Scheme-independent calculation of γ_{ψ[over ¯]ψ,IR} for an SU(3) gauge theory

Thomas A. Ryttov and Robert Shrock

Phys. Rev. D **94**, 105014 — Published 21 November 2016

DOI: 10.1103/PhysRevD.94.105014

Scheme-Independent Calculation of $\gamma_{\bar\psi\psi,IR}$ for an SU(3) Gauge Theory

Thomas A. Ryttov[a] and Robert Shrock[b]
(a) CP^3-Origins and Danish Institute for Advanced Study
University of Southern Denmark, Campusvej 55, Odense, Denmark and
(b) C. N. Yang Institute for Theoretical Physics
Stony Brook University, Stony Brook, NY 11794, USA

We present a scheme-independent calculation of the infrared value of the anomalous dimension of the fermion bilinear, $\gamma_{\bar\psi\psi,IR}$ in an SU(3) gauge theory as a function of the number of fermions, N_f, via a series expansion in powers of Δ_f, where $\Delta_f = (16.5 - N_f)$, to order Δ_f^4. We perform an extrapolation to obtain the first determination of the exact $\gamma_{\bar\psi\psi,IR}$ from continuum field theory. The results are compared with calculations of the n-loop values of this anomalous dimension from series in powers of the coupling and from lattice measurements.

A fundamental problem in quantum field theory concerns the evolution of an asymptotically free gauge theory from large Euclidean momentum scales μ in the ultraviolet (UV), where it is weakly coupled, to small μ in the infrared (IR). The dependence of the running gauge coupling $g = g(\mu)$ on μ is determined by the beta function [1], $\beta = d\alpha/dt$, where $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $dt = d\ln\mu$ (we often suppress the argument μ in the notation). Here we consider an asymptotically free (AF) vectorial gauge theory with gauge group $G = \mathrm{SU}(3)$ and N_f fermions ψ_i, $i = 1, ..., N_f$ in the fundamental (F) representation. The fermions are taken to be massless, since a fermion with mass m is integrated out of the effective theory for $\mu < m$ and hence does not affect the evolution to the IR with $\mu < m$. This theory corresponds to quantum chromodynamics (QCD) with N_f massless quarks.

The beta function of this theory has the series expansion

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ , \qquad (1)$$

where $a = g^2/(16\pi^2) = \alpha/(4\pi)$, b_ℓ is the ℓ-loop coefficient, $\bar{b}_\ell = b_\ell/(4\pi)^\ell$, and we extract an overall minus sign in Eq. (1). The n-loop ($n\ell$) beta function, denoted $\beta_{n\ell}$, is given by Eq. (1) with the upper limit on the ℓ-loop summation changed from $\ell = \infty$ to $\ell = n$. The one-loop and two-loop coefficients are independent of the scheme used for regularization and renormalization (i.e., scheme-independent, SI), while the b_ℓ with $\ell \geq 3$ are scheme-dependent (SD) [2]; these are $b_1 = 11 - (2/3)N_f$ [3] and $b_2 = 102 - (38/3)N_f$ [4]. In our analysis, we formally extend N_f to nonnegative real numbers, understanding that the physical values are nonnegative integers. Since b_1 vanishes as N_f increases through the value $N_{f,b1z} = 33/2$, the AF property implies the upper bound $N_f < N_{f,b1z} = 33/2$, which we assume. The interval $0 \leq N_f < N_{f,b1z}$ is denoted I_{AF}. We define

$$\Delta_f = N_{f,b1z} - N_f = \frac{33}{2} - N_f \ . \qquad (2)$$

The coefficients b_3 and b_4 were calculated in [5] and [6] (and checked in [7]), in the $\overline{\mathrm{MS}}$ scheme [8]; e.g., $b_3 = (2857/2) - (5033/18)N_f + (325/54)N_f^2$.

