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The prospects are explored for testing Lorentz- and CPT-violating quantum electrodynamics in experiments with Penning traps. We present the Lagrange density of Lorentz-violating spinor electrodynamics with operators of mass dimensions up to six, and we discuss some of its properties. The theory is used to derive Lorentz- and CPT-violating perturbative shifts of the energy levels of a particle confined to a Penning trap. Observable signals are discussed for trapped electrons, positrons, protons, and antiprotons. Existing experimental measurements on anomaly frequencies are used to extract new or improved bounds on numerous coefficients for Lorentz and CPT violation, using sidereal variations of observables and comparisons between particles and antiparticles.

I. INTRODUCTION

A powerful approach to investigating the fundamental properties of a stable particle is to trap it for an extended period, which allows probing it in detail. Electromagnetic traps operate by taking advantage of a charge or magnetic moment to confine the particle using a suitable field configuration. For charged particles, the Penning trap is a standard tool. An idealized Penning trap involves a uniform magnetic field bounding the particle motion in the perpendicular plane, together with a quadrupole electrostatic field preventing escape along the axis. Penning traps can be used to achieve impressive sensitivities to properties of fundamental particles, as originally demonstrated by Dehmelt *et al.* in measurements of the electron $g-2$ factor and in a comparison of the electron and positron g factors to parts in a trillion [1, 2].

The high sensitivity offered by experiments with Penning traps implies they are well suited to precision studies of fundamental symmetries. This includes the foundational Lorentz and CPT invariances of relativity. Studies of these invariances have undergone a renaissance in recent years, following the observation that tiny violations of Lorentz symmetry could emerge in models unifying gravity with quantum physics such as string theory [3]. The potential opportunity to detect experimentally a physical effect arising from the Planck scale $M_P \simeq 10^{19}$ GeV has stimulated many new high-precision searches for relativity violations across various subfields of physics [4]. Here, we advance this active area of research by investigating the prospects for searches for Lorentz and CPT violation via spectroscopy of particles in Penning traps.

One possible approach to studying Lorentz and CPT violation is to propose a specific model and investigate its implications. However, given the current absence of compelling experimental evidence for Lorentz and CPT violation, it is advantageous to work within a general and realistic framework allowing for all possible types of violations, thereby offering a comprehensive treatment for prospective searches.

A general methodology for studying tiny signals arising as suppressed effects from an inaccessible sector is provided by effective field theory [5]. For Lorentz violation, the comprehensive realistic effective field theory can be constructed from General Relativity and the Standard Model of particle physics by adding to the action all Lorentz-violating operators, each contracted with a controlling coefficient that maintains coordinate independence of the physics [6, 7]. In this framework, known as the Standard-Model Extension (SME), Lorentz-violating operators of larger mass dimension d can be interpreted as higher-order effects appearing in the low-energy limit. The SME also describes general CPT-violating physics because the breaking of CPT symmetry in the context of effective field theory is accompanied by Lorentz violation [6, 8]. Restricting attention to operators of renormalizable dimension $d \leq 4$ yields the minimal SME, which in Minkowski spacetime is power-counting renormalizable. The experimental implications of any desired specific model that is compatible with effective field theory can be obtained from the SME framework by matching the model parameters to a suitable subset of the SME coefficients and adopting the corresponding experimental constraints [4, 9].

The minimal SME reveals that Lorentz and CPT violation can induce a variety of subtle but measurable effects in experiments studying the anomalous magnetic moment or charge-to-mass ratio of a particle confined to a Penning trap [10, 11]. These effects include shifts in the anomaly and cyclotron frequencies that can differ between particles and antiparticles and that can vary with sidereal time. Experimental searches for these SME effects that have been published to date compare the electron and positron anomaly frequencies [12], constrain sidereal signals in the electron anomaly and cyclotron frequencies [13], and measure the cyclotron frequency of the H^- ion relative to the antiproton [14, 15]. On the theory side, several treatments have been given of Penning-trap sensitivities to Lorentz and CPT signals both in the minimal SME and also for certain nonminimal SME terms involving interactions at $d = 5$ [10, 11, 16–18].

In the present work, we further the theoretical basis for studies of Lorentz and CPT symmetry in Penning traps by developing the relevant nonminimal sector of the SME, studying its properties, and determining its predicted signals for trapped electrons, positrons, protons, and antiprotons. The recent characterization and enumeration of effects arising when a Dirac fermion propagates in the presence of Lorentz-violating operators of arbitrary mass dimension d [19] provides a partial guide for investigations of nonminimal effects on particles in a Penning trap. However, the interactions of the particle with the electromagnetic fields in the trap can introduce additional types of nonminimal Lorentz violations beyond those associated with propagation, and these additional effects lack a systematic treatment in the literature to date. One goal of this work is to address this gap, by presenting and investigating the explicit Lagrange density for Lorentz-violating spinor electrodynamics that describes the behavior of a fermion coupled to the electromagnetic field in the presence of both minimal and nonminimal Lorentz and CPT violation with $d \leq 6$. More generally, investigations of nonminimal SME effects are of significance to various aspects of Lorentz and CPT violation, ranging from phenomenological implications of specific models involving noncommutative quantum field theory [20, 21] or supersymmetry [22] to more formal issues such as the stability and causality of Lorentz-violating quantum field theories [23] or their mathematical foundations in Riemann-Finsler geometry [24]. The theoretical aspects discussed here are thus of relevance beyond the immediate implications for experiments.

Another major goal of this work is to establish specific observables for both minimal and nonminimal Lorentz and CPT violation that are relevant to existing or near-future experiments on particles in Penning traps. We use perturbation theory to determine the dominant Lorentz- and CPT-violating shifts in the anomaly and cyclotron frequencies of electrons, positrons, protons, and antiprotons. Armed with this information, we revisit published experimental studies of Lorentz and CPT symmetry with Penning traps [12, 13] and extract some additional constraints. The perturbative analysis also reveals bounds on SME coefficients arising from other data, including measurements of the electron anomaly frequency [25] and of the proton and antiproton magnetic moments [26–28], and it permits identification of potential signals in forthcoming experiments with positrons [29] and antiprotons [30, 31]. Here, we extract constraints on SME coefficients from available data and provide tools for the analysis of future experiments. The results are complementary to existing and proposed studies of Lorentz and CPT violation involving measurements of the muon anomalous magnetic moment via magnetic confinement in a ring accelerator [32, 33], and more generally to constraints on nonminimal coefficients in the electron and proton sectors from experiments on hydrogen, antihydrogen, and related systems [34].

This work is organized as follows. In Sec. II, we present and investigate some properties of quantum electrodynamics (QED) with Lorentz- and CPT-violating operators of dimensions $d \leq 6$. The Lagrange density is given in Sec. II A, along with its relation to some special models in the literature. The issue of field redefinitions and physical observables is tackled in Sec. II B. We consider gauge-covariant invertible fermion redefinitions in Sec. II B 1, tabulating the effects of each possibility. In Sec. II B 2, the issue of absorbing a given fermion coupling to the electromagnetic field into other terms in the Lagrange density is addressed. The special case of field redefinitions and observables in the presence of a constant electromagnetic field, which is of prime importance in the context of Penning traps, is treated in Sec. II B 3. The experimental observables are affected by the noninertial nature of any laboratory frame on the Earth, and the necessary generic frame changes to convert results to the canonical Sun-centered frame are described in Sec. II C.

We next turn in Sec. III to applications of the theory to Penning-trap experiments. Theoretical aspects of this subject are addressed in Sec. III A in the context of trapped electrons, positrons, protons, and antiprotons. We begin in Sec. III A 1 by deriving the dominant Lorentz- and CPT-violating perturbative shifts of the energy levels of the trapped fermion, and then turn in Sec. III A 2 to a derivation of the effects on the cyclotron and anomaly frequencies of trapped particles. The experimental implications of these results are the subject of Sec. III B. Some conceptual issues for experimental analyses are considered in Sec. III B 1. In Sec. III B 2, we investigate existing and prospective signals for experiments, and we use the results together with published data to extract bounds on various SME coefficients, including some that were previously unconstrained. In Sec. IV, we summarize the work and provide some outlook. Finally, Appendix A contains some detailed results for the perturbative Lorentz- and CPT-violating energy shifts. The notation and conventions in this work follow those of Ref. [19], except as otherwise indicated. In particular, we work in natural units with $c = \hbar = 1$.

II. THEORY

In this section, we present the Lagrange density for the fermion sector of Lorentz-violating QED, incorporating operators with $d \leq 6$. The procedure for using field redefinitions to identify physical observables is discussed, and the effects of a key set of redefinitions are tabulated. Particular attention is paid to the special case of constant external field relevant to many experimental configurations, including those using a Penning trap discussed in this work.

A. Lagrange density

The Lorentz-violating QED for a single Dirac fermion field ψ of mass m_ψ and charge q coupled to the photon field A_μ can be constructed by adding to the action of conventional QED all terms that preserve U(1) gauge invariance formed from contractions of Lorentz-violating operators with coefficients for Lorentz violation [6]. The coefficients can be viewed as background fields that induce coordinate-independent Lorentz- and CPT-violating effects. For operators of arbitrary mass dimension d, the fermion sector of this theory can be specified via a Lagrange density of the form

$$\mathcal{L}_\psi = \tfrac{1}{2}\overline{\psi}(\gamma^\mu iD_\mu - m_\psi + \widehat{\mathcal{Q}})\psi + \text{h.c.}, \tag{1}$$

where $\widehat{\mathcal{Q}}$ is a 4×4 spinor matrix depending on the coefficients for Lorentz violation, the covariant derivative iD_α and the electromagnetic field strength $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$. The covariant derivative acting on the spinor takes the standard form $iD_\alpha\psi = (i\partial_\alpha - qA_\alpha)\psi$. Note that $\widehat{\mathcal{Q}}$ satisfies the hermiticity condition $\widehat{\mathcal{Q}} = \gamma_0\widehat{\mathcal{Q}}^\dagger\gamma_0$. In the limit of vanishing photon field A_α, the explicit form of $\widehat{\mathcal{Q}}$ for arbitrary d has been presented and studied in Ref. [19]. The analogous Lagrange density for the quadratic part of the pure-photon sector at arbitrary d is the subject of Ref. [35]. Similar treatments exist for the nonminimal neutrino [36] and gravity sectors [37].

In the present work, our focus is on operators having mass dimensions $d \le 6$, which are expected to generate the dominant physical effects beyond the minimal SME. The Lagrange density (1) can be decomposed as the sum of the usual Dirac Lagrange density $\mathcal{L}_0$ and a series of terms $\mathcal{L}^{(d)}$ arising from the expansion of $\widehat{\mathcal{Q}}$ in operators of mass dimension d,

$$\mathcal{L}_\psi = \mathcal{L}_0 + \mathcal{L}^{(3)} + \mathcal{L}^{(4)} + \mathcal{L}^{(5)} + \mathcal{L}^{(6)} + \ldots. \tag{2}$$

The explicit forms of the terms $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$ are given in the original papers constructing the minimal SME [6] and are reproduced here for convenience,

$$\mathcal{L}^{(3)} = -a^\mu\overline{\psi}\gamma_\mu\psi - b^\mu\overline{\psi}\gamma_5\gamma_\mu\psi - \tfrac{1}{2}H^{\mu\nu}\overline{\psi}\sigma_{\mu\nu}\psi, \tag{3}$$

and

$$\begin{aligned}\mathcal{L}^{(4)} = {} & \tfrac{1}{2}c^{\mu\alpha}\overline{\psi}\gamma_\mu iD_\alpha\psi + \text{h.c.}\\ & +\tfrac{1}{2}d^{\mu\alpha}\overline{\psi}\gamma_5\gamma_\mu iD_\alpha\psi + \text{h.c.}\\ & +\tfrac{1}{2}e^{\alpha}\overline{\psi}iD_\alpha\psi + \text{h.c.}\\ & +\tfrac{1}{2}if^{\alpha}\overline{\psi}\gamma_5 iD_\alpha\psi + \text{h.c.}\\ & +\tfrac{1}{4}g^{\mu\nu\alpha}\overline{\psi}\sigma_{\mu\nu} iD_\alpha\psi + \text{h.c.}\end{aligned} \tag{4}$$

These terms have been the subject of numerous investigations, and experimental constraints have been placed on many of the corresponding coefficients in several sectors of the SME [4].

At $d = 5$, two kinds of terms enter the Lagrange density $\mathcal{L}^{(5)}$, one involving only symmetrized covariant derivatives D_α and one involving the electromagnetic field strength $F_{\alpha\beta}$,

$$\mathcal{L}^{(5)} = \mathcal{L}^{(5)}_D + \mathcal{L}^{(5)}_F. \tag{5}$$

The former is given explicitly by

$$\begin{aligned}\mathcal{L}^{(5)}_D = {} & -\tfrac{1}{2}m^{(5)\alpha\beta}\overline{\psi}iD_{(\alpha}iD_{\beta)}\psi + \text{h.c.}\\ & -\tfrac{1}{2}im_5^{(5)\alpha\beta}\overline{\psi}\gamma_5 iD_{(\alpha}iD_{\beta)}\psi + \text{h.c.}\\ & -\tfrac{1}{2}a^{(5)\mu\alpha\beta}\overline{\psi}\gamma_\mu iD_{(\alpha}iD_{\beta)}\psi + \text{h.c.}\\ & -\tfrac{1}{2}b^{(5)\mu\alpha\beta}\overline{\psi}\gamma_5\gamma_\mu iD_{(\alpha}iD_{\beta)}\psi + \text{h.c.}\\ & -\tfrac{1}{4}H^{(5)\mu\nu\alpha\beta}\overline{\psi}\sigma_{\mu\nu} iD_{(\alpha}iD_{\beta)}\psi + \text{h.c.}\end{aligned} \tag{6}$$

The remaining piece is

$$\begin{aligned}\mathcal{L}^{(5)}_F = {} & -\tfrac{1}{2}m_F^{(5)\alpha\beta}F_{\alpha\beta}\overline{\psi}\psi - \tfrac{1}{2}im_{5F}^{(5)\alpha\beta}F_{\alpha\beta}\overline{\psi}\gamma_5\psi\\ & -\tfrac{1}{2}a_F^{(5)\mu\alpha\beta}F_{\alpha\beta}\overline{\psi}\gamma_\mu\psi - \tfrac{1}{2}b_F^{(5)\mu\alpha\beta}F_{\alpha\beta}\overline{\psi}\gamma_5\gamma_\mu\psi\\ & -\tfrac{1}{4}H_F^{(5)\mu\nu\alpha\beta}F_{\alpha\beta}\overline{\psi}\sigma_{\mu\nu}\psi.\end{aligned} \tag{7}$$

The two pieces $\mathcal{L}^{(5)}_D$ and $\mathcal{L}^{(5)}_F$ can be constructed as the symmetric and antisymmetric combinations involving two covariant derivatives because the electromagnetic field strength $F_{\alpha\beta}$ is obtained by commutation of covariant derivatives,

$$[iD_\alpha, iD_\beta] = -iqF_{\alpha\beta}. \tag{8}$$

Within each piece $\mathcal{L}^{(5)}_D$ and $\mathcal{L}^{(5)}_F$, the convenient separation of terms displayed in Eqs. (6) and (7) reflects the decomposition of a 4×4 spinor matrix using the standard 16-component gamma-matrix basis.

For the Lagrange density at $d = 6$, three types of terms appear,

$$\mathcal{L}^{(6)} = \mathcal{L}^{(6)}_D + \mathcal{L}^{(6)}_F + \mathcal{L}^{(6)}_{\partial F}. \tag{9}$$

The first involves only totally symmetrized combinations of three covariant derivatives,

$$\begin{aligned}\mathcal{L}^{(6)}_D = {} & \tfrac{1}{2}c^{(6)\mu\alpha\beta\gamma}\overline{\psi}\gamma_\mu iD_{(\alpha}iD_\beta iD_{\gamma)}\psi + \text{h.c.}\\ & +\tfrac{1}{2}d^{(6)\mu\alpha\beta\gamma}\overline{\psi}\gamma_5\gamma_\mu iD_{(\alpha}iD_\beta iD_{\gamma)}\psi + \text{h.c.}\\ & +\tfrac{1}{2}e^{(6)\alpha\beta\gamma}\overline{\psi}iD_{(\alpha}iD_\beta iD_{\gamma)}\psi + \text{h.c.}\\ & +\tfrac{1}{2}if^{(6)\alpha\beta\gamma}\overline{\psi}\gamma_5 iD_{(\alpha}iD_\beta iD_{\gamma)}\psi + \text{h.c.}\\ & +\tfrac{1}{4}g^{(6)\mu\nu\alpha\beta\gamma}\overline{\psi}\sigma_{\mu\nu} iD_{(\alpha}iD_\beta iD_{\gamma)}\psi + \text{h.c.}\end{aligned} \tag{10}$$

The second involves the field strength $F_{\alpha\beta}$,

$$\begin{aligned}\mathcal{L}^{(6)}_F = {} & \tfrac{1}{4}c_F^{(6)\mu\alpha\beta\gamma}F_{\beta\gamma}\big(\overline{\psi}\gamma_\mu iD_\alpha\psi + \text{h.c.}\big)\\ & +\tfrac{1}{4}d_F^{(6)\mu\alpha\beta\gamma}F_{\beta\gamma}\big(\overline{\psi}\gamma_5\gamma_\mu iD_\alpha\psi + \text{h.c.}\big)\\ & +\tfrac{1}{4}e_F^{(6)\alpha\beta\gamma}F_{\beta\gamma}\big(\overline{\psi}iD_\alpha\psi + \text{h.c.}\big)\\ & +\tfrac{1}{4}if_F^{(6)\alpha\beta\gamma}F_{\beta\gamma}\big(\overline{\psi}\gamma_5 iD_\alpha\psi + \text{h.c.}\big)\\ & +\tfrac{1}{8}g_F^{(6)\mu\nu\alpha\beta\gamma}F_{\beta\gamma}\big(\overline{\psi}\sigma_{\mu\nu} iD_\alpha\psi + \text{h.c.}\big).\end{aligned} \tag{11}$$

The remaining contributions involve the derivative $\partial_\alpha F_{\beta\gamma}$

of the field strength, and they take the form

$$\begin{aligned}
\mathcal{L}^{(6)}_{\partial F} &= -\tfrac{1}{2} m^{(6)\alpha\beta\gamma}_{\partial F} \partial_\alpha F_{\beta\gamma}\ \overline{\psi}\psi \\
&\quad -\tfrac{1}{2} i m^{(6)\alpha\beta\gamma}_{5\partial F} \partial_\alpha F_{\beta\gamma}\ \overline{\psi}\gamma_5\psi \\
&\quad -\tfrac{1}{2} a^{(6)\mu\alpha\beta\gamma}_{\partial F} \partial_\alpha F_{\beta\gamma}\ \overline{\psi}\gamma_\mu\psi \\
&\quad -\tfrac{1}{2} b^{(6)\mu\alpha\beta\gamma}_{\partial F} \partial_\alpha F_{\beta\gamma}\ \overline{\psi}\gamma_5\gamma_\mu\psi \\
&\quad -\tfrac{1}{4} H^{(6)\mu\nu\alpha\beta\gamma}_{\partial F} \partial_\alpha F_{\beta\gamma}\ \overline{\psi}\sigma_{\mu\nu}\psi. \qquad (12)
\end{aligned}$$

In constructing the above contributions to the Lagrange density $\mathcal{L}_\psi$, all of which are U(1) gauge invariant, the coefficients for Lorentz violation are assumed to be real and can be taken as constant in an inertial frame in the vicinity of the Earth [6, 7]. The dimension superscript (d) is suppressed on minimal-SME coefficients. Coefficients with subscript F or ∂F are associated with interactions directly involving the electromagnetic field strength or its derivative, and they can be present even if the particle has zero charge. The notation is chosen so that the indices μ, ν are associated with spin properties, while α, β, γ are associated with covariant momenta including field strengths. Parentheses on n indices imply symmetrization with a factor of $1/n!$. The index symmetries of the coefficients are otherwise evident by inspection.

