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The Embedding of Gauged STU Supergravity in Eleven Dimensions

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ABSTRACT

The consistency of the embedding of four-dimensional $SO(8)$ gauged $\mathcal{N}=8$ supergravity into eleven-dimensional supergravity, where the internal directions are compactified on a seven-sphere, was established by de Wit and Nicolai in the 1980s. The reduction ansatz for the eleven-dimensional metric, and for some of the components of the 4-form field strength, were found at that time, and recently the complete expression for the 4-form reduction has been obtained. The expressions are quite complicated, and in many practical applications it would be sufficient to know the ansatz for a subset of the four-dimensional fields. In this paper, we obtain explicit expressions for the embedding of the truncation of the full $\mathcal{N}=8$ gauged theory to the $\mathcal{N}=2$ gauged STU supergravity. This corresponds, in the bosonic sector, to a consistent truncation of the $\mathcal{N}=8$ supergravity fields to those that are singlets under the $U(1)^4$ Cartan subalgebra of $SO(8)$. This truncation to STU supergravity, which comprises $\mathcal{N}=2$ supergravity coupled to three vector multiplets, suffices, for example, for lifting the general 8-charge asymptotically-AdS rotating black holes to eleven dimensions. We also give two distinct further truncations to $\mathcal{N}=2$ supergravities coupled to single vector multiplets.

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1 Introduction

The idea that the four-dimensional $\mathcal{N} = 8$ gauged $SO(8)$ supergravity of de Wit and Nicolai [1] could be obtained by means of a dimensional reduction of eleven-dimensional supergravity on S^7 dates back to the early 1980s [2, 3]. Originally, this reduction was discussed just at the level of the linearisation of the field equations around the Freund-Rubin [4] $\mathrm{AdS}_4 \times S^7$ ground state. A reduction in which all the massive Kaluza-Klein towers as well as the massless $\mathcal{N} = 8$ supergravity subsector is retained can obviously be consistently extended, in principle, to the full non-linear order. However, one cannot in general expect to be able to perform a consistent truncation of this full theory in which the towers of massive multiplets are set to zero, leaving only the massless four-dimensional gauged $\mathcal{N} = 8$ supergravity [5]. The key issue here is that one would generically expect that in the full theory, non-linear "currents" built from powers of the massless fields would act as sources in the equations of motion of the massive fields that one wishes to set to zero. Indeed, this is exactly what would happen in a sphere or coset-space reduction of any generic theory, and thus a reduction in which just a "massless sector" (including the gauge bosons of the full isometry group of the compactifying manifold) was retained would be inconsistent beyond the linearised level. The first indication that the S^7 reduction of eleven-dimensional supergravity might be an exception to the general rule was found in ref. [6], where it is shown that a crucial trilinear coupling of two $SO(8)$ gauge bosons to a massive spin-2 multiplet of fields is absent. This satisfies a first, necessary, condition for the retained gauge bosons not in fact to act as sources for the massive fields that one wishes to set to zero. This happens because the gauge bosons enter not only in the metric uplift ansatz but also in the uplift ansatz for the 4-form field strength of eleven-dimensional supergravity. This, together with a certain identity obeyed by the Killing vectors on the 7-sphere, conspires to remove the trilinear couplings that would otherwise be the first signal of the inconsistency of the truncated reduction.

The possibility of a dimensional reduction of a higher-dimensional theory on a sphere, in which a finite number of lower-dimensional fields including the gauge bosons of the isometry group were retained, was in fact conceived by Pauli in 1953 (in the context of an S^2 reduction of six-dimensional Einstein gravity) [7–9], but he recognised that the consistency problems mentioned above would be an obstacle to realising his idea. In fact, the S^7 reduction of eleven-dimensional supergravity was the first non-trivial example in which the idea of a "Pauli reduction" [10] actually works.

An indirect, but nevertheless complete, demonstration of the consistency of the S^7

reduction was provided by de Wit and Nicolai in ref. [11] (see also [12]). They obtained an explicit expression for the metric uplift ansatz, and also partial results for the uplift ansatz for the 4-form field strength. Their construction, making extensive use of the supersymmetry of the theory, essentially proves that the reduction is necessarily a consistent one. Much more recently, further work has provided more complete expressions for the 4-form uplift ansatz, and has also provided further insights into the structure of the reduction [13–16].

Although the consistency of the S^7 Pauli reduction of eleven-dimensional supergravity has now been fully established, and the uplift ansätze for the metric and 4-form field strength are explicitly known, these expressions are in practice somewhat unwieldy and complicated to use in full generality. In many cases, when for example lifting a solution of the four-dimensional gauged supergravity to $D = 11$, it may suffice to have explicit expressions for the uplift ansatz for only a subset of the fields in the full four-dimensional supergravity theory. The uplift ansatz for such a truncation of the full gauged supergravity theory may be much simpler and more manageable. Examples of this kind that have been obtained previously include the embedding of four-dimensional $SO(4)$-gauged $\mathcal{N} = 4$ supergravity, for which the complete and explicit bosonic S^7 uplift ansatz is given in ref. [17]. The four-dimensional $\mathcal{N} = 4$ gauged theory is of course itself a consistent truncation of the $\mathcal{N} = 8$ gauged theory.

Another example that has proved to be of considerable utility is given in ref. [18]. This describes the embedding of a truncation of the full $SO(8)$ gauged theory in which only the gauge bosons of the $U(1)^4$ abelian subgroup of $SO(8)$ are retained. The truncation in ref. [18] retains also the four-dimensional metric, and three dilatonic scalar fields. The associated four-dimensional theory in this case is not a fully consistent truncation of $\mathcal{N} = 8$ gauged $SO(8)$ supergravity: to be consistent, one should include a total of six scalar fields rather than just the three that are retained in ref. [18], comprising three axions as well as the three dilatonic scalars. However, if one restricts attention to four-dimensional solutions in which the $U(1)$ gauge fields are essentially purely electric or purely magnetic, but not both, then the three axions can be consistently set to zero and the ansatz in ref. [18] can then be used in order to lift such solutions to eleven dimensions.[1] The uplift ansatz found in ref. [18], with its restricted notion of consistency, is sufficient for the purposes of lifting the static four-charge black hole solutions to eleven dimensions.

The purpose of the present paper is to obtain explicit expressions for the bosonic uplift

[1]To be more precise, the three axions can be consistently set to zero if the wedge products $F^\alpha \wedge F^\beta$ of the four $U(1)$ field strengths all vanish.

ansatz for the embedding of the fully consistent truncation of the $SO(8)$ gauged supergravity to its abelian $U(1)^4$ subsector. This theory, comprising the metric, four $U(1)$ gauge fields, and the three dilatons and three axions mentioned above, is the bosonic sector of the $\mathcal{N}=2$ supersymmetric gauged STU supergravity theory. Its field content comprises the $\mathcal{N}=2$ supergravity multiplet coupled to three vector multiplets. The inclusion of the additional three axionic scalar fields makes the uplift ansatz considerably more complicated than the restricted uplift ansatz that was found in ref. [18]. In fact an attempt to construct the ansatz for the embedding of the bosonic sector of the gauged STU supergravity was made in ref. [19], but at that time only the partial results for the 4-form uplift ansatz that had been obtained in ref. [11] were available, and the results in ref. [19] are for that reason incomplete. With the recent advances in the construction of the internal 4-form uplift ansatz for the $SO(8)$ gauged supergravity that have been achieved in refs. [13, 15, 20], we are now in a position to complete the job that was left unfinished in ref. [19].

The ansatz that we shall present in this paper is considerably more complicated than the restricted one without the three axionic scalars that was obtained in ref. [18]. It is, however, still considerably simpler, for practical purposes, than the complete ansatz for the reduction to the full $\mathcal{N}=8$ gauged $SO(8)$ supergravity theory. Uplifting four-dimensional solutions of $\mathcal{N}=2$ theories to eleven dimensions has been of recent interest in the context of AdS/CFT, in particular, the ABJM proposal [21]. An interesting example has been the study of flow solutions as a new way of understanding strongly-coupled phases of M2 brane dynamics [22, 23]. More generally, uplifting to eleven dimensions may facilitate a clearer understanding of the four-dimensional physics, such as the microscopic counting of black hole solutions in $\mathcal{N}=2$ gauged supergravities, which has recently been studied in Ref. [24]. The ansatz we obtain allows one to lift, for example, any black hole solution of gauged $\mathcal{N}=8$ supergravity carrying abelian charges to eleven dimensions. The most general such solution would have four electric and four magnetic charges (although global symmetries could be used in order to rotate to a duality complexion where a total of 5 independent non-zero charges remain).

2 Gauged STU supergravity

The gauged STU supergravity is a consistent truncation of the $\mathcal{N}=8$ $SO(8)$ gauged theory with $\mathcal{N}=2$ supersymmetry and residual gauge group $U(1)^4$, the maximal abelian subgroup of $SO(8)$ [18, 25]. In particular, the truncation leaves a total of six non-trivial scalars, three dilatons and three axions parameterised by λ_i and σ_i, respectively, with $i = 1, 2, 3$, and four $U(1)$ gauge fields A^α ($\alpha = 1, 2, 3, 4$) with associated field strengths F^α. The bosonic Lagrangian, in the notation of ref. [19], is

$$L = R *1\!\!1 + L_{KinS} - V *1\!\!1 + L_{KinA} + L_{CS}, \tag{2.1}$$

where the scalar kinetic terms are simply [19]

$$L_{KinS} = -\frac{1}{2}\sum_i \left((\partial\lambda_i)^2 + \sinh^2\lambda_i\,(\partial\sigma_i)^2 \right) *1\!\!1 \tag{2.2}$$

and the scalar potential [2]

$$V = -4\,g^2 \sum_i (Y_i^2 + \widetilde{Y}_i^2), \tag{2.3}$$

where [19]

$$\cosh\lambda_i = \frac{1}{2}(Y_i^2 + \widetilde{Y}_i^2), \qquad \cos\sigma_i\,\sinh\lambda_i = \frac{1}{2}(Y_i^2 - \widetilde{Y}_i^2). \tag{2.4}$$