As $N_f \in I_{AF}$ increases from 0, b_2 decreases, vanishing at $N_{f,b2z} = 153/19 = 8.0526$, and is negative in the interval $153/19 < N_f < 33/2$, which is denoted I_{IRZ}. If $N_f \in I_{IRZ}$, then the two-loop beta function $\beta_{2\ell}$ has an IR zero (IRZ), at $\alpha = \alpha_{IR,2\ell} = -4\pi b_1/b_2$. Here the IR zero of the n-loop beta function $\beta_{n\ell}$ is denoted $\alpha_{IR,n\ell}$. As $N_f \nearrow N_{f,b1z}$ at the upper end of I_{IRZ}, $\alpha_{IR,2\ell} \to 0$, enabling a perturbative study of the IR behavior [4, 9]. As $N_f \in I_{IRZ}$ decreases below $N_{f,b1z}$, $\alpha_{IR,2\ell}$ increases, eventually becoming O(1). Therefore, the perturbative study of IR behavior for N_f toward the middle and lower part of I_{IRZ}, necessitates higher-loop calculations. These were performed to four-loop order in [10]-[15]. For $n \geq 3$ loops, $\alpha_{IR,n\ell}$ is scheme-dependent, and the effect of this was studied in [16]. For sufficiently large $N_f \in I_{IRZ}$, the theory evolves to an exact IR fixed point (IRFP) of the renormalization group (RG) in a chirally symmetric non-Abelian Coulomb phase (NACP). For sufficiently small N_f, spontaneous chiral symmetry breaking (SχSB) occurs, the fermions gain dynamical masses, and they are integrated out of the low-energy effective theory that is applicable at lower scales in the IR. In this latter case, the IR zero is only an approximate IRFP. The lowest value of N_f in the NACP is denoted as $N_{f,cr}$. The UV to IR flow in the chirally broken phase near to this lower boundary of the NACP can exhibit quasiconformal behavior, which might be relevant to physics beyond the Standard Model. It is of great interest to elucidate the properties of the theory at the IRFP.

In this letter we report a significant advance toward the achievement of this goal, namely a new scheme-independent calculation of the anomalous dimension of the fermion bilinear, $\bar\psi\psi \equiv \bar\psi_i\psi_i$ (no sum on i), evaluated at the IR zero of the beta function. We denote this as $\gamma_{\bar\psi\psi,IR}$ [17]. As a physical quantity, this is clearly scheme-independent [2]. The full scaling dimension of the $\bar\psi\psi$ operator is $D(\bar\psi_i\psi_i) = 3 - \gamma_{\bar\psi\psi}$, with the anomalous dimension $\gamma_{\bar\psi\psi} = -d\ln Z_{\bar\psi\psi}/dt$, where $Z_{\bar\psi\psi}$ is the renormalization constant for this operator. For brevity, we set $\gamma_{\bar\psi\psi} \equiv \gamma$ and $\gamma_{\bar\psi\psi,IR} \equiv \gamma_{IR}$. In a usual perturbative calculation, γ is expressed as the series

$$\gamma = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \alpha^\ell \ , \tag{3}$$

where c_ℓ is the ℓ-loop term and $\bar{c}_\ell = c_\ell/(4\pi)^\ell$. The coefficient $c_1 = 8$ is scheme-independent, while the c_ℓ with $\ell \ge 2$ are scheme-dependent and have been calculated to $\ell = 4$ loop order in [18]. The n-loop result for γ is defined by replacing $\ell = \infty$ by $\ell = n$ as the upper limit on the sum in (3), and the n-loop approximation to the exact γ_{IR}, denoted $\gamma_{IR,n\ell}$, is then obtained by setting $\alpha = \alpha_{IR,n\ell}$ in $\gamma_{n\ell}$. A rigorous upper bound is

$$\gamma_{IR} < 2 \tag{4}$$

in both the NACP and the chirally broken phase [19].

The quantities $\alpha_{IR,n\ell}$ and $\gamma_{IR,n\ell}$ were calculated to $n = 4$ loop order in [12, 13]. Although b_5 and c_5 have not yet been calculated for general G and fermion representation R, c_5 is known [20] and b_5 has recently been calculated [21] in the $\overline{\rm MS}$ scheme for the present theory, $G = {\rm SU}(3)$, $R = F$. Using these results, we have computed $\alpha_{IR,5\ell}$ and $\gamma_{IR,5\ell}$ in this scheme [22].

It is highly desirable to construct a calculational framework in which γ_{IR} can be expressed as a series expansion such that at every order in this expansion, the result is scheme-independent. One of us (T.A.R.) recently achieved this goal in [23], expressing γ_{IR} as

$$\gamma_{IR} = \sum_{k=1}^{\infty} \kappa_k \Delta_f^k \ , \tag{5}$$

where each κ_k is scheme-independent. The inputs for the calculation of κ_k are the b_ℓ at loop order $1 \le \ell \le k+1$ and the c_ℓ at loop order $1 \le \ell \le k$. For the finite series approximation we denote $\gamma_{IR,\Delta^p} = \sum_{k=1}^{p} \kappa_k \Delta_f^k$. Ref. [23] gave γ_{IR,Δ^p} for the powers $1 \le p \le 3$ for general G and R.

