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# Open Effective Field Theories from Deeply Inelastic Reactions


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Effective field theories have often been applied to systems with deeply inelastic reactions that produce particles with large momenta outside the domain of validity of the effective theory. The effects of the deeply inelastic reactions have been taken into account in previous work by adding local anti-Hermitian terms to the effective Hamiltonian. Here we show that when multi-particle systems are considered, an additional modification is required in equations governing the density matrix. We define an effective density matrix by tracing over the states containing high-momentum particles, and show that it satisfies a Lindblad equation, with local Lindblad operators determined by the anti-Hermitian terms in the effective Hamiltonian density.


## I. INTRODUCTION

Elementary particles are accurately described by local quantum field theories. Some of the most fascinating phenomena in condensed matter physics and in atomic physics can also be described by local quantum field theories. The modern understanding of local quantum field theories is based largely on *effective field theory* [1–7]. This approach provides physical interpretations of the mathematical singularities that are ubiquitous in local quantum field theories. Effective field theory provides a systematic framework for quantifying the effects of higher-momentum physics that can not be described explicitly within a low-energy effective theory. Effective field theory also provides a systematic framework for developing low-energy approximations for phenomena that are described by a local quantum field theory.

One area where effective field theories have proven useful is in the analysis of the impact of high-momentum decays on nonrelativistic field theories. Examples include positronium decay into photons, analyzed in nonrelativistic QED [8, 9], and the decays of quarkonium states into gluons, analyzed in nonrelativistic QCD [10]. Processes such as these, whose final-state particles have much larger three-momenta than the inital-state particles, are mimicked in nonrelativistic effective field theories by local non-Hermitian corrections to the effective Lagrangian.

Previous analyses have focused on systems consisting of a single atom or a single meson, where the treatment of the non-Hermitian corrections is straightforward. In this paper, we extend this earlier work to include multi-particle systems. We show that the non-Hermitian terms in the effective Lagrangian lead to modifications in the evolution equation of the effective density matrix that describes multi-particle systems. In particular, we show that the effective density matrix satisfies a Lindblad equation [11–13]. This is true provided the decay products escape from the system or otherwise decouple, so they cannot influence it later.

In Section II, we review the ideas behind effective field theories and their use for deeply inelastic processes. We then outline how these ideas must be adapted for use in multi-particle systems and the role played by the Lindblad equation. In Section III, we show explicitly how the Lindblad equation emerges from a perturbative analysis of a simple model. Finally in Section IV we summarize our results and discuss possible applications.

## II. EFFECTIVE THEORIES AND DEEPLY INELASTIC PROCESSES

### A. Two Types of Locality

An effective field theory is obtained by removing (*integrating out*) states from a field theory. The simplest applications involve removing very massive particles. A muon, for example, decays into a $\nu_\mu$ neutrino and a $W$ boson. The $W$ is almost a thousand times more massive than the muon and so is highly virtual. It decays almost immediately ($\Delta t \approx 1/M_W$) into an electron and the antineutrino $\bar{\nu}_e$. This process is very accurately modeled by the Fermi interaction (Fig. 1), where the decay occurs at a point rather than spread over space-time distances of order $1/M_W$:

$$\frac{G_F}{\sqrt{2}}\bar{\nu}_\mu\gamma_\alpha(1-\gamma_5)\mu\ \bar{e}\gamma^\alpha(1-\gamma_5)\nu_e. \tag{1}$$

We have integrated the $W$ out of the theory.

The Fermi interaction is the leading term in a series of local operators that can be used to mimic the decay process to arbitrary precision. This series is ob-

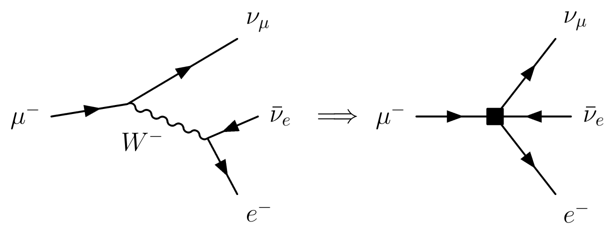


FIG. 1. In the amplitude for muon decay, the $W$ can propagate only over short distances. Its exchange can therefore be approximated by a contact interaction.