In addition, we define [19]

$$\sin\sigma_i\,\sinh\lambda_i = b_i. \tag{2.5}$$

It will sometimes be more convenient to use an $SL(2,\mathbb{R})$ parameterisation for the scalar fields rather than an $SO(2,1)$ parameteristion, by defining dilaton/axion pairs (φ_i, χ_i) by

$$e^{\varphi_i} = \cosh\lambda_i + \sinh\lambda_i\,\cos\sigma_i\,, \qquad \chi_i\,e^{\varphi_i} = \sinh\lambda_i\,\sin\sigma_i\,, \tag{2.6}$$

in terms of which the scalar kinetic Lagrangian becomes

$$L_{KinS} = -\frac{1}{2}\sum_i \left((\partial\varphi_i)^2 + e^{2\varphi_i}\,(\partial\chi_i)^2 \right) *1\!\!1\,. \tag{2.7}$$

The quantities Y_i, $\widetilde{Y}_i$ and b_i defined above are now given by

$$Y_i^2 = e^{\varphi_i}\,, \qquad \widetilde{Y}_i^2 = e^{-\varphi_i} + \chi_i^2\,e^{\varphi_i}\,, \qquad b_i = \chi_i\,e^{\varphi_i}\,. \tag{2.8}$$

The kinetic terms for the gauge fields $\mathcal{L}_{KinA}$ and the Chern-Simons terms $\mathcal{L}_{CS}$, given by equations (36) and (38) of ref. [19], are

$$\begin{aligned}
\mathcal{L}_{KinA} = -\tfrac{1}{2}|W|^{-2} \Big[& P_0 \Big(\tilde{Y}_1^2\,\tilde{Y}_2^2\,\tilde{Y}_3^2 *F^1_{(2)} \wedge F^1_{(2)} + \tilde{Y}_1^2\,Y_2^2\,Y_3^2 *F^2_{(2)} \wedge F^2_{(2)} \\
& + Y_1^2\,\tilde{Y}_2^2\,Y_3^2 *F^3_{(2)} \wedge F^3_{(2)} + Y_1^2\,Y_2^2\,\tilde{Y}_3^2 *F^4_{(2)} \wedge F^4_{(2)} \Big) \\
& + 2P_1\,b_2\,b_3\,(\tilde{Y}_1^2 *F^1_{(2)} \wedge F^2_{(2)} - Y_1^2 *F^3_{(2)} \wedge F^4_{(2)}) \\
& + 2P_2\,b_1\,b_3\,(\tilde{Y}_2^2 *F^1_{(2)} \wedge F^3_{(2)} - Y_2^2 *F^2_{(2)} \wedge F^4_{(2)}) \\
& + 2P_3\,b_1\,b_2\,(\tilde{Y}_3^2 *F^1_{(2)} \wedge F^4_{(2)} - Y_3^2 *F^2_{(2)} \wedge F^3_{(2)}) \Big]
\end{aligned} \tag{2.9}$$

[2]Note that this potential is a particular example of the Fayet-Iliopoulos potentials of four-dimensional gauged $\mathcal{N} = 2$ supergravities. See, for example, Ref. [26].

and

$$
\begin{aligned}
\mathcal{L}_{CS} = -|W|^{-2} \Big[& b_1\, b_2\, b_3 \Big(\tilde{Y}_1^2\, \tilde{Y}_2^2\, \tilde{Y}_3^2\, F^1_{(2)} \wedge F^1_{(2)} + \tilde{Y}_1^2\, Y_2^2\, Y_3^2\, F^2_{(2)} \wedge F^2_{(2)} \\
& \qquad + Y_1^2\, \tilde{Y}_2^2\, Y_3^2\, F^3_{(2)} \wedge F^3_{(2)} + Y_1^2\, Y_2^2\, \tilde{Y}_3^2\, F^4_{(2)} \wedge F^4_{(2)} \Big) \\
& + b_1\, (P_0 + 2b_2^2\, b_3^2)\, (\tilde{Y}_1^2\, F^1_{(2)} \wedge F^2_{(2)} - Y_1^2\, F^3_{(2)} \wedge F^4_{(2)}) \\
& + b_2\, (P_0 + 2b_1^2\, b_3^2)\, (\tilde{Y}_2^2\, F^1_{(2)} \wedge F^3_{(2)} - Y_2^2\, F^2_{(2)} \wedge F^4_{(2)}) \\
& + b_3\, (P_0 + 2b_1^2\, b_2^2)\, (\tilde{Y}_3^2\, F^1_{(2)} \wedge F^4_{(2)} - Y_3^2\, F^2_{(2)} \wedge F^3_{(2)}) \Big] \,, \qquad (2.10)
\end{aligned}
$$

where

$$
\begin{aligned}
& P_0 \equiv 1 + b_1^2 + b_2^2 + b_3^2\,, \quad W \equiv P_0 - 2\mathrm{i}\; b_1\, b_2\, b_3\,, \\
& P_1 \equiv 1 - b_1^2 + b_2^2 + b_3^2\,, \quad P_2 \equiv 1 + b_1^2 - b_2^2 + b_3^2\,, \quad P_3 \equiv 1 + b_1^2 + b_2^2 - b_3^2\,. \qquad (2.11)
\end{aligned}
$$

3 Complete ansätze for $\mathcal{N} = 8$ gauged $SO(8)$ supergravity

In this section we summarise briefly the uplift ansätze for the full $\mathcal{N} = 8$ gauged $SO(8)$ supergravity [13, 15, 16, 20, 27]. [3] We remark that the ansätze presented below are not unique, and in fact there are many possible ways of writing them down [28, 29]. Of course, these ansätze should all be equivalent and the consistency of both the $D = 11$ and $\mathcal{N} = 8$ theories as well as the reduction will guarantee that this will indeed be the case. Here, we present the ansätze in what we believe to be their simplest known form.

The metric ansatz [27]

$$
\Delta^{-1} g^{mn}(x, y) = \frac{1}{8} K^{m\,IJ}(y) K^{n\,KL}(y) \Big[\left(u^{MN}{}_{IJ} + v^{MNIJ} \right) \left(u_{MN}{}^{KL} + v_{MNKL} \right) \Big](x), \quad (3.1)
$$

where $K^{m\,IJ}$ are the 28 Killing vectors on the round S^7 as defined in equation (4.5), has been known for some time and is in fact applied in ref. [19] to determine the internal metric for the full gauged STU supergravity. The recent progress concerns uplift ansätze for components of the field strength F_{MNPQ} (or equivalently the three form potential A_{MNP}), *viz.*

$$
F_{mnpq}, \quad F_{\mu mnp}, \quad F_{\mu\nu mn}, \quad F_{\mu\nu\rho m} \quad \text{and} \quad F_{\mu\nu\rho\sigma}. \qquad (3.2)
$$

Note that the components as presented above do not quite correspond in a direct manner to the notation used in ref. [19]. This is because, in line with the whole spirit of a Kaluza-Klein reduction, all fields in ref. [19] are defined with the Kaluza-Klein gauge fields included;

[3] We simply state the uplift ansätze here and do not give an account of how they are derived. For such an account, the reader is encouraged to consult Refs. [14–16].

i.e. the seven-dimensional coordinate differentials dy^m always appear in the Kaluza-Klein covariantised form

$$dy^m \longrightarrow \Big(dy^m - K^{mIJ}(y)\, A_\mu^{IJ}(x)\, dx^\mu\Big)\,. \tag{3.3}$$

The Kaluza-Klein redefinition above is easy to account for and we shall do so when presenting the full set of uplift ansätze (see section 5).

The F_{mnpq} components of the field strength can be obtained simply from the ansatz for A_{mnp} [13, 15, 20]

$$A_{mnp}(x,y) = -\frac{i\sqrt{2}}{96}\,\Delta g_{pq}\, K_{mn}^{IJ}(y) K^{q\,KL}(y) \Big[\, \big(u^{MN}{}_{IJ} - v^{MNIJ}\big)\, \big(u_{MN}{}^{KL} + v_{MNKL}\big)\,\Big](x), \tag{3.4}$$

where K_{mn}^{IJ} are defined in equation (4.7). In particular,

$$F_{mnpq} = 4\partial_{[m} A_{npq]}. \tag{3.5}$$

Similarly,

$$F_{\mu mnp} = \partial_\mu A_{mnp} - 3\partial_{[m} A_{|\mu| np]} \tag{3.6}$$

with A_{mnp} as determined by equation (3.4) and $A_{\mu mn}$ given by the following exact vector ansatz [15]

$$A_{\mu mn}(x,y) = \frac{1}{24} K_{mn}{}^{IJ}(y) A_{\mu\, IJ}(x) - \frac{\sqrt{2}}{4} A_{mnp}(x,y) K^{p\,IJ}(y) A_\mu{}^{IJ}(x). \tag{3.7}$$

The ansatz for $F_{\alpha\beta ab}$ [16] is particularly simple

$$e_\mu{}^\alpha e_\nu{}^\beta e_m{}^a e_n{}^b F_{\alpha\beta ab} = \frac{\sqrt{2}}{8}\left(K_{mn}{}^{IJ}(y)\, \mathcal{G}_{\mu\nu\, IJ}(x) - 12\Delta^{-1} A_{mnp}(x,y)\, K^p{}_{IJ}(y)\, \mathcal{H}_{\mu\nu}{}^{IJ}(x)\right), \tag{3.8}$$

where $\mathcal{G}_{\mu\nu\, IJ}$ and $\mathcal{H}_{\mu\nu}{}^{IJ}$ are covariantised field strengths of the electric and magnetic vectors and $e_\mu{}^\alpha$ and $e_m{}^a$ are defined by

$$E_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & B_\mu{}^p e_p{}^a \\ 0 & e_m{}^a \end{pmatrix}. \tag{3.9}$$

The Kaluza-Klein vectors $B_\mu{}^m$ in the STU truncation are given by the abelian $U(1)^4$ truncation of $K^{mIJ}\, A_\mu^{IJ}$. In particular, from the above ansatz we deduce that $F_{\alpha\beta ab}$ vanishes for four-dimensional solutions with vanishing vector expectations values.