Here we report two new results: (i) the calculation of κ_4 and hence γ_{IR,Δ^4}, and (ii) using the γ_{IR,Δ^p} with p up to 4, an extrapolation to the exact γ_{IR} for $G = {\rm SU}(3)$, $R = F$, and $N_f \in I_{IRZ}$. The lower-order coefficients for this SU(3) theory are [24]

$$\kappa_1 = \frac{16}{3 \cdot 107} = 4.9844 \times 10^{-2} \tag{6}$$

$$\kappa_2 = \frac{125452}{(3 \cdot 107)^3} = 3.7928 \times 10^{-3} \tag{7}$$

and

$$\kappa_3 = \frac{972349306}{(3 \cdot 107)^5} - \frac{140800}{3^3 \cdot (107)^4}\zeta(3) = 2.3747 \times 10^{-4} \tag{8}$$

Using the SI method of [23] together with b_5 from [21] (and lower-loop b_ℓ and c_ℓ), we find

$$\kappa_4 = \frac{33906710751871}{2^2(3 \cdot 107)^7} - \frac{1684980608}{3^5 \cdot (107)^6}\zeta(3) + \frac{59840000}{(3 \cdot 107)^5}\zeta(5)$$

$$= 3.6789 \times 10^{-5} \ , \tag{9}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

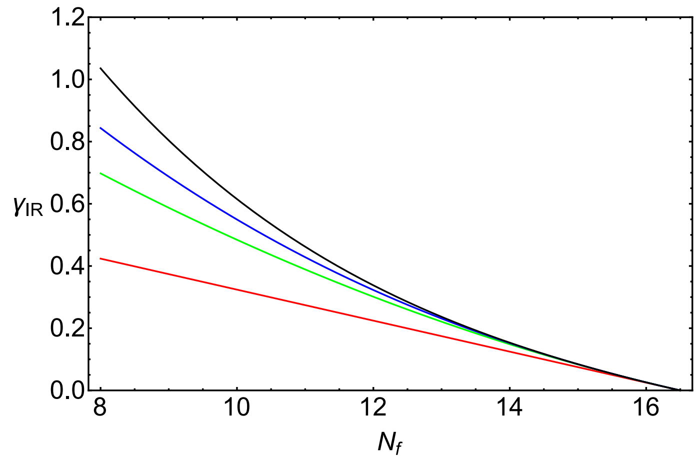

FIG. 1: Plot of γ_{IR,Δ^p} for $1 \le p \le 4$ as a function of N_f. From bottom to top, the curves (with colors online) refer to $\gamma_{IR,\Delta}$ (red), γ_{IR,Δ^2} (green), γ_{IR,Δ^3} (blue), and γ_{IR,Δ^4} (black).

In Fig. 1 we show a plot of γ_{IR,Δ^p} and in Table I we list numerical results for $1 \le p \le 4$, with $N_f \in I_{IRZ}$. For comparison, this table also lists results for $\gamma_{IR,n\ell}$ at n-loop level for $1 \le n \le 5$ from [12, 22]. The values of $\gamma_{IR,2\ell}$ for $N_f = 9, \ 10$ exceed the upper bound (4) and hence, as noted in [12], we regard these N_f values at the lower end of I_{IRZ} to be beyond reliable perturbative analysis via the series (3). The estimates of $\gamma_{IR,5\ell}$ for $N_f = 9, \ 10$ were not given in [22]; they use the IR zero from the $[3,1]$ Padé approximants for $\beta_{5\ell}$. Here we see another merit of the SI expansion (5), namely that it allows us to study the IR behavior closer to the lower end of the interval I_{IRZ}. Although $N_f = 8 < N_{f,b2z}$ is below the lower end of I_{IRZ}, we mention that $\gamma_{IR,\Delta^p} = 0.424, \ 0.698, \ 0.844, \ 1.04$ for $1 \le p \le 4$.