Table I lists some properties of the terms appearing in the expansion of the Lagrange density (2) for $d \leq 6$. The first column gives the dimension of the Lorentz-violating operator, while the second lists the corresponding coefficient. The units of each coefficient are GeV^{4-d}. The CPT parity of the operator is presented in the third column. The final column displays the number of independent operators. Note that this counting incorporates the constraints from the Bianchi identity, which here is equivalent to the usual homogeneous Maxwell equations $\epsilon^{\alpha\beta\gamma\delta}\partial_\gamma F_{\alpha\beta} = 0$.

In the limit of vanishing A_α, the contributions (6) and (10) are the leading-order nonminimal terms in the general treatment of Dirac fermions in the presence of Lorentz-violating operators at arbitrary d, which includes various special models as limiting cases [19]. Experimental constraints on some of the corresponding coefficients in the electron, proton, and muon sectors have been obtained [4]. The same coefficients control all terms in $\mathcal{L}^{(5)}_D$ and $\mathcal{L}^{(6)}_D$, even when A_α is nonzero, so the corresponding constraints hold in any models built from these terms as well.

In contrast, the literature lacks a systematic treatment of $d \leq 6$ Lorentz-violating spinor couplings to the field strength $F_{\alpha\beta}$ and its derivative $\partial_\gamma F_{\alpha\beta}$. The set of operators involving these couplings in the above expressions for $\mathcal{L}^{(5)}_F$, $\mathcal{L}^{(6)}_F$, and $\mathcal{L}^{(6)}_{\partial F}$ provides a complete enumeration for $d \leq 6$ and encompasses all possible models of this type. A subset of these terms appears naturally in noncommutative QED [21], where the noncommutativity parameter $\theta^{\alpha\beta} \equiv -i[x^\alpha, x^\beta]$ generates coefficients for

TABLE I: Properties of terms in $\mathcal{L}^{(d)}$ for $d \leq 6$.

d	Coefficient	CPT	Number
3	a^μ	odd	4
	b^μ	odd	4
	$H^{\mu\nu}$	even	6
4	$c^{\mu\alpha}$	even	10
	$d^{\mu\alpha}$	even	10
	e^α	odd	4
	f^α	odd	4
	$g^{\mu\nu\alpha}$	odd	24
5	$m^{(5)\alpha\beta}$	even	10
	$m_5^{(5)\alpha\beta}$	even	10
	$a^{(5)\mu\alpha\beta}$	odd	40
	$b^{(5)\mu\alpha\beta}$	odd	40
	$H^{(5)\mu\nu\alpha\beta}$	even	60
	$m_F^{(5)\alpha\beta}$	even	6
	$m_{5F}^{(5)\alpha\beta}$	even	6
	$a_F^{(5)\mu\alpha\beta}$	odd	24
	$b_F^{(5)\mu\alpha\beta}$	odd	24
	$H_F^{(5)\mu\nu\alpha\beta}$	even	36
6	$c^{(6)\mu\alpha\beta\gamma}$	even	80
	$d^{(6)\mu\alpha\beta\gamma}$	even	80
	$e^{(6)\alpha\beta\gamma}$	odd	20
	$f^{(6)\alpha\beta\gamma}$	odd	20
	$g^{(6)\mu\nu\alpha\beta\gamma}$	odd	120
	$c_F^{(6)\mu\alpha\beta\gamma}$	even	96
	$d_F^{(6)\mu\alpha\beta\gamma}$	even	96
	$e_F^{(6)\alpha\beta\gamma}$	odd	24
	$f_F^{(6)\alpha\beta\gamma}$	odd	24
	$g_F^{(6)\mu\nu\alpha\beta\gamma}$	odd	144
	$m_{\partial F}^{(6)\alpha\beta\gamma}$	odd	20
	$m_{5\partial F}^{(6)\alpha\beta\gamma}$	odd	20
	$a_{\partial F}^{(6)\mu\alpha\beta\gamma}$	even	80
	$b_{\partial F}^{(6)\mu\alpha\beta\gamma}$	even	80
	$H_{\partial F}^{(6)\mu\nu\alpha\beta\gamma}$	odd	120

Lorentz violation at $d = 5$ and $d = 6$ according to

$$\begin{aligned}
m^{(5)\alpha\beta}_{5F} &\to -\tfrac{1}{2} m q \theta^{\alpha\beta}, \\
c^{(6)\mu\alpha\beta\gamma}_{F} &\to -\tfrac{1}{2} q (\eta^{\mu\alpha}\theta^{\beta\gamma} + 2\eta^{\mu[\beta}\theta^{\gamma]\alpha}). \qquad (13)
\end{aligned}$$

The work of Belich, Costa-Soares, Ferreira, and Helayël-Neto [38], which studies the special Lorentz-violating limits

$$a^{(5)\mu\alpha\beta}_{F} \to g\epsilon^{\mu\alpha\beta}{}_{\gamma} v^\gamma, \quad b^{(5)\mu\alpha\beta}_{F} \to -g_a \epsilon^{\mu\alpha\beta}{}_{\gamma} v^\gamma, \quad (14)$$

spawned numerous followup investigations of models restricted to specifically chosen Lorentz-violating operators with spinor couplings to $F_{\alpha\beta}$ [39]. Also, a model containing all $d=5$ operators that cannot be reduced via equations of motion to ones with $d<5$ has been given in Ref. [40], using a different organization of terms than adopted here.

Actual constraints on physical effects from $\mathcal{L}_F^{(5)}$, $\mathcal{L}_F^{(6)}$, and $\mathcal{L}_{\partial F}^{(6)}$ have so far been obtained on only a small part of the available coefficient space displayed in Table I [17, 18, 21, 41]. For anomalous magnetic moments, which are the focus of the sections that follow, the sole limits to date have been reported recently by Araujo, Casana, and Ferreira [17, 18], who consider in turn several special Lorentz-violating limits of the coefficient $H_F^{(5)\mu\nu\alpha\beta}$ given by

$$\begin{aligned}
H_F^{(5)\mu\nu\alpha\beta} &\to -2\lambda(K_F)^{\mu\nu\alpha\beta},\\
H_F^{(5)\mu\nu\alpha\beta} &\to -\lambda_A\epsilon_{\rho\sigma}{}^{\mu\nu}(K_F)^{\rho\sigma\alpha\beta},\\
H_F^{(5)\mu\nu\alpha\beta} &\to -2\lambda_1'(\eta^{\alpha[\mu}T^{\nu]\beta}-\eta^{\beta[\mu}T^{\nu]\alpha}),\\
H_F^{(5)\mu\nu\alpha\beta} &\to \tfrac{3}{4}\lambda_3(\eta^{\mu[\alpha}T^{\beta]\nu}-\eta^{\nu[\alpha}T^{\beta]\mu}),
\end{aligned} \tag{15}$$

where $(K_F)^{\rho\sigma\alpha\beta}$ is taken to have the symmetries of the Riemann tensor.

The above discussions of both the theoretical and experimental implications of terms with spinor couplings to $F_{\alpha\beta}$ are further convoluted by the possibility of removing the corresponding Lorentz-violating operators from the Lagrange density using field redefinitions. We show in Sec. II B below that this possibility, which has been overlooked in the literature to date, implies that only certain combinations of these terms can produce observable effects in experiments. Remarkably, it turns out that many specific spinor couplings to $F_{\alpha\beta}$ at finite d can be removed from observables in favor of other terms, including in particular the coefficient $H_F^{(5)\mu\nu\alpha\beta}$.

We emphasize that the Lagrange density (2) also contains all Lorentz-*invariant* fermion-photon couplings. These terms arise from components of the SME coefficients that are proportional to the Minkowski-metric or Levi-Civita tensors, both of which are Lorentz invariant. The only Lorentz-invariant terms arising in the minimal SME are

$$\mathcal{L}_{\rm LI}^{(4)}=\tfrac{1}{2}c^{(4)}\overline{\psi}\gamma^\mu iD_\mu\psi+\tfrac{1}{2}d^{(4)}\overline{\psi}\gamma_5\gamma^\mu iD_\mu\psi+{\rm h.c.} \tag{16}$$

These terms can be absorbed into the normalizations of the left- and right-handed components of the spinor field ψ. Both are typically assumed to vanish in the literature on Lorentz violation.

The Lorentz-invariant terms of mass dimension $d=5$ can be written explicitly as

$$\begin{aligned}
\mathcal{L}_{\rm LI}^{(5)} &= -\tfrac{1}{2}m^{(5)}\overline{\psi}(iD)^2\psi+{\rm h.c.}\\
&\quad -\tfrac{1}{2}im_5^{(5)}\overline{\psi}\gamma_5(iD)^2\psi+{\rm h.c.}\\
&\quad -H_{F,1}^{(5)}F^{\mu\nu}\overline{\psi}\sigma_{\mu\nu}\psi-H_{F,2}^{(5)}\widetilde{F}^{\mu\nu}\overline{\psi}\sigma_{\mu\nu}\psi,
\end{aligned} \tag{17}$$

where $\widetilde{F}^{\mu\nu}=\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}/2$. These terms include Lorentz-invariant contributions to the anomalous magnetic and electric moments of the spinor field ψ involving the coefficients $H_{F,1}^{(5)}$ and $H_{F,2}^{(5)}$. Finally, the Lorentz-invariant terms with $d=6$ are

$$\begin{aligned}
\mathcal{L}_{\rm LI}^{(6)} &= \tfrac{1}{6}c^{(6)}\overline{\psi}\gamma^\mu[iD_\mu(iD)^2+iD^\alpha iD_\mu iD_\alpha\\
&\qquad +(iD)^2 iD_\mu]\psi+{\rm h.c.}\\
&\quad +\tfrac{1}{2}c_{F,1}^{(6)}F^{\mu\nu}(\overline{\psi}\gamma_\mu iD_\nu\psi+{\rm h.c.})\\
&\quad +\tfrac{1}{2}c_{F,2}^{(6)}\widetilde{F}^{\mu\nu}(\overline{\psi}\gamma_\mu iD_\nu\psi+{\rm h.c.})\\
&\quad +\tfrac{1}{6}d^{(6)}\overline{\psi}\gamma_5\gamma^\mu[iD_\mu(iD)^2+iD^\alpha iD_\mu iD_\alpha\\
&\qquad +(iD)^2 iD_\mu]\psi+{\rm h.c.}\\
&\quad +\tfrac{1}{2}d_{F,1}^{(6)}F^{\mu\nu}(\overline{\psi}\gamma_5\gamma_\mu iD_\nu\psi+{\rm h.c.})\\
&\quad +\tfrac{1}{2}d_{F,2}^{(6)}\widetilde{F}^{\mu\nu}(\overline{\psi}\gamma_5\gamma_\mu iD_\nu\psi+{\rm h.c.})\\
&\quad -a_{\partial F,1}^{(6)}\partial_\alpha F^{\alpha\beta}\overline{\psi}\gamma_\beta\psi-a_{\partial F,2}^{(6)}\partial_\alpha\widetilde{F}^{\alpha\beta}\overline{\psi}\gamma_\beta\psi\\
&\quad -b_{\partial F,1}^{(6)}\partial_\alpha F^{\alpha\beta}\overline{\psi}\gamma_5\gamma_\beta\psi-b_{\partial F,2}^{(6)}\partial_\alpha\widetilde{F}^{\alpha\beta}\overline{\psi}\gamma_5\gamma_\beta\psi,
\end{aligned} \tag{18}$$

where the homogeneous Maxwell equations have been used.

Note that the possibility of using field redefinitions to remove some terms in the Lagrange density in favor of others also applies to the Lorentz-invariant operators in $\mathcal{L}_{\rm LI}^{(5)}$ and $\mathcal{L}_{\rm LI}^{(6)}$. One example in the next subsection illustrates this by absorbing the conventional couplings $H_{F,1}^{(5)}$ and $H_{F,2}^{(5)}$ for the anomalous magnetic and electric moments into other coefficients.

B. Field redefinitions

The freedom to choose canonical dynamical variables via suitable field redefinitions often implies that two seemingly different theories in fact describe the same physics. For example, in the context of the standard kinetic term for a Dirac fermion, a chiral rotation of the field ψ can absorb a possible term $-im_5\overline{\psi}\gamma_5\psi$ into the usual mass term modulo anomaly considerations, leaving m_ψ as the fermion mass.

In the context of the SME, field redefinitions reveal that some terms that naively appear to violate Lorentz symmetry have no measurable implications, while others are observable only in certain specific combinations [6, 7, 19, 42]. The simplest example involving Lorentz and CPT violation is a linear phase redefinition of the form $\psi=\exp(-ia^\mu x_\mu)\chi$, which physically redefines the zero of energy and momentum and can be used to eliminate the term $a^\mu\overline{\psi}\gamma_\mu\psi$ from $\mathcal{L}$ at leading order in Lorentz violation. We remark in passing that the contribution from a_μ is distinct from that due to a constant 4-potential A_μ because a_μ is gauge invariant and so cannot be removed by a gauge transformation.

In this subsection, we examine the effects of certain field redefinitions on the terms in $\mathcal{L}_\psi$ with $d\le 6$. Specific results are extracted for a constant electromagnetic field, which is the scenario of relevance for many experimental applications.

1. Fermion redefinitions

We consider here gauge-covariant field redefinitions amounting to renormalizations of ψ taking the form

$$\psi = (1+\widehat{\sigma})\psi', \tag{19}$$

where we allow $\widehat{\sigma}$ to depend on covariant derivatives. Under this transformation, the physics is invariant provided the Lorentz-violating terms in $\mathcal{L}$ remain perturbative, which holds if $\widehat{\sigma}$ itself is perturbative [19]. Note that this implies both the field strength $F_{\alpha\beta}$ and the coefficients for Lorentz violation must be small on the scale of the energies and momenta of interest.

For notational simplicity, it is convenient to work in momentum space, writing $p_\alpha = iD_\alpha$ and

$$[p_\alpha, p_\beta] = -iqF_{\alpha\beta}. \tag{20}$$

The redefinition (19) induces a new operator $\widehat{\mathcal{Q}}'$ from the Lagrange density (2),

$$\psi^\dagger\gamma_0(\gamma^\mu p_\mu - m_\psi + \widehat{\mathcal{Q}})\psi \approx \psi'^\dagger\gamma_0(\gamma^\mu p_\mu - m_\psi + \widehat{\mathcal{Q}}')\psi', \tag{21}$$

where

$$\widehat{\mathcal{Q}}' = \widehat{\mathcal{Q}} + (\gamma^\mu p_\mu - m_\psi)\widehat{\sigma} + \gamma_0\widehat{\sigma}^\dagger\gamma_0(\gamma^\mu p_\mu - m_\psi). \tag{22}$$

For convenience, we can separate $\widehat{\sigma}$ into a hermitian piece $\widehat{X}$ and an antihermitian piece $\widehat{Y}$ given by

$$\widehat{\sigma} = \widehat{X} + i\widehat{Y}, \ \ \widehat{X} = \tfrac{1}{2}(\widehat{\sigma} + \gamma_0\widehat{\sigma}^\dagger\gamma_0), \ \ \widehat{Y} = \tfrac{1}{2i}(\widehat{\sigma} - \gamma_0\widehat{\sigma}^\dagger\gamma_0). \tag{23}$$

Note that $\widehat{X}$ and $\widehat{Y}$ satisfy the hermiticity conditions $\widehat{X} = \gamma_0\widehat{X}^\dagger\gamma_0$, $\widehat{Y} = \gamma_0\widehat{Y}^\dagger\gamma_0$, in parallel with the hermiticity of the $\widehat{\mathcal{Q}}$ operator. Using these definitions, the shift $\delta\widehat{\mathcal{Q}} = \widehat{\mathcal{Q}}' - \widehat{\mathcal{Q}}$ in the modified Dirac operator $\widehat{\mathcal{Q}}$ arising from the field redefinition is found to be

$$\begin{aligned}
\delta\widehat{\mathcal{Q}} &= -2m_\psi\widehat{X} + \{p_\mu\gamma^\mu, \widehat{X}\} + i[p_\mu\gamma^\mu, \widehat{Y}] \\
&= -2m_\psi\widehat{X} + p_\mu\{\gamma^\mu, \widehat{X}\} + ip_\mu[\gamma^\mu, \widehat{Y}] \\
&\quad -[p_\mu, \widehat{X}]\gamma^\mu + i[p_\mu, \widehat{Y}]\gamma^\mu.
\end{aligned} \tag{24}$$

Explicit expressions for the shift $\delta\widehat{\mathcal{Q}}$ can be found via decomposition of $\widehat{X}$ and $\widehat{Y}$ in terms of the basis of 16 Dirac matrices and power series in p_α. First, we define

$$\begin{aligned}
\widehat{X} &= \widehat{X}_I\Gamma^I \\
&\equiv \widehat{X}_S + i\widehat{X}_P\gamma_5 + \widehat{X}_V^\mu\gamma_\mu + \widehat{X}_A^\mu\gamma_5\gamma_\mu + \tfrac{1}{2}\widehat{X}_T^{\mu\nu}\sigma_{\mu\nu}, \\
\widehat{Y} &= \widehat{Y}_I\Gamma^I \\
&\equiv \widehat{Y}_S + i\widehat{Y}_P\gamma_5 + \widehat{Y}_V^\mu\gamma_\mu + \widehat{Y}_A^\mu\gamma_5\gamma_\mu + \tfrac{1}{2}\widehat{Y}_T^{\mu\nu}\sigma_{\mu\nu}.
\end{aligned} \tag{25}$$

Here, the index I takes values S, P, V, A, T and is summed. Each component in these expressions can then in turn be expanded in powers of p_α,

$$\begin{aligned}
\widehat{X}_I^\varsigma &= X_I^\varsigma + X_I^{\varsigma\alpha}p_\alpha + X_I^{\varsigma\alpha\beta}p_\alpha p_\beta + X_I^{\varsigma\alpha\beta\gamma}p_\alpha p_\beta p_\gamma + \ldots, \\
\widehat{Y}_I^\varsigma &= Y_I^\varsigma + Y_I^{\varsigma\alpha}p_\alpha + Y_I^{\varsigma\alpha\beta}p_\alpha p_\beta + Y_I^{\varsigma\alpha\beta\gamma}p_\alpha p_\beta p_\gamma + \ldots,
\end{aligned} \tag{26}$$

where the index ς takes values that are null, μ, or $\mu\nu$ according to the Lorentz properties of the corresponding spinor matrix. Note that the ordering of the momenta in this expression is significant because they have nonzero commutators. Via this procedure, all the spin and momentum dependence is explicitly extracted and so the components appearing in the decomposition (26) are merely constants.