The ansatz for $F_{\alpha\beta\gamma a}$ is perhaps the most involved [16]

$$
\begin{aligned}
& e_\mu{}^\alpha e_\nu{}^\beta e_\rho{}^\gamma e_m{}^a F_{\alpha\beta\gamma a} \\
& \quad = -\frac{\sqrt{2}}{48}\eta_{\mu\nu\rho}{}^\sigma K^{n\,IJ}(y)\, K_{mn}{}^{KL}(y)\, \Big[\big(u^{ij}{}_{IJ} + v^{ijIJ}\big)\big(u^{kl}{}_{KL} - v^{klKL}\big)\Big](x) \\
& \qquad \times \Big(\hat{\mathcal{V}}^{\mathcal{M}}{}_{ij}\partial_\sigma \hat{\mathcal{V}}_{\mathcal{M}\,kl} - 2\sqrt{2}m_7 A_\sigma{}^{MN}\hat{\mathcal{V}}^{MP}{}_{ij}\hat{\mathcal{V}}_{NP\,kl} - 2\sqrt{2}m_7 A_\sigma{}^{MN}\hat{\mathcal{V}}^{MP}{}_{kl}\hat{\mathcal{V}}_{NP\,ij}\Big)(x),
\end{aligned} \tag{3.10}
$$

where $\hat{\mathcal{V}}_{\mathcal{M}\,ij}$ is usual four-dimensional $E_{7(7)}/SU(8)$ coset element parameterising the 70 scalars.

The ansatz for the $F_{\mu\nu mn}$ components of the four-form are

$$
\begin{aligned}
F_{\mu\nu mn} &= E_\mu{}^A E_\nu{}^B E_m{}^C E_n{}^D F_{ABCD} \\
&= e_m{}^a e_n{}^b e_\mu{}^\alpha e_\nu{}^\beta F_{\alpha\beta ab} + 2B_\nu{}^p F_{\mu mnp} - B_\mu{}^p B_\nu{}^q F_{mnpq},
\end{aligned} \tag{3.11}
$$

where the first term is given in equation (3.8). Similarly, the $F_{\mu\nu\rho m}$ are given by equation (3.10) and appropriate contractions of the $F_{\mu\nu mn}$, $F_{\mu mnp}$ and F_{mnpq} components with the Kaluza-Klein vectors $B_\mu{}^m$. [4]

On the other hand, the *conjectured* ansatz for the Freund-Rubin term [5] is perhaps the simplest and most elegant [16] [6]

$$
\mathfrak{f}_{FR} = -\frac{i}{24}\eta^{\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta} = -\frac{m_7}{96\sqrt{2}g^2}\left(V(x) - \frac{g^2}{6}\Big(Q^{ijkl}(x)\hat{\Sigma}_{ijkl}(x,y) + \text{h.c.}\Big)\right), \tag{3.12}
$$

where $V(x)$ is the four-dimensional scalar potential, Q^{ijkl} is simply given in terms of the T-tensor

$$
Q^{ijkl} = \frac{3}{4}A_{2\ m}{}^{n[ij}A_{2\ n}{}^{kl]m} - A_1{}^{m[i}A_{2\ m}{}^{jkl]} \tag{3.13}
$$

and complex self-dual tensor

$$
\hat{\Sigma}_{ijkl}(x,y) = \big(u_{ij}{}^{IJ}u_{kl}{}^{KL} - v_{ij\,IJ}v_{kl\,KL}\big)(x)\; K^{m\,[IJ}(y)K_m{}^{KL]}(y). \tag{3.14}
$$

Note that the only components of the four-form field strength for which an ansatz is not given in ref. [19] are the ones related to the *internal* components A_{mnp} of A_{MNP}, which, as explained above, are given by the ansatz in equation (3.4), after including the Kaluza-Klein redefinitions explained above and in section 5.

[4] Note that in ref. [16], the components on the left-hand side of equations (3.8) and (3.10) are denoted by $F_{\mu\nu mn}$ and $F_{\mu\nu\rho m}$, respectively.

[5] There are other established ansätze for the Freund-Rubin term (see ref. [16]). However, it has not yet been possible to show that those ansätze have the following simple form.

[6] In this paper, we use the conventions of ref. [19], which are related to the conventions of ref. [16] as follows: $g^{\text{ref. [16]}} = \sqrt{2}\,g^{\text{ref. [19]}}$ and $V^{\text{ref. [16]}} = 1/2\,V^{\text{ref. [19]}}$.

4 Derivation of A_{mnp}

In this section, we outline the derivation of the purely internal (7-sphere) components A_{mnp} using ansatz (3.4), which we repeat here for convenience

$$A_{mnp} = -\frac{i\sqrt{2}}{96}\,\Delta g_{pq}\,K^{IJ}_{mn}K^{q\,KL}\left(u^{MN}{}_{IJ} - v^{MNIJ}\right)\left(u_{MN}{}^{KL} + v_{MNKL}\right). \qquad (4.1)$$

As is clear from the ansatz above, the first step is to compute the u and v tensors. These tensors have already been computed in ref. [19] (see also ref. [30]), but here, we express them in terms of $U(1)^4$-invariant tensors. In this way the u and v are expressed more covariantly with respect to the $U(1)^4$ symmetry.

Without loss of generality, we can choose $U(1)^4 \subset SO(8)$ to act on the index pairs $\{12\},\{34\},\{56\}$ and $\{78\}$, where, for example the first $U(1)$ rotates 1 and 2 into each other. Introducing

$$(X^{IJKL})_{I'J'K'L'} = 4!\,\delta^{IJKL}_{I'J'K'L'}\,, \qquad (4.2)$$

we define $C^i_\pm$ as

$$C^1_\pm = X^{1234} \pm X^{5678}\,, \qquad C^2_\pm = X^{1256} \pm X^{3478}\,, \qquad C^3_\pm = X^{1278} \pm X^{3456}\,. \qquad (4.3)$$

It is clear that $C^i_\pm$ are the unique rank-4 $U(1)^4$-invariant tensors, up to redefinitions of the embedding of $U(1)^4 \subset SO(8)$. Furthermore, there are also four rank-2 invariant tensors

$$F^{(1)}_{IJ} = 2\,\delta^{12}_{IJ}, \qquad F^{(2)}_{IJ} = 2\,\delta^{34}_{IJ}, \qquad F^{(3)}_{IJ} = 2\,\delta^{56}_{IJ}, \qquad F^{(4)}_{IJ} = 2\,\delta^{78}_{IJ}. \qquad (4.4)$$

The 28 Killing vectors of the $SO(8)$ symmetry of the round S^7 are written in the spinor representation, i.e. with

$$K^{IJ} = \tfrac{1}{2}(\Gamma_{ab})^{IJ}\left(x^a\frac{\partial}{\partial x^b} - x^b\,\frac{\partial}{x^a}\right), \qquad (4.5)$$

where x^a are the Cartesian coordinates on $\mathbb{R}^8$ that are employed in equation (A.2). We then define the following vectors and tensors:

$$\begin{gathered}
\xi^{(i)}_m = \tfrac{1}{16}C^{(i)}_{+IJKL}\,K^{IJ}_{mn}\,K^{nKL}\,, \qquad \xi^{(i)}_{mn} = -\tfrac{1}{16}C^{(i)}_{+IJKL}\,K^{IJ}_m\,K^{KL}_n\,,\\
S^{(i)}_{mnp} = \tfrac{1}{16}C^{(i)}_{-IJKL}\,K^{IJ}_{mn}\,K^{KL}_p\,,\\
F^{(\alpha)}_m = F^{(\alpha)}_{IJ}\,K^{IJ}_m\,, \qquad F^{(\alpha)}_{mn} = F^{(\alpha)}_{IJ}\,K^{IJ}_{mn}\,,
\end{gathered} \qquad (4.6)$$

where

$$K_{mn}{}^{IJ} = K_{[mn]}{}^{IJ} = -\frac{1}{m_7}\mathring{D}_m K_n{}^{IJ}. \qquad (4.7)$$

Indices on $K_m{}^{IJ}$ and $K_{mn}{}^{IJ}$ as well as all $U(1)^4$-invariant tensors are raised and lowered with the background (inverse) metric $\mathring{g}^{mn}$ and $\mathring{g}_{mn}$, respectively.

The 56-bein $\mathcal{V} = \mathcal{V}_1\mathcal{V}_2\mathcal{V}_3$, where

$$\mathcal{V}_i = \begin{pmatrix} u^{(i)}{}_{ij}{}^{IJ} & v^{(i)}{}_{ijKL} \\ v^{(i)klIJ} & u^{(i)kl}{}_{KL} \end{pmatrix}. \tag{4.8}$$

In symmetric gauge each 56-bein is expressed as

$$\mathcal{V}_i = \exp\begin{pmatrix} 0 & \phi^{(i)} \\ \bar{\phi}^{(i)} & 0 \end{pmatrix}, \tag{4.9}$$

where $\bar{\phi}^{(i)}$ is the complex conjugate of $\phi^{(i)}$. In terms of the $U(1)^4$-invariant tensors above, the scalar expectation values are

$$\phi^{(i)} = -2\lambda_i \left(\cos\sigma_i\, C^i_+ - i\,\sin\sigma_i\, C^i_-\right). \tag{4.10}$$

Hence by exponentiation,

$$u^{(i)}{}_{IJ}{}^{KL} = \delta^{KL}_{IJ} + \frac{1}{4}\left(\cosh\frac{\lambda_i}{2} - 1\right)\left(C^i_+ C^i_+\right)_{IJKL}, \tag{4.11}$$

$$v^{(i)}{}_{IJKL} = -\frac{1}{2}\sinh\frac{\lambda_i}{2}\left(\cos\sigma_i\, C^i_+ - i\,\sin\sigma_i\, C^i_-\right)_{IJKL}. \tag{4.12}$$

The u and v matrices corresponding to the full 56-bein $\mathcal{V}$ can then be found from

$$\begin{aligned} u_{IJ}{}^{KL} &= u^{(1)}{}_{IJ}{}^{MN}\left(u^{(2)}{}_{MN}{}^{PQ}u^{(3)}{}_{PQ}{}^{KL} + v^{(2)}{}_{MNPQ}v^{(3)PQKL}\right) \\ &\quad + v^{(1)}{}_{IJMN}\left(v^{(2)MNPQ}u^{(3)}{}_{PQ}{}^{KL} + u^{(2)MN}{}_{PQ}v^{(3)PQKL}\right), \end{aligned} \tag{4.13}$$

$$\begin{aligned} v^{IJKL} &= v^{(1)IJMN}\left(u^{(2)}{}_{MN}{}^{PQ}u^{(3)}{}_{PQ}{}^{KL} + v^{(2)}{}_{MNPQ}v^{(3)PQKL}\right) \\ &\quad + u^{(1)IJ}{}_{MN}\left(v^{(2)MNPQ}u^{(3)}{}_{PQ}{}^{KL} + u^{(2)MN}{}_{PQ}v^{(3)PQKL}\right). \end{aligned} \tag{4.14}$$