Having the four SI values γ_{IR,Δ^p} with $1 \le p \le 4$, we can carry out a polynomial (in $1/p$) extrapolation to estimate the exact $\gamma_{IR} = \lim_{p \to \infty} \gamma_{IR,\Delta^p}$ for each N_f. We have investigated two such extrapolations, one of which uses all four terms and the other of which uses the three highest-order terms, i.e. $p = 2, \ 3, \ 4$. These two types of extrapolations give consistent results. We report the values obtained with the second extrapolation method here. For example, for $N_f = 12$, we obtain the fitting polynomial $\gamma_{IR,\Delta^p,fit} = 0.2048p^{-2} - 0.3005p^{-1} + 0.400$, from which we get $\gamma_{IR} = \lim_{p\to\infty} \gamma_{IR,\Delta^p,fit} = 0.400$ for this N_f. We list our results for γ_{IR} as a function of N_f from this extrapolation in Table I. For N_f values near the upper end of the interval I_{IRZ}, where Δ_f is small, our γ_{IR,Δ^4} and extrapolation to the exact γ_{IR} (both of which are SI) are very close to the value of $\gamma_{IR,4\ell}$ calculated in the $\overline{\rm MS}$ scheme [12, 13] and in other schemes [16] and to the value of $\gamma_{IR,5\ell}$ in [22]. As N_f decreases in I_{IRZ} , our γ_{IR,Δ^5} and extrapolated exact γ_{IR} become progressively larger than the corresponding values of $\gamma_{IR,n\ell}$ for $3 \le n \le 5$. If we were to apply the same extrapolation procedure at $N_f = 8$ (below $N_{f,b2z}$), we would get an unphysical value of γ_{IR} slightly above 2.

An important general result concerns the monotonicity of γ_{IR} as a function of N_f. We find that for $G = \mathrm{SU}(N_c)$ for general N_c and for R equal to the fundamental, adjoint, and symmetric and antisymmetric rank-2 tensor representations, the κ_p for $p = 1,\ 2,\ 3$ given in [23] are positive. Hence, for all of these cases, for $p = 1, 2, 3$, γ_{IR,Δ^p} is a monotonically increasing function of Δ_f, i.e., a monotonically decreasing function of N_f in the range where this Δ_f expansion applies, which includes the interval I_{IRZ}. Further, our Eq. (9) shows that $\kappa_4 > 0$ $G = \mathrm{SU}(3)$ and $R = F$, so for this case γ_{IR,Δ^4} and our extrapolated exact γ_{IR} are also monotonically increasing functions of Δ_f, i.e. decreasing functions of N_f, throughout I_{IRZ}. A plausible conjecture, based on these results, is that for $G = \mathrm{SU}(N_c)$ with general N_c and for $R = F$, $\kappa_p > 0$ for all $p \ge 1$. Assuming this conjecture is correct, then the inequality $\gamma_{IR,\Delta^p} \le \gamma_{IR}$ follows (realized as a strict inequality except at $N_f = N_{f,b1z}$ where $\gamma_{IR} = 0$). We note that $\kappa_p > 0$ for all $p \ge 1$ in QCD with $\mathcal{N} = 1$ supersymmetry (SQCD) [23, 25].

We next compare our results for γ_{IR,Δ^4} and extrapolation for γ_{IR} with lattice measurements of γ_{IR} [26]. The most extensive measurements have been carried out for the case $N_f = 12$ and include the following values: $\gamma_{IR} \sim 0.414 \pm 0.016$ [27], $\gamma_{IR} \simeq 0.35$ [28], $\gamma_{IR} \simeq 0.4$ [29], $\gamma_{IR} = 0.27(3)$ [30], $\gamma_{IR} \simeq 0.25$ [31], $\gamma_{IR} = 0.235(46)$ [32], and $0.2 \lesssim \gamma_{IR} \lesssim 0.4$ [33] (see [26]-[33] for discussions of estimates of overall uncertaintites in these measurements). Our value $\gamma_{IR,\Delta^4} = 0.338$ and our extrapolated $\gamma_{IR} = 0.40$ are consistent with this range of lattice measurements and are somewhat higher than the five-loop value $\gamma_{IR,5\ell} = 0.255$ from the conventional α series that we obtained in [22]. There is also consistency between our determinations of γ_{IR} and rough estimates that $\gamma_{IR} \sim 1$ from lattice studies for $N_f = 10$ [34] and $N_f = 8$ [35, 36].