In studying the possible shifts $\delta\widehat{\mathcal{Q}}$ induced by field redefinitions, each of the constant components can be treated as inducing an independent field redefinition. Each of these is U(1) gauge covariant by construction. It suffices for present purposes to keep terms up to third order in p_α. Since there are two pieces $\widehat{X}$, $\widehat{Y}$, each of which has five spin components, each of which has four momentum components, we see that the above decomposition allows for 40 distinct field redefinitions in this language. Note that some redefinitions duplicate effects and some redefinitions induce multiple coefficient shifts. The redefinition X_S introduces an irrelevant scaling of the usual Dirac action, while the redefinition Y_S has no effect.

As a simple example, consider the field redefinition associated with $\widehat{Y} \supset Y_S^\alpha p_\alpha$. The result (24) implies

$$\delta\widehat{\mathcal{Q}} = -qY_S^{[\alpha}\eta^{\beta]\mu}F_{\alpha\beta}\gamma_\mu \leftrightarrow -\tfrac{1}{2}a_F^{(5)\mu\alpha\beta}F_{\alpha\beta}\gamma_\mu, \tag{27}$$

where the brackets around index pairs indicate antisymmetrization with a factor of 1/2. The last part of this expression gives the match to the corresponding term in the Lagrange density (7). The shift $\delta a_F^{(5)\mu\alpha\beta}$ induced in the coefficient $a_F^{(5)\mu\alpha\beta}$ via this field redefinition is therefore $\delta a_F^{(5)\mu\alpha\beta} = 2qY_S^{[\alpha}\eta^{\beta]\mu}$. One consequence of this result is that the trace of the mixed-symmetry representation in $a_F^{(5)\mu\alpha\beta}$ has no independent physical content and hence cannot be measured independently in experiments.

As a more involved example, consider the redefinition associated with $\widehat{X} \supset X_V^{\mu\alpha}p_\alpha\gamma_\mu$. We obtain

$$\begin{aligned}
\delta\widehat{\mathcal{Q}} &= -2m_\psi X_V^{\mu\alpha}p_\alpha\gamma_\mu + 2X_V^{\mu\alpha}p_{(\mu}p_{\alpha)} \\
&\quad -\tfrac{1}{2}q(X_V^{\mu[\alpha}\eta^{\beta]\nu} - X_V^{\nu[\alpha}\eta^{\beta]\mu})F_{\alpha\beta}\sigma_{\mu\nu}.
\end{aligned} \tag{28}$$

The correspondence to terms in the Lagrange density (2) yields

$$\begin{aligned}
\delta c^{\mu\alpha} &= -2m_\psi X_V^{\mu\alpha}, \\
\delta m^{(5)\alpha\beta} &= -2X_V^{(\alpha\beta)}, \\
\delta H_F^{(5)\mu\nu\alpha\beta} &= 2q(X_V^{\mu[\alpha}\eta^{\beta]\nu} - X_V^{\nu[\alpha}\eta^{\beta]\mu}).
\end{aligned} \tag{29}$$

The parameter $X_V^{\mu\alpha}$ can itself be decomposed into symmetric traceless, antisymmetric, and trace pieces, each of which can also be viewed as an independent redefinition. The above equations therefore reproduce the known result that the antisymmetric part of $c^{\mu\alpha}$ is unphysical [6] and reveal that the coefficient $m^{(5)\alpha\beta}$ can be removed by absorption into $X_V^{\mu[\alpha}\eta^{\beta]\nu}$.

We provide here one final explicit example, based on the redefinition associated with $\widehat{X} \supset \frac{1}{2}\sigma_{\mu\nu}X_T^{\mu\nu\alpha\beta}p_\alpha p_\beta$. Some calculation yields

$$\begin{aligned}
\delta\widehat{\mathcal{Q}} \;=\; & -m_\psi X_T^{\mu\nu(\alpha\beta)}\sigma_{\mu\nu}p_{(\alpha}p_{\beta)} \\
& +\tfrac{1}{2}im_\psi qX_T^{\mu\nu[\alpha\beta]}F_{\alpha\beta}\sigma_{\mu\nu} \\
& +X_T^{\rho\sigma(\alpha\beta}\epsilon^{\gamma)\mu}{}_{\rho\sigma}p_{(\alpha}p_\beta p_{\gamma)}\gamma_5\gamma_\mu \\
& +q(X_T^{\beta\mu(\alpha\gamma)}-X_T^{\gamma\mu(\alpha\beta)})F_{\beta\gamma}p_\alpha\gamma_\mu \\
& -\tfrac{1}{2}iqX_T^{\rho\sigma[\beta\gamma]}\epsilon^{\alpha\mu}{}_{\rho\sigma}F_{\beta\gamma}p_\alpha\gamma_5\gamma_\mu. \qquad (30)
\end{aligned}$$

This generates coefficient shifts given by

$$\begin{aligned}
\delta H^{(5)\mu\nu\alpha\beta} &= 2m_\psi X_T^{\mu\nu(\alpha\beta)}, \\
\delta H_F^{(5)\mu\nu\alpha\beta} &= -2im_\psi qX_T^{\mu\nu[\alpha\beta]}, \\
\delta d^{(6)\mu\alpha\beta\gamma} &= X_T^{\rho\sigma(\alpha\beta}\epsilon^{\gamma)\mu}{}_{\rho\sigma}, \\
\delta c_F^{(6)\mu\alpha\beta\gamma} &= 2q(X_T^{\beta\mu(\alpha\gamma)}-X_T^{\gamma\mu(\alpha\beta)}), \\
\delta d_F^{(6)\mu\alpha\beta\gamma} &= iqX_T^{\rho\sigma[\beta\gamma]}\epsilon^{\mu\alpha}{}_{\rho\sigma}. \qquad (31)
\end{aligned}$$

Among the implications of these equations is that the coefficient $H_F^{(5)\mu\nu\alpha\beta}$, which controls $d=5$ spinor couplings to $F_{\alpha\beta}$ and has been a popular subject of investigation in the literature, can be absorbed into the coefficient $d_F^{(6)\mu\alpha\beta\gamma}$. This point is discussed further in a more general context in the following subsection.

A related and striking observation is that the standard Lorentz-invariant terms describing anomalous magnetic and electric dipole moments can be removed from the Lagrange density by using a special limit of the redefinition (31). Suppose a fermion is described by the conventional Dirac Lagrange density plus the specific coupling in Eq. (17) involving $H_{F,1}^{(5)}$. Choosing

$$X_T^{\mu\nu[\alpha\beta]} = -\frac{i}{m_\psi q}H_{F,1}^{(5)}(\eta^{\mu[\alpha}\eta^{\beta]\nu}-\eta^{\nu[\alpha}\eta^{\beta]\mu}) \qquad (32)$$

and performing the corresponding redefinition removes the operator $F^{\mu\nu}\overline{\psi}\sigma_{\mu\nu}\psi$ with coupling $H_{F,1}^{(5)}$ in favor of the operator with coupling $d_{F,2}^{(6)}$ in the Lagrange density (18). With a similar redefinition, the coupling $H_{F,2}^{(5)}$ in $\mathcal{L}_{\rm LI}^{(5)}$ can be absorbed into the coupling $d_{F,1}^{(6)}$ in $\mathcal{L}_{\rm LI}^{(6)}$. Explicitly, these redefinitions implement the transformations

$$\begin{aligned}
(H_{F,1}^{(5)}, d_{F,2}^{(6)}\equiv 0) &\;\to\; (H_{F,1}^{(5)}\equiv 0, d_{F,2}^{(6)}=2H_{F,1}^{(5)}/m_\psi), \\
(H_{F,2}^{(5)}, d_{F,1}^{(6)}\equiv 0) &\;\to\; (H_{F,2}^{(5)}\equiv 0, d_{F,1}^{(6)}=-2H_{F,2}^{(5)}/m_\psi), \qquad (33)
\end{aligned}$$

TABLE II: Effects of field redefinitions for $d\leq 6$.

d	Shift	Field redefinition
3	δa^μ	$2m_\psi X_V^\mu$
	δb^μ	$2m_\psi X_A^\mu$
	$\delta H^{\mu\nu}$	$2m_\psi X_T^{\mu\nu}$
4	$\delta c^{\mu\alpha}$	$2X_S\eta^{\mu\alpha}, -2m_\psi X_V^{\mu\alpha}$
	$\delta d^{\mu\alpha}$	$X_T^{\nu\rho}\epsilon^{\mu\alpha}{}_{\nu\rho}, 2Y_P\eta^{\mu\alpha}, -iY_T^{\nu\rho}\epsilon^{\mu\alpha}{}_{\nu\rho}$
	δe^α	$-2m_\psi X_S^\alpha, 2X_V^\alpha$
	δf^α	$-2m_\psi X_P^\alpha$
	$\delta g^{\mu\nu\alpha}$	$2X_A^\rho\epsilon_\rho{}^{\mu\nu\alpha}, 2m_\psi X_T^{[\mu\nu]\alpha},$ $-4Y_V^{[\mu}\eta^{\nu]\alpha}, 2iY_A^\rho\epsilon_\rho{}^{\mu\nu\alpha}$
5	$\delta m^{(5)\alpha\beta}$	$2m_\psi X_S^{(\alpha\beta)}, -2X_V^{(\alpha\beta)}$
	$\delta m_5^{(5)\alpha\beta}$	$2m_\psi X_P^{(\alpha\beta)}, 2Y_A^{(\alpha\beta)}$
	$\delta a^{(5)\mu\alpha\beta}$	$2X_S^{(\alpha}\eta^{\beta)\mu}, 2m_\psi X_V^{\mu(\alpha\beta)}, 2Y_T^{\mu(\alpha\beta)}$
	$\delta b^{(5)\mu\alpha\beta}$	$2m_\psi X_A^{\mu(\alpha\beta)}, X_T^{\nu\rho(\alpha}\epsilon^{\beta)\mu}{}_{\nu\rho}, iY_P^{(\alpha}\eta^{\beta)\mu}$
	$\delta H^{(5)\mu\nu\alpha\beta}$	$2m_\psi X_T^{\mu\nu(\alpha\beta)}, 4Y_V^{[\mu\vert(\alpha}\eta^{\beta)\vert\nu]}$
	$\delta m_F^{(5)\alpha\beta}$	$-2im_\psi qX_S^{[\alpha\beta]}, -2qY_V^{[\alpha\beta]}$
	$\delta m_{5F}^{(5)\alpha\beta}$	$-2im_\psi qX_P^{[\alpha\beta]}, -2qX_A^{[\alpha\beta]}$
	$\delta a_F^{(5)\mu\alpha\beta}$	$-2iqm_\psi X_V^{\mu[\alpha\beta]}, 2qX_T^{\mu[\alpha\beta]}, 2qY_S^{[\alpha}\eta^{\beta]\mu}$
	$\delta b_F^{(5)\mu\alpha\beta}$	$-2qX_P^{[\alpha}\eta^{\beta]\mu}, -2im_\psi qX_A^{\mu[\alpha\beta]}, qY_T^{\nu\rho[\alpha}\epsilon^{\beta]\mu}{}_{\nu\rho}$
	$\delta H_F^{(5)\mu\nu\alpha\beta}$	$2q(X_V^{\mu[\alpha}\eta^{\beta]\nu}-X_V^{\nu[\alpha}\eta^{\beta]\mu}), 4X_A^{\rho[\alpha}\epsilon^{\beta]\mu\nu}{}_\rho,$ $-2im_\psi qX_T^{\mu\nu[\alpha\beta]}, -2qY_A^{\rho[\alpha}\epsilon^{\beta]\mu\nu}{}_\rho$
6	$\delta c^{(6)\mu\alpha\beta\gamma}$	$2X_S^{(\alpha\beta}\eta^{\gamma)\mu}, -2m_\psi X_V^{\mu(\alpha\beta\gamma)}, Y_T^{\mu(\alpha\beta\gamma)}$
	$\delta d^{(6)\mu\alpha\beta\gamma}$	$-2m_\psi X_A^{\mu(\alpha\beta\gamma)}, X_T^{\nu\rho(\alpha\beta}\epsilon^{\gamma)\mu}{}_{\nu\rho}, -2iY_P^{(\alpha\beta}\eta^{\gamma)\mu}$
	$\delta e^{(6)\alpha\beta\gamma}$	$-2m_\psi X_S^{(\alpha\beta\gamma)}, 4X_V^{(\alpha\beta\gamma)}$
	$\delta f^{(6)\alpha\beta\gamma}$	$-2m_\psi X_P^{(\alpha\beta\gamma)}, -2Y_A^{\alpha\beta\gamma}$
	$\delta g^{(6)\mu\nu\alpha\beta\gamma}$	$-2X_A^{\rho(\alpha\beta}\epsilon^{\gamma)\mu\nu}{}_\rho, -2m_\psi X_T^{\mu\nu(\alpha\beta\gamma)}$ $-4Y_V^{[\mu\vert(\alpha\beta}\eta^{\gamma)\vert\nu]}$
	$\delta c_F^{(6)\mu\alpha\beta\gamma}$	$-2iqX_S^{[\beta\gamma]}\eta^{\alpha\mu}, 4im_\psi qX_V^{\mu\langle\alpha[\beta\gamma]\rangle},$ $2q(X_T^{\beta\mu(\alpha\gamma)}-X_T^{\gamma\mu(\alpha\beta)}),$ $-2q(Y_S^{(\alpha\beta)}\eta^{\gamma\mu}-Y_S^{(\alpha\gamma)}\eta^{\beta\mu}), -2iqY_T^{\mu\alpha[\beta\gamma]}$
	$\delta d_F^{(6)\mu\alpha\beta\gamma}$	$2q(X_P^{(\alpha\beta)}\eta^{\gamma\mu}-X_P^{(\alpha\gamma)}\eta^{\beta\mu}), 4im_\psi qX_A^{\mu\langle\alpha[\beta\gamma]\rangle},$ $-iqX_T^{\nu\rho[\beta\gamma]}\epsilon^{\alpha\mu}{}_{\nu\rho}, 2qY_P^{[\beta\gamma]}\eta^{\alpha\mu},$ $-q(Y_T^{\nu\rho(\alpha\beta)}\epsilon^{\gamma\mu}{}_{\nu\rho}-Y_T^{\nu\rho(\alpha\gamma)}\epsilon^{\beta\mu}{}_{\nu\rho})$
	$\delta e_F^{(6)\alpha\beta\gamma}$	$4im_\psi qX_S^{\langle\alpha[\beta\gamma]\rangle}, -4iqX_V^{\alpha[\beta\gamma]},$ $2q(Y_V^{\beta(\alpha\gamma)}-Y_V^{\gamma(\alpha\beta)})$
	$\delta f_F^{(6)\alpha\beta\gamma}$	$4im_\psi qX_P^{\langle\alpha[\beta\gamma]\rangle}, 2q(X_A^{\beta(\alpha\gamma)}-X_A^{\gamma(\alpha\beta)}),$ $2iqY_A^{\alpha[\beta\gamma]}$
	$\delta g_F^{(6)\mu\nu\alpha\beta\gamma}$	$-4q(X_V^{[\mu\vert(\alpha\beta)}\eta^{\gamma\vert\nu]}-X_V^{[\mu\vert(\alpha\gamma)}\eta^{\beta\vert\nu]}),$ $2iqX_A^{\rho[\beta\gamma]}\epsilon^{\alpha\mu\nu}{}_\rho, 4im_\psi qX_T^{\mu\nu\langle\alpha[\beta\gamma]\rangle},$ $4iqY_V^{\mu[\beta\gamma]}\eta^{\alpha\nu},$ $-2q(Y_A^{\rho(\alpha\beta)}\epsilon^{\gamma\mu\nu}{}_\rho-Y_A^{\rho(\alpha\gamma)}\epsilon^{\beta\mu\nu}{}_\rho)$

thereby moving the usual $d=5$ Lorentz-invariant anomalous magnetic and electric dipole moments to the $d=6$ nonminimal Lorentz-invariant sector.

For the general case, the results of each field redefinition considered in turn are displayed in Table II. The first column gives the dimension of the operator. The second column shows the coefficient shift being considered. The third column indicates the structure of the field redefinition implementing the coefficient shift. Most coefficients are shifted by more than one field redefinition, and the different redefinitions are separated by commas. Notable exceptions are the coefficients appearing in $\mathcal{L}^{(6)}_{\partial F}$, for which the corresponding operators play no role in the field redefinitions due to the Bianchi identity and which are therefore omitted from the table. As before, parentheses and brackets around n indices imply symmetrization and antisymmetrization, respectively, with a factor of $1/n!$ included. A few terms involve the specific index combination of three indices that we denote by chevrons,

$$\langle\alpha\beta\gamma\rangle \equiv \tfrac{1}{2}(\alpha\beta\gamma + \beta\gamma\alpha - \gamma\alpha\beta). \tag{34}$$

Following standard convention, vertical bars are used to denote indices omitted from the symmetrization or antisymmetrization.

2. Absorption of couplings to $F_{\alpha\beta}$

As an interesting and potentially useful example of the application of field redefinitions, we investigate in this subsection the possibility of absorbing a given spinor coupling to $F_{\alpha\beta}$ into other terms in the Lagrange density. The discussion considers operators of any mass dimension d.

The terms of primary interest for this illustrative calculation are the gauge-invariant spinor couplings to the field strength $F_{\alpha\beta}$, where $F_{\alpha\beta}$ may be nonconstant. For definiteness, we focus here on terms involving exactly one power of $F_{\alpha\beta}$, rather than those nonlinear in $F_{\alpha\beta}$ or those involving derivatives of $F_{\alpha\beta}$. In momentum space, the corresponding operators can be collected in a quantity $\widehat{\mathcal{Q}}_F$ taking the form

$$\widehat{\mathcal{Q}}_F = \sum_{d>4} k_I^{(d)\varsigma\alpha\beta\alpha_1\ldots\alpha_{d-5}} F_{\alpha\beta}\; p_{(\alpha_1}\ldots p_{\alpha_{d-5})}\Gamma^I_\varsigma, \tag{35}$$

where $k_I^{\varsigma\alpha\beta\alpha_1\ldots\alpha_j}$ are F-type coefficients for Lorentz violation. As before, the index I takes values S, P, V, A, T, while ς is null or takes values μ or $\mu\nu$. For example, the operators appearing in the expression (7) for $\mathcal{L}^{(5)}_F$ and the expression (11) for $\mathcal{L}^{(6)}_F$ are reproduced by particular terms in the series (35).