Now that we have the u and v tensors, we can compute $\Delta^{-1}g^{pq}A_{mnp}$ by contracting them with K^{IJ}_{mn} and $K^{q\,KL}$ as prescribed by the formula (4.1). In terms of the $U(1)^4$-invariant vectors and tensors defined in equations (4.6),[7]

$$\Delta^{-1}g^{pq}A_{mnp} = \sum_{i=1}^{3} b_i\, S^{(i)q}{}_{mn} + \frac{1}{64}\sum_{\alpha,\beta=1}^{4} f_{\alpha\beta}\; F^{(\alpha)q}\, F^{(\beta)}_{mn}, \tag{4.15}$$

where, defining

$$y_i^2 = Y_i^2 - \widetilde{Y}_i^2, \qquad \tilde{y}_i^2 = Y_i^2 + \widetilde{Y}_i^2, \tag{4.16}$$

[7]Recall that indices on $U(1)^4$-invariant tensors are raised/lowered using the background (inverse) metric on the round seven-sphere, $\mathring{g}^{mn}/\mathring{g}_{mn}$.

$$
\begin{aligned}
f_{11} &= y_1^2(b_3 y_2^2 + b_2 y_3^2) + b_1 y_2^2 y_3^2 - 4b_1 b_2 b_3, & f_{22} &= -y_1^2(b_3 y_2^2 + b_2 y_3^2) + b_1 y_2^2 y_3^2 - 4b_1 b_2 b_3,\\
f_{33} &= -y_1^2(b_3 y_2^2 - b_2 y_3^2) - b_1 y_2^2 y_3^2 - 4b_1 b_2 b_3, & f_{44} &= y_1^2(b_3 y_2^2 - b_2 y_3^2) - b_1 y_2^2 y_3^2 - 4b_1 b_2 b_3,\\
f_{12} &= -\tilde{y}_1^2(b_3 y_2^2 + b_2 y_3^2) + b_1 \tilde{y}_2^2 \tilde{y}_3^2 - 4b_1, & f_{21} &= \tilde{y}_1^2(b_3 y_2^2 + b_2 y_3^2) + b_1 \tilde{y}_2^2 \tilde{y}_3^2 - 4b_1,\\
f_{13} &= -\tilde{y}_2^2(b_3 y_1^2 + b_1 y_3^2) + b_2 \tilde{y}_1^2 \tilde{y}_3^2 - 4b_2, & f_{31} &= \tilde{y}_2^2(b_3 y_1^2 + b_1 y_3^2) + b_2 \tilde{y}_1^2 \tilde{y}_3^2 - 4b_2,\\
f_{14} &= -\tilde{y}_3^2(b_2 y_1^2 + b_1 y_2^2) + b_3 \tilde{y}_1^2 \tilde{y}_2^2 - 4b_3, & f_{41} &= \tilde{y}_3^2(b_2 y_1^2 + b_1 y_2^2) + b_3 \tilde{y}_1^2 \tilde{y}_2^2 - 4b_3,\\
f_{23} &= \tilde{y}_3^2(b_2 y_1^2 - b_1 y_2^2) - b_3 \tilde{y}_1^2 \tilde{y}_2^2 + 4b_3, & f_{32} &= -\tilde{y}_3^2(b_2 y_1^2 - b_1 y_2^2) - b_3 \tilde{y}_1^2 \tilde{y}_2^2 + 4b_3,\\
f_{24} &= \tilde{y}_2^2(b_3 y_1^2 - b_1 y_3^2) - b_2 \tilde{y}_1^2 \tilde{y}_3^2 + 4b_2, & f_{42} &= -\tilde{y}_2^2(b_3 y_1^2 - b_1 y_3^2) - b_2 \tilde{y}_1^2 \tilde{y}_3^2 + 4b_2,\\
f_{34} &= \tilde{y}_1^2(b_3 y_2^2 - b_2 y_3^2) - b_1 \tilde{y}_2^2 \tilde{y}_3^2 + 4b_1, & f_{43} &= -\tilde{y}_1^2(b_3 y_2^2 - b_2 y_3^2) - b_1 \tilde{y}_2^2 \tilde{y}_3^2 + 4b_1.
\end{aligned}
\tag{4.17}
$$

The $U(1)^4$-invariant vectors and tensors are given explicitly in terms of a set of adapted coordinates (μ_α, ϕ_ρ) on S^7 in appendix A, where μ_α are subject to constraint (A.1). Thus, all that is left to do in order to find A_{mnp} is to contract Δg_{pq}, which we know from the metric ansatz [19], with the expression found above written in adapted coordinates, i.e.

$$
A_{mnp} = \Delta g_{pq} \left(\sum_{i=1}^{3} b_i \, S^{(i)q}{}_{mn} + \frac{1}{64} \sum_{\alpha,\beta=1}^{4} f_{\alpha\beta} \; F^{(\alpha)q} \, F^{(\beta)}_{mn} \right). \tag{4.18}
$$

The resulting expression is [8]

$$
\begin{aligned}
A_{\alpha 56}\, d\mu_\alpha \;&=\; \frac{b_1}{2\,\Xi\, g^3} \Big[\mu_1^2\, W_2\, d(\mu_2^2) - \mu_2^2\, W_1\, d(\mu_1^2) - \mu_1^2\, \mu_2^2\, (Y_2^2 \widetilde{Y}_2^2\, d\alpha_2 + Y_3^2 \widetilde{Y}_3^2\, d\alpha_3) \Big]\,,\\
A_{\alpha 78}\, d\mu_\alpha \;&=\; \frac{b_1}{2\,\Xi\, g^3} \Big[\mu_4^2\, W_3\, d(\mu_3^2) - \mu_3^2\, W_4\, d(\mu_4^2) + \mu_3^2\, \mu_4^2\, (Y_2^2 \widetilde{Y}_2^2\, d\alpha_2 - Y_3^2 \widetilde{Y}_3^2\, d\alpha_3) \Big]\,,\\
A_{\alpha 57}\, d\mu_\alpha \;&=\; \frac{b_2}{2\,\Xi\, g^3} \Big[\mu_1^2\, W_3\, d(\mu_3^2) - \mu_3^2\, W_1\, d(\mu_1^2) - \mu_1^2\, \mu_3^2\, (Y_1^2 \widetilde{Y}_1^2\, d\alpha_1 + Y_3^2 \widetilde{Y}_3^2\, d\alpha_3) \Big]\,,\\
A_{\alpha 68}\, d\mu_\alpha \;&=\; \frac{b_2}{2\,\Xi\, g^3} \Big[\mu_4^2\, W_2\, d(\mu_2^2) - \mu_2^2\, W_4\, d(\mu_4^2) + \mu_2^2\, \mu_4^2\, (Y_1^2 \widetilde{Y}_1^2\, d\alpha_1 - Y_3^2 \widetilde{Y}_3^2\, d\alpha_3) \Big]\,,\\
A_{\alpha 58}\, d\mu_\alpha \;&=\; \frac{b_3}{2\,\Xi\, g^3} \Big[\mu_1^2\, W_4\, d(\mu_4^2) - \mu_4^2\, W_1\, d(\mu_1^2) - \mu_1^2\, \mu_4^2\, (Y_1^2 \widetilde{Y}_1^2\, d\alpha_1 + Y_2^2 \widetilde{Y}_2^2\, d\alpha_2) \Big]\,,\\
A_{\alpha 67}\, d\mu_\alpha \;&=\; \frac{b_3}{2\,\Xi\, g^3} \Big[\mu_3^2\, W_2\, d(\mu_2^2) - \mu_2^2\, W_3\, d(\mu_3^2) + \mu_2^2\, \mu_3^2\, (Y_1^2 \widetilde{Y}_1^2\, d\alpha_1 - Y_2^2 \widetilde{Y}_2^2\, d\alpha_3) \Big]\,,
\end{aligned}
\tag{4.19}
$$

[8]The factor of g^{-3} is introduced in the expressions below because we are using dimensionless coordinates on the unit sphere.

where

$$\begin{aligned}
W_1 &= \widetilde{Y}_2^2 \widetilde{Y}_3^2\, \mu_2^2 + \widetilde{Y}_1^2 \widetilde{Y}_3^2\, \mu_3^2 + \widetilde{Y}_1^2 \widetilde{Y}_2^2\, \mu_4^2\,, \\
W_2 &= Y_2^2 Y_3^2\, \mu_1^2 + \widetilde{Y}_1^2 Y_2^2\, \mu_3^2 + \widetilde{Y}_1^2 Y_3^2\, \mu_4^2\,, \\
W_3 &= Y_1^2 Y_3^2\, \mu_1^2 + Y_1^2 \widetilde{Y}_2^2\, \mu_2^2 + \widetilde{Y}_2^2 Y_3^2\, \mu_4^2\,, \\
W_4 &= Y_1^2 Y_2^2\, \mu_1^2 + Y_1^2 \widetilde{Y}_3^2\, \mu_2^2 + Y_2^2 \widetilde{Y}_3^2\, \mu_3^2
\end{aligned} \tag{4.20}$$

and

$$\alpha_1 = \mu_1^2 + \mu_2^2 \qquad \alpha_2 = \mu_1^2 + \mu_3^2\,, \qquad \alpha_3 = \mu_1^2 + \mu_4^2\,. \tag{4.21}$$

Note that the W_α defined above are related to the Z_α defined in ref. [19] as follows

$$W_\alpha = Z_\alpha - \mu_\alpha^2\,. \tag{4.22}$$

The coordinate indices $(5,6,7,8)$ on the A_{mnp} refer to the directions $(\phi_1, \phi_2, \phi_3, \phi_4)$ respectively, so $A_{(3)} = A_{\alpha 56}\, d\mu_\alpha \wedge d\phi_1 \wedge d\phi_2 + \cdots$.