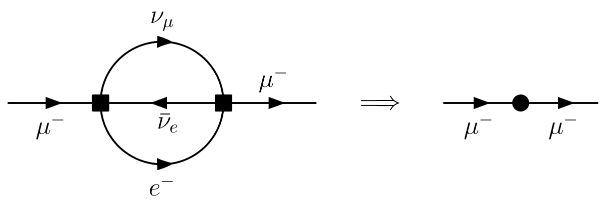


FIG. 2. In the amplitude for $\mu^- \to \nu_\mu e^- \bar\nu_e \to \mu^-$, the high-momentum intermediate-state particles are created and annihilated in a localized region. Their effects can therefore be reproduced by local operators.

tained at tree-level by Taylor expanding the $W$ propagator $1/(q_W^2 - M_W^2)$ in powers of $q_W^2/m_W^2$ to obtain

$$\delta\mathcal{L}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{n=0}^{\infty} \bar\nu_\mu \gamma_\alpha (1-\gamma_5)\mu \left(\frac{-\partial^2}{M_W^2}\right)^n \bar e \gamma^\alpha (1-\gamma_5)\nu_e. \tag{2}$$

In practice only one or two terms in this series are needed to account for experiment. The individual operators are renormalized when loop corrections are included, but the effective theory is still capable of reproducing the original theory to arbitrary precision provided operators of sufficiently high dimension are retained. Operators of dimension $n$ correct the theory at order $(p/M_W)^{n-4}$, where $p$ is the muon's momentum.

A less obvious opportunity to remove states arises when a particle decays to particles with much smaller masses. Such a decay is an example of a *deeply inelastic reaction*, where a large fraction of the initial state's rest mass is converted into large kinetic energies for the final-state particles. Muon decay is again a good example. The amplitude for $\mu^- \to \nu_\mu e^- \bar\nu_e \to \mu^-$ on the left side of Fig. 2 is analytic in the muon's three-momentum $\boldsymbol{p}$ provided that momentum is nonrelativistic. This is because the nearest nonanalyticity in the amplitude is at the threshold energy for the $\nu_\mu e^- \bar\nu_e$ state, which is effectively zero and far below the nonrelativistic muon's energy ($\approx m_\mu$). As a result, we can Taylor expand the amplitude in powers of $\boldsymbol{p}^2/m_\mu^2$:

$$T\big[\mu \to \nu_\mu e \bar\nu_e \to \mu\big] = T_0 \left(1 + \sum_{n=1}^{\infty} b_{2n} \left(\frac{-\boldsymbol{p}^2}{m_\mu^2}\right)^n\right). \tag{3}$$

In practice, only a few terms need to be retained, depending upon how nonrelativistic the muon is. These corrections can be incorporated into an effective field theory by discarding the high-energy $\nu_\mu e^- \bar\nu_e$ states and introducing new correction terms in the effective theory's Lagrangian:

$$\delta\mathcal{L}_{\text{eff}} = T_0\, \psi_\mu^\dagger \left(1 + \sum_{n=1}^{\infty} b_{2n} \left(\frac{\nabla^2}{m_\mu^2}\right)^n\right) \psi_\mu, \tag{4}$$

where $\psi_\mu$ is the (2-component) nonrelativistic muon field.

Here we are particularly interested in the imaginary part of this series, coming from the muon's deeply inelastic decay reaction. We can write that part of the effective Lagrangian as

$$\mathcal{L}_{\text{deep}} = \frac{i}{2}\Gamma_\mu \psi_\mu^\dagger \left(1 + \sum_{n=1}^{\infty} c_{2n} \left(\frac{\nabla^2}{m_\mu^2}\right)^n\right) \psi_\mu, \tag{5}$$

where $\Gamma_\mu$ is the muon's decay rate to $\nu_\mu e^- \bar\nu_e$. (The subscript "deep" stands for "deeply inelastic reactions".) These terms mimic the effects of muon decay in the effective theory.

Given our first example, it seems nonintuitive that the effects of a decay to on-shell particles can be mimicked by local operators. In fact the decay process is quite local. This is because the location of the decay can be reconstructed by tracking the decay products back to their origin. The decay products have relatively short wavelengths of order $1/m_\mu$ (because of their high momenta, of order $m_\mu$), and so can locate the decay with a resolution of order $\Delta x \approx 1/m_\mu$. So the decay is localized over a region of size $\Delta x$, which is very small compared to the wavelength of a nonrelativistic muon ($\gg 1/m_\mu$).