For definiteness, we examine here the term $-2m_\psi\widehat{X}$ in the shift (24) of $\delta\widehat{\mathcal{Q}}$ and investigate its implications for terms in the series (35). The quantity $\widehat{X}$ can be expanded according to the expression (25), and then each resulting piece $\widehat{X}^\varsigma_I$ can itself be expanded in covariant momenta following the definition (26). A given term in the expansion (26) of $\widehat{X}^\varsigma_I$ can be viewed as a sum of irreducible representations obtained by decomposing the product of momenta. The commutator (20) of any two momenta produces a factor of $F_{\alpha\beta}$, so the expansion (26) can be seen to contain a series of terms involving powers of $F_{\alpha\beta}$. Here, the interest lies in terms with a single factor of $F_{\alpha\beta}$, corresponding to the representation with all momenta symmetrized except for a single pair. Denoting this representation by $\{\alpha\beta\ldots\}$, the expansion (26) contains

$$\begin{aligned}\widehat{X}^\varsigma_I \;\supset\; & -\tfrac{1}{2}iq(X_I^{\varsigma\alpha\beta}F_{\alpha\beta} + X_I^{\varsigma\alpha\beta\gamma}F_{\{\alpha\beta}p_{\gamma\}} \\ & \qquad + X_I^{\varsigma\alpha\beta\gamma\delta}F_{\{\alpha\beta}p_\gamma p_{\delta\}} + \ldots) \\ = & -\tfrac{1}{2}iq(X_I^{\varsigma\alpha\beta}F_{\alpha\beta} + X_I^{\varsigma\{\alpha\beta\gamma\}}F_{\alpha\beta}p_\gamma \\ & \qquad + X_I^{\varsigma\{\alpha\beta\gamma\delta\}}F_{\alpha\beta}p_{(\gamma}p_{\delta)} + \ldots).\end{aligned} \tag{36}$$

It follows that

$$\widehat{X} \supset -\tfrac{1}{2}iq\sum_{d>4} X_I^{\varsigma\{\alpha\beta\alpha_1\ldots\alpha_{d-5}\}}F_{\alpha\beta}\; p_{(\alpha_1}\ldots p_{\alpha_{d-5})}\Gamma^I_\varsigma. \tag{37}$$

Comparison of this expression with the form (35) of the operator $\widehat{\mathcal{Q}}_F$ confirms that $\delta\widehat{\mathcal{Q}}$ and $\widehat{\mathcal{Q}}_F$ contain operators of the same general form. So a choice of $\widehat{X}$ exists at each d that creates from the conventional Dirac equation a variety of linear spinor couplings to $F_{\alpha\beta}$. Equivalently, the corresponding terms in $\widehat{\mathcal{Q}}_F$ can be absorbed into other terms in the Lagrange density.

The other terms in $\mathcal{L}_\psi$ that are associated with the redefinition (37) include ones involving different spinor couplings to $F_{\alpha\beta}$. Notice that no operators involving only symmetrized covariant momenta can appear, as $\widehat{X}$ is already linear in $F_{\alpha\beta}$. Since $\widehat{Y} = 0$ for the redefinition (37) and since the commutators (20) imply that the result of $[p_\mu, \widehat{X}]$ is second order in $F_{\alpha\beta}$, the terms involving different linear spinor couplings to $F_{\alpha\beta}$ arise from the anticommutator $p_\mu\{\gamma^\mu, \widehat{X}\}$ with the extra momentum p_μ symmetrized with those in $\widehat{X}$. This reveals that the resulting shift (24) in $\widehat{\mathcal{Q}}$ contains terms at first order in $F_{\alpha\beta}$ given by

$$\begin{aligned}\delta\widehat{\mathcal{Q}} \;\supset\; & im_\psi q\sum_{d>4} X_I^{\varsigma\{\alpha\beta\alpha_1\ldots\alpha_{d-5}\}}F_{\alpha\beta}\Big(p_{(\alpha_1}\ldots p_{\alpha_{d-5})}\Gamma^I_\varsigma \\ & \qquad -\frac{1}{2m_\psi}p_{(\mu}p_{\alpha_1}\ldots p_{\alpha_{d-5})}\{\gamma^\mu, \Gamma^I_\varsigma\}\Big).\end{aligned} \tag{38}$$

Any given linear spinor coupling to $F_{\alpha\beta}$ at dimension d is therefore paired with another at dimension $d+1$. This implies that certain linear $F_{\alpha\beta}$ couplings of mass dimension d can be absorbed into others of dimension $d+1$. The results are analogous to those obtained for the noninteracting case in Sec. II B of Ref. [19].

Note that other choices of $\widehat{X}$ can mix linear spinor couplings to $F_{\alpha\beta}$ and operators with symmetrized covariant momenta, via the commutator $[p_\mu, \widehat{X}]\gamma^\mu$ and anticommutator $p_\mu\{\gamma^\mu, \widehat{X}\}$ terms in Eq. (24). This can be seen directly from Table II. It implies more than one type of redefinition can be used to absorb certain linear spinor couplings to $F_{\alpha\beta}$, which has potential consequences for the interpretation of models involving these couplings.

In the applications below to studies of Lorentz and CPT violation with Penning traps, we keep all relevant terms rather than simplifying calculations by absorbing some couplings via field redefinitions. Although more labor intensive, this reveals directly the combinations of measurable coefficients and has the added benefit of permitting an extra check on calculations by verifying consistency with the redefinitions shown in Table II.

3. *Scenario with constant $F_{\alpha\beta}$*

For many experimental applications, including those to Penning traps discussed in the sections to follow, the predominant part of the electromagnetic field strength is constant in magnitude and direction in the laboratory frame. In this scenario, the Lagrange density $\mathcal{L}_\psi$ presented in Sec. II A reduces to a simpler form for calculational purposes. We remark in passing that a similar interpretation to what follows can also be envisaged for more general scenarios involving nonconstant $F_{\alpha\beta}$, whenever $F_{\alpha\beta}$ plays the role of a fixed background rather than a dynamical field.

The requirement of constant $F_{\alpha\beta}$,

$$D_\gamma F_{\alpha\beta} \equiv \partial_\gamma F_{\alpha\beta} = 0, \qquad (39)$$

immediately eliminates the contributions $\mathcal{L}^{(6)}_{\partial F}$ to $\mathcal{L}_\psi$ presented in Eq. (12). Moreover, it also implies that the linear couplings to $F_{\alpha\beta}$ in the Lagrange densities (7) and (11) can be reinterpreted in terms of simpler couplings in the laboratory frame. As an explicit example, consider the coefficient $a_F^{(5)\mu\alpha\beta}$ appearing in $\mathcal{L}_F^{(5)}$. This coefficient is contracted with $F_{\alpha\beta}$, so when $F_{\alpha\beta}$ is constant the combination $a_F^{(5)\mu\alpha\beta}F_{\alpha\beta}$ effectively behaves like a contribution to the coefficient a^μ in the minimal Lagrange density (3), involving a coupling of mass dimension three instead of five.

This line of reasoning shows that most of the Lagrange densities $\mathcal{L}_F^{(5)}$ and $\mathcal{L}_F^{(6)}$ can be absorbed into the terms $\mathcal{L}^{(3)}$ and $\mathcal{L}^{(4)}$ when applied to scenarios with constant $F_{\alpha\beta}$, via the replacements

$$\begin{aligned}
a^\mu &\rightarrow a^\mu + \tfrac{1}{2}a_F^{(5)\mu\alpha\beta}F_{\alpha\beta},\\
b^\mu &\rightarrow b^\mu + \tfrac{1}{2}b_F^{(5)\mu\alpha\beta}F_{\alpha\beta},\\
H^{\mu\nu} &\rightarrow H^{\mu\nu} + \tfrac{1}{2}H_F^{(5)\mu\nu\alpha\beta}F_{\alpha\beta},\\
c^{\mu\alpha} &\rightarrow c^{\mu\alpha} + \tfrac{1}{2}c_F^{(6)\mu\alpha\beta\gamma}F_{\beta\gamma},\\
d^{\mu\alpha} &\rightarrow d^{\mu\alpha} + \tfrac{1}{2}d_F^{(6)\mu\alpha\beta\gamma}F_{\beta\gamma},\\
e^{\alpha} &\rightarrow e^{\alpha} + \tfrac{1}{2}e_F^{(6)\alpha\beta\gamma}F_{\beta\gamma},\\
f^{\alpha} &\rightarrow f^{\alpha} + \tfrac{1}{2}f_F^{(6)\alpha\beta\gamma}F_{\beta\gamma},\\
g^{\mu\nu\alpha} &\rightarrow g^{\mu\nu\alpha} + \tfrac{1}{2}g_F^{(6)\mu\nu\alpha\beta\gamma}F_{\beta\gamma}.
\end{aligned} \qquad (40)$$

The remaining terms involving the coefficients $m_F^{(5)\alpha\beta}$ and $m_{5F}^{(5)\alpha\beta}$ can also be absorbed, but into the fermion mass instead. When $F_{\alpha\beta}$ is constant, the combination $m_F^{(5)\alpha\beta}F_{\alpha\beta}$ represents a contribution to the Dirac mass, while $m_{5F}^{(5)\alpha\beta}F_{\alpha\beta}$ acts as a chiral mass term. The latter can be removed by a chiral transformation with parameter θ determined by

$$\psi \rightarrow e^{-i\theta\gamma_5}\psi, \quad \tan\theta = \frac{m_{5F}^{(5)\alpha\beta}F_{\alpha\beta}}{2m_\psi + m_F^{(5)\alpha\beta}F_{\alpha\beta}}. \qquad (41)$$

This transformation leaves invariant the usual Dirac kinetic term, and it has no leading-order effect on other Lorentz-violating terms because it differs from the identity only by powers of coefficients for Lorentz violation. The absorption of the coefficients $m_F^{(5)\alpha\beta}$ and $m_{5F}^{(5)\alpha\beta}$ is thereby found to be equivalent to the replacement

$$\begin{aligned}
m_\psi &\rightarrow \sqrt{(m_\psi + \tfrac{1}{2}m_F^{(5)\alpha\beta}F_{\alpha\beta})^2 + (\tfrac{1}{2}m_{5F}^{(5)\alpha\beta}F_{\alpha\beta})^2}\\
&\approx m_\psi + \tfrac{1}{2}m_F^{(5)\alpha\beta}F_{\alpha\beta}
\end{aligned} \qquad (42)$$

at leading order in Lorentz violation. Note that the coefficient $m_{5F}^{(5)\alpha\beta}$ is unobservable at this order.

The above discussion demonstrates that the spinor couplings to constant $F_{\alpha\beta}$ in Lorentz-violating QED with $d \leq 6$ reduce to terms in the minimal QED extension of Ref. [6]. However, the fermion mass and the minimal coefficients for Lorentz violation become dependent on $F_{\alpha\beta}$ according to the results (40) and (42). The operators with symmetrized covariant momenta appearing in the Lagrange densities $\mathcal{L}_D^{(5)}$ and $\mathcal{L}_D^{(6)}$ are unaffected by this argument.

In the discussions below analyzing experiments with Penning traps, the method described in this subsection is used to simplify calculations with terms containing a factor of $F_{\alpha\beta}$. As expected, the results obtained are consistent with direct perturbative calculations that explicitly keep the terms in $\mathcal{L}_\psi$ involving spinor couplings to constant $F_{\alpha\beta}$.

C. Frame changes

This subsection outlines some generic considerations involving the frame changes that appear in performing an analysis for violations of rotation invariance. More specific details for these and also other types of searches for Lorentz and CPT violation using Penning traps are provided in subsequent parts of this work.

Tests of Lorentz and CPT symmetry with a trapped particle effectively investigate its properties under rotations or boosts, or compare its behavior to that of a trapped antiparticle. Since boosts close under commutation into rotations, it is impossible to break Lorentz invariance without also breaking rotation invariance: even if the physics predicted by a particular model is isotropic in a special frame, any boost to another frame reintroduces anisotropic effects. Also, as CPT violation in realistic effective field theory is accompanied by Lorentz violation [6, 8], it follows that CPT violation comes with rotation violation as well. Tests of rotation symmetry are therefore of particular importance in the search for Lorentz and CPT violation.

The explicit form of a coefficient for Lorentz violation depends on the inertial frame of the observer. Comparing different experiments thus involves comparing results in a standard frame. The canonical frame adopted in the literature is the Sun-centered celestial-equatorial frame [43], which has the origin of its time coordinate T defined as the 2000 vernal equinox. The cartesian coordinates $X^J \equiv (X, Y, Z)$ in this frame are specified as having the Z axis aligned along the rotation axis of the Earth and the X axis pointing from the Earth to the Sun, with the Y axis completing a right-handed coordinate system. The Sun-centered frame is well suited as a standard frame because it is essentially inertial during typical experimental time scales and because its axes are conveniently chosen for laboratory studies.

In any inertial frame in the vicinity of the Earth, including the canonical Sun-centered frame, the coefficients for Lorentz violation can be assumed to be constants in time and space [6, 7]. However, the Earth rotates in this frame, and so the coefficients for Lorentz violation change with sidereal time when observed in the laboratory [44]. As a result, experimental observables for Lorentz violation can oscillate in time at harmonics of the Earth's sidereal frequency $\omega_\oplus \simeq 2\pi/(23\text{ h } 56\text{ min})$, with their amplitudes and phases controlled by the coefficients.

To establish the time dependence of the coefficients appearing in an experiment located on the Earth's surface, it is useful to introduce a standard laboratory frame with time coordinate t and cartesian coordinates $x^j \equiv (x, y, z)$ [43]. The origin of t can be defined conveniently for a given laboratory. A useful choice is to match t with the local sidereal time $T_\oplus$, defined to have origin at a chosen moment when the y axis lies along the Y axis. This is offset from the time T in the Sun-centered frame by any chosen integer number of sidereal rotations of the Earth and by an additional shift

$$T_0 \equiv T - T_\oplus \simeq \frac{(66.25^\circ - \lambda)}{360^\circ}(23.934\text{ hr}), \qquad (43)$$

where λ is the longitude of the laboratory in degrees. The spatial axes in the standard laboratory frame are defined with the x-axis pointing to local south, the y-axis pointing to local east, and the z-axis pointing to the local zenith. To obtain dominant effects, both the boost $\beta_\oplus \simeq 10^{-4}$ of the Earth relative to the Sun-centered frame and the boost $\beta_L \simeq 10^{-6}$ of the laboratory due to the rotation of the Earth can be treated as negligible. The relationship $x^j = R^{jJ}x^J$ between the coordinates x^j in the laboratory frame and the coordinates x^J in the Sun-centered frame is then given by the $T_\oplus$-dependent rotation matrix [43]

$$R^{jJ} = \begin{pmatrix} \cos\chi\cos\omega_\oplus T_\oplus & \cos\chi\sin\omega_\oplus T_\oplus & -\sin\chi \\ -\sin\omega_\oplus T_\oplus & \cos\omega_\oplus T_\oplus & 0 \\ \sin\chi\cos\omega_\oplus T_\oplus & \sin\chi\sin\omega_\oplus T_\oplus & \cos\chi \end{pmatrix}. \qquad (44)$$

This matrix generates the harmonic time dependences of the coefficients for Lorentz violation observed in the laboratory frame.

For many laboratory experiments, it is also convenient to introduce an apparatus frame with cartesian coordinates $x^a \equiv (x^1, x^2, x^3)$. We denote the corresponding unit vectors by $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$. For example, in the experiments with Penning traps discussed below, the x^3 axis is taken to be aligned with the uniform trapping magnetic field. This may subtend a nonzero angle to the local zenith specified in the standard laboratory frame by $\hat{z}$, so that $\hat{x}_3 \cdot \hat{z} \neq 0$. The relationship $x^a = R^{aj}x^j$ connecting the standard laboratory coordinates (x, y, z) to the apparatus coordinates (x^1, x^2, x^3) then involves a rotation matrix R^{aj}, which can be specified in general as the product of three Euler rotations for suitable Euler angles α, β, and γ,

$$\begin{aligned} R^{aj}(T) &= \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \\ &\quad \times \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \qquad (45)$$

Combining the above results reveals that the coordinates in the Sun-centered frame are related to those in the apparatus frame by

$$x^a(T_\oplus) = R^{aj}R^{jJ}(T_\oplus)X^J. \qquad (46)$$

Expressions relating any given coefficients for Lorentz violation in the two frames can be obtained from this result.

In addition to its sidereal rotation, the Earth's revolution about the Sun induces further time variations in the coefficients for Lorentz violation in the laboratory and apparatus frames. These variations occur at harmonics of the annual frequency, and they arise due to the boost $\beta_\oplus \simeq 10^{-4}$ of the Earth in the Sun-centered frame. Effects also arise from the boost $\beta_L \simeq 10^{-6}$ of the laboratory due to the rotation of the Earth. All these effects are suppressed by one or more powers of the boost. Nonetheless, they typically introduce experimental sensitivities to coefficients for Lorentz violation beyond those observable via pure rotations, as has been demonstrated in the literature [33, 34, 45–47]. They can also be of larger magnitude than effects suppressed by other mechanisms, such as those involving couplings to electromagnetic fields in the apparatus. However, to retain a reasonable scope for the present work, we disregard boost effects in our analysis below of experiments with Penning traps, focusing instead on sidereal signals arising from rotations. A treatment of boost effects for trapped particles is feasible in principle and would make an excellent subject for a future work.

III. APPLICATION TO PENNING TRAPS

In this section, the Lagrange density (1) is used as the starting point for an analysis of the sensitivity to Lorentz and CPT violation attainable in experiments with Penning traps. We apply perturbation theory to determine the shifts in energy levels for electrons, positrons, protons, and antiprotons arising in the presence of coefficients for Lorentz violation. This leads to expressions for the dominant shifts in the cyclotron and anomaly frequencies of trapped particles and permits studies of experimental effects. Observable signals arise from sidereal variations and comparisons of particle and antiparticle properties. We apply the results to published data from Penning traps to obtain first constraints on several SME coefficients, and we investigate the potential signals in some forthcoming experiments.

A. Theory

For precision experiments on particles in Penning traps, the transitions of primary interest involve the energy levels created by the constant magnetic field of the trap. It is therefore appropriate to base the theoretical analysis on the idealized scenario of a relativistic charged quantum particle moving in a uniform magnetic field, for which the unperturbed eigenenergies are the relativistic Landau levels. The signals of experimental interest are energy-level shifts rather than transition probabilities. The dominant effects from Lorentz and CPT violation can thus be treated as perturbative energy-level shifts, to be added to the independent perturbations generated by radiative corrections in conventional QED such as the splitting induced by the anomalous magnetic moment of the trapped particle.

The unperturbed eigenenergies and eigenfunctions in the absence of Lorentz violation or radiative corrections can be found by solving the minimally coupled Dirac equation for a spin-1/2 fermion of mass m and charge $q \equiv \sigma|q|$ of fixed sign σ in a constant magnetic field. For definiteness, we choose the apparatus frame such that the magnetic field $\boldsymbol{B} = B\hat{x}_3$ lies along the positive x^3 axis, and we fix the gauge so that the electromagnetic potential is $A^\mu = (0, x_2 B, 0, 0) = (0, -x^2 B, 0, 0)$. The spin-up and spin-down eigenstates form the two stacks of relativistic Landau levels, which are degenerate except for the ground state. We denote the level number as $n = 0, 1, 2, 3, \ldots$ and label the fermion spin relative to the magnetic field by $s = +1$ and $s = -1$ for up and down, respectively. The stacks of levels are similar for the antifermion, but with spin labels reversed.