The function Ξ is defined in equation (21) of ref. [19] as follows

$$\begin{aligned}
\Xi &= Y_1^2 Y_2^2 Y_3^2 \mu_1^4 + Y_1^2 \widetilde{Y}_2^2 \widetilde{Y}_3^2 \mu_2^4 + \widetilde{Y}_1^2 Y_2^2 \widetilde{Y}_3^2 \mu_3^4 + \widetilde{Y}_1^2 \widetilde{Y}_2^2 Y_3^2 \mu_4^4 \\
&\quad + (Y_2^2 \widetilde{Y}_2^2 + Y_3^2 \widetilde{Y}_3^2)(Y_1^2 \mu_1^2 \mu_2^2 + \widetilde{Y}_1^2 \mu_3^2 \mu_4^2) \\
&\quad + (Y_1^2 \widetilde{Y}_1^2 + Y_3^2 \widetilde{Y}_3^2)(Y_2^2 \mu_1^2 \mu_3^2 + \widetilde{Y}_2^2 \mu_2^2 \mu_4^2) \\
&\quad + (Y_1^2 \widetilde{Y}_1^2 + Y_2^2 \widetilde{Y}_2^2)(Y_3^2 \mu_1^2 \mu_4^2 + \widetilde{Y}_3^2 \mu_2^2 \mu_3^2).
\end{aligned} \tag{4.23}$$

5 Uplifting gauged STU supergravity to eleven dimensions

The uplift ansatz for A_{mnp}, given in (4.19) in the previous section, along with the previous results of ref. [19], allows us to complete the uplift of gauged STU supergravity to eleven dimensions. The uplift of the four-dimensional metric ds_4^2 was obtained in ref. [19], and is given by

$$\begin{aligned}
d\hat{s}_{11}^2 &= \Xi^{\frac13}\, ds_4^2 + \Xi^{\frac13}\, d\hat{s}_7^2 \\
&= \Xi^{\frac13}\, ds_4^2 + g^{-2}\, \Xi^{-\frac23} \Big[\sum_\alpha Z_\alpha\, (d\mu_\alpha^2 + \mu_\alpha^2\, D\phi_\alpha^2) + 2b_2\, b_3\, (\mu_1^2\, \mu_2^2\, D\phi_1\, D\phi_2 - \mu_3^2\, \mu_4^2\, D\phi_3\, D\phi_4) \\
&\quad + 2b_1\, b_3\, (\mu_1^2\, \mu_3^2\, D\phi_1\, D\phi_3 - \mu_2^2\, \mu_4^2\, D\phi_2\, D\phi_4) + 2b_1\, b_2\, (\mu_1^2\, \mu_4^2\, D\phi_1\, D\phi_4 - \mu_2^2\, \mu_3^2\, D\phi_2\, D\phi_3) \\
&\quad + \tfrac12 b_1^2 \Big((\mu_1\, d\mu_1 + \mu_2\, d\mu_2)^2 + (\mu_3\, d\mu_3 + \mu_4\, d\mu_4)^2 \Big) \\
&\quad + \tfrac12 b_2^2 \Big((\mu_1\, d\mu_1 + \mu_3\, d\mu_3)^2 + (\mu_2\, d\mu_2 + \mu_4\, d\mu_4)^2 \Big) \\
&\quad + \tfrac12 b_3^2 \Big((\mu_1\, d\mu_1 + \mu_4\, d\mu_4)^2 + (\mu_2\, d\mu_2 + \mu_3\, d\mu_3)^2 \Big) \Big]\,,
\end{aligned} \tag{5.1}$$

where ds_4^2 is the four-dimensional metric, the Z_α are defined in (4.20) and (4.22), and Ξ is defined in (4.23). The 1-forms $D\phi_\alpha$, which are defined by

$$D\phi_\alpha \equiv d\phi_\alpha - g\, A^\alpha_{(1)}\,, \tag{5.2}$$

manifest the expected $U(1)^4$ gauge invariance $A^\alpha_{(1)}(x) \longrightarrow A^\alpha_{(1)}(x) + d\Lambda^\alpha(x)$ that originates via the subset $\phi_\alpha \longrightarrow \phi_\alpha + g\,\Lambda^\alpha(x)$ of the eleven-dimensional coordinate transformations.

The 4-form field strength is given in equation (41) of ref. [19], which we reproduce here for convenience

$$\hat{F}_{(4)} \;=\; -2gU\epsilon_{(4)} + \hat{F}'_{(4)} + \hat{F}''_{(4)} + \hat{G}_{(4)}\,, \tag{5.3}$$

where $\epsilon_{(4)}$ is the volume form in the four-dimensional spacetime metric ds_4^2. The first term in (5.3) corresponds to the Freund-Rubin term, with

$$U = Y_1^2\,(\mu_1^2+\mu_2^2)+\widetilde{Y}_1^2\,(\mu_3^2+\mu_4^2)+Y_2^2\,(\mu_1^2+\mu_3^2)+\widetilde{Y}_2^2\,(\mu_2^2+\mu_4^2)+Y_3^2\,(\mu_1^2+\mu_4^2)+\widetilde{Y}_3^2\,(\mu_2^2+\mu_3^2)\,. \tag{5.4}$$

In terms of the components F_{ABCD}, this corresponds to $F_{\alpha\beta\gamma\delta}$.

The next term in the expression above is $\hat{F}'_{(4)} = d\hat{A}'_{(3)}$, where $\hat{A}'_{(3)}$ is obtained by making the replacements $d\phi_\alpha \to D\phi_\alpha$ in the expression for the internal projection of the 3-form potential (4.19). Thus we have [9]

$$\hat{A}'_{(3)} = \tfrac{1}{2} A_{\alpha\hat{\beta}\hat{\gamma}}\, d\mu_\alpha \wedge (d\phi_\beta - g\, A^\beta_{(1)}) \wedge (d\phi_\gamma - g\, A^\gamma_{(1)})\,, \tag{5.5}$$

where the components $A_{\alpha\hat{\beta}\hat{\gamma}}$ are given in (4.19).

The term $\hat{F}''_{(4)}$, which is given in equation (43) of ref. [19], is

$$\hat{F}''_{(4)} = -\frac{1}{2g^2}\,|W|^{-2} \sum_\alpha d\mu_\alpha^2 \wedge (d\phi_\alpha - g\, A^\alpha_{(1)}) \wedge R_\alpha\,, \tag{5.6}$$

where

$$\begin{aligned}
R_1 \;=\;& \tilde{Y}_1^2\,\tilde{Y}_2^2\,\tilde{Y}_3^2\,[P_0 *F^1_{(2)} + 2b_1\,b_2\,b_3\,F^1_{(2)}] + \tilde{Y}_1^2\,[P_1\,b_2\,b_3 *F^2_{(2)} + b_1\,(P_0 + 2b_2^2\,b_3^2)\,F^2_{(2)}] \\
&+\tilde{Y}_2^2\,[P_2\,b_1\,b_3 *F^3_{(2)} + b_2\,(P_0 + 2b_1^2\,b_3^2)\,F^3_{(2)}] + \tilde{Y}_3^2\,[P_3\,b_1\,b_2 *F^4_{(2)} + b_3\,(P_0 + 2b_1^2\,b_2^2)\,F^4_{(2)}]\,, \\
R_2 \;=\;& \tilde{Y}_1^2\,Y_2^2\,Y_3^2\,[P_0 *F^2_{(2)} + 2b_1\,b_2\,b_3\,F^2_{(2)}] + \tilde{Y}_1^2\,[P_1\,b_2\,b_3 *F^1_{(2)} + b_1\,(P_0 + 2b_2^2\,b_3^2)\,F^1_{(2)}] \\
&-Y_2^2\,[P_2\,b_1\,b_3 *F^4_{(2)} + b_2\,(P_0 + 2b_1^2\,b_3^2)\,F^4_{(2)}] - Y_3^2\,[P_3\,b_1\,b_2 *F^3_{(2)} + b_3\,(P_0 + 2b_1^2\,b_2^2)\,F^3_{(2)}]\,, \\
R_3 \;=\;& Y_1^2\,\tilde{Y}_2^2\,Y_3^2\,[P_0 *F^3_{(2)} + 2b_1\,b_2\,b_3\,F^3_{(2)}] - Y_1^2\,[P_1\,b_2\,b_3 *F^4_{(2)} + b_1\,(P_0 + 2b_2^2\,b_3^2)\,F^4_{(2)}] \\
&+\tilde{Y}_2^2\,[P_2\,b_1\,b_3 *F^1_{(2)} + b_2\,(P_0 + 2b_1^2\,b_3^2)\,F^1_{(2)}] - Y_3^2\,[P_3\,b_1\,b_2 *F^2_{(2)} + b_3\,(P_0 + 2b_1^2\,b_2^2)\,F^2_{(2)}]\,, \\
R_4 \;=\;& Y_1^2\,Y_2^2\,\tilde{Y}_3^2\,[P_0 *F^4_{(2)} + 2b_1\,b_2\,b_3\,F^4_{(2)}] - Y_1^2\,[P_1\,b_2\,b_3 *F^3_{(2)} + b_1\,(P_0 + 2b_2^2\,b_3^2)\,F^3_{(2)}] \\
&-Y_2^2\,[P_2\,b_1\,b_3 *F^2_{(2)} + b_2\,(P_0 + 2b_1^2\,b_3^2)\,F^2_{(2)}] + \tilde{Y}_3^2\,[P_3\,b_1\,b_2 *F^1_{(2)} + b_3\,(P_0 + 2b_1^2\,b_2^2)\,F^1_{(2)}]\,,
\end{aligned} \tag{5.7}$$

[9] Note that we have introduced the hatted indices $\hat{\alpha}$ such that $(\hat{1},\hat{2},\hat{3},\hat{4}) = (5,6,7,8)$.

and W and P_α are defined in (2.11).

Finally, the remaining term in (5.3) is given by

$$\begin{aligned}\hat{G}_{(4)} &= \frac{1}{2g}\,(2Y_1^{-1}\,{*dY_1} - \chi_1\,Y_1^4\,{*d\chi_1}) \wedge d(\mu_1^2+\mu_2^2)\\ &\quad+\frac{1}{2g}\,(2Y_2^{-1}\,{*dY_2} - \chi_2\,Y_2^4\,{*d\chi_2}) \wedge d(\mu_1^2+\mu_3^2)\\ &\quad+\frac{1}{2g}\,(2Y_3^{-1}\,{*dY_3} - \chi_3\,Y_3^4\,{*d\chi_3}) \wedge d(\mu_1^2+\mu_4^2)\,. \end{aligned} \tag{5.8}$$

Note that, as observed in ref. [19], the equations of motion for the $U(1)$ gauge fields in the STU theory are simply given by $d(|W|^{-2}\,R_\alpha) = 0$.

5.1 Consistency of the other uplift formulae with the general uplift ansätze

In this section, we revisit the uplift ansätze for some of the other components of the 4-form field strength. In particular, in light of the recent developments [16], we show that the ansatz for, in particular, the Freund-Rubin term takes a simple form.