Combining the upper bound $\gamma_{IR} < 2$ with the monotonicity of γ_{IR}, we infer that if γ_{IR} saturates its upper bound as $N_f \searrow N_{f,cr}$ at the lower end of the NACP [37], then we would conclude that $8 < N_{f,cr} < 9$. However, we stress that it is not known if, in fact, γ_{IR} saturates its upper bound in this way as $N_f \searrow N_{f,cr}$.

In contrast to γ_{IR}, the IR zero of β, α_{IR}, is scheme-dependent. Nevertheless, one can use the Δ_f expansion to obtain an estimate of α_{IR} that is complementary to the estimate from the calculation of the zero of β expressed as a series expansion in powers of α. We write

$$\alpha_{IR} = 4\pi \sum_{n=1} \tilde{a}_n \Delta_f^n \tag{10}$$

and give our results in Table II. We have calculated the $\tilde{a}_n$ for general G and R for $1 \le n \le 3$. Using b_5 from [21] for $G = \mathrm{SU}(3)$ and $R = F$, we have also calculated $\tilde{a}_4$ for this case, for which we find

$$\tilde{a}_1 = \frac{2}{3 \cdot 107} = 0.62305 \times 10^{-2} \tag{11}$$

$$\tilde{a}_2 = \frac{11675}{2(3 \cdot 107)^3} = 1.7649 \times 10^{-4} \tag{12}$$

$$\tilde{a}_3 = \frac{145645559}{2^2 \cdot 3^4 \cdot (107)^5} + \frac{170720}{3^3 \cdot (107)^4}\,\zeta(3) = 0.90035 \times 10^{-4} \tag{13}$$

$$\begin{aligned} \tilde{a}_4 &= \frac{119816461287557}{2^5 \cdot 3^8 \cdot (107)^7} + \frac{15442747864}{3^7 \cdot (107)^6}\,\zeta(3) - \frac{24534400}{(3 \cdot 107)^5}\,\zeta(5) \\ &= 1.7453 \times 10^{-6}\ . \end{aligned} \tag{14}$$

In summary, using the recently calculated b_5 from [21], we have presented a scheme-independent calculation of γ_{IR,Δ^4} and an extrapolation to estimate the exact anomalous dimension of the fermion bilinear at the IR zero of the beta function, γ_{IR}, as a function of N_f in a QCD-like gauge theory. We have compared the results with n-loop calculations obtained from power series in the coupling and with lattice measurements.

The research of T.A.R. and R.S. was supported in part by the Danish National Research Foundation grant DNRF90 to CP^3-Origins at SDU and by the U.S. NSF Grant NSF-PHY-13-16617, respectively.

[1] Some early works include E. C. G. Stueckelberg and A. Peterman, Helv. Phys. Acta **26**, 499 (1953); M. Gell-Mann and F. Low, Phys. Rev. **95**, 1300 (1954); N. N. Bogolubov and D. V. Shirkov, Doklad. Akad. Nauk SSSR **103**, 391 (1955); C. G. Callan, Phys. Rev. D **2**, 1541 (1970); K. Symanzik, Commun. Math. Phys. **18**, 227 (1970); K. Wilson, Phys. Rev. D **3**, 1818 (1971).

[2] D. J. Gross, in R. Balian and J. Zinn-Justin, eds. *Methods in Field Theory*, Les Houches 1975 (North Holland, Amsterdam, 1976), p. 141.

[3] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973); G. 't Hooft, unpublished.

[4] W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974); D. R. T. Jones, Nucl. Phys. B **75**, 531 (1974).

[5] O. V. Tarasov, A. A. Vladimirov, and A. Yu. Zharkov, Phys. Lett. B **93**, 429 (1980); S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B **303**, 334 (1993).

[6] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. B **400**, 379 (1997).

[7] M. Czakon, Nucl. Phys. B **710**, 485 (2005).

[8] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, Phys. Rev. D **18**, 3998 (1978).

[9] T. Banks and A. Zaks, Nucl. Phys. B **196**, 189 (1982).

[10] E. Gardi and M. Karliner, Nucl. Phys. B **529**, 383 (1998); E. Gardi and G. Grunberg, JHEP 03, 024 (1999).

[11] F. A. Chishtie, V. Elias, V. A. Miransky, and T. G. Steele, Prog. Theor. Phys. **104**, 603 (2000).