The utility of the effective theory is easily illustrated by adding QED effects. Corrections are needed in $\mathcal{L}_{\text{deep}}$ to account for photons radiated by the decay products (the $W$ or the electron). Gauge invariance requires the following form:

$$\begin{aligned}\mathcal{L}_{\text{deep}} = \frac{i}{2}\Gamma_\mu \psi_\mu^\dagger \bigg\{ 1 &+ c_2 \frac{\boldsymbol{D}^2}{m_\mu^2} + c_4 \frac{\boldsymbol{D}^4}{m_\mu^4} + \cdots \\ &+ f_2 \frac{e\boldsymbol{\sigma}\cdot\boldsymbol{B}}{m_\mu^2} + f_3 \frac{e\boldsymbol{\nabla}\cdot\boldsymbol{E}}{m_\mu^3} + \cdots \bigg\} \psi_\mu,\end{aligned} \tag{6}$$

where $\boldsymbol{D}$ is the QED gauge-covariant derivative and $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields. The coefficients $c_{2n}$ are the same (to leading order) as in the theory without QED corrections. This formula shows, for example, that the lifetime of a $\mu^- e^+$ atom equals the muon's lifetime up to corrections of order $\alpha^2 (m_e/m_\mu)^2 \Gamma_\mu$ (due to the $c_2$ term); in particular there are no corrections to the binding energy of order $\alpha^2 (m_e/m_\mu)\Gamma_\mu$ [14].

These same ideas apply to deeply inelastic scattering reactions. For example, the final-state neutrinos in $\mu^- e^+ \to \nu_\mu \bar\nu_e$ have momenta of order $m_\mu$ when the initial positron and muon are nonrelativistic. The amplitude for $\mu^- e^+ \to \nu_\mu \bar\nu_e \to \mu^- e^+$ on the left side of Fig. 3 is analytic in the momenta of $\mu^-$ and $e^+$. The imaginary part

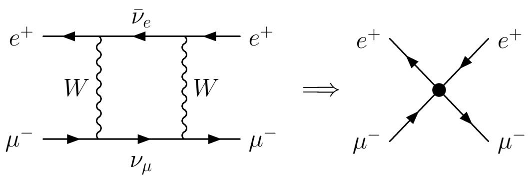


FIG. 3. In the amplitude for $\mu^- e^+ \to \nu_\mu \bar{\nu}_e \to \mu^- e^+$, the high-momentum intermediate-state neutrinos are created and annihilated in a localized region. Their effects can therefore be reproduced by local operators.

of the amplitude comes from the deeply inelastic scattering reaction. We can again mimic the effects of the high-momentum final states using local interactions:

$$\mathcal{L}_{\text{deep}} = iB_0\, \psi_\mu^\dagger \psi_\mu\, \psi_{\bar{e}}^\dagger \psi_{\bar{e}} + iB_1\, \psi_\mu^\dagger \boldsymbol{\sigma} \psi_\mu \cdot \psi_{\bar{e}}^\dagger \boldsymbol{\sigma} \psi_{\bar{e}} + \cdots\,, \quad (7)$$

where $B_0$ and $B_1$ are obtained from the imaginary part of the amplitude on the left side of Fig. 3. It is also straightforward to add QED effects here.

### B. Multi-particle systems

The Hamiltonian that follows from the effective theory described in the previous section has both a Hermitian piece $H_{\text{eff}}$, associated with conventional dynamics, and an anti-Hermitian piece $-iK_{\text{deep}}$, coming from the deeply inelastic reactions whose products have been removed from the theory.[1] In the case of our nonrelativistic muon,

$$H_{\text{eff}} = \int d^3\boldsymbol{r}\, \psi_\mu^\dagger \left\{ eA_0 - \frac{\boldsymbol{D}^2}{2m_\mu} + \cdots \right\} \psi_\mu, \quad (8a)$$

$$K_{\text{deep}} = \frac{1}{2}\Gamma_\mu \int d^3\boldsymbol{r}\, \psi_\mu^\dagger \left\{ 1 + c_2 \frac{\boldsymbol{D}^2}{m_\mu^2} + \cdots \right\} \psi_\mu, \quad (8b)$$

where $A_\alpha$ is the photon field. The leading term in $K_{\text{deep}}$ is $\frac{1}{2}\Gamma_\mu\, \hat{N}_\mu$, where $\hat{N}_\mu$ is the muon number operator:

$$\hat{N}_\mu = \int d^3\boldsymbol{r}\, \psi_\mu^\dagger \psi_\mu. \quad (9)$$

This Hamiltonian applies to both single-muon and multi-muon systems.