In ref. [16], an uplift ansatz is given for the Freund-Rubin term and it is, moreover, conjectured that the term can be expressed in terms of the sum of the potential and its derivative according to the conjectured formula (3.12). In order to express the Freund-Rubin term in this form, we introduce the following paramaterisation of the scalars:

$$\eta_i = \lambda_i\,\cos\sigma_i\,, \qquad \zeta_i = \lambda_i\,\sin\sigma_i\,. \tag{5.9}$$

The scalars η_i are the coefficients of the self-dual tensors in the parameterisation of ϕ_{ijkl} in the unitary gauge, and ζ_i are the coefficients of the anti-self dual tensors.

It can now be seen that the function U given in (5.4) can be written as

$$U = -\frac{1}{8g^2}\,(V + \sum_i \frac{\partial V}{\partial \eta_i}\,\xi_i)\,, \tag{5.10}$$

where

$$\xi_1 = \mu_1^2+\mu_2^2-\mu_3^2-\mu_4^2\,, \qquad \xi_2 = \mu_1^2+\mu_3^2-\mu_2^2-\mu_4^2\,, \qquad \xi_3 = \mu_1^2+\mu_4^2-\mu_2^2-\mu_3^2\,. \tag{5.11}$$

Note also that, in terms of tensors ξ_i, $\hat{G}_{(4)}$ given in (5.8) can be written as

$$\hat{G}_{(4)} = \frac{1}{4g}\,\sum_i(\cos\sigma_i\,{*d\lambda_i} - \sin\sigma_i\,\sinh\lambda_i\,\cosh\lambda_i\,{*d\sigma_i}) \wedge d\xi_i\,. \tag{5.12}$$

The form of this is consistent with the ansatz for the $F_{\alpha\beta\gamma a}$ component of the field strength proposed in ref. [16], equation (3.10). Furthermore, it is now straightforward to see that the

Bianchi identity $d\hat{F}_{(4)} = 0$ is satisfied provided that the scalar fields satisfy their equations of motion.

The $\hat{F}'_{(4)}$ term is given by the Kaluza-Klein covariantisation of F_{mnpq}, $F_{\mu mnp}$ and, since $\hat{F}'_{(4)} = d\hat{A}'_{(3)}$ with $\hat{A}'_{(3)}$ given by equation (5.5), the second term on the right-hand side of the expression for $F_{\alpha\beta ab}$, (3.8).

Moreover, the $\hat{F}''_{(4)}$ is given by the Kaluza-Klein covariantisation of the first term on the right-hand side of the expression for $F_{\alpha\beta ab}$, (3.8).

Thus, we have established the direct, if somewhat intricate, relationship between the uplift ansätze of the gauged STU supergravity with the general ansätze for the full $\mathcal{N} = 8$ supergravity, given in ref. [16].

6 Consistent truncations of the STU embedding

There are two inequivalent consistent truncations of STU supergravity that are sometimes useful in their own right, and have the merit of being considerably simpler than the full STU theory. In each case, the bosonic sector of the truncated supergravity comprises gravity coupled to two $U(1)$ gauge fields, a dilatonic scalar and an axionic scalar. The two truncations, which we shall refer to as the $2+2$ truncation and the $3+1$ truncation, are implemented by setting

$$\begin{aligned} 2+2: \qquad & \lambda_1 = \lambda \qquad \sigma_1 = \sigma\,, \qquad \lambda_2 = \lambda_3 = \sigma_2 = \sigma_3 = 0\,, \\ & A^1_\mu = A^2_\mu = A_\mu\,, \qquad A^3_\mu = A^4_\mu = \widetilde{A}_\mu\,, \end{aligned} \tag{6.1}$$

$$\begin{aligned} 3+1: \qquad & \lambda_1 = \lambda_2 = \lambda_3 = \lambda\,, \qquad \sigma_1 = \sigma_2 = \sigma_3 = \sigma\,, \\ & A^1_\mu = \widetilde{A}_\mu\,, \qquad A^2_\mu = A^3_\mu = A^4_\mu = A_\mu\,. \end{aligned} \tag{6.2}$$

Note that we are not rescaling the gauge potentials in these truncations, so A_μ and $\widetilde{A}_\mu$ in the $2+2$ truncation and A_μ in the $3+1$ truncation will have non-canonically normalised kinetic terms. Also, in the $3+1$ truncation the scalar fields λ and σ will have non-canonically normalised kinetic terms. We have chosen not to rescale the truncated fields in order to avoid the occurrence of many $\sqrt{2}$ or $\sqrt{3}$ factors. It will be convenient to parameterise the scalar fields in the standard $SL(2,\mathbb{R})$ form, for which their kinetic terms are proportional to $-\partial\varphi^2 - e^{2\varphi}\,\partial\chi^2$, by introducing φ and χ that are related to λ and σ by

$$e^\varphi = \cosh\lambda + \sinh\lambda\,\cos\sigma\,, \qquad \chi\, e^\varphi = \sinh\lambda\,\sin\sigma\,, \tag{6.3}$$

in terms of which we have

$$Y^2 = e^{\varphi}\,, \qquad \widetilde{Y}^2 = e^{-\varphi} + \chi^2\, e^{\varphi}\,, \qquad b = \chi\, e^{\varphi}\,. \tag{6.4}$$

6.1 $2+2$ truncation

Here, with the fields truncated as in (6.1), we choose an adapted parameterisation for the the μ_α coordinates in which we take

$$\mu_1 = c\,\cos\tfrac{1}{2}\theta\,, \qquad \mu_2 = c\,\sin\tfrac{1}{2}\theta\,, \qquad \mu_3 = s\,\cos\tfrac{1}{2}\tilde{\theta}\,, \qquad \mu_4 = s\,\sin\tfrac{1}{2}\tilde{\theta}\,, \tag{6.5}$$

where we have also defined

$$c = \cos\xi\,, \qquad s = \sin\xi\,. \tag{6.6}$$

The four azimuthal angles ϕ_α will be parameterised by defining

$$\phi_1 = \tfrac{1}{2}(\psi+\phi)\,, \qquad \phi_2 = \tfrac{1}{2}(\psi-\phi)\,, \qquad \phi_3 = \tfrac{1}{2}(\widetilde{\psi}+\tilde{\phi})\,, \qquad \phi_4 = \tfrac{1}{2}(\widetilde{\psi}-\tilde{\phi})\,. \tag{6.7}$$

The unit 7-sphere metric then takes the form

$$d\Omega_7^2 = \sum_{\alpha=1}^{4}(d\mu_\alpha^2 + \mu_\alpha^2\, d\phi_\alpha^2) = d\xi^2 + \cos^2\xi\, d\Omega_3^2 + \sin^2\xi\, d\widetilde{\Omega}_3^2\,, \tag{6.8}$$

where

$$d\Omega_3^2 = \tfrac{1}{4}\Big[d\theta^2 + \sin^2\theta\, d\phi^2 + (d\psi + \cos\theta\, d\phi)^2\Big]\,, \quad d\widetilde{\Omega}_3^2 = \tfrac{1}{4}\Big[d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\phi}^2 + (d\widetilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi})^2\Big] \tag{6.9}$$

are the metrics on two unit 3-spheres.

The metric reduction ansatz (5.1) in the $2+2$ truncation thus reduces to

$$\begin{aligned} d\hat{s}_{11}^2 \; &= \Xi^{\frac{1}{3}}\, ds_4^2 + \frac{\Xi^{\frac{1}{3}}}{g^2}\Big\{d\xi^2 + \frac{\cos^2\xi}{4Z_3}\Big[d\theta^2 + \sin^2\theta\, d\phi^2 + (d\psi + \cos\theta\, d\phi - 2gA_{(1)})^2\Big] \\ &\qquad + \frac{\sin^2\xi}{4Z_1}\Big[d\tilde{\theta}^2 + \sin^2\tilde{\theta}\, d\tilde{\phi}^2 + (d\widetilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi} - 2g\widetilde{A}_{(1)})^2\Big]\Big\}\,, \end{aligned} \tag{6.10}$$

where, from (4.20) and (4.22), we now have

$$Z_1 \;=\; Z_2 = \widetilde{Y}^2\, \sin^2\xi + \cos^2\xi\,, \quad Z_3 = Z_4 = Y^2\, \cos^2\xi + \sin^2\xi\,, \quad \Xi = Z_1\, Z_3\,. \tag{6.11}$$

The 4-form reduction ansatz is now given by

$$\begin{aligned} \hat{F}_{(4)} \;&=\; -2g\, U\, \epsilon_{(4)} - \frac{sc}{g}\,({*d\varphi} - \chi\, e^{2\varphi}\, {*d\chi}) \wedge d\xi + d\hat{A}'_{(3)} + \hat{F}''_{(4)}\,, \\ \hat{A}'_{(3)} \;&=\; \frac{1}{g^3}\,\chi\, e^{\varphi}\Big[\frac{c^4}{Z_3}\,\Omega(A) - \frac{s^4}{Z_1}\,\widetilde{\Omega}(\widetilde{A})\Big]\,, \\ \hat{F}''_{(4)} \;&=\; \frac{c}{2g^2 Y^2}\Big[s d\xi \wedge (d\psi + \cos\theta\, d\phi - 2gA_{(1)}) + \tfrac{1}{2}c\,\sin\theta\, d\theta \wedge d\phi\Big] \wedge ({*F_{(2)}} + \chi\, Y^2\, F_{(2)}) \\ &\qquad - \frac{s}{2g^2\widetilde{Y}^2}\Big[c\, d\xi \wedge (d\widetilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi} - 2g\widetilde{A}_{(1)}) - \tfrac{1}{2}s\,\sin\tilde{\theta}\, d\tilde{\theta} \wedge d\tilde{\phi}\Big] \wedge ({*\widetilde{F}_{(2)}} - \chi\, Y^2\, \widetilde{F}_{(2)})\,, \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} U &= c^2\, Y^2 + s^2\, \widetilde{Y}^2 + 2\,, \\ \Omega(A) &= \tfrac{1}{8}\sin\theta\,(d\psi + \cos\theta\, d\phi - 2gA_{(1)}) \wedge d\theta \wedge d\phi\,, \\ \widetilde{\Omega}(\widetilde{A}) &= \tfrac{1}{8}\sin\tilde{\theta}\,(d\tilde{\psi} + \cos\tilde{\theta}\, d\tilde{\phi} - 2g\widetilde{A}_{(1)}) \wedge d\tilde{\theta} \wedge d\tilde{\phi}\,. \end{aligned} \tag{6.13}$$

The bosonic sector of the STU supergravity Lagrangian given in section 3 reduces under the present $2+2$ truncation to

$$\begin{aligned} \mathcal{L} = &\ R\,{*\mathbb{1}} - \tfrac{1}{2}{*d\varphi} \wedge d\varphi - \tfrac{1}{2}e^{2\varphi}\,{*d\chi} \wedge d\chi - V\,{*\mathbb{1}} \\ &- Y^{-2}\,{*F_{(2)}} \wedge F_{(2)} - \widetilde{Y}^{-2}\,{*\widetilde{F}_{(2)}} \wedge \widetilde{F}_{(2)} \\ &- \chi\, F_{(2)} \wedge F_{(2)} + \chi Y^2\, \widetilde{Y}^{-2}\, \widetilde{F}_{(2)} \wedge \widetilde{F}_2\,, \end{aligned} \tag{6.14}$$

where

$$V = -4g^2\,(Y^2 + \widetilde{Y}^2 + 4)\,. \tag{6.15}$$

It can be verified that the uplift of the 2+2 truncation that we obtained in this subsection agrees with the abelian truncation of the uplift of the $\mathcal{N}=4$ gauged $SO(4)$ supergravity that was obtained in ref. [17].

