+(2))\partial_1^r \delta(1-2)\\
&= i g_A{}^M g_B{}^N f_{MN}{}^O (g^{-1})_O{}^C \triangleright_C \delta(1-2) + i g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-2)\\
&\quad + i\left(g_A{}^M g_B{}^N g_a{}^p (g^{-1})_r{}^a \eta_{MNp}\right)((1)+(2))\partial_1^r \delta(1-2)\\
&= i f_{AB}{}^C \triangleright_C \delta(1-2) + i g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-2)\\
&\quad + i\eta_{ABa}(g^{-1})_r{}^a((1)+(2))\partial_1^r \delta(1-2)\\
&= i f_{AB}{}^C \triangleright_C \delta(1-2) + i g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-2)\\
&\quad + i\eta_{ABa}\triangleright^a((1)-(2))\delta(1-2),
\end{aligned}$$

where $f_{AB}{}^C = g_A{}^M g_B{}^N f_{MN}{}^O (g^{-1})_O{}^C$, $\eta_{ABa} = g_A{}^M g_B{}^N g_a{}^r \eta_{MNr}$, and

$$\triangleright^a = (g^{-1})_r{}^a \partial^r$$

Since both f's and η's are invariant under H-group, $f_{AB}{}^C$ and η_{ABa} are numerically equal to $f_{MN}{}^O$ and η_{MNr} respectively. The term $g_{[A|}{}^M \partial^r g_{|B)}{}^N \eta_{MNr}\delta(1-2)$ is in fact a torsion term:

$$\begin{aligned}
g_{[A|}{}^{M}\partial^r g_{|B)}{}^{N}\eta_{MNr} &= g_{[A|}{}^{M}\partial^r g_{|B)}{}^{O}(g^{-1})_O{}^C g_C{}^N \eta_{MNr} \\
&= g_{[A|}{}^{M}\left(\partial^r\alpha^I\right)\left(\frac{\delta}{\delta\alpha^I} g_{|B)}{}^{O}\right)(g^{-1})_O{}^C g_C{}^N \eta_{MNr} \\
&= g_{[A|}{}^{M}\left(\partial^r\alpha^I\right)(e^{-1})_I{}^S \left(G_S\right)_{|B)}{}^{C} g_C{}^N \eta_{MNr} \\
&= -\left(\triangleright^a\alpha^I\right)(e^{-1})_I{}^S f_{S[A|}{}^C \eta_{C|B)a} \\
&= \triangleright^{Sa} f_{S[A|}{}^C \eta_{C|B)a} \\
&= \triangleright_\Sigma f_{AB}{}^\Sigma,
\end{aligned}$$

where

$$\triangleright_\Sigma = \triangleright^{Sa} \equiv (g^{-1})_r{}^a \mathring{\triangleright}^{Sr}.$$

The third equality comes from $(\delta g)g^{-1} = (\delta\alpha)e^{-1}G$. For the fourth equality we use the fact that $\left(G_a\right)_b{}^c = f_{ab}{}^c$ in adjoint representation. The $\left(\triangleright^a\alpha^I\right)(e^{-1})_I{}^S$ in the fourth line is the covariant "dual" of $\triangleright_S$ ($\triangleright_\Sigma = \triangleright^{\widetilde{S}a}$). Using that fact that $\eta_{S\Sigma a} = \eta_S{}^{\widetilde{Sb}}{}_a = -\frac{1}{2}\delta_S^{\widetilde{S}}\delta_a^b$ (the $-\frac{1}{2}$ comes from the definition of equation (3.2)), we get

$$f_{AB}{}^\Sigma \eta_{\Sigma Sa} = \frac{1}{2} f_{S[A|}{}^C \eta_{C|B)a}. \tag{3.3}$$

We close this section by calculating the commutation relation between $\rhd_S$ and $\rhd_\Sigma$:

$$\begin{aligned}
&\Big[\rhd_S(1), \rhd_\Sigma(2)\Big] = \Big[\rhd_{S_1}(1), \rhd^{S_2 a}(2)\Big] \\
&= \Big[ie_{S_1}{}^I \frac{\delta}{\delta\alpha^I}(1), (g^{-1})_r{}^a \left(\partial^r\alpha^J\right) e_J{}^{S_2}(2)\Big] \\
&= if_{bS_1}{}^a (g^{-1})_r{}^b \left(\partial^r\alpha^I\right) e_I{}^{S_2}\delta(1-2) + i\left(e_{S_1}{}^I \frac{\partial}{\partial\alpha^I}(e^{-1})_J{}^{S_2}\right)(g^{-1})_r{}^a\left(\partial^r\alpha^J\right)\delta(1-2) \\
&\quad + ie_{S_1}{}^I(1)(g^{-1})_r{}^a(2)(e^{-1})_I{}^{S_2}(2)\partial_2^r\delta(1-2) \\
&= if_{bS_1}{}^a (g^{-1})_r{}^b \left(\partial^r\alpha^I\right) e_I{}^{S_2}\delta(1-2) + i\left(e_{S_1}{}^I \frac{\partial}{\partial\alpha^I}(e^{-1})_J{}^{S_2}\right)(g^{-1})_r{}^a\left(\partial^r\alpha^J\right)\delta(1-2) \\
&\quad + i\left(\partial^r e_{S_1}{}^I\right)(g^{-1})_r{}^a(e^{-1})_I{}^{S_2}\delta(1-2) + i\left(e_{S_1}{}^I(g^{-1})_r{}^a(e^{-1})_I{}^{S_2}\right)(2)\partial_2^r\delta(1-2) \\
&= if_{bS_1}{}^a (g^{-1})_r{}^b \left(\partial^r\alpha^I\right) e_I{}^{S_2}\delta(1-2) \\
&\quad - ie_{S_1}{}^I(e^{-1})_J{}^{S_3}\left(\frac{\partial}{\partial\alpha^I}e_{S_3}{}^K\right)(e^{-1})_K{}^{S_2}(g^{-1})_r{}^a\left(\partial^r\alpha^J\right)\delta(1-2) \\
&\quad + ie_{S_3}{}^I(e^{-1})_J{}^{S_3}\left(\frac{\partial}{\partial\alpha^I}e_{S_1}{}^K\right)(e^{-1})_K{}^{S_2}(g^{-1})_r{}^a\left(\partial^r\alpha^J\right)\delta(1-2) + i\delta_{S_1}^{S_2}\rhd^a(2)\delta(1-2) \\
&= if_{bS_1}{}^a (g^{-1})_r{}^b \left(\partial^r\alpha^I\right) e_I{}^{S_2}\delta(1-2) + if_{S_3S_1}{}^{S_2}(e^{-1})_J{}^{S_3}\left(\partial^r\alpha^J\right)\delta(1-2) \\
&\quad + i\delta_{S_1}^{S_2}\rhd^a(2)\delta(1-2) \\
&= if_{bS_1}{}^a\rhd^{S_2 b}\delta(1-2) + if_{S_3S_1}{}^{S_2}\rhd^{S_3 a}\delta(1-2) + i\delta_{S_1}^{S_2}\rhd^a(2)\delta(1-2) \\
&= if_{S\Sigma}{}^{\Sigma'}\rhd_{\Sigma'}\delta(1-2) + i\eta_{S\Sigma a}\rhd^a(2)\delta(1-2).
\end{aligned}$$

4 Jacobi Identity

In the last section, we have constructed a very general mechanism to find all the currents. We will now check the Jacobi identity between currents and check if they are consistent with the method described above. Here we mention some results in [17]: The worldvolume currents are $\{\mathring{\rhd}_D,\ \mathring{\rhd}_P,\ \mathring{\rhd}_\Omega\}$ with the following nonvanishing commutation relations:

$$\begin{aligned}
\left\{\overset{\circ}{\triangleright}_{D_1}(1), \overset{\circ}{\triangleright}_{D_2}(2)\right\} &= i f_{D_1 D_2}{}^{P} \overset{\circ}{\triangleright}_{P} \delta(1-2), \\
\left[\overset{\circ}{\triangleright}_{D}(1), \overset{\circ}{\triangleright}_{P}(2)\right] &= i f_{D_1 P}{}^{\Omega} \overset{\circ}{\triangleright}_{\Omega} \delta(1-2), \\
\left[\overset{\circ}{\triangleright}_{P_1}(1), \overset{\circ}{\triangleright}_{P_2}(2)\right] &= 2i \eta_{P_1 P_2 r} \partial_1^r \delta(1-2) \\
\left\{\overset{\circ}{\triangleright}_{D}(1), \overset{\circ}{\triangleright}_{\Omega}(2)\right\} &= 2i \eta_{D\Omega r} \partial_2^r \delta(1-2).
\end{aligned}$$

We now generalize the above commutation relations using the method mentioned in Section 3 to include H group; we get:

$$\begin{aligned}
\left\{\triangleright_{D_1}(1), \triangleright_{D_2}(2)\right\} &= i f_{D_1 D_2}{}^{P} \triangleright_{P} \delta(1-2), \\
\left[\triangleright_{D}(1), \triangleright_{P}(2)\right] &= i f_{D_1 P}{}^{\Omega} \triangleright_{\Omega} \delta(1-2), \\
\left[\triangleright_{P_1}(1), \triangleright_{P_2}(2)\right] &= i f_{P_1 P_2}{}^{\Sigma} \triangleright_{\Sigma} + i \eta_{P_1 P_2 a} \triangleright^{a}((1)-(2)) \delta(1-2) \\
\left\{\triangleright_{D}(1), \triangleright_{\Omega}(2)\right\} &= i f_{D\Omega}{}^{\Sigma} \triangleright_{\Sigma} + i \eta_{D\Omega a} \triangleright^{a}((1)-(2)) \delta(1-2), \\
\left[\triangleright_{S}(1), \triangleright_{D}(2)\right] &= i f_{SD}{}^{D'} \triangleright_{D'} \delta(1-2), \\
\left[\triangleright_{S}(1), \triangleright_{P}(2)\right] &= i f_{SP}{}^{P'} \triangleright_{P'} \delta(1-2), \\
\left[\triangleright_{S}(1), \triangleright_{\Omega}(2)\right] &= i f_{S\Omega}{}^{\Omega'} \triangleright_{\Omega'} \delta(1-2), \\
\left[\triangleright_{S}(1), \triangleright_{S}(2)\right] &= i f_{SS}{}^{S'} \triangleright_{S'} \delta(1-2), \\
\left[\triangleright_{S}(1), \triangleright_{\Sigma}(2)\right] &= i f_{S\Sigma}{}^{\Sigma'} \triangleright_{\Sigma'} \delta(1-2) + 2i \eta_{S\Sigma a} \triangleright^{a}(2) \delta(1-2).
\end{aligned}$$

We can find all the relations between f's and η's by plugging in the above commutation relations into the Jacobi identities, which are listed in Appendix B. Here we point out some of the interesting ones. The first example is combining equation (B.19) and (B.16),

$$\left\{ \begin{array}{l} 0 = f_{SP_1}{}^{P'} \eta_{P' P_2 a} + f_{SP_2}{}^{P'} \eta_{P_1 P' a} + f_{Sa}{}^{b} \eta_{P_1 P_2 b} \\ 0 = f_{P_1 S}{}^{P'} \eta_{P' P_2 a} + f_{P_1 P_2}{}^{\Sigma'} \eta_{S \Sigma' a} - \frac{1}{2} f_{Sa}{}^{b} \eta_{P_1 P_2 b} \end{array} \right.$$

gives

$$f_{P_1 P_2}{}^{\Sigma} \eta_{\Sigma S a} = \frac{1}{2} f_{S[P_1|}{}^{P'} \eta_{P'|P_2]a},$$

which is exactly the result in equation (3.3). It is worthwhile to point out that equation (B.17) or (B.21) together with (B.15) gives the same result as equation (3.3).

And another interesting result is equation (B.14),

$$0 = f_{S_1 S_2}{}^{S'} \eta_{S'\Sigma a} + f_{S_1 \Sigma}{}^{\Sigma'} \eta_{S_2 \Sigma' a} + f_{S_1 a}{}^{b} \eta_{S_2 \Sigma b}.$$

If we write the above equation explicit in $\widetilde{S}$ and b (worldvolume) indices, we get:

$$\begin{aligned}
0 &= f_{S_1 S_2}{}^{S'} \eta_{S'}{}^{\widetilde{S}b}{}_a + f_{S_1}{}^{\widetilde{S}b}{}_{\widetilde{S}'b'} \eta_{S_2}{}^{\widetilde{S}'b'}{}_a + f_{S_1 a}{}^{a'} \eta_{S_2}{}^{\widetilde{S}b}{}_{a'} \\
&= f_{S_1 S_2}{}^{S'} \eta_{S_2}{}^{\widetilde{S}b}{}_a + f_{\widetilde{S}' S_1}{}^{\widetilde{S}} \eta_{S_2}{}^{\widetilde{S}'b}{}_a + f_{b' S_1}{}^{b} \eta_{S_2}{}^{\widetilde{S}b'}{}_a + f_{S_1 a}{}^{a'} \eta_{S_2}{}^{\widetilde{S}b}{}_{a'} \\
&= f_{S_1 S_2}{}^{S'} \eta_{S'}{}^{\widetilde{S}} \delta^b_a + f_{\widetilde{S}' S_1}{}^{\widetilde{S}} \eta_{S_2}{}^{\widetilde{S}'} \delta^b_a + f_{b' S_1}{}^{b} \eta_{S_2}{}^{\widetilde{S}} \delta^{b'}_a + f_{S_1 a}{}^{a'} \eta_{S_2}{}^{\widetilde{S}} \delta^b_{a'} \\
&= f_{S_1 S_2}{}^{S'} \eta_{S'}{}^{\widetilde{S}} \delta^b_a + f_{\widetilde{S}' S_1}{}^{\widetilde{S}} \eta_{S'}{}^{\widetilde{S}'} \delta^b_b \\
\Rightarrow 0 &= f_{S_1 S_2}{}^{S'} \eta_{S'}{}^{\widetilde{S}} + f_{\widetilde{S}' S_1}{}^{\widetilde{S}} \eta_{S_2}{}^{\widetilde{S}'},
\end{aligned}$$

i.e. $\eta_S{}^{\widetilde{S}}$ is H-invariant by itself.