[12] T. A. Ryttov, R. Shrock, Phys. Rev. D **83**, 056011 (2011) [arXiv:1011.4542].

[13] C. Pica, F. Sannino, Phys. Rev. D **83**, 035013 (2011) [arXiv1011.5917].

[14] R. Shrock, Phys. Rev. D **87**, 105005 (2013); R. Shrock, Phys. Rev. D **87**, 116007 (2013).

[15] Refs. [12]-[14] studied the IR zero in $\beta_{n\ell}$ and resultant $\gamma_{IR,n\ell}$ for general G and fermion representation R.

[16] T. A. Ryttov and R. Shrock, Phys. Rev. D **86**, 065032 (2012); T. A. Ryttov and R. Shrock, Phys. Rev. D **86**, 085005 (2012); R. Shrock, Phys. Rev. D **88**, 036003 (2013); R. Shrock, Phys. Rev. D **90**, 045011 (2014); R. Shrock, Phys. Rev. D **91**, 125039 (2015); G. Choi and R. Shrock, Phys. Rev. D **90**, 125029 (2014); G. Choi and R. Shrock, arXiv:1607.03500; T. A. Ryttov, Phys. Rev. D **89**, 016013 (2014); T. A. Ryttov, Phys. Rev. D **89**, 056001 (2014); T. A. Ryttov, Phys. Rev. D **90**, 056007 (2014). J. A. Gracey and R. M. Simms, Phys. Rev. D **91**, 085037 (2015); P. M. Stevenson, arXiv:1607.01670.

[17] The anomalous dimension $\gamma_{\bar\psi\psi}$ is often denoted γ_m; we use the more general notation $\gamma_{\bar\psi\psi}$ here, since no mass m is generated in the non-Abelian Coulomb phase.

[18] K. G. Chetyrkin, Phys. Lett. B **404**, 161 (1997); J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, Phys. Lett. B **405**, 327 (1997).

[19] In the conformal phase, see S. Ferrara, R. Gatto, A. F. Grillo, Phys. Rev. D **9**, 3564 (1974); G. Mack, Commun. Math. Phys. **55**, 1 (1977); B. Grinstein, K. Intriligator, and I. Rothstein, Phys. Lett. B **662**, 367 (2008). In the phase with SχSB, the dynamically generated momentum-dependent fermion mass is $m(k) \sim \Lambda(\Lambda/k)^{2-\gamma_{\bar\psi\psi}}$ up to logs, and the requirement that $\lim_{k\to\infty} m(k) = 0$ (where k = Euclidean momentum) yields the same bound.

[20] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, JHEP 10, 076 (2014).

[21] P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, arXiv:1606.08659.

[22] T. A. Ryttov and R. Shrock, arXiv:1607.06866.

[23] T. A. Ryttov, Phys. Rev. Lett. **117**, 071601 (2016) [arXiv:1604.00687].

[24] Aside from κ_1, the numerators of terms in these κ_p do not have similarly simple factorizations; e.g., in κ_2, $125452 = 2^2 \cdot 79 \cdot 397$, in κ_3, $972349306 = 2 \cdot 17 \cdot 569 \cdot 50261$, etc.

[25] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B **229**, 381, 407 (1983); Phys. Lett. B **166**, 329 (1986).

[26] See, e.g., talks in the CP3 Workshop at http://cp3-origins.dk/events/meetings/mass2013; Lattice-2014 at https://www.bnl.gov/lattice2014; SCGT15 at http://www.kmi.nagoya-u.ac.jp/workshop/SCGT15; and Lattice-2015 at http://www.aics.riken.jp/sympo/lattice2015 and There is not yet a consensus among all lattice groups concerning the value of $N_{f,cr}$ for this SU(3) theory.

[27] T. Appelquist et al. (LSD Collab.), Phys. Rev. D **84**, 054501 (2011).

[28] T. DeGrand, Phys. Rev. D **84**, 116901 (2011); Rev. Mod. Phys. **88**, 015001 (2016).

[29] Y. Aoki et al. (LatKMI Collab.), Phys. Rev. D **86**, 054506 (2012).

[30] A. Hasenfratz, A. Cheng, G. Petropoulos, and D. Schaich, in *Proc. Lattice-2012* [arXiv:1207.7162].

[31] A. Hasenfratz, A. Cheng, G. Petropoulos, and D. Schaich, in *Proc. Lattice-2013* [arXiv:1310.1124].