6.2 $3+1$ truncation

For the $3+1$ truncation of the fields of STU supergravity, given by equation (6.2), an appropriate adapted parameterisation for the μ_α coordinates is given by taking

$$\mu_1 = \cos\xi\,, \qquad \mu_a = \nu_a\,\sin\xi\,, \quad a = 2,3,4\,, \qquad \sum_a \nu_a^2 = 1\,. \tag{6.16}$$

The unit S^7 metric then takes the form

$$d\Omega_7^2 = d\xi^2 + \cos^2\xi\, d\phi_1^2 + \sin^2\xi\, d\Omega_5^2\,, \tag{6.17}$$

where

$$d\Omega_5^2 = \sum_{a=2}^{4}(d\nu_a^2 + \nu_a^2\, d\phi_a^2) = d\Sigma_2^2 + (d\psi + B)^2 \tag{6.18}$$

is the metric on the unit 5-sphere, and $d\Sigma_2^2$ is the standard "unit" Fubini-Study metric[10] on $\mathbb{CP}^2$, with $J = \frac{1}{2}dB$ being the Kähler form on $\mathbb{CP}^2$. In terms of ν_a and ϕ_a we have

$$(d\psi + B) = \sum_a \nu_a^2\, d\phi_a\,. \tag{6.19}$$

[10]The Fubini-Study metric is given in terms of the complex coordinates $z_a = \nu_a\, e^{\mathrm{i}\,\phi_a}$ on S^5 by $d\Sigma_2^2 = \sum_a d\bar{z}_a dz_a - |\sum_a \bar{z}_a dz_a|^2$. This "unit" metric is Einstein with $R_{ij} = 6g_{ij}$.

It will be convenient to define the two functions

$$\beta = Y^2\,(Y^2\,c^2 + \widetilde{Y}^2\,s^2)\,,\qquad \gamma = Y^4\,c^2 + s^2\,,\tag{6.20}$$

where, as before, we have defined $c = \cos\xi$ and $s = \sin\xi$. From (4.20), (4.22) and (4.23) we find

$$Z_1 = \widetilde{Y}^4\,s^2 + c^2\,,\qquad Z_a = \beta - b^2\,s^2\,\nu_a^2\,,\qquad \Xi = \beta^2\,Y^{-2}\,,\tag{6.21}$$

where, as before, $b = \chi\,e^{\varphi} = \chi\,Y^2$. The eleven-dimensional metric (5.1) now becomes, in the $3+1$ truncation we are considering here,

$$\begin{aligned} d\hat{s}_{11}^2 &= \Xi^{\frac13}\,ds_4^2 + g^{-2}\,\Xi^{-\frac23}\,\Big[\frac{\beta^2}{Y^4}\,d\xi^2 + \gamma\,s^2\,\Big((d\psi + B - gA_{(1)}) + \frac{b^2\,c^2}{\gamma}\,(d\phi_1 - g\widetilde{A}_{(1)})\Big)^2 \\ &\qquad + \beta\,s^2\,d\Sigma_2^2 + \frac{\beta^2\,c^2}{\gamma\,Y^4}\,(d\phi_1 - g\widetilde{A}_{(1)})^2\Big]\,. \end{aligned}\tag{6.22}$$

Substituting the $3+1$ truncation into the uplift formula for the 4-form in STU supergravity given in section 5, we now find

$$\hat{F}_{(4)} = -2gU\,\epsilon_{(4)} + \hat{G}_{(4)} + d\hat{A}'_{(3)} + \hat{F}''_{(4)}\,,\tag{6.23}$$

with

$$\begin{aligned} U &= 2(Y^2\,c^2 + \widetilde{Y}^2\,s^2) + Y^2\,, \\ \hat{G}_{(4)} &= -\frac{2sc}{g}\,({*d\varphi} - \chi\,e^{2\varphi}\,{*d\chi})\wedge d\xi\,, \\ \hat{A}'_{(3)} &= \frac{sc\,\chi}{g^3}\,d\xi\wedge(d\phi_1 - g\widetilde{A}_{(1)})\wedge(d\psi + B - gA_{(1)}) - \frac{s^2c^2}{\beta g^3}\,\chi\,e^{2\varphi}\,(d\phi_1 - g\widetilde{A}_{(1)})\wedge J \\ &\quad + \frac{s^4}{\beta g^3}\,\chi\,e^{2\varphi}\,(d\psi + B - gA_{(1)})\wedge J\,, \\ \hat{F}''_{(4)} &= \frac{sc}{g^2\,|W|^2}\,d\xi\wedge\widetilde{R}\wedge(d\phi_1 - g\widetilde{A}_{(1)}) - \frac{sc}{g^2\,|W|^2}\,d\xi\wedge R\wedge(d\psi + B - gA_{(1)}) \\ &\quad - \frac{s^2}{g^2\,|W|^2}\,R\wedge J\,, \end{aligned}\tag{6.24}$$

where, from (2.11),

$$|W|^2 = (1+4b^2)(1+b^2)^2\tag{6.25}$$

and from (5.7),

$$\widetilde{R} = R_1\,,\qquad R = R_2 = R_3 = R_4\,,\tag{6.26}$$

with

$$\begin{aligned} \widetilde{R} &= \widetilde{Y}^6\,[(1+3b^2)\,{*\widetilde{F}_{(2)}} + 2b^3\,\widetilde{F}_{(2)}] + 3b\,(1+b^2)\,\widetilde{Y}^2\,[b\,{*F_{(2)}} + (1+2b^2)\,F_{(2)}]\,, \\ R &= Y^2\,(1+b^2)^2\,[{*F_{(2)}} - 2b\,F_{(2)}] + b\,(1+b^2)\,\widetilde{Y}^2\,[b\,{*\widetilde{F}_{(2)}} + (1+2b^2)\,\widetilde{F}_{(2)}]\,. \end{aligned}\tag{6.27}$$

The bosonic sector of the STU supergravity Lagrangian given in section 3 reduces, under the $3+1$ truncation, to

$$\mathcal{L}_4 = R\,{*\mathbb{1}} - \tfrac{3}{2}\left(*d\varphi\wedge d\varphi + e^{2\varphi}\,{*d\chi}\wedge d\chi\right) - V\,{*\mathbb{1}} + \mathcal{L}_{KinA} + \mathcal{L}_{CS}\,, \qquad (6.28)$$

with

$$\begin{aligned} V &= -12g^2(Y^2+\widetilde{Y}^2),\\ \mathcal{L}_{KinA} &= -\frac{1}{2(1+4\chi^2\,e^{2\varphi})}\big[6\chi^2\,e^{\varphi}\,{*F_{(2)}}\wedge\widetilde{F}_{(2)} + e^{-3\varphi}(1+3\chi^2e^{2\varphi})(1+\chi^2e^{2\varphi})\,{*\widetilde{F}_{(2)}}\wedge\widetilde{F}_{(2)}\\ &\qquad\qquad +3e^{\varphi}\,{*F_{(2)}}\wedge F_{(2)}\big],\\ \mathcal{L}_{CS} &= -\frac{\chi}{(1+4\chi^2\,e^{2\varphi})}\big[-3e^{2\varphi}F_{(2)}\wedge F_{(2)} + 3(1+2\chi^2\,e^{2\varphi})\,\widetilde{F}_{(2)}\wedge F_{(2)}\\ &\qquad\qquad +\chi^2(1+\chi^2\,e^{2\varphi})\,\widetilde{F}_{(2)}\wedge\widetilde{F}_{(2)}\big]. \end{aligned} \qquad (6.29)$$

7 Conclusions

The existence of a consistent reduction of eleven-dimensional supergravity on S^7, to give four-dimensional $\mathcal{N}=8$ $SO(8)$ gauged supergravity, was first established in [11]. In that paper, the complete expression for the eleven-dimensional metric ansatz was obtained, and partial expressions also for the four-form field strength. The complete expressions for the uplifted four-form field strength were obtained recently [13–16]. Although the results are now complete, they are not necessarily convenient to use in practice, if, as is commonly the case, one is interested in uplifting four-dimensional configurations that involve only a subset of the full set of $\mathcal{N}=8$ supergravity fields. In the context, for example, of four-dimensional black hole solutions, it is almost always the case that only an abelian subsector of the $SO(8)$ gauge fields is turned on. Thus for many practical purposes, it suffices to know the uplift formulae for the truncation of the $SO(8)$ gauged supergravity to its abelian $U(1)^4$ subsector. The consistent truncation of the maximal gauged supergravity to this abelian subsector corresponds to the gauged $\mathcal{N}=2$ STU supergravity theory, whose bosonic sector comprises the metric, the four abelian $U(1)$ gauge fields, three dilatonic scalars and three axionic scalars. This theory is, for example, sufficient in order to describe the general class of rotating asymptotically AdS black holes, which can carry four electric and four magnetic charges.

In this paper, we have carried out the abelian truncation of the results for the uplifting of the gauged $\mathcal{N}=8$ theory, thereby obtaining fully explicit expressions for the lifting of the bosonic sector of the four-dimensional gauged STU supergravity to eleven dimensions.