5 Example: 5-brane

F-theory on the 5-brane has been investigated quite intensively, e.g. [15–18, 20]. We go along with the trend and apply the above method to this case. It is shown in section 2 that the bosonic sector lives in $SL(5)/Spin(3,2)$. In order to be generalized to supersymmetry we impose

$$\left\{ \mathring{\triangleright}_{D_1}, \mathring{\triangleright}_{D_2} \right\} = i f_{D_1 D_2}{}^{P} \mathring{\triangleright}_P .$$

The only invariant tensors that are symmetric in the two symmetric spinor indices are the Dirac γ-matrices with two antisymmetric vector indices, i.e. $(\gamma^{mn})_{\alpha\beta}$. Hence,

$$\left\{ \mathring{\triangleright}_{D_1}(1), \mathring{\triangleright}_{D_2}(2) \right\} = \left\{ \mathring{\triangleright}_{\alpha_1}(1), \mathring{\triangleright}_{\alpha_2}(2) \right\} = i \left(\gamma^{mn}\right)_{\alpha_1 \alpha_2} \mathring{\triangleright}_{mn} \delta(1-2),$$

i.e. $\mathring{\triangleright}_P = \mathring{\triangleright}_{mn} = -\mathring{\triangleright}_{nm}$. This then leads to

$$\left[\mathring{\triangleright}_{P_1}(1), \mathring{\triangleright}_{P_2}(2) \right] = \left[\mathring{\triangleright}_{m_1 n_1}(1), \mathring{\triangleright}_{m_2 n_2}(2) \right] = 2i \eta_{m_1 n_1 m_2 n_2 r} \partial^r_1 \delta(1-2).$$

Again, $SL(5)$ invariant tensors are proportional to 5 dimensional Levi-Civita tensors or their combinations. Since $\mathring{\triangleright}_P = \frac{1}{2}\mathring{\triangleright}_{[mn]}$ and $\eta_{P_1P_2r}$ should be symmetric in P_1 and P_2, it can be chosen to be

$$\eta_{P_1P_2r} = \eta_{m_1n_1m_2n_2r} = \epsilon_{m_1n_1m_2n_2r}.$$

By construction,

$$\begin{aligned} \left\{\mathring{\triangleright}_D, \mathring{\triangleright}_\Omega\right\} = \left\{\mathring{\triangleright}_\alpha, \mathring{\triangleright}^{\beta r}\right\} = 2i\delta_\alpha{}^\beta \partial_1^r \delta(1-2) \\ \Rightarrow \eta_{D\Omega q} = \eta_\alpha{}^{\beta r}{}_q = \delta_\alpha{}^\beta \delta_q^r. \end{aligned}$$

As explained in Section 3, the above structure constants and metrics are numerically the same as before and after introducing H group element g. All we have to put in is $\eta_{S\Sigma a}$, and the rest of the structure constants and the metrics can be found by using the equations in Appendix B. By construction, $\triangleright_S$ transforms all the indices the same way as usual $Spin(3,2)$ indices.

We first use equation B.18 to find the only one that doesn't involve Σ that's left, $f_{DP}{}^\Omega$:

$$\begin{aligned} 0 &= f_{D_1D_2}{}^{P'}\eta_{P'Pa} + f_{D_1P}{}^{\Omega'}\eta_{D_2\Omega'a} \\ &= f_{\alpha_1\alpha_2}{}^{c'd'}\eta_{c'd'cda} + f_{\alpha_1 cd\beta b}\eta_{\alpha_2}{}^{\beta b}{}_a \\ \Rightarrow f_{DP}{}^\Omega &= f_{\alpha cd\beta a} = \left(\gamma^{ef}\right)_{\alpha\beta}\epsilon_{efcda}. \end{aligned}$$

We now determine what $\eta_{S\Sigma a}$ is. As explained in Section 3, $\eta_{S\Sigma a} = \eta_S{}^{\widetilde{S}b}{}_a = \eta_S{}^{\widetilde{S}}\delta_a^b$. In the 5-brane case, $\triangleright_S$ is $Spin(3,2)$ generators, hence $\mathbf{S}$ is antisymmetric in its two indices. Using this property, we can conclude $\eta_S{}^{\widetilde{S}} = \frac{1}{2}\delta_{cd}^{ef}$. Using equation (3.3) we found the rest of two unsolved structure constants:

$$\begin{aligned} & f_{P_1P_2}{}^\Sigma\eta_{\Sigma Sa} = \frac{1}{2}f_{S[P_1|}{}^{P'}\eta_{P'|P_2]a} \\ \Rightarrow\ & f_{P_1P_2}{}^\Sigma = f_{c_1d_1c_2d_2efa} = -\left(\mathring{\eta}_{[c_1|[e}\epsilon_{f]|d_1]c_2d_2a} - \mathring{\eta}_{[c_2|[e}\epsilon_{f]|d_2]c_1d_1a}\right), \\ & f_{D\Omega}{}^\Sigma\eta_{\Sigma Sa} = \frac{1}{2}f_{SD}{}^{D'}\eta_{D'\Omega a} + \frac{1}{2}f_{S\Omega}{}^{\Omega'}\eta_{\Omega' Da} \\ \Rightarrow\ & f_{D\Omega}{}^\Sigma = f_\alpha{}^{\beta b}{}_{efa} = \frac{1}{2}\left(\gamma_{ef}\right)_\alpha{}^\beta\delta_a^b. \end{aligned}$$

The $\mathring{\eta}$ above is $SO(3,2)$ metric.