At the nth level, the stationary eigenstates $\chi_{n,s}$ for the positive-energy fermion are associated with eigenenergies

$$E_{n,s} = \sqrt{m^2 + p_3^2 + (2n + 1 - \sigma s)|qB|}. \tag{47}$$

The energy eigenvalues for the corresponding antifermion eigenstates $\chi^c_{n,s}$ take the same form, but with σ being the opposite sign. For example, this equation encodes the information that the ground state for electrons or antiprotons is $E_{0,-1}$ with spin down, while that for positrons or protons is $E_{0,+1}$ with spin up.

The four-component eigenspinors $\chi_{n,s}$ are given by

$$\begin{aligned}
\chi_{n,+1} &= N_{n,+1} \begin{pmatrix} (m + E_{n,+1}) u_n \\ 0 \\ p_3 u_n \\ -\sqrt{|qB|}\; u_{n+1} \end{pmatrix}, \\
\chi_{n,-1} &= N_{n,-1} \begin{pmatrix} 0 \\ (m + E_{n,-1}) u_n \\ -2n\sqrt{|qB|}\; u_{n-1} \\ -p_3 u_n \end{pmatrix},
\end{aligned} \tag{48}$$

where the functions $u_n(\zeta)$ are defined as

$$u_n(\zeta) = \exp\left(ip_1 x^1 + ip_3 x^3\right) \exp\left(-\zeta^2/2\right) H_n(\zeta), \tag{49}$$

in terms of the Hermite polynomials $H_n(\zeta)$, with

$$\zeta = \sqrt{|qB|} \left(x^2 + \frac{p_1}{\sigma|qB|} \right). \tag{50}$$

The normalization factors $N_{n,s}$ are

$$N_{n,s} = \sqrt{\frac{\sqrt{|qB|}}{\sqrt{\pi}\; 2^{n+1} n!\; E_{n,s}(m + E_{n,s}) L^2}}, \tag{51}$$

where a cutoff L has been adopted along the x^1 and x^3 directions. The corresponding antifermion eigenstates $\chi^c_{n,s}$ are found to be

$$\begin{aligned}
\chi^c_{n,+1} &= N_{n,+1} \begin{pmatrix} (m + E_{n,+1}) u_n \\ 0 \\ p_3 u_n \\ 2n\sqrt{|qB|}\; u_{n-1} \end{pmatrix}, \\
\chi^c_{n,-1} &= N_{n,-1} \begin{pmatrix} 0 \\ (m + E_{n,-1}) u_n \\ \sqrt{|qB|}\; u_{n+1} \\ -p_3 u_n \end{pmatrix},
\end{aligned} \tag{52}$$

where the various quantities are defined as before but involve the opposite value of σ. These positive-energy antifermion eigenstates can be obtained from the negative-energy fermion solutions by charge conjugation in the usual way.

1. Perturbative energy shift

The unperturbed eigenstates (48) can be used to calculate the perturbative shifts of the particle eigenenergies once the perturbation hamiltonian $\delta\mathcal{H}$ is known. However, a direct construction of $\delta\mathcal{H}$ is challenging due to the higher powers of momenta appearing in the Lorentz-violating operator $\widehat{\mathcal{Q}}$. Following Ref. [36], we can instead adopt a procedure that yields an approximation to $\delta\mathcal{H}$ valid at leading order in Lorentz violation.

The exact hamiltonian $\mathcal{H}$ can be defined from the modified Dirac equation via

$$(p^0 - \mathcal{H})\psi = \gamma_0(p\cdot\gamma - m + \widehat{\mathcal{Q}})\psi = 0, \tag{53}$$

where p^0 is the exact energy. This gives

$$\mathcal{H} = \gamma_0(\boldsymbol{p}\cdot\boldsymbol{\gamma} + m - \widehat{\mathcal{Q}}) \equiv \mathcal{H}_0 + \delta\mathcal{H}, \tag{54}$$

where $\delta\mathcal{H} = -\gamma_0\widehat{\mathcal{Q}}$ is the exact perturbation hamiltonian. This form cannot be used directly in a perturbative calculation because $\delta\mathcal{H}$ depends on the eigenenergies of $\mathcal{H}$ and therefore requires prior knowledge of the energy shifts. However, the energy shifts are perturbative, so their contributions to $\delta\mathcal{H}$ lead to corrections at second or higher order in coefficients for Lorentz violation. This means that leading-order results can be derived by evaluating $\delta\mathcal{H}$ using the unperturbed eigenenergies,

$$\delta\mathcal{H} \approx -\gamma_0\widehat{\mathcal{Q}}|_{p^0\to E_{n,s}}. \tag{55}$$

For a trapped fermion, the dominant perturbative energy shifts due to Lorentz and CPT violation are therefore given by the matrix elements

$$\delta E_{n,s} = \langle\chi_{n,s}|\delta\mathcal{H}|\chi_{n,s}\rangle. \tag{56}$$

The corresponding perturbation hamiltonian $\delta\mathcal{H}^c$ for antiparticles is obtained from $\delta\mathcal{H}$ by reversing the sign of the charge q and the spin orientation s and changing the sign of all coefficients for Lorentz violation that control CPT-odd operators. These coefficients are identified in Table I. The shifts in the antiparticle energy levels can then be obtained using the unperturbed eigenstates (52),

$$\delta E^c_{n,s} = \langle\chi^c_{n,s}|\delta\mathcal{H}^c|\chi^c_{n,s}\rangle. \tag{57}$$

To obtain explicit results, we can take advantage of the constancy of the magnetic field and adopt the approach presented in Sec. II B 3. It therefore suffices to limit attention to the operators appearing in $\mathcal{L}^{(3)}$, $\mathcal{L}^{(4)}$, $\mathcal{L}^{(5)}_D$, and $\mathcal{L}^{(6)}_D$. After calculation with these terms, we obtain the corresponding perturbative energy shifts. The results are somewhat lengthy, so here we report them only for $d = 3$, 4, and 5 and relegate them to Appendix A. They hold at leading order in coefficients for Lorentz violation but are exact in other quantities. To obtain the additional contributions from operators in $\mathcal{L}^{(5)}_F$ and $\mathcal{L}^{(6)}_F$, it suffices to apply the substitutions listed in Sec. II B 3, while keeping only terms linear in coefficients for Lorentz violation. The corresponding energy-level shifts for a trapped antiparticle can be obtained from those for the particle by reversing the spin s and changing the signs σ of the charge and of all coefficients controlling CPT-odd operators.

The scales of all the energy shifts are set by the coefficients, which are therefore the appropriate targets for experimental measurements. However, some contributions are suppressed. Among these are corrections proportional to any nonzero power of $|qB|$, all of which arise from operators involving covariant derivatives. Even the comparatively large magnetic fields of $B \simeq 5$ T often found in Penning-trap experiments produce only effects suppressed by $|eB|/m_e^2 \simeq 10^{-9}$ for electrons or positrons and by $|eB|/m_p^2 \simeq 10^{-16}$ for protons or antiprotons. This means that the results presented in Appendix A could be used together with experimental data to obtain constraints on many coefficients associated with a factor of $|qB|$, albeit yielding weaker sensitivities. However, since these coefficients are associated with covariant-derivative couplings, they are also accessible in unsuppressed experimental studies of the behavior of free particles. We therefore disregard effects proportional to $|qB|$ in what follows. In contrast, terms proportional to B without a factor of q, which arise from operators in $\mathcal{L}^{(5)}_F$ and $\mathcal{L}^{(6)}_F$, represent Lorentz-violating couplings that are independent of free-particle motion and hence can only be detected in the presence of an electromagnetic field. Comparatively few investigations of these terms have been performed to date. We therefore include these effects in this work, placing first constraints on some of the coefficients appearing in $\mathcal{L}^{(5)}_F$ and $\mathcal{L}^{(6)}_F$.

A Penning trap includes not only the radially confining magnetic field of uniform magnitude B but also an axially confining electric field of varying magnitude E. The Landau momentum p_3 appearing in the expressions in Appendix A therefore physically represents an effective momentum for the axial motion. In the presence of the electric field, terms involving powers of p_3 become expectation values of the physical axial momentum. For a trapped particle the odd powers must vanish, but the even powers can be expected to contribute. When the axial quantum number is low, neglecting energy shifts from the even powers is a reasonable approximation because the ratio of axial to cyclotron frequencies is typically much less than one. Some cooling procedures may equipartition the axial and cyclotron energies and thus lead to large axial quantum numbers, which could produce Lorentz-violating perturbative shifts proportional to $|qE|$ comparable to those proportional to $|qB|$. For coefficients associated with covariant-derivative couplings, neglecting both effects is therefore consistent. For the F-type coefficients involving E, the effects are interesting in principle because they cannot be studied in the absence of the electric field. However, they are more challenging to analyze because E varies with position, and moreover the sensitivity to these coefficients is typically weaker by the ratio $|E/B|$. For example, for a typical configuration with 100 V applied over about 5 mm in a 5 T magnetic field, the ratio is about 10^{-5} in natural units. We therefore choose to disregard effects involving these coefficients in what follows.

With the above choices, the perturbative energy shift $\delta E_{n,s}$ for a fermion of species w, charge sign σ, and spin sign s in a magnetic field of magnitude B oriented along $\hat{x}_3$ is found to have the form

$$\delta E^w_{n,s} = \widetilde{a}^0_w - \sigma s \widetilde{b}^3_w - \widetilde{m}^3_{F,w} B + \sigma s \widetilde{b}^{33}_{F,w} B \qquad (58)$$

in the noninertial laboratory frame, where the tilde quantities are convenient combinations of the cartesian coefficients, defined by

$$\begin{aligned}
\widetilde{a}^0_w &= a^0_w - m_w c^{00}_w - m_w e^0_w + m_w^2 m^{(5)00}_w + m_w^2 a^{(5)000}_w \\
&\quad - m_w^3 c^{(6)0000}_w - m_w^3 e^{(6)000}_w, \\
\widetilde{b}^3_w &= b^3_w + H^{12}_w - m_w d^{30}_w - m_w g^{120}_w \\
&\quad + m_w^2 b^{(5)300}_w + m_w^2 H^{(5)1200}_w \\
&\quad - m_w^3 d^{(6)3000}_w - m_w^3 g^{(6)12000}_w, \\
\widetilde{m}^3_{F,w} &= m^{(5)12}_{F,w} + a^{(5)012}_{F,w} - m_w c^{(6)0012}_{F,w} - m_w e^{(6)012}_{F,w}, \\
\widetilde{b}^{33}_{F,w} &= b^{(5)312}_{F,w} + H^{(5)1212}_{F,w} - m_w d^{(6)3012}_{F,w} - m_w g^{(6)12012}_{F,w}.
\end{aligned} \qquad (59)$$

The fermion-flavor dependence of the coefficients is reflected in the subscript w, which can take the values e, p, and in principle others as well. The indices 0, 3, and 33 on these tilde quantities correctly reflect their properties under spatial rotations, as the index pair 12 is antisymmetric wherever it appears on the right-hand side and hence transforms like a single 3 index. The dependence on only the $\hat{x}_3$ direction is due to the cylindrical symmetry of the Penning trap.

We remark in passing that the first four terms of the quantity $\widetilde{b}^3_w$ form a widely used coefficient in studies of the minimal SME, also denoted $\widetilde{b}^3_w$ [4], which here is extended to include $d = 5$ and 6 effects. In fact, judicious use of Eqs. (26)-(28) of Ref. [19] permits a further generalization of this coefficient to include effects arising at arbitrary d, giving

$$\begin{aligned}
\widetilde{b}^3_w &= \sum_d m_w^{d-3} (b^{(d)30^{d-3}}_w + H^{(d)120^{d-3}}_w \\
&\qquad - d^{(d)30^{d-3}}_w - g^{(d)120^{d-3}}_w),
\end{aligned} \qquad (60)$$

where the sum is over odd values of d for the b- and H-type coefficients and over even values of d for the d- and g-type coefficients, and where the index 0^{d-3} denotes $d-3$ timelike indices. A similar result can be obtained for $\widetilde{a}^0_w$. Obtaining the analogous expressions for the F-type coefficients requires the Lagrange density for F-type couplings at arbitrary d, which remains unexplored to date.

The perturbative energy shift (58) is the key to extracting dominant signals for Lorentz and CPT violation in Penning-trap experiments. It reveals that only four quantities in the noninertial laboratory frame, the tilde coefficients $\widetilde{a}^0_w$, $\widetilde{b}^3_w$, $\widetilde{m}^3_{F,w}$, and $\widetilde{b}^{33}_{F,w}$, govern all the dominant Lorentz-violating energy shifts for a given fermion in an idealized Penning trap. However, the isotropic coefficient $\widetilde{a}^0_w$ provides the same instantaneous shift for all energy levels, which cancels in all frequencies and is therefore unobservable. Moreover, the coefficient $\widetilde{m}^3_{F,w}$ also provides an identical instantaneous shift to all energy levels, despite the shift being dependent on the magnetic field and ultimately also dependent on sidereal time due to the coefficient anisotropy. In contrast, the other two coefficients $\widetilde{b}^3_w$ and $\widetilde{b}^{33}_{F,w}$ can in principle be detected in suitable experiments. They contribute with opposite signs for spin up and spin down and therefore shift the two Landau-level stacks relative to each other, which is a measurable effect. This shift preserves the level spacing within each stack because the perturbation (58) is independent of the level number n. Notice that the magnitude $|q|$ of the fermion charge plays no role here.

The expression for the perturbative energy shift $\delta E^{\overline{w}}_{n,s}$ for the corresponding antifermion is obtained by reversing the orientation of the spin s and the signs of the coefficients controlling CPT-odd operators in the energy shift (58),

$$\begin{aligned}
\delta E^{\overline{w}}_{n,s} &= \delta E_{n,-s}|_{(a,b,e,g)\to(-a,-b,-e,-g)} \\
&\equiv -\widetilde{a}^{*0}_w + \sigma s \widetilde{b}^{*3}_w - \widetilde{m}^{*3}_{F,w} B - \sigma s \widetilde{b}^{*33}_{F,w} B,
\end{aligned} \qquad (61)$$

where the set of four starred tilde coefficients is defined by

$$\begin{aligned}
\widetilde{a}^{*0}_w &= a^0_w + m_w c^{00}_w - m_w e^0_w - m_w^2 m^{(5)00}_w + m_w^2 a^{(5)000}_w \\
&\quad + m_w^3 c^{(6)0000}_w - m_w^3 e^{(6)000}_w, \\
\widetilde{b}^{*3}_w &= b^3_w - H^{12}_w + m_w d^{30}_w - m_w g^{120}_w \\
&\quad + m_w^2 b^{(5)300}_w - m_w^2 H^{(5)1200}_w \\
&\quad + m_w^3 d^{(6)3000}_w - m_w^3 g^{(6)12000}_w, \\
\widetilde{m}^{*3}_{F,w} &= m^{(5)12}_{F,w} - a^{(5)012}_{F,w} - m_w c^{(6)0012}_{F,w} + m_w e^{(6)012}_{F,w}, \\
\widetilde{b}^{*33}_{F,w} &= b^{(5)312}_{F,w} - H^{(5)1212}_{F,w} + m_w d^{(6)3012}_{F,w} - m_w g^{(6)12012}_{F,w}.
\end{aligned} \qquad (62)$$

In deriving the result (61), the sign σ of the fermion charge is understood to change, the orientation of the magnetic field is assumed constant, and the direction s of the spin is still taken relative to the magnetic field. In parallel with the fermion case, only the combinations $\widetilde{b}^{*3}_w$ and $\widetilde{b}^{*33}_{F,w}$ are observable.

2. *Cyclotron and anomaly frequencies*

In Penning-trap experiments, the primary observables are frequencies. Two key frequencies are the cyclotron frequency $\nu_c = \omega_c/2\pi$ and the Larmor spin-precession frequency $\nu_L = \nu_a + \nu_c$, where $\nu_a = \omega_a/2\pi$ is the anomaly frequency [50]. In the presence of Lorentz and CPT violation, these frequencies can become shifted. For experiments with a fixed magnetic field and trapped fermions or antifermions of a given flavor w, the dominant shifts depend on only the four combinations $\widetilde{b}_w^3$, $\widetilde{b}_{F,w}^{33}$, $\widetilde{b}_w^{*3}$, and $\widetilde{b}_{F,w}^{*33}$ of cartesian coefficients in the noninertial laboratory frame. In this subsection, we use the results (58) and (61) to determine these shifts for the cyclotron and anomaly frequencies of trapped electrons, positrons, protons, and antiprotons. We show that the shifts are governed by a total of 36 independent inertial-frame observables in Penning-trap experiments, formed as combinations of 432 independent components of cartesian coefficients in the Sun-centered frame.

The cyclotron frequency ω_c is in natural units the energy difference between the ground-state $n=0$ level and the $n=1$ level in the same Landau stack, which for the particles of interest here is the stack with $s=\sigma$. Since the perturbations (58) and (61) are are independent of n and therefore constant for fixed s and σ, no change in the cyclotron frequency appears at leading order,

$$\begin{aligned}\delta\omega_c^w &= \delta E_{1,\sigma}^w - \delta E_{0,\sigma}^w \approx 0,\\ \delta\omega_c^{\overline{w}} &= \delta E_{1,\sigma}^{\overline{w}} - \delta E_{0,\sigma}^{\overline{w}} \approx 0,\end{aligned} \tag{63}$$

for either a fermion $w=e^-$, p or for an antifermion $\overline{w}=e^+$, $\overline{p}$. Note that the exact expressions for the energy shifts in Appendix A reveal the existence of subleading effects suppressed by $|qB|$ that do vary with n and therefore can produce subleading shifts in the cyclotron frequency, but these can be neglected here in accordance with the discussion in the previous subsection.

The dominant Lorentz-violating effects thus appear as shifts in the anomaly frequency ω_a. In natural units and for the particles relevant here, this is the energy difference between the $n=1$ level in the Landau stack with $s=\sigma$ and the $n=0$ level in the stack with $s=-\sigma$. Using the perturbative corrections (58) and (61) reveals that the anomaly frequencies for either a fermion $w=e^-$, p or for an antifermion $\overline{w}=e^+$, $\overline{p}$ are shifted according to

$$\begin{aligned}\delta\omega_a^w &= \delta E_{0,-\sigma}^w - \delta E_{1,\sigma}^w = 2\widetilde{b}_w^3 - 2\widetilde{b}_{F,w}^{33}B,\\ \delta\omega_a^{\overline{w}} &= \delta E_{0,-\sigma}^{\overline{w}} - \delta E_{1,\sigma}^{\overline{w}} = -2\widetilde{b}_w^{*3} + 2\widetilde{b}_{F,w}^{*33}B.\end{aligned} \tag{64}$$

Note that for each flavor all four tilde coefficients in the laboratory frame appear in these expressions. Note also that the antifermion result can be obtained from the fermion one by changing the signs of all the basic coefficients associated with CPT-odd operators, as might be expected.