We formulated the truncation in a notation that is adapted to the $U(1)^4$ isometries in the internal directions on the seven-sphere. In special cases where the gauge fields are either purely electric or purely magnetic, one can make a further consistent truncation (to a non-supersymmetric theory) in which the three axionic scalars vanish. This leads to enormous simplifications in the formulae, and they then reduce to ones that are given in ref. [18].

We also considered two distinct supersymmetric truncations of the STU supergravity, where, in each case, two gauge fields, a dilatonic scalar and an axionic scalar survive. The first, which we referred to as the $2+2$ truncation, is achieved by setting the four original gauge fields to be pairwise equal. The second, which we call the $3+1$ truncation, is achieved instead by setting three of the original four gauge fields equal. The geometric structure of the internal seven-sphere becomes particularly simple in these two truncations. In the $2+2$ truncation, the seven-sphere is described as a foliation by $S^3 \times S^3$ factors, with the two surviving gauge fields being associated with the two $U(1)$ isometries acting on the Hopf fibres in the two S^3 factors, viewed as $U(1)$ bundles over S^2. In the $3+1$ truncation the seven-sphere is instead described as a foliation by $S^5 \times S^1$ factors, with the two surviving gauge fields being associated with the $U(1)$ isometry acting on the Hopf fibres of S^5 viewed as a $U(1)$ bundle over $\mathbb{CP}^2$, and the $U(1)$ isometry of the S^1 factor. Consequently, the uplift formulae are much simpler for these truncations.

We hope that the new uplift ansätze presented in this paper for what are particularly interesting truncations of maximal $SO(8)$ gauged supergravity will be of use in future applications.

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A $U(1)^4$-invariant tensors in adapted coordinates

We shall use an adapted coordinate system $y^m = (\mu_\alpha, \phi_\rho)$ for S^7, with the direction cosines μ_α satisfying

$$\sum_{\alpha=1}^{4} \mu_\alpha^2 = 1. \tag{A.1}$$

These coordinates are related to the $\mathbb{R}^8$ coordinates as follows

$$x^1 + \mathrm{i}\, x^2 = \mu_1\, e^{\mathrm{i}\,\phi_1}\,, \quad x^3 + \mathrm{i}\, x^4 = \mu_2\, e^{\mathrm{i}\,\phi_2}\,, \quad x^5 + \mathrm{i}\, x^6 = \mu_3\, e^{\mathrm{i}\,\phi_3}\,, \quad x^7 + \mathrm{i}\, x^8 = \mu_4\, e^{\mathrm{i}\,\phi_4}\,. \tag{A.2}$$

The expressions for the various $U(1)^4$-invariant tensors become much more elegant in terms of these coordinates. First of all, we have the background round S^7 metric

$$d\Omega_7^2 = \mathring{g}_{mn} dy^m dy^n = \sum_\alpha (d\mu_\alpha^2 + \mu_\alpha^2\, d\phi_\alpha^2)\,. \tag{A.3}$$

Note that the inverse background metric in these coordinates is

$$\mathring{\partial}_7^2 = \mathring{g}^{mn} \partial_{y^m} \partial_{y^n} = \sum_{\alpha<\beta} (\mu_\alpha \partial_{\mu_\beta} - \mu_\beta \partial_{\mu_\alpha})^2 + \sum_\alpha \mu_\alpha^{-2}\, \partial_{\phi_\alpha}^2\,. \tag{A.4}$$

The vectors $\xi_m^{(i)}$, viewed as 1-forms, are then given by

$$\begin{aligned}
\xi_m^{(1)} dy^m &= \tfrac{3}{2} d(\mu_1^2 + \mu_2^2) = 3(\mu_1\, d\mu_1 + \mu_2\, d\mu_2) = -\tfrac{3}{2} d(\mu_3^2 + \mu_4^2) = -3(\mu_3\, d\mu_3 + \mu_4\, d\mu_4)\,, \\
\xi_m^{(2)} dy^m &= \tfrac{3}{2} d(\mu_1^2 + \mu_3^2) = 3(\mu_1\, d\mu_1 + \mu_3\, d\mu_3) = -\tfrac{3}{2} d(\mu_2^2 + \mu_4^2) = -3(\mu_2\, d\mu_2 + \mu_4\, d\mu_4)\,, \\
\xi_m^{(3)} dy^m &= \tfrac{3}{2} d(\mu_1^2 + \mu_4^2) = 3(\mu_1\, d\mu_1 + \mu_4\, d\mu_4) = -\tfrac{3}{2} d(\mu_2^2 + \mu_3^2) = -3(\mu_2\, d\mu_2 + \mu_3\, d\mu_3)\,.
\end{aligned} \tag{A.5}$$

The 2-index symmetric tensors $\xi_{mn}^{(i)}$ are given by

$$\begin{aligned}
\xi_{mn}^{(1)}\, dy^m dy^n &= -\tfrac{1}{4}(\mu_1^2 + \mu_2^2)(d\mu_1^2 + d\mu_2^2 + \mu_1^2 d\phi_1^2 + \mu_2^2 d\phi_2^2) \\
&\qquad + \tfrac{1}{4}(\mu_3^2 + \mu_4^2)(d\mu_3^2 + d\mu_4^2 + \mu_3^2 d\phi_3^2 + \mu_4^2 d\phi_4^2)\,, \\
\xi_{mn}^{(2)}\, dy^m dy^n &= -\tfrac{1}{4}(\mu_1^2 + \mu_3^2)(d\mu_1^2 + d\mu_3^2 + \mu_1^2 d\phi_1^2 + \mu_3^2 d\phi_3^2) \\
&\qquad + \tfrac{1}{4}(\mu_2^2 + \mu_4^2)(d\mu_2^2 + d\mu_4^2 + \mu_2^2 d\phi_2^2 + \mu_4^2 d\phi_4^2)\,, \\
\xi_{mn}^{(3)}\, dy^m dy^n &= -\tfrac{1}{4}(\mu_1^2 + \mu_4^2)(d\mu_1^2 + d\mu_4^2 + \mu_1^2 d\phi_1^2 + \mu_4^2 d\phi_4^2) \\
&\qquad + \tfrac{1}{4}(\mu_2^2 + \mu_3^2)(d\mu_2^2 + d\mu_3^2 + \mu_2^2 d\phi_2^2 + \mu_3^2 d\phi_3^2)\,.
\end{aligned} \tag{A.6}$$

The vectors $F_m^{(\alpha)}$ are given by

$$\begin{aligned}
F_m^{(1)} dy^m &= \mu_1^2\, d\phi_1 + \mu_2^2\, d\phi_2 + \mu_3^2\, d\phi_3 + \mu_4^2\, d\phi_4\,, \\
F_m^{(2)} dy^m &= \mu_1^2\, d\phi_1 + \mu_2^2\, d\phi_2 - \mu_3^2\, d\phi_3 - \mu_4^2\, d\phi_4\,, \\
F_m^{(3)} dy^m &= \mu_1^2\, d\phi_1 - \mu_2^2\, d\phi_2 + \mu_3^2\, d\phi_3 - \mu_4^2\, d\phi_4\,, \\
F_m^{(4)} dy^m &= \mu_1^2\, d\phi_1 - \mu_2^2\, d\phi_2 - \mu_3^2\, d\phi_3 + \mu_4^2\, d\phi_4\,.
\end{aligned} \tag{A.7}$$

The 2-forms $F^{(\alpha)}_{(2)} = dF^{(\alpha)}_{(1)}$ are given by

$$\begin{aligned}
\tfrac{1}{2}F^{(1)}_{mn}dy^m \wedge dy^n &= 2\mu_1 d\mu_1 \wedge d\phi_1 + 2\mu_2 d\mu_2 \wedge d\phi_2 + 2\mu_3 d\mu_3 \wedge d\phi_3 + 2\mu_4 d\mu_4 \wedge d\phi_4\,, \\
\tfrac{1}{2}F^{(2)}_{mn}dy^m \wedge dy^n &= 2\mu_1 d\mu_1 \wedge d\phi_1 + 2\mu_2 d\mu_2 \wedge d\phi_2 - 2\mu_3 d\mu_3 \wedge d\phi_3 - 2\mu_4 d\mu_4 \wedge d\phi_4\,, \\
\tfrac{1}{2}F^{(3)}_{mn}dy^m \wedge dy^n &= 2\mu_1 d\mu_1 \wedge d\phi_1 - 2\mu_2 d\mu_2 \wedge d\phi_2 + 2\mu_3 d\mu_3 \wedge d\phi_3 - 2\mu_4 d\mu_4 \wedge d\phi_4\,, \\
\tfrac{1}{2}F^{(4)}_{mn}dy^m \wedge dy^n &= 2\mu_1 d\mu_1 \wedge d\phi_1 - 2\mu_2 d\mu_2 \wedge d\phi_2 - 2\mu_3 d\mu_3 \wedge d\phi_3 + 2\mu_4 d\mu_4 \wedge d\phi_4\,.
\end{aligned} \tag{A.8}$$

The 3-forms $S^{(i)}$ are given by

$$\begin{aligned}
S^{(1)} &= -\tfrac{1}{2}\mu_1\mu_2\,(\mu_1 d\mu_2 - \mu_2 d\mu_1) \wedge d\phi_1 \wedge d\phi_2 + \tfrac{1}{2}\mu_3\mu_4\,(\mu_3 d\mu_4 - \mu_4 d\mu_3) \wedge d\phi_3 \wedge d\phi_4\,, \\
S^{(2)} &= -\tfrac{1}{2}\mu_1\mu_3\,(\mu_1 d\mu_3 - \mu_3 d\mu_1) \wedge d\phi_1 \wedge d\phi_3 + \tfrac{1}{2}\mu_2\mu_4\,(\mu_2 d\mu_4 - \mu_4 d\mu_2) \wedge d\phi_2 \wedge d\phi_4\,, \\
S^{(3)} &= -\tfrac{1}{2}\mu_1\mu_4\,(\mu_1 d\mu_4 - \mu_4 d\mu_1) \wedge d\phi_1 \wedge d\phi_4 + \tfrac{1}{2}\mu_2\mu_3\,(\mu_2 d\mu_3 - \mu_3 d\mu_2) \wedge d\phi_2 \wedge d\phi_3\,.
\end{aligned} \tag{A.9}$$

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