The full commutation relations are listed in Appendix C.

6 Conclusion

The method presented in this paper gives a consistent mathematical structure for higher dimensional brane current algebra (higher than 1) by construction, as opposed to the usual Jacobi identity method used in string theory [19,21]. The main reason is that for 1-brane or string, on the "metric" the additional worldvolume index can take only one value, which is inert under H transformation. For (higher dimensional) brane current algebra, the additional worldvolume indices can have more than one choice. However, in **F**-theory worldvolume indices are also spacetime indices, therefore they all have to react to H-group transformations the same way spacetime indices do. This property makes the original construction unsuitable for the higher dimensional brane algebra (mentioned in the beginning of Section 3). The method presented in this paper can be used in any finite dimensional brane. We've worked out the 5-brane case in detail.

The method used in this paper is not just interesting by itself but also can be utilized for the following subjects:

i) Generalize the method to curved spacetime (**F**-gravity).

ii) Analyze massive modes.

iii) Understand string field theory.

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A Notations

Instead of explaining our notations all over the paper, some common notations are defined in this section so that the readers don't have to hunt for them.

i) $f(1) \equiv f(\sigma_1)$, where σ is worldvolume coordinates, $f(1-2) \equiv f(\sigma_1 - \sigma_2)$, $f((1)+(2)) \equiv f(1) + f(2)$, and, similarly, $f((1)-(2)) \equiv f(1) - f(2)$.

ii) Worldvolume vector indices are denoted as q, r, $\cdots$; spacetime spinor indices are α, β, $\cdots$; superspace indices (which include $\{D, P, \Omega\}$) are M, N, O, $\cdots$;

covariantized superspace indices are A, B, C, $\cdots$ (also include $\{D, P, \Omega\}$); group coordinate indices are denoted as I, J, K, $\cdots$; the covariantized index for H group is S, and the full set of covariantized superspace indices, including all "A" indices, S, and Σ, are $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\cdots$.

iii) $\overset{\circ}{\rhd}_M(\sigma)$ = worldvolume current: e.g. $\overset{\circ}{\rhd}_D(\sigma) = \overset{\circ}{\rhd}_\alpha$, $\overset{\circ}{\rhd}_\Omega(\sigma) = \overset{\circ}{\rhd}^{\alpha r}$.

iv) η_{MNr} is the generalized constant metric, $f_{MN}{}^O$ is the structure constants.

v) $\alpha^I(\sigma)$ is the coordinates of H group (a function of the worldvolume).

vi) $e_S{}^I(\sigma)$ is the vielbein that converts functional derivatives $\left(\frac{\delta}{\delta\alpha^I(\sigma)}\right)$ into symmetry generators ($\rhd_S(\sigma)$).

vii) $\partial^r = \frac{\partial}{\partial\sigma_r}$, a worldvolume coordinate derivative. Sometimes we have to specify which coordinate we act on: Then we add an additional index $\partial_1^r = \frac{\partial}{\partial\sigma_{1r}}$.

viii) $\rhd_A(\sigma)$ = covariantized worldvolume current.

ix) $\rhd_{\mathcal{A}}(\sigma)$ = the full set of covariant worldvolume currents.

x) $g_A{}^M(\sigma)$ is a worldvolume field and is an element of H group.

xi) Parenthesis [ ) in $f_{[m|n|o)}$ is the graded (anti)symmetrization, i.e. sum of index permutation (with a minus sign if not interchanging two spinor indices) in the parenthesis but not the ones in between the two vertical lines, | |.

B Relating f's and η's Using Jacobi

The following are the complete list of all the relations between f's and η's. The "zero modes" means no derivative on delta function ones (δ^2) and the "oscillating modes" means the ones that have a derivative on a delta function ($\delta\partial\delta$). Equations (B.1 $\sim$ B.11) show that f's are invariant under group H, and equation (B.12) gives nothing new.