The above formulae for the shifts in the anomaly frequencies involve coefficients controlling a mixture of CPT-even and CPT-odd effects. However, comparisons between particles and antiparticles can in principle permit the independent extraction of the CPT-odd contributions. For simplicity, suppose the magnetic fields in the two measurements have the same magnitude and orientation. Given the shifts in the anomaly frequencies $\delta\omega_a^w$ for a fermion and $\delta\omega_a^{\overline{w}}$ for its antifermion, we can take the difference to obtain

$$\begin{aligned}\Delta\omega_a^w &\equiv \tfrac{1}{2}(\delta\omega_a^w - \delta\omega_a^{\overline{w}})\\ &= \widetilde{b}_w^3 - \widetilde{b}_{F,w}^{33}B + \widetilde{b}_w^{*3} - \widetilde{b}_{F,w}^{*33}B\\ &= 2b_w^3 - 2m_w g_w^{120} + 2m_w^2 b_w^{(5)300} - 2m_w^3 g_w^{(6)12000}\\ &\quad -2b_{F,w}^{(5)312}B + 2m_w g_{F,w}^{(6)12012}B\\ &= 2\Delta\widetilde{b}_w^3 + 2\Delta\widetilde{b}_{F,w}^{33},\end{aligned} \tag{65}$$

where in the last expression we have introduced the convenient definitions

$$\begin{aligned}\Delta\widetilde{b}_w^3 &\equiv \tfrac{1}{2}(\widetilde{b}_w^3 - \widetilde{b}_w^{*3})\\ &= b_w^3 - m_w g_w^{120} + m_w^2 b_w^{(5)300} - m_w^3 g_w^{(6)12000},\\ \Delta\widetilde{b}_{F,w}^{33} &\equiv \tfrac{1}{2}(\widetilde{b}_{F,w}^{33} - \widetilde{b}_{F,w}^{*33})\\ &= -b_{F,w}^{(5)312} + m_w g_{F,w}^{(6)12012}.\end{aligned} \tag{66}$$

The result (65) shows explicitly that only coefficients for CPT violation appear in $\Delta\omega_a^w$. In fact, all of the CPT-odd effects are encoded in the difference $\Delta\omega_a^w$, as the orthogonal combination

$$\begin{aligned}\Sigma\omega_a^w &\equiv \tfrac{1}{2}(\delta\omega_a^w + \delta\omega_a^{\overline{w}})\\ &= \widetilde{b}_w^3 - \widetilde{b}_{F,w}^{33}B - \widetilde{b}_w^{*3} + \widetilde{b}_{F,w}^{*33}B\\ &= 2H_w^{12} - 2m_w d_w^{30} + 2m_w^2 H_w^{(5)1200} - 2m_w^3 d_w^{(6)3000}\\ &\quad +2H_{F,w}^{(5)1212}B - 2m_w d_{F,w}^{(6)3012}B\end{aligned} \tag{67}$$

contains only coefficients for CPT-even Lorentz violation. We remark in passing that each term contributing to the CPT violation in the result (65) is also CT violating, as predicted by the discussion in Sec. II C of Ref. [11].

To express the shifts (64) in the anomaly frequencies and the difference (65) in terms of constant coefficients in the Sun-centered frame requires applying the methods described in Sec. II C. This thereby reveals the sidereal-time and geometric dependences of the laboratory-frame tilde coefficients. As a simple example, consider a scenario having the laboratory located at colatitude χ with the magnetic field pointing to the local zenith so that $\hat{x}_3=\hat{z}$, and focus on the single-index laboratory-frame coefficient $\widetilde{b}_w^3$. In this special case, application of the rotation matrices given in Sec. II C and the transformation (46) yields the result

$$\widetilde{b}_w^3 = \widetilde{b}_w^Z\cos\chi + (\widetilde{b}_w^X\cos\omega_\oplus T_\oplus + \widetilde{b}_w^Y\sin\omega_\oplus T_\oplus)\sin\chi, \tag{68}$$

expressing the noninertial-frame quantity $\widetilde{b}_w^3$ in terms of the three independent quantities $\widetilde{b}_w^J$, $J=X,Y,Z$, in the canonical inertial frame. More generally, when the magnetic field points along a generic direction in the laboratory frame, trigonometric functions of the extra Euler angles α, β, γ in Eq. (45) appear as well.

In an analogous fashion, the laboratory-frame tilde coefficient $\widetilde{b}_{F,w}^{33}$ is associated with the six independent combinations $\widetilde{b}_{F,w}^{(JK)}$ in the Sun-centered frame. These produce up to second harmonics in the sidereal frequency, due to the nature of $\widetilde{b}_{F,w}^{(JK)}$ as an observer 2-tensor. For example, in the above simple scenario at colatitude χ with the magnetic field pointing to the local zenith, we find

$$\begin{aligned}\widetilde{b}_{F,w}^{33} \;=\; & \widetilde{b}_{F,w}^{ZZ} + \tfrac{1}{2}(\widetilde{b}_{F,w}^{XX} + \widetilde{b}_{F,w}^{YY} - 2\widetilde{b}_{F,w}^{ZZ}) \sin^2\chi \\ & + (\widetilde{b}_{F,w}^{(XZ)} \cos\omega_\oplus T_\oplus + \widetilde{b}_{F,w}^{(YZ)} \sin\omega_\oplus T_\oplus) \sin 2\chi \\ & + [\tfrac{1}{2}(\widetilde{b}_{F,w}^{XX} - \widetilde{b}_{F,w}^{YY}) \cos 2\omega_\oplus T_\oplus \\ & \qquad + \widetilde{b}_{F,w}^{(XY)} \sin 2\omega_\oplus T_\oplus] \sin^2\chi. \qquad (69)\end{aligned}$$

Taking into account the relevant two fermion flavors w and including also experiments with antiparticles, which can access the nine additional independent combinations $\widetilde{b}_w^{*J}$ and $\widetilde{b}_{F,w}^{*(JK)}$, we can conclude that there are 36 independent tilde observables in the Sun-centered frame. Each of these observables is a linear combination of cartesian coefficients, of which 12 independent components appear in the perturbative corrections (58) and (61). Various combinations such as the 18 independent differences

$$\begin{aligned}\Delta\widetilde{b}_w^J \;&\equiv\; \tfrac{1}{2}(\widetilde{b}_w^J - \widetilde{b}_w^{*J}), \\ \Delta\widetilde{b}_{F,w}^{(JK)} \;&\equiv\; \tfrac{1}{2}(\widetilde{b}_{F,w}^{(JK)} - \widetilde{b}_{F,w}^{*(JK)}) \qquad (70)\end{aligned}$$

may also appear in performing experimental analyses. We thus see that the 36 independent observables in Penning-trap experiments are formed as linear combinations of 432 independent components of cartesian coefficients in the Sun-centered frame. Each observable corresponds to a physically distinct and dominant Lorentz-violating effect, so Penning traps offer excellent coverage of the available coefficient space, and moreover coverage at high sensitivity.

Any single Penning-trap experiment with fixed magnetic field and a given particle can in principle access four harmonics and a constant term, although the latter is time independent and hence challenging to measure. This means that at most five of the 36 independent pieces of information are accessible in any given experiment. A joint analysis of data from multiple experiments is therefore required to explore fully the available coefficient space. Complete coverage can be obtained only if experiments are performed with all relevant particle and antiparticles and if different experimental geometries are adopted. The experimental conditions can be changed by changing the orientation or magnitude of the magnetic field, or by performing the experiment at a different colatitude.

B. Experiments

In this subsection, we first discuss some concepts essential to studies of Lorentz and CPT violation in Penning-trap experiments. These concepts and the results obtained above are then used to extract estimated constraints on coefficients for Lorentz and CPT violation and to predict potential future signals in some existing and forthcoming experiments.

1. Concepts

Studies of the anomalous magnetic moment and the g factor of a particle in a Penning trap can be idealized as measurements of the ratio of the anomaly frequency ν_a to the cyclotron frequency ν_c, linked to g in a Lorentz- and CPT-invariant scenario by

$$\frac{\nu_a}{\nu_c} \equiv \frac{\omega_a}{\omega_c} = \frac{g}{2} - 1 \quad \text{(Lorentz/CPT invariance).} \qquad (71)$$

In this conventional Lorentz- and CPT-invariant case, g is a numerical scalar quantity that is an intrinsic property of the particle. The predicted value of g can in principle be calculated in a suitable theoretical framework such as Lorentz-invariant quantum field theory, and it is related to fundamental quantities such as the fine structure constant. Radiative corrections modify the theoretical tree-level value of g [48], and real measurements must take into account various experimental effects involving the axial frequency, the relativistic shift, the cavity shift, and more [49], but g remains an intrinsic numerical property of the particle.

In the presence of Lorentz and CPT violation, this scenario is drastically changed because the energies and hence the anomaly frequency ω_a are directly shifted, as is evident from Eq. (58). The portion of the shift associated with the coefficient $\widetilde{b}_w^3$ is independent of the magnitude of B but depends on geometric factors such as the local sidereal time $T_\oplus$, the colatitude χ of the experiment, and the direction $\hat{x}_3$ of the magnetic field, while the part involving $\widetilde{b}_{F,w}^{33}$ depends both on B and on the geometric factors. In short, the anomaly frequency can be viewed as a function of these variables,

$$\omega_a = \omega_a(T_\oplus, \chi, \hat{x}_3, B) \quad \text{(Lorentz/CPT violation).} \quad (72)$$

An immediate consequence is that *the experimental ratio ω_a/ω_c is no longer an intrinsic property of the particle* and instead becomes an experiment-dependent quantity. Reported values of g obtained using the result (71) therefore cannot be directly compared between experiments in a meaningful way because they depend on the local experimental conditions: the local sidereal time, the colatitude of the laboratory, and the direction and magnitude of the magnetic field. Instead, the intrinsic quantities that provide experiment-independent measures of Lorentz and CPT violation are SME coefficients expressed in the canonical Sun-centered frame. In the present context of Penning-trap experiments with electrons, positrons, protons, and antiprotons, these intrinsic quantities can be taken as the 36 tilde coefficients $\widetilde{b}_e^J$, $\widetilde{b}_e^{*J}$, $\widetilde{b}_{F,e}^{(JK)}$, $\widetilde{b}_{F,e}^{*(JK)}$ and $\widetilde{b}_p^J$, $\widetilde{b}_p^{*J}$, $\widetilde{b}_{F,p}^{(JK)}$, $\widetilde{b}_{F,p}^{*(JK)}$ in the Sun-centered frame. They can be extracted from the ratios ω_a/ω_c obtained for the different species under various laboratory conditions, by matching to the predicted dependences on the geometrical factors relevant for each given experiment.

In the conventional context with Lorentz and CPT invariance, one experimental advantage of extracting the ratio (71) is that both ω_a and ω_c are proportional to B, so B cancels in the determination of g. For example, if the measurements can be performed quasi-simultaneously, then accurate knowledge of B is unnecessary to achieve a high-precision measurement of g. However, in the presence of Lorentz and CPT violation, the ratio (71) is no longer independent of B because the coefficients $\widetilde{b}_w^J$, $\widetilde{b}_w^{*J}$ appear without an accompanying factor of B. A precision measurement of these coefficients therefore requires continous calibration of B as implemented, for instance, in a sidereal-variation analysis performed at the University of Washington [13]. However, for measurements restricting attention to the F-type coefficients $\widetilde{b}_{F,w}^{(JK)}$ and $\widetilde{b}_{F,w}^{*(JK)}$, which always come with a factor of B, the cancellation remains in force and accurate knowledge of B is again unnecessary.

Implications related to the above conceptual points also arise for comparative tests involving particles and antiparticles. Suppose one experiment measures the ratio ω_a^w/ω_c^w for a particle of species w, while a second experiment measures the ratio $\omega_a^{\overline{w}}/\omega_c^{\overline{w}}$ for the corresponding antiparticle. We are allowing here for the possibility that the cyclotron frequencies ω_c^w, $\omega_c^{\overline{w}}$ of the two measurements may differ due to different magnitudes of the experimental magnetic fields. In a Lorentz- and CPT-invariant scenario, the difference between these two measurements is

$$\frac{\omega_a^w}{\omega_c^w} - \frac{\omega_a^{\overline{w}}}{\omega_c^{\overline{w}}} = \tfrac{1}{2}(g - \overline{g}) \quad \text{(CPT invariance)} \tag{73}$$

according to Eq. (71). The CPT theorem guarantees that this quantity is identically zero.

However, in the presence of CPT violation, the picture again changes drastically due to the qualitatively different nature of the anomaly frequency (72). Using Eq. (64), the difference between the two measurements is found to be

$$\frac{\omega_a^w}{\omega_c^w} - \frac{\omega_a^{\overline{w}}}{\omega_c^{\overline{w}}} = \frac{\delta\omega_a^w}{\omega_c^w} - \frac{\delta\omega_a^{\overline{w}}}{\omega_c^{\overline{w}}} \quad \text{(CPT violation)}, \tag{74}$$

since the CPT theorem guarantees the cancellation of all Lorentz- and CPT-invariant contributions. We see from the result (72) that *the experimental difference* $(\omega_a^w/\omega_c^w) - (\omega_a^{\overline{w}}/\omega_c^{\overline{w}})$ *depends on the local experimental conditions:* the local sidereal time, the colatitudes of the laboratories where the two experiments are performed, and the directions and magnitudes of the magnetic fields.

More insight can be gained by algebraically expressing the difference (74) in terms of sums and differences of the anomaly and cyclotron frequencies, defined as

$$\begin{aligned} \Delta\omega_a^w &= \tfrac{1}{2}(\delta\omega_a^w - \delta\omega_a^{\overline{w}}), \quad \Sigma\omega_a^w = \tfrac{1}{2}(\delta\omega_a^w + \delta\omega_a^{\overline{w}}), \\ \Delta\omega_c^w &= \tfrac{1}{2}(\omega_c^w - \omega_c^{\overline{w}}), \quad \Sigma\omega_c^w = \tfrac{1}{2}(\omega_c^w + \omega_c^{\overline{w}}). \end{aligned} \tag{75}$$

This gives

$$\frac{\omega_a^w}{\omega_c^w} - \frac{\omega_a^{\overline{w}}}{\omega_c^{\overline{w}}} = \frac{2}{\omega_c^w \omega_c^{\overline{w}}} \left(\Sigma\omega_c^w \Delta\omega_a^w - \Delta\omega_c^w \Sigma\omega_a^w\right). \tag{76}$$

We have seen in the previous section that no leading-order changes in the cyclotron frequencies occur in the presence of Lorentz and CPT violation, so any difference $\Delta\omega_c^w$ is purely due to experimental magnetic fields of different magnitude. For magnetic fields of identical orientation the theoretical predictions (65) for $\Delta\omega_a^w$ and (67) for $\Sigma\omega_a^w$ show that the first term of the result (76) involves CPT violation, while the second involves CPT-invariant Lorentz violation. These points reveal that *the experimental difference* $(\omega_a^w/\omega_c^w) - (\omega_a^{\overline{w}}/\omega_c^{\overline{w}})$ *is a clean measure of CPT violation only if both measurements use magnetic fields of identical strength and orientation.*

In the event that indeed both ratio measurements are made using the same $\boldsymbol{B}$, which implies $\omega_c^w = \omega_c^{\overline{w}}$, then the explicit form of the difference (76) reduces to

$$\begin{aligned} \frac{\omega_a^w}{\omega_c^w} - \frac{\omega_a^{\overline{w}}}{\omega_c^{\overline{w}}} &= \frac{2\Delta\omega_a^w}{\omega_c^w} \quad \text{(CPT violation, } \omega_c^{\overline{w}} = \omega_c^w\text{)} \\ &= \frac{4}{\omega_c^w}(\Delta\widetilde{b}_w^3 + \Delta\widetilde{b}_{F,w}^{33} B) \end{aligned} \tag{77}$$

by using the result (65). This is is indeed a pure CPT test, as only coefficients for CPT violation enter the definitions (66) for $\Delta\widetilde{b}_w^3$ and $\Delta\widetilde{b}_{F,w}^{33}$. Conversion of this expression from the noninertial laboratory frame to the canonical inertial Sun-centered frame using the transformation (46) displays the dependence on the 18 intrinsic experiment-independent observables $\Delta\widetilde{b}_w^J$, $\Delta\widetilde{b}_{F,w}^{(JK)}$ for CPT violation and exposes the explicit dependence on the local sidereal time $T_\oplus$, the colatitude χ of the laboratory, and the direction $\hat{z}$ of the magnetic field.

2. *Sensitivities and signals*

The above discussion shows that the experiment-independent observables relevant for studies of the anomaly frequency of a trapped electron, positron, proton, or antiproton are the 36 quantities $\widetilde{b}_e^J$, $\widetilde{b}_e^{*J}$, $\widetilde{b}_{F,e}^{(JK)}$, $\widetilde{b}_{F,e}^{*(JK)}$ and $\widetilde{b}_p^J$, $\widetilde{b}_p^{*J}$, $\widetilde{b}_{F,p}^{(JK)}$, $\widetilde{b}_{F,p}^{*(JK)}$. In the special case of comparative tests between particles and antiparticles performed with magnetic fields of the same magnitude B, these observables reduce to the 18 differences $\Delta\widetilde{b}_e^J$, $\Delta\widetilde{b}_{F,e}^{(JK)}$ and $\Delta\widetilde{b}_p^J$, $\Delta\widetilde{b}_{F,p}^{(JK)}$. As a guide to existing and prospective sensitivities to Lorentz and CPT violation that could be obtained, we consider next a subset of sensitive Penning-trap experiments measuring the anomaly frequency for these species.

The experiments chosen for the discussion here are listed in Table III. For each experiment, we show the species involved, the colatitude χ of the laboratory, the direction $\hat{x}_3$ of the magnetic field, and its magnitude B.

TABLE III: Geometrical quantities for some experiments.

Experiment	Species	χ	$\hat{x}_3$	B	λ	T_0
Washington [12]	e^-, e^+	42.5°	upward	5.85 T	−122.3°	12.54 h
Washington [13]	e^-	42.5°	upward	5.85 T	−122.3°	12.54 h
Harvard [25]	e^-	47.6°	upward	5.36 T	−71.1°	9.13 h
Harvard [29]	e^+	47.6°	upward	$\simeq 6$ T	−71.1°	9.13 h
ATRAP [27]	$\overline{p}$	43.8°	upward	5.2 T	6.1°	4.00 h
BASE [28]	p	40.0°	south	1.90 T	8.3°	3.85 h
BASE [31]	$\overline{p}$	43.8°	60° west of north	1.95 T	6.1°	4.00 h

TABLE IV: Analysis for the electron sector.