[32] M. P. Lombardo, K. Miura, T. J. Nunes da Silva, and E. Pallante, JHEP 12, 183 (2014) [arXiv:1410.0298].

[33] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, C. Schroeder, and C.H. Wong, in *Proc. Lattice-2012* [arXiv:1211.4238]; Z. Fodor, K. Holland, J. Kuti, S. Mondal, D. Nogradi, and C. H. Wong, arXiv:1607.06121 .

[34] T. Appelquist et al. (LSD Collab.), arXiv:1204.6000.

[35] Y. Aoki et al. (LatKMI Collab.), Phys. Rev. D **87**, 094511 (2013); Y. Aoki et al. (LatKMI Collab.), Phys. Rev. D **89**, 111502 (2014).

[36] T. Appelquist et al. (LSD Collab.), Phys. Rev. D **90**, 114502 (2014).

[37] In SQCD, the upper bound on γ_{IR} is $\gamma_{IR} < 1$ in the NACP, and γ_{IR} does saturate this upper bound as the the number N_f of pairs of chiral superfields $\hat\Phi$ and $\tilde{\hat\Phi}$ decreases toward the lower end of the NACP.

TABLE I: Values of the scheme-independent IR anomalous dimension for the fermion bilinear, γ_{IR,Δ^p} for $1 \le p \le 4$ as a function of $N_f \in I_{IRZ}$, and the extrapolated values of the exact γ_{IR}, where the number in parentheses is an estimate of the uncertainty in the last significant figure in the extrapolated value. For comparison, we also include $\overline{\rm MS}$ calculations of $\gamma_{IR,n\ell}$ at the $2 \le n \le 5$ loop level from [12, 22].

N_f	$\gamma_{IR,2\ell}$	$\gamma_{IR,3\ell}$	$\gamma_{IR,4\ell}$	$\gamma_{IR,5\ell}$	$\gamma_{IR,\Delta}$	γ_{IR,Δ^2}	γ_{IR,Δ^3}	γ_{IR,Δ^4}	γ_{IR}
9	> 2	1.062	< 0	< 0	0.374	0.587	0.687	0.804	1.4(2)
10	> 2	0.647	0.156	0.211	0.324	0.484	0.549	0.615	0.95(6)
11	1.61	0.439	0.250	0.294	0.274	0.389	0.428	0.462	0.62(2)
12	0.773	0.312	0.253	0.255	0.224	0.301	0.323	0.338	0.400(5)
13	0.404	0.220	0.210	0.239	0.174	0.221	0.231	0.237	0.257(5)
14	0.212	0.146	0.147	0.154	0.125	0.148	0.152	0.153	0.154(4)
15	0.0997	0.0826	0.0836	0.0843	0.0748	0.0833	0.0841	0.0843	0.0841(2)
16	0.0272	0.0258	0.0259	0.0259	0.0249	0.0259	0.0259	0.0259	0.0259(1)

TABLE II: Values of α_{IR,Δ^p} with $1 \le p \le 4$ as functions of $N_f \in I_{IRZ}$, together with $\alpha_{IR,2\ell}$ and $\overline{\rm MS}$ values of n-loop $\alpha_{IR,n\ell}$ with $3 \le n \le 5$ for comparison. The values of $\alpha_{IR,5\ell}$ for $9 \le N_f \le 12$ are from the $[3,1]$ Padé approximants (PAs) to the respective beta functions in [22].

N_f	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$	$\alpha_{IR,5\ell}$	$\alpha_{IR,\Delta}$	α_{IR,Δ^2}	α_{IR,Δ^3}	α_{IR,Δ^4}
9	5.24	1.028	1.072	1.02_{PA}	0.587	0.712	1.19	1.26
10	2.21	0.764	0.815	0.756_{PA}	0.509	0.603	0.913	0.952
11	1.23	0.578	0.626	0.563_{PA}	0.431	0.498	0.686	0.706
12	0.754	0.435	0.470	0.4075_{PA}	0.352	0.397	0.500	0.509
13	0.468	0.317	0.337	0.406	0.274	0.301	0.350	0.353
14	0.278	0.215	0.224	0.233	0.196	0.210	0.227	0.228
15	0.143	0.123	0.126	0.127	0.117	0.122	0.126	0.126
16	0.0416	0.0397	0.0398	0.0398	0.0391	0.0397	0.0398	0.0398