Zero modes:

$$0 = f_{S_1S_2}{}^{S'} f_{S'S_3}{}^{S_4} + f_{S_2S_3}{}^{S'} f_{S'S_1}{}^{S_4} + f_{S_1S_3}{}^{S'} f_{S_2S'}{}^{S_4}, \tag{B.1}$$
$$0 = f_{S_1S_2}{}^{S'} f_{S'D_1}{}^{D_2} + f_{S_2D_1}{}^{D'} f_{D'S_1}{}^{D_2} + f_{S_1D_1}{}^{D'} f_{S_2D'}{}^{D_2}, \tag{B.2}$$
$$0 = f_{S_1S_2}{}^{S'} f_{S'P_1}{}^{P_2} + f_{S_2P_1}{}^{P'} f_{P'S_1}{}^{P_2} + f_{S_1P_1}{}^{P'} f_{S_2P'}{}^{P_2}, \tag{B.3}$$
$$0 = f_{S_1S_2}{}^{S'} f_{S'\Omega_1}{}^{\Omega_2} + f_{S_2\Omega_1}{}^{\Omega'} f_{\Omega'S_1}{}^{\Omega_2} + f_{S_1\Omega_1}{}^{\Omega'} f_{S_2\Omega'}{}^{\Omega_2}, \tag{B.4}$$
$$0 = f_{S_1S_2}{}^{S'} f_{S'\Sigma_1}{}^{\Sigma_2} + f_{S_2\Sigma_1}{}^{\Sigma'} f_{\Sigma'S_1}{}^{\Sigma_2} + f_{S_1\Sigma_1}{}^{\Sigma'} f_{S_2\Sigma'}{}^{\Sigma_2}, \tag{B.5}$$
$$0 = f_{SD_1}{}^{D'} f_{D'D_2}{}^{P} + f_{D_1D_2}{}^{P'} f_{P'S}{}^{P} + f_{SD_2}{}^{D'} f_{D_1D'}{}^{P}, \tag{B.6}$$
$$0 = f_{SD}{}^{D'} f_{D'P}{}^{\Omega} + f_{DP}{}^{\Omega'} f_{\Omega'S}{}^{\Omega} + f_{SP}{}^{P'} f_{DP'}{}^{\Omega}, \tag{B.7}$$
$$0 = f_{SD}{}^{D'} f_{D'\Omega}{}^{\Sigma} + f_{D\Omega}{}^{\Sigma'} f_{\Sigma'S}{}^{\Sigma} + f_{S\Omega}{}^{\Omega'} f_{D\Omega'}{}^{\Sigma}, \tag{B.8}$$
$$0 = f_{SP}{}^{P'} f_{P'D}{}^{\Omega} + f_{PD}{}^{\Omega'} f_{\Omega'S}{}^{\Omega} + f_{SD}{}^{D'} f_{PD'}{}^{\Omega}, \tag{B.9}$$
$$0 = f_{SP_1}{}^{P'} f_{P'P_2}{}^{\Sigma} + f_{P_1P_2}{}^{\Sigma'} f_{\Sigma'S}{}^{\Sigma} + f_{SP_2}{}^{P'} f_{P_1P'}{}^{\Sigma}, \tag{B.10}$$
$$0 = f_{S\Omega}{}^{\Omega'} f_{\Omega'D}{}^{\Sigma} + f_{\Omega D}{}^{\Sigma'} f_{\Sigma'S}{}^{\Sigma} + f_{SD}{}^{D'} f_{\Omega D'}{}^{\Sigma}, \tag{B.11}$$
$$0 = f_{D_1D_2}{}^{P'} f_{P'D_3}{}^{\Omega} + f_{D_2D_3}{}^{P'} f_{P'D_1}{}^{\Omega} + f_{D_3D_1}{}^{P'} f_{P'D_2}{}^{\Omega}, \tag{B.12}$$
$$0 = f_{D_1D_2}{}^{P'} f_{P'P}{}^{\Sigma} - f_{D_2P}{}^{\Omega'} f_{\Omega'D_1}{}^{\Sigma} + f_{PD_1}{}^{\Omega'} f_{\Omega'D_2}{}^{\Sigma}. \tag{B.13}$$

Oscillating modes:

$$0 = f_{S_1S_2}{}^{S'} \eta_{S'\Sigma a} + f_{S_1\Sigma}{}^{\Sigma'} \eta_{S_2\Sigma' a} + f_{S_1a}{}^{b} \eta_{S_2\Sigma b}, \tag{B.14}$$
$$0 = f_{SD}{}^{D'} \eta_{D'\Omega a} + f_{S\Omega}{}^{\Omega'} \eta_{D\Omega' a} + f_{Sa}{}^{b} \eta_{D\Omega b}, \tag{B.15}$$
$$0 = f_{SP_1}{}^{P'} \eta_{P'P_2 a} + f_{SP_2}{}^{P'} \eta_{P_1P' a} + f_{Sa}{}^{b} \eta_{P_1P_2 b}, \tag{B.16}$$
$$0 = f_{DS}{}^{D'} \eta_{D'\Omega a} + f_{D\Omega}{}^{\Sigma'} \eta_{S\Sigma' a} - \frac{1}{2} f_{Sa}{}^{b} \eta_{D\Omega b}, \tag{B.17}$$
$$0 = f_{D_1D_2}{}^{P'} \eta_{P'Pa} + f_{D_1P}{}^{\Omega'} \eta_{D_2\Omega' a}, \tag{B.18}$$
$$0 = f_{P_1S}{}^{P'} \eta_{P'P_2 a} + f_{P_1P_2}{}^{\Sigma'} \eta_{S\Sigma' a} - \frac{1}{2} f_{Sa}{}^{b} \eta_{P_1P_2 b}, \tag{B.19}$$
$$0 = f_{PD_1}{}^{\Omega'} \eta_{\Omega'D_2 a} + f_{PD_2}{}^{\Omega'} \eta_{D_1\Omega' a}, \tag{B.20}$$
$$0 = f_{\Omega S}{}^{\Omega'} \eta_{\Omega'Da} + f_{\Omega D}{}^{\Sigma'} \eta_{S\Sigma' a} - \frac{1}{2} f_{Sa}{}^{b} \eta_{\Omega D b}, \tag{B.21}$$
$$0 = f_{\Sigma S_1}{}^{\Sigma'} \eta_{\Sigma'S_2 a} + f_{\Sigma S_2}{}^{\Sigma'} \eta_{S_1\Sigma' a} - f_{S_1a}{}^{b} \eta_{S_2\Sigma b} - f_{S_2a}{}^{b} \eta_{S_1\Sigma b}. \tag{B.22}$$

C 5-Brane Commutation Relations

This appendix shows all the nonvanishing commutation relations for the 5-brane.

1. $\left\{ \triangleright_{D_1}(1), \triangleright_{D_2}(2) \right\} = \left\{ \triangleright_{\alpha_1}(1), \triangleright_{\alpha_2}(2) \right\}$
$= i \left(\gamma^{mn}\right)_{\alpha_1\alpha_2} \mathring{\triangleright}_{mn}\delta(1-2) = if_{D_1D_2}{}^{P} \triangleright_P \delta(1-2).$

2. $\left[\triangleright_D(1), \triangleright_P(2) \right] = \left[\triangleright_\alpha(1), \triangleright_{cd}(2) \right]$
$= i \left(\gamma^{ef}\right)_{\alpha\beta} \epsilon_{efcda} \triangleright^{\beta a}\delta(1-2) = if_{D_1P}{}^{\Omega} \triangleright_\Omega \delta(1-2).$

3. $\left[\triangleright_{P_1}(1), \triangleright_{P_2}(2) \right] = \left[\triangleright_{c_1d_1}(1), \triangleright_{c_2d_2}(2) \right]$
$= i \left(-\mathring{\eta}_{[c_1|[e}\epsilon_{f]|d_1]c_2d_2a} + \mathring{\eta}_{[c_2|[e}\epsilon_{f]|d_2]c_1d_1a} \right) \triangleright^{efa}\delta(1-2)$
$+ i\epsilon_{c_1d_1c_2d_2a} \triangleright^a((1)-(2)\delta(1-2)$
$= if_{P_1P_2}{}^{\Sigma} \triangleright_\Sigma \delta(1-2) + i\eta_{P_1P_2a} \triangleright^a((1)-(2))\delta(1-2).$

4. $\left\{ \triangleright_D(1), \triangleright_\Omega(2) \right\} = \left\{ \triangleright_\alpha(1), \triangleright^{\beta b}(2) \right\}$
$= \frac{i}{2} \left(\gamma_{ef}\right)_\alpha{}^\beta \triangleright^{efb}\delta(1-2) + i\delta_\alpha{}^\beta \delta_a^b \triangleright^a((1)-(2))\delta(1-2)$
$= if_{D\Omega}{}^{\Sigma} \triangleright_\Sigma \delta(1-2) + i\eta_{D\Omega a} \triangleright^a((1)-(2))\delta(1-2).$

5. $\left[\triangleright_S(1), \triangleright_D(2) \right] = \left[\triangleright_{ef}(1), \triangleright_\alpha(2) \right]$
$= \frac{i}{4} \left(\gamma_{ef}\right)_\alpha{}^\rho \triangleright_\rho \delta(1-2) = if_{SD}{}^{D'} \triangleright_{D'} \delta(1-2).$

6. $\left[\triangleright_S(1), \triangleright_P(2) \right] = \left[\triangleright_{ef}(1), \triangleright_{cd}(2) \right]$
$= -i\mathring{\eta}_{[c|[e}\delta^{c'd'}_{f]|d]} \triangleright_{c'd'}\delta(1-2) = if_{SP}{}^{P'} \triangleright_{P'} \delta(1-2).$

7. $\left[\triangleright_S(1), \triangleright_\Omega(2) \right] = \left[\triangleright_{ef}(1), \triangleright^{\beta b}(2) \right]$
$= i \left[-\frac{1}{4} \left(\gamma_{ef}\right)_\rho{}^\beta \delta_a^b + \delta_\rho{}^\beta \delta^b_{[e}\mathring{\eta}_{f]a} \right] \triangleright^{\rho a}\delta(1-2) = if_{S\Omega}{}^{\Omega'} \triangleright_{\Omega'} \delta(1-2).$