Experiment	Lab. frame	Sun-centered frame	Harmonic
Washington [12]	$\Delta\widetilde{b}_e^3$	$0.7\Delta\widetilde{b}_e^Z$	1
	$\Delta\widetilde{b}_{F,e}^{33}$	$0.2(\Delta\widetilde{b}_{F,e}^{XX}+\Delta\widetilde{b}_{F,e}^{YY})+0.5\Delta\widetilde{b}_{F,e}^{ZZ}$	1
Washington [13]	$\widetilde{b}_e^3$	$0.7\widetilde{b}_e^X$	$\cos\omega_\oplus T_\oplus$
		$0.7\widetilde{b}_e^Y$	$\sin\omega_\oplus T_\oplus$
	$\widetilde{b}_{F,e}^{33}$	$\widetilde{b}_{F,e}^{(XZ)}$	$\cos\omega_\oplus T_\oplus$
		$\widetilde{b}_{F,e}^{(YZ)}$	$\sin\omega_\oplus T_\oplus$
Harvard [25]	$\widetilde{b}_e^3$	$0.7\widetilde{b}_e^X$	$\cos\omega_\oplus T_\oplus$
		$0.7\widetilde{b}_e^Y$	$\sin\omega_\oplus T_\oplus$
	$\widetilde{b}_{F,e}^{33}$	$\widetilde{b}_{F,e}^{(XZ)}$	$\cos\omega_\oplus T_\oplus$
		$\widetilde{b}_{F,e}^{(YZ)}$	$\sin\omega_\oplus T_\oplus$
		$0.3(\widetilde{b}_{F,e}^{XX}-\widetilde{b}_{F,e}^{YY})$	$\cos 2\omega_\oplus T_\oplus$
		$0.5\widetilde{b}_{F,e}^{(XY)}$	$\sin 2\omega_\oplus T_\oplus$
Harvard [29]	$\widetilde{b}_e^{*3}$	$-0.7\widetilde{b}_e^{*X}$	$\cos\omega_\oplus T_\oplus$
		$-0.7\widetilde{b}_e^{*Y}$	$\sin\omega_\oplus T_\oplus$
	$\widetilde{b}_{F,e}^{*33}$	$-\widetilde{b}_{F,e}^{*(XZ)}$	$\cos\omega_\oplus T_\oplus$
		$-\widetilde{b}_{F,e}^{*(YZ)}$	$\sin\omega_\oplus T_\oplus$
		$-0.3(\widetilde{b}_{F,e}^{*XX}-\widetilde{b}_{F,e}^{*YY})$	$\cos 2\omega_\oplus T_\oplus$
		$-0.5\widetilde{b}_{F,e}^{*(XY)}$	$\sin 2\omega_\oplus T_\oplus$
	$\Delta\widetilde{b}_e^3$	$0.7\Delta\widetilde{b}_e^Z$	1
	$\Delta\widetilde{b}_{F,e}^{33}$	$0.3(\Delta\widetilde{b}_{F,e}^{XX}+\Delta\widetilde{b}_{F,e}^{YY})+0.5\Delta\widetilde{b}_{F,e}^{ZZ}$	1

For completeness and reference, we also provide the longitude λ of the laboratory and the offset time T_0 relating the local sidereal time $T_\oplus$ to the canonical time T in the Sun-centered frame according to Eq. (43).

Consider first experiments sensitive to the electron sector. A 1999 experiment at the University of Washington [12] compared the anomaly frequencies of electrons and positrons with a precision of about 2 ppt. The data were analysed for an effect independent of sidereal time, so the reported results can be viewed as time-averaged measurements of the predicted difference (74). The cyclotron frequencies used for the two species were almost identical, so the form (77) can be adopted to obtain sensitivities to the coefficients $\Delta\widetilde{b}_e^J$, $\Delta\widetilde{b}_{F,e}^{(JK)}$. Using the geometric factors in Table III and the rotation transformation (46), the expression (77) can be converted to the Sun-centered frame. The relevant combinations of tilde coefficients are shown in the first two lines of Table IV. We conservatively take the constraint $b \lesssim 50$ rad/s reported in Ref. [12] to represent the limit $|b_e^Z| \lesssim 5\times 10^{-24}$ GeV in the present notation. Averaging the result over time and substituting for the values in Ref. [12] then gives the bound

$$\Big|\Delta\widetilde{b}_e^Z + (4\times 10^{-16}\ \mathrm{GeV}^2)(\Delta\widetilde{b}_{F,e}^{XX} + \Delta\widetilde{b}_{F,e}^{YY}) + (8\times 10^{-16}\ \mathrm{GeV}^2)\Delta\widetilde{b}_{F,e}^{ZZ}\Big| \lesssim 7\times 10^{-24}\ \mathrm{GeV}. \quad (78)$$

The result (78) can be viewed as generalizing the published result to the tilde coefficients $\Delta\widetilde{b}_e^Z$ and $\Delta\widetilde{b}_{F,e}^{JJ}$ or, equivalently, as incorporating also the basic coefficients g_e^{XYT}, $b_e^{(5)ZTT}$, $g_e^{(6)XYTTT}$, $b_{F,e}^{(5)XYZ}$, $b_{F,e}^{(5)YZX}$, $b_{F,e}^{(5)ZXY}$, $g_{F,e}^{(6)XYTXY}$, $g_{F,e}^{(6)YZTYZ}$, and $g_{F,e}^{(6)ZXTZX}$.

Another 1999 analysis at the University of Washington [13] reported results from an analysis of the sidereal variation of the anomaly frequency of a trapped electron measured over a two-month period, with the magnetic field continuously calibrated. Fitting the data to a sinusoid at the sidereal frequency $\omega_\oplus$ and constraining its amplitude yielded a 2σ limit of $|\delta\omega_a^e| \le 8\times 10^{-25}$ GeV in the present notation. The combinations of tilde coefficients $\widetilde{b}_e^J$ and $\widetilde{b}_{F,e}^{(JK)}$ relevant for this experiment are shown in Table IV. These expressions reveal that the experiment places the constraint

$$\Big(\big[0.7\widetilde{b}_e^X + (10^{-15}\ \mathrm{GeV}^2)\widetilde{b}_{F,e}^{(XZ)}\big]^2 + \big[0.7\widetilde{b}_e^Y + (10^{-15}\ \mathrm{GeV}^2)\widetilde{b}_{F,e}^{(YZ)}\big]^2\Big)^{1/2} \lesssim 4\times 10^{-25}\ \mathrm{GeV}. \quad (79)$$

Comparing this result to the bound (78) obtained using the 1999 comparison of electron and positron anomaly frequencies shows that the sidereal analysis constrains different spatial components of the tilde coefficients. Moreover, CPT-even effects are also contained in the result (79) via H- and d-type basic coefficients.

A more recent Penning-trap experiment at Harvard University [25] measured the g factor of the electron to 0.28 ppt. This impressive precision offers in principle improved sensitivity to several components of the tilde coefficients $\widetilde{b}_e^J$, $\widetilde{b}_{F,e}^{(JK)}$ via the study of sidereal variations in analogy to the 1999 work discussed above [13]. No such analysis has been performed to date, but we present in Table IV the relevant combinations of tilde coefficients for the first and second harmonics required for this study and specific to the geometric factors of the experiment.

A similar sidereal analysis for a trapped positron could be performed using another experiment under development at Harvard University [29]. This would offer not only the first sensitivity to components of the tilde coefficients $\widetilde{b}_e^{*J}$, $\widetilde{b}_{F,e}^{*(JK)}$ but would also permit improved measurements of the experiment-independent CPT-odd observables $\Delta\widetilde{b}_e^Z$ and $\Delta\widetilde{b}_{F,e}^{JJ}$ by comparison with measurements for the electron. Table IV contains the combinations of tilde coefficients for the first and second harmonics for the planned positron experiment. We also list the components of the CPT-odd observables that could be constrained by comparing the anomaly frequencies for the electron and positron at the same magnetic field strength. Note that sidereal variations of these difference coefficients are also of interest. The corresponding expressions in the Sun-centered frame follow from the definitions (66), so they appear in the same linear combinations up to an overall sign for the antiparticle coefficients.

Next, we consider experiments sensitive to the proton sector. In an experiment located at CERN, the ATRAP collaboration has measured the magnetic moment of the antiproton to 4.4 ppm [27]. In principle, an analysis of sidereal variations using this experiment could yield measurements of some components of the tilde coefficients $\widetilde{b}_p^{*J}$ and $\widetilde{b}_{F,p}^{*(JK)}$, which would represent the first sensitivity achieved to these physical effects. The components accessible to the geometry of this experiment via first and second harmonics of the sidereal frequency are listed in the first few lines of Table V.

A measurement of the magnetic moment of the proton at the record sensitivity of 3.3 ppb has recently been performed by the BASE collaboration in an experiment located in Mainz [28]. In this case, a search for sidereal variations could in principle provide sensitivity to certain components of the tilde coefficients $\widetilde{b}_p^J$ and $\widetilde{b}_{F,p}^{(JK)}$. Table V shows the combinations of coefficients that would be accessible to this experimental geometry. The BASE collaboration also plans to perform a version of this experiment at CERN, which is at a different colatitude, using a different orientation and strength of the bore of the primary magnet and ultimately using a quantum logic readout that will permit rapid measurements of the proton and antiproton anomaly frequencies [31]. This offers the opportunity to measure many components of the tilde coefficients $\widetilde{b}_p^J$, $\widetilde{b}_{F,p}^{(JK)}$, $\widetilde{b}_p^{*J}$, and $\widetilde{b}_{F,p}^{*(JK)}$ via sidereal-variation studies. The constant parts and the sidereal variations of the differences $\Delta\widetilde{b}_p^J$ and $\Delta\widetilde{b}_{F,p}^{(JK)}$ would also be measurable with this setup. Using the geometrical factors listed in Table III reveals that this future experiment can access the combinations of difference components shown in Table V. Modulo an overall sign for the antiproton case, the same linear combinations of tilde coefficients appear in sidereal studies, as can be deduced by inspection of the definitions (66).

Taken together, the published results [27, 28] from the ATRAP and BASE experiments can be combined to extract estimated constraints on experiment-independent observables for Lorentz and CPT violation. The methodology to derive these constraints is of potential interest for future experiments as well, so we outline it here. Consider the comparison (76), recalling that all anomaly frequencies are functions of the form (72). Since both experiments took data over an extended time period, we can plausibly approximate the sidereal variations as averaging to zero, leaving only the constant shifts. This means only the dependence on the colatitude and on the direction and strength of the magnetic fields needs to be

TABLE V: Analysis for the proton sector.

Experiment	Lab. frame	Sun-centered frame	Harmonic
ATRAP [27]	$\widetilde{b}_p^{*3}$	$-0.7\widetilde{b}_p^{*X}$	$\cos\omega_\oplus T_\oplus$
		$-0.7\widetilde{b}_p^{*Y}$	$\sin\omega_\oplus T_\oplus$
	$\widetilde{b}_{F,p}^{*33}$	$-\widetilde{b}_{F,p}^{*(XZ)}$	$\cos\omega_\oplus T_\oplus$
		$-\widetilde{b}_{F,p}^{*(YZ)}$	$\sin\omega_\oplus T_\oplus$
		$-0.2(\widetilde{b}_{F,p}^{*XX}-\widetilde{b}_{F,p}^{*YY})$	$\cos 2\omega_\oplus T_\oplus$
		$-0.5\widetilde{b}_{F,p}^{*XY}$	$\sin 2\omega_\oplus T_\oplus$
BASE [28]	$\widetilde{b}_p^{3}$	$0.8\widetilde{b}_p^{X}$	$\cos\omega_\oplus T_\oplus$
		$0.8\widetilde{b}_p^{Y}$	$\sin\omega_\oplus T_\oplus$
	$\widetilde{b}_{F,p}^{33}$	$-\widetilde{b}_{F,p}^{(XZ)}$	$\cos\omega_\oplus T_\oplus$
		$-\widetilde{b}_{F,p}^{(YZ)}$	$\sin\omega_\oplus T_\oplus$
		$0.3(\widetilde{b}_{F,p}^{XX}-\widetilde{b}_{F,p}^{YY})$	$\cos 2\omega_\oplus T_\oplus$
		$0.6\widetilde{b}_{F,p}^{XY}$	$\sin 2\omega_\oplus T_\oplus$
BASE [31]	$\Delta\widetilde{b}_p^{3}$	$0.3\Delta\widetilde{b}_p^{Z}$	1
		$-0.4\Delta\widetilde{b}_p^{X}-0.9\Delta\widetilde{b}_p^{Y}$	$\cos\omega_\oplus T_\oplus$
		$0.9\Delta\widetilde{b}_p^{X}+0.4\Delta\widetilde{b}_p^{Y}$	$\sin\omega_\oplus T_\oplus$
	$\Delta\widetilde{b}_{F,p}^{33}$	$0.4(\Delta\widetilde{b}_{F,p}^{XX}+\Delta\widetilde{b}_{F,p}^{YY})+0.1\Delta\widetilde{b}_{F,p}^{ZZ}$	1
		$-0.2\Delta\widetilde{b}_{F,p}^{(XZ)}-0.6\Delta\widetilde{b}_{F,p}^{(YZ)}$	$\cos\omega_\oplus T_\oplus$
		$0.6\Delta\widetilde{b}_{F,p}^{(XZ)}-0.2\Delta\widetilde{b}_{F,p}^{(YZ)}$	$\sin\omega_\oplus T_\oplus$
		$-0.3(\Delta\widetilde{b}_{F,p}^{XX}-\Delta\widetilde{b}_{F,p}^{YY})+0.6\Delta\widetilde{b}_{F,p}^{(XY)}$	$\cos 2\omega_\oplus T_\oplus$
		$-0.3(\Delta\widetilde{b}_{F,p}^{XX}-\Delta\widetilde{b}_{F,p}^{YY})-0.6\Delta\widetilde{b}_{F,p}^{(XY)}$	$\sin 2\omega_\oplus T_\oplus$

considered. For BASE, the colatitude is $\chi \simeq 40.0^\circ$ and the magnetic field $B \simeq 1.9$ T points to local south, corresponding to the $\hat{x}$ direction in the standard laboratory frame discussed in Sec. II C. For ATRAP, the colatitude is $\chi^* \simeq 43.8^\circ$ and the magnetic field $B^* \simeq 5.2$ T points upwards, along the $\hat{z}$ direction in the standard laboratory frame. The expressions (64) for the frequency shifts can then be combined to yield

$$\begin{aligned}
\Delta\omega_a^p &\equiv \tfrac{1}{2}(\delta\omega_a^p - \delta\omega_a^{\overline{p}}) \\
&= \widetilde{b}_p^x - \widetilde{b}_{F,p}^{xx}B + \widetilde{b}_p^{*z} - \widetilde{b}_{F,p}^{*zz}B^* \\
&= -\widetilde{b}_p^Z \sin\chi + \widetilde{b}_p^{*Z}\cos\chi^* \\
&\quad -\tfrac{1}{2}(\widetilde{b}_{F,p}^{XX}+\widetilde{b}_{F,p}^{YY})B\cos^2\chi - \widetilde{b}_{F,p}^{ZZ}B\sin^2\chi \\
&\quad -\tfrac{1}{2}(\widetilde{b}_{F,p}^{*XX}+\widetilde{b}_{F,p}^{*YY})B^*\sin^2\chi^* - \widetilde{b}_{F,p}^{*ZZ}B^*\cos^2\chi^*, \\
\Sigma\omega_a^p &\equiv \tfrac{1}{2}(\delta\omega_a^w + \delta\omega_a^{\overline{w}}) \\
&= \widetilde{b}_p^x - \widetilde{b}_{F,p}^{xx}B - \widetilde{b}_p^{*z} + \widetilde{b}_{F,p}^{*zz}B^* \\
&= -\widetilde{b}_p^Z \sin\chi - \widetilde{b}_p^{*Z}\cos\chi^* \\
&\quad -\tfrac{1}{2}(\widetilde{b}_{F,p}^{XX}+\widetilde{b}_{F,p}^{YY})B\cos^2\chi - \widetilde{b}_{F,p}^{ZZ}B\sin^2\chi \\
&\quad +\tfrac{1}{2}(\widetilde{b}_{F,p}^{*XX}+\widetilde{b}_{F,p}^{*YY})B^*\sin^2\chi^* + \widetilde{b}_{F,p}^{*ZZ}B^*\cos^2\chi^*.
\end{aligned} \tag{80}$$

These results can be entered on the right-hand side of the comparison (76). Using the numerical values of the other quantities reported by the ATRAP and BASE measurements and keeping only one significant figure in light of the approximations made, we obtain the 2-sigma limit

$$\begin{aligned}
\Big|\widetilde{b}_p^Z - 0.4\widetilde{b}_p^{*Z} \\
&+(2\times 10^{-16}\text{ GeV}^2)(\widetilde{b}_{F,p}^{XX}+\widetilde{b}_{F,p}^{YY}) \\
&+(2\times 10^{-16}\text{ GeV}^2)\widetilde{b}_{F,p}^{ZZ} \\
&+(1\times 10^{-16}\text{ GeV}^2)(\widetilde{b}_{F,p}^{*XX}+\widetilde{b}_{F,p}^{*YY}) \\
&+(3\times 10^{-16}\text{ GeV}^2)\widetilde{b}_{F,p}^{*ZZ}\Big| \lesssim 2\times 10^{-21}\text{ GeV}.
\end{aligned} \tag{81}$$

This is the desired experiment-independent measure of Lorentz and CPT violation in the proton sector, which is specific to the comparison of the BASE proton and ATRAP antiproton results.

Each of the three constraints (78), (79), and (81) obtained above involves several tilde coefficients. Some intuition for the scope of these constraints can be obtained by assuming each coefficient in turn to be the only nonzero one and determining its bound. This procedure, which is common practice across many subfields search-

TABLE VI: Constraints on tilde coefficients.

Coefficient	Constraint	Ref.
$\lvert\widetilde{b}_e^X\rvert$	$< 6 \times 10^{-25}$ GeV	[13]
$\lvert\widetilde{b}_e^Y\rvert$	$< 6 \times 10^{-25}$ GeV	[13]
$\lvert\widetilde{b}_e^Z\rvert$	$< 7 \times 10^{-24}$ GeV	[12]
$\lvert\widetilde{b}_e^{*Z}\rvert$	$< 7 \times 10^{-24}$ GeV	[12]
$\lvert\widetilde{b}_{F,e}^{XX} + \widetilde{b}_{F,e}^{YY}\rvert$	$< 2 \times 10^{-8}$ GeV^{-1}	[12]
$\lvert\widetilde{b}_{F,e}^{ZZ}\rvert$	$< 8 \times 10^{-9}$ GeV^{-1}	[12]
$\lvert\widetilde{b}_{F,e}^{(XZ)}\rvert$	$< 4 \times 10^{-10}$ GeV^{-1}	[13]
$\lvert\widetilde{b}_{F,e}^{(YZ)}\rvert$	$< 4 \times 10^{-10}$ GeV^{-1}	[13]
$\lvert\widetilde{b}_{F,e}^{*XX} + \widetilde{b}_{F,e}^{*YY}\rvert$	$< 2 \times 10^{-8}$ GeV^{-1}	[12]
$\lvert\widetilde{b}_{F,e}^{*ZZ}\rvert$	$< 8 \times 10^{-9}$ GeV^{-1}	[12]
$\lvert\widetilde{b}_p^Z\rvert$	$< 2 \times 10^{-21}$ GeV	[27, 28]
$\lvert\widetilde{b}_p^{*Z}\rvert$	$< 6 \times 10^{-21}$ GeV	[27, 28]
$\lvert\widetilde{b}_{F,p}^{XX} + \widetilde{b}_{F,p}^{YY}\rvert$	$< 1 \times 10^{-5}$ GeV^{-1}	[27, 28]
$\lvert\widetilde{b}_{F,p}^{ZZ}\rvert$	$< 1 \times 10^{-5}$ GeV^{-1}	[27, 28]
$\lvert\widetilde{b}_{F,p}^{*XX} + \widetilde{b}_{F,p}^{*YY}\rvert$	$< 2 \times 10^{-5}$ GeV^{-1}	[27, 28]
$\lvert\widetilde{b}_{F,p}^{*ZZ}\rvert$	$< 8 \times 10^{-6}$ GeV^{-1}	[27, 28]

ing for Lorentz and CPT violation [4], neglects any cancellation or interference among different coefficients but does offer insight and a reasonable measure of the sensitivity to individual coefficients provided no signal has been observed. The resulting constraints on each tilde coefficient are displayed in Table VI. All 16 of these bounds are new in detail because they include effects from $d = 4$, 5, and 6 that are analyzed for the first time in the present work. As described above, some of them reduce in an appropriate limit to results already reported in a suitable minimal-SME limit. Note that a large number of the 36 independent observables remain unexplored in Penning-trap experiments to date.