8. $\Big[\rhd_S(1), \rhd_S(2)\Big] = \Big[\rhd_{e_1 f_1}(1), \rhd_{e_2 f_2}(2)\Big]$
$= -i\mathring{\eta}_{[e_2|[e_1}\delta^{e'f'}_{f_1]|f_2]}\rhd_{e'f'}\delta(1-2) = if_{SS}{}^{S'}\rhd_{S'}\delta(1-2).$

9. $\Big[\rhd_S(1), \rhd_\Sigma(2)\Big] = \Big[\rhd_{ef}(1), \rhd^{gha}(2)\Big]$
$= i\mathring{\eta}_{[g'|[e}\delta^{gh}_{f]|h']}\delta^a_b\rhd^{g'h'b}\delta(1-2) + i\mathring{\eta}_{b'[e}\delta^a_{f]}\rhd^{ghb'}\delta(1-2) + 2i\delta^{gh}_{ef}\rhd^a(2)\delta(1-2)$
$= if_{S\Sigma}{}^{\Sigma'}\rhd_{\Sigma'}\delta(1-2) + 2i\eta_{S\Sigma a}\rhd^a(2)\delta(1-2).$

References

[1] A. Giveon, M. Porrati, and E. Rabinovici, "Target space duality in string theory," *Phys. Rept.* **244** (1994) 77–202, arXiv:hep-th/9401139 [hep-th].

[2] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories," *Nucl. Phys.* **B435** (1995) 129–146, arXiv:hep-th/9411149 [hep-th].

[3] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," *Nucl. Phys.* **B438** (1995) 109–137, arXiv:hep-th/9410167 [hep-th].

[4] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys.* **B443** (1995) 85–126, arXiv:hep-th/9503124 [hep-th].

[5] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven-dimensions," *Nucl. Phys.* **B460** (1996) 506–524, arXiv:hep-th/9510209 [hep-th].

[6] E. Cremmer, B. Julia, H. Lu, and C. N. Pope, "Dualization of dualities. 1.," *Nucl. Phys.* **B523** (1998) 73–144, arXiv:hep-th/9710119 [hep-th].

[7] N. A. Obers and B. Pioline, "U duality and M theory," *Phys. Rept.* **318** (1999) 113–225, arXiv:hep-th/9809039 [hep-th].

[8] C. Vafa, "Evidence for F theory," *Nucl. Phys.* **B469** (1996) 403–418, arXiv:hep-th/9602022 [hep-th].

[9] W. Siegel, "Two vierbein formalism for string inspired axionic gravity," *Phys. Rev.* **D47** (1993) 5453–5459, arXiv:hep-th/9302036 [hep-th].

[10] W. Siegel, "Superspace duality in low-energy superstrings," *Phys. Rev.* **D48** (1993) 2826–2837, arXiv:hep-th/9305073 [hep-th].

[11] W. Siegel, "Manifest duality in low-energy superstrings," in *In *Berkeley 1993, Proceedings, Strings '93* 353-363, and State U. New York Stony Brook - ITP-SB-93-050 (93,rec.Sep.) 11 p. (315661)*. 1993. arXiv:hep-th/9308133 [hep-th].

[12] C. Hull and B. Zwiebach, "Double Field Theory," *JHEP* **09** (2009) 099, arXiv:0904.4664 [hep-th].

[13] M. Hatsuda and T. Kimura, "Canonical approach to Courant brackets for D-branes," *JHEP* **06** (2012) 034, arXiv:1203.5499 [hep-th].

[14] M. Hatsuda and K. Kamimura, "M5 algebra and SO(5,5) duality," *JHEP* **06** (2013) 095, arXiv:1305.2258 [hep-th].

[15] W. D. Linch and W. Siegel, "F-theory Superspace," arXiv:1501.02761 [hep-th].

[16] W. D. Linch, III and W. Siegel, "F-theory from Fundamental Five-branes," arXiv:1502.00510 [hep-th].

[17] W. D. Linch and W. Siegel, "Critical Super F-theories," arXiv:1507.01669 [hep-th].

[18] W. D. Linch and W. Siegel, "F-theory with Worldvolume Sectioning," arXiv:1503.00940 [hep-th].

[19] M. Poláček and W. Siegel, "T-duality off shell in 3D Type II superspace," *JHEP* **06** (2014) 107, arXiv:1403.6904 [hep-th].

[20] J.-H. Park and Y. Suh, "U-geometry: SL(5)," *JHEP* **04** (2013) 147, arXiv:1302.1652 [hep-th]. [Erratum: JHEP11,210(2013)].

[21] M. Poláček and W. Siegel, "Natural curvature for manifest T-duality," *JHEP* **01** (2014) 026, arXiv:1308.6350 [hep-th].