IV. SUMMARY AND OUTLOOK

In this work, we explore the prospects for searching for Lorentz- and CPT-violating effects using experiments with Penning traps. We begin in Sec. II by presenting the Lagrange density for Lorentz-violating spinor QED with operators of mass dimensions up to six. The minimal-SME terms in this theory are given in Eqs. (3) and (4), while the complete set of terms at $d = 5$ and 6 is displayed in Eqs. (6), (7), (10), (11), and (12). The basic properties of the corresponding coefficients for Lorentz violation are compiled in Table I.

Determining the observables in the theory requires investigating the interplay between different operators under field redefinitions. We perform a general fermion field redefinition (19) and list the resulting transformations in Table II. Among other results, this analysis demonstrates that many terms in the Lagrange density that couple spinors to the electromagnetic field strength can be absorbed into other terms in the theory via suitable field redefinitions. A result of practical utility in this work involves the case of a constant electromagnetic field, for which the piece (12) of the Lagrange density vanishes, while all the F-type coupling terms in Eqs. (7) and (11) can be generated by the replacements (40) and (42) in the Lagrange-density terms (6) and (10).

Another issue in characterizing the observables for Lorentz and CPT violation is the noninertial nature of any laboratory on the surface of the Earth. In Sec. II C, we discuss three relevant frames for experimental analysis: the inertial Sun-centered frame, the standard noninertial laboratory frame, and a noninertial apparatus frame. Allowing for the rotation of the Earth, the transformations required to achieve the inertial Sun-centered frame are given by Eqs. (44) and (45). This analysis neglects the suppressed boost effects arising from the revolution of the Earth about the Sun, which would be an interesting subject for a separate work.

Applications of the theory to experiments with Penning traps are discussed in Sec. III. We use perturbation theory to determine the effects of Lorentz and CPT violation on the relativistic Landau levels of a particle in a uniform magnetic field. The results obtained are at leading order in Lorentz violation but exact in other quantities. They are found to be lengthy and are presented in the Appendix. The dominant Lorentz- and CPT-violating perturbative shifts of the energy levels are given in Eq. (58), while the corresponding results for antiparticles are presented in Eq. (61). These expressions permit the derivation of the dominant Lorentz- and CPT-violating shifts of the cyclotron and anomaly frequencies of trapped particles and antiparticles. At leading order, the cyclotron-frequency shifts (63) are found to vanish. The leading-order shifts in anomaly frequencies for particles and antiparticles of species w are given explicitly by Eq. (64). We use the latter expressions to show that the difference (65) between these anomaly frequencies is a measure of pure CPT violation in idealized comparative experiments with the same orientation and magnitude of the trapping magnetic field, while the sum (67) involves only CPT-even effects.

Turning next to issues closer to experiment, we discuss observable signals for trapped electrons, positrons, protons, and antiprotons. Since the anomaly frequency (72) depends on the magnitude of the magnetic field and on geometric factors including the local sidereal time, the colatitude of the experiment, and the local direction of the magnetic field, it follows that the ratio of the anomaly to cyclotron frequencies is no longer an intrinsic property of the particle but becomes an experiment-dependent quantity. We prove that the intrinsic observables providing experiment-independent measures of Lorentz and CPT violation are instead the 36 tilde coefficients $\widetilde{b}_e^J$, $\widetilde{b}_e^{*J}$,

$\widetilde{b}_{F,e}^{(JK)}$, $\widetilde{b}_{F,e}^{*(JK)}$ and $\widetilde{b}_{p}^{J}$, $\widetilde{b}_{p}^{*J}$, $\widetilde{b}_{F,p}^{(JK)}$, $\widetilde{b}_{F,p}^{*(JK)}$. Comparisons of results for particles and antiparticles must also take this into account. One consequence is that the difference (76) between the ratios of the anomaly to cyclotron frequencies for a particle and an antiparticle typically contains both CPT-odd and CPT-even effects.

The above results make feasible the analysis of existing and future experiments for sensitivity to experiment-independent observables for Lorentz and CPT violation. The theory predicts oscillations of all observables at specific harmonics of the sidereal frequency, along with time-independent signals that can be detected in comparative experiments with particles and antiparticles. To illustrate the methodology for the analysis, we consider the sensitive experiments listed in Table III and examine some of their implications. The key information permitting the extraction of constraints on observables is derived and tabulated for electrons and positrons in Table IV and for protons and antiprotons in Table V. Existing experimental measurements are used to extract new and improved constraints on numerous tilde coefficients for Lorentz and CPT violation, using the sidereal variation of observables and comparisons between particles and antiparticles. In the electron sector, we obtain the bounds (78) and (79) using results from experiments at the University of Washington [12, 13], while in the proton sector we combine independent results from the ATRAP [27] and BASE [28] experiments to obtain the bound (81). Table VI lists the ensuing 16 constraints obtained when a single tilde coefficient is taken to be nonzero at a time.

We close this work with a brief outlook on some open and feasible projects that would further enhance the role of Penning traps in studying the foundational Lorentz and CPT symmetry of nature. Each of the following five general topics represents an open challenge for theory and experiment, the resolution of which will ultimately require disentangling conceptual and calculational issues and performing analyses to extract constraints from experimental data.

1. Boost effects. A comparatively direct extension of the present work would involve investigation of suppressed effects neglected here. The largest of these effects comes from the revolution of the Earth about the Sun, which introduces harmonics of the annual revolution frequency and corresponding sidebands near the sidereal harmonics. The new observables come with a suppression factor of the Earth's boost $\beta_\oplus \simeq 10^{-4}$, but they include coefficient combinations that are unobservable without the boost. Additional smaller effects associated with the boost of the laboratory due to the rotation of the Earth, which are suppressed by $\beta_L \simeq 10^{-6}$, are also of potential interest. While boosts can generate sensitivity to coefficients otherwise unobservable in Penning-trap experiments, the corresponding shifts in the Landau levels remain independent of the level number, so much of the conceptual structure for the treatment of signals given in the present work remains in force. The techniques for handling the boosts have been developed in several prior contexts [33, 34, 45–47] and could be transferred to Penning-trap analyses.

2. Cyclotron-frequency shifts. Qualitatively different suppressed effects arise from subleading Lorentz- and CPT-violating contributions to the energy shifts that are proportional to $|qB|$. These contributions can be extracted from the expressions for the energy shifts for $d \leq 5$ given in Appendix A. The suppression factors are stronger than those for boosts, being of order 10^{-9} for electrons or positrons and of order 10^{-16} for protons or antiprotons. However, many of the contributions produce energy shifts that depend on the level number, so they can change the relative spacing of the lowest-lying levels in a single Landau stack and hence affect the cyclotron frequency as well as the anomaly frequency. This implies that signals for Lorentz and CPT violation can appear not only in measurements of anomalous magnetic moments but also in measurements of charge-to-mass ratios. Signals of this type have been studied theoretically in the minimal SME [11], and they have led to constraints using experiments comparing the cyclotron frequencies of antiprotons and H^- ions [14, 15]. Revisiting the theoretical basis for these works while including effects at $d = 5$ and 6 can be expected to yield interesting new constraints and stimulate further experiments.

3. Field effects. Additional sensitivities to Lorentz and CPT violation could be obtained by refining the analysis of the electromagnetic fields in a realistic Penning trap. For example, the presence of the electric field that restricts the axial motion of the trapped particle produces several types of novel and potentially interesting effects. If the experimental procedure equipartitions the axial and cyclotron energies, then effects from the axial motion will be comparable to those proportional to $|qB|$ mentioned above and so will permit suppressed sensitivities to additional coefficients for Lorentz and CPT violation associated with covariant-derivative couplings of the trapped fermion. These kinds of effects correspond to terms involving powers of p_3 in the energy shifts given in Appendix A. The electric field also introduces new sensitivities to the F-type coefficients, which are associated with electromagnetic couplings of the fermion that vanish in the absence of the electric field. For a constant electric field, these effects can be derived from the energy shifts given in Appendix A by performing the replacements (40) and (42). Moreover, the spatially varying electric field in a realistic Penning trap could offer sensitivity to the terms in the Lagrange density (12) that otherwise are inaccessible. Control of the magnetic field also implies interesting prospects for studying independent observables. For example, the dependence on B of the anomaly frequency (72) shows that two experiments differing only in the magnitudes of the magnetic fields can yield sensitivities to coefficients for Lorentz violation.

4. Other species. Trapping and studying the magnetic moments of species other than electrons, positrons, protons, and antiprotons could provide additional sensitivities to coefficients for Lorentz and CPT violation beyond those discussed here. For example, experiments on any ion with magnetic moment influenced by the neutrons in its nucleus could offer sensitivities to coefficients in the neutron sector. A theoretical treatment of this possibility along the lines in the present work would make an interesting project with the potential to influence experimental discovery. The coefficients for Lorentz violation for composite species are combinations of those in the electron, proton, and neutron sectors, and determining the relationship a crucial part of this type of investigation. For H^- ions in the context of the minimal SME, the link has been established at leading order in Lorentz violation and shown to imply experimental sensitivities differing from those for trapped electrons or protons [11]. Inclusion of operators with $d = 5$, 6 would introduce unique dependences on the momenta of the particles forming the composite species. For nuclear components with comparatively high momenta, this implies a potential increase in the experimental reach by several orders of magnitude, in line with results from atomic spectroscopy [34].

5. Additional SME sectors. Efforts to extend the theoretical scope of our analysis can also be expected to provide interesting and novel results. One option would be to extend the results in this work to operators of arbitrary d. Partial results along these lines are given in Eq. (60). Another line of investigation would consider effects from other SME sectors. For example, contributions from Lorentz and CPT violation in the photon sector are known to modify the Maxwell equations and hence could in principle affect the behavior of trapped particles, although most effects are tightly constrained by other experiments [4, 51, 52]. Effects on trapped particles from Lorentz and CPT violation in the strong, electroweak, or gravitational sectors could also be envisaged. Many of these are likely to be suppressed in typical scenarios. For example, effects proportional to the local gravitational acceleration in the laboratory must come with a numerical suppression factor of order 10^{-32}. In light of the current reach of experiments with Penning traps, countershaded Lorentz and CPT violation [53] may be the most interesting possibility to pursue in this context.

In conclusion, this work presents the general theory for Lorentz- and CPT-violating QED including operators of mass dimensions $d \leq 6$ and offers a guide to the prospects for detecting dominant effects from Lorentz and CPT violation in precision experiments on particles and antiparticles confined to a Penning trap. We have used the methodology developed here and existing experimental data to constrain 16 of the 36 experiment-independent observables for Lorentz and CPT violation in the electron and proton sectors, but much work remains before a complete coverage of all predicted dominant effects can be achieved. The many prospective effects in current and future Penning-trap experiments provide strong inducement for continuing these types of efforts to investigate Lorentz and CPT symmetry, with promising potential for uncovering violations of these basic spacetime symmetries.

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Appendix A: Perturbative energy shifts

In this Appendix, we present the results of perturbative calculations for the energy levels of a fermion of mass m, charge $q = \sigma|q|$, and spin orientation $s = \pm 1$. The analysis is performed using Eq. (56) with Lorentz- and CPT-violating operators appearing in $\mathcal{L}^{(3)}$, $\mathcal{L}^{(4)}$, and $\mathcal{L}_D^{(5)}$. As discussed in Sec. III A 1, the contributions from $\mathcal{L}_F^{(5)}$ can be obtained via the substitutions presented in Sec. II B 3. The expressions below are valid at leading order in Lorentz violation but are otherwise exact. They are presented in the apparatus frame having coordinates (x^1, x^2, x^3) described in Sec. II C, with the magnetic field aligned along $\hat{x}_3$.

At $d = 3$, calculation with $\mathcal{L}^{(3)}$ reveals contributions to the energy shift given by

$$\delta E_{n,s}^{(3)} = a^0 - a^3 \frac{p_3}{E_{n,s}} + \sigma s b^0 \frac{p_3}{E_{n,s}} - \sigma s b^3 \left(1 - \frac{(2n+1-\sigma s)|qB|}{E_{n,s}(E_{n,s}+m)}\right) - \sigma s H^{12} \left(1 - \frac{p_3^2}{E_{n,s}(E_{n,s}+m)}\right). \qquad \text{(A1)}$$

For $d=4$, we obtain from $\mathcal{L}^{(4)}$ the results

$$\begin{aligned}
\delta E^4_{n,s} \;=\; & -c^{00}E_{n,s}+(c^{03}+c^{30})p_3-(c^{11}+c^{22})\frac{(2n+1-\sigma s)|qB|}{2E_{n,s}}-c^{33}\frac{p_3^2}{E_{n,s}} \\
& -\sigma s d^{00}p_3+\sigma s d^{30}m\left(1-\frac{p_3^2}{E_{n,s}(E_{n,s}+m)}\right)+\sigma s(d^{03}+d^{30})\frac{p_3^2}{E_{n,s}} \\
& -\sigma s(d^{11}+d^{22})p_3\frac{(2n+1-\sigma s)|qB|}{2E_{n,s}(E_{n,s}+m)}-\sigma s d^{33}p_3\left(1-\frac{(2n+1-\sigma s)|qB|}{E_{n,s}(E_{n,s}+m)}\right)-e^0m+e^3p_3\frac{m}{E_{n,s}} \\
& +\sigma s g^{120}\left(m+\frac{(2n+1-\sigma s)|qB|}{E_{n,s}+m}\right)-\sigma s g^{123}p_3\left(\frac{m}{E_{n,s}}+\frac{(2n+1-\sigma s)|qB|}{E_{n,s}(E_{n,s}+m)}\right) \\
& +\sigma s(g^{231}-g^{132})p_3\frac{(2n+1-\sigma s)|qB|}{2E_{n,s}(E_{n,s}+m)}+\sigma s(g^{012}-g^{021})\frac{(2n+1-\sigma s)|qB|}{2E_{n,s}}. \qquad \text{(A2)}
\end{aligned}$$

The contributions from $\mathcal{L}^{(5)}_D$ at $d=5$ are found to be

$$\begin{aligned}
\delta E^{(5)}_{n,s} \;=\; & m^{(5)00}mE_{n,s}-2m^{(5)03}p_3m+(m^{(5)11}+m^{(5)22})|qB|\left(\frac{(2n+1)m}{2E_{n,s}}+\sigma s\frac{(2n+1-\sigma s)|qB|}{2E_{n,s}(E_{n,s}+m)}\right)+m^{(5)33}p_3^2\frac{m}{E_{n,s}} \\
& +a^{(5)000}{E_{n,s}}^2-2a^{(5)003}p_3E_{n,s}+(a^{(5)011}+a^{(5)022})\left(\frac{2n+1}{2}-\sigma s\frac{(2n+1-\sigma s)|qB|}{2E_{n,s}(E_{n,s}+m)}\right)|qB| \\
& +a^{(5)033}p_3^2+(a^{(5)101}+a^{(5)202})(2n+1-\sigma s)|qB|-(a^{(5)113}+a^{(5)223})p_3\frac{(2n+1-\sigma s)|qB|}{E_{n,s}} \\
& -a^{(5)300}p_3E_{n,s}+2a^{(5)303}p_3^2-(a^{(5)311}+a^{(5)322})p_3\frac{(2n+1)|qB|}{2E_{n,s}}-a^{(5)333}p_3^3\frac{1}{E_{n,s}} \\
& +\sigma s b^{(5)000}p_3E_{n,s}-2\sigma s b^{(5)003}p_3^2+\sigma s(b^{(5)011}+b^{(5)022})p_3\frac{(2n+1)|qB|}{2E_{n,s}}+\sigma s b^{(5)033}p_3^2\frac{1}{E_{n,s}} \\
& +\sigma s(b^{(5)101}+b^{(5)202})p_3\frac{(2n+1-\sigma s)|qB|}{(E_{n,s}+m)}-\sigma s(b^{(5)113}+b^{(5)223})p_3^2\frac{(2n+1-\sigma s)|qB|}{E_{n,s}(E_{n,s}+m)} \\
& -\sigma s b^{(5)300}E_{n,s}\left(E_{n,s}-\frac{(2n+1-\sigma s)|qB|}{E_{n,s}+m}\right)+2\sigma s b^{(5)303}p_3\left(E_{n,s}-\frac{(2n+1-\sigma s)|qB|}{E_{n,s}+m}\right) \\
& -\sigma s(b^{(5)311}+b^{(5)322})|qB|\left(\frac{2n+1}{2}+\sigma s\frac{(2n+1-\sigma s)|qB|}{E_{n,s}(E_{n,s}+m)}\right)-\sigma s b^{(5)333}p_3^2\left(1-\frac{(2n+1-\sigma s)|qB|}{2E_{n,s}(E_{n,s}+m)}\right) \\
& +\sigma s(H^{(5)0102}-H^{(5)0201})(2n+1-\sigma s)|qB|-\sigma s(H^{(5)0123}-H^{(5)0212})p_3\frac{(2n+1-\sigma s)|qB|}{E_{n,s}} \\
& -\sigma s H^{(5)1200}{E_{n,s}}^2\left(1-\frac{p_3^2}{E_{n,s}(E_{n,s}+m)}\right)+2\sigma s H^{(5)1203}p_3\left(E_{n,s}-\frac{p_3^2}{E_{n,s}+m}\right) \\
& -\sigma s(H^{(5)1211}+H^{(5)1222})|qB|\left(\frac{(2n+1)m}{2E_{n,s}}+\frac{(2n+1-\sigma s)^2|qB|}{2E_{n,s}(E_{n,s}+m)}\right)-\sigma s H^{(5)1233}p_3^2\left(1-\frac{p_3^2}{E_{n,s}(E_{n,s}+m)}\right) \\
& +\sigma s(H^{(5)1302}-H^{(5)2301})p_3\frac{(2n+1-\sigma s)|qB|}{(E_{n,s}+m)}-\sigma s(H^{(5)1323}-H^{(5)2313})p_3^2\frac{(2n+1-\sigma s)|qB|}{E_{n,s}(E_{n,s}+m)}. \qquad \text{(A3)}
\end{aligned}$$

The corresponding energy shifts for the antiparticle can be obtained as described in Sec. III A 1.

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