The quantum mechanics of such a theory is unconventional because probability is not obviously conserved. The Hamiltonian $H_{\text{eff}} - iK_{\text{deep}}$ does *not* change the number of muons in a state, because it commutes with $\hat{N}_\mu$. Instead it accounts for the effects of muon decay by reducing the probability carried by each state: the norm of a state that starts with $n$ muons decays to zero with the decay rate $n\Gamma_\mu$ (in leading order). This is the correct result — the probability for $n$ muons to still be $n$ muons after time $t$ is $\exp(-n\Gamma_\mu t)$.

We typically want more information about where the probability goes. In the effective theory, an $n$-muon state evolves into a mixture of states with $n$, $n-1$, $n-2\ldots$ muons that is most naturally described by a density matrix. We can construct an *effective density matrix* $\hat{\rho}_{\text{eff}}$ from the density matrix $\hat{\rho}$ of the full theory by tracing over the deeply inelastic decay products:

$$\hat{\rho}_{\text{eff}}(t) = \text{Tr}_{\text{deep}}\big(\hat{\rho}(t)\big). \quad (10)$$

More precisely, we trace out any state containing a particle with momentum exceeding the ultraviolet cutoff $\Lambda_{\text{UV}}$ of the effective field theory. In the case of our nonrelativistic muon theory, this cutoff is some fraction of the muon mass $m_\mu$.

The effective density matrix defined by Eq. (10), like the density matrix of the full theory, is Hermitian and non-negative and it has unit trace: $\text{Tr}(\hat{\rho}_{\text{eff}}) = 1$. Fourier modes with large frequencies of order $m_\mu$ cannot be described accurately in the effective theory. Thus the definition of the effective density matrix should also involve a time average that eliminates high frequencies. Such a time average is implicit in Eq. (10).

The density matrix defined by the partial trace in Eq. (10) is in general non-Markovian. The time derivative $(d/dt)\hat{\rho}_{\text{eff}}(t)$ at time $t$ is determined not only by $\hat{\rho}_{\text{eff}}(t)$ but also by its past history: $\hat{\rho}_{\text{eff}}(t')$, $t' < t$. The non-Markovian behavior arises because a high-momentum particle created by a decay at time $t'$ can interact with a low-energy particle at a later time $t$. We make an additional physical assumption that eliminates this possibility. We assume the high-energy particles from the deeply inelastic reactions interact so weakly with the low-energy particles that their subsequent interactions can be ignored. This would certainly be the case if the high-momentum particles escape from the system. Given this assumption, the effective density matrix should be Markovian.

Given that $\hat{\rho}_{\text{eff}}$ is Markovian, we might naively expect its time evolution equation to be

$$i\frac{d}{dt}\hat{\rho}_{\text{eff}} \overset{?}{=} \big[H_{\text{eff}}, \hat{\rho}_{\text{eff}}\big] - i\big\{K_{\text{deep}}, \hat{\rho}_{\text{eff}}\big\}, \quad (11)$$

but this equation does *not* conserve the total probability $\text{Tr}(\hat{\rho}_{\text{eff}})$. The correct evolution equation has the structure of the Lindblad equation [11, 12]: if the local operator $K_{\text{deep}}$ can be written in the form

$$K_{\text{deep}} = \int d^3\boldsymbol{r} \sum_n L_n^\dagger(\boldsymbol{r}) L_n(\boldsymbol{r}), \quad (12)$$

[1] Contributions to $K_{\text{deep}}$ mimic the anti-Hermitian parts of scattering amplitudes $\langle b|T|a\rangle$, where $|a\rangle$ and $|b\rangle$ are states in the effective theory that are connected by intermediate deeply inelastic reaction channels. The Hermitian parts of these amplitudes are absorbed into $H_{\text{eff}}$.

the Lindblad equation is

$$i\frac{d}{dt}\hat{\rho}_{\rm eff} = \left[H_{\rm eff}, \hat{\rho}_{\rm eff}\right] - i\left\{K_{\rm deep}, \hat{\rho}_{\rm eff}\right\} + 2i\int d^3\boldsymbol{r}\sum_n L_n(\boldsymbol{r})\,\hat{\rho}_{\rm eff}\,L_n^\dagger(\boldsymbol{r}). \tag{13}$$

The additional term makes $\mathrm{Tr}(\hat{\rho}_{\rm eff})$ time independent, since the trace of a commutator is zero and the traces of the last two terms in Eq. (13) cancel. The Lindblad equation is a necessary consequence of our physical requirements on the effective density matrix: $\hat{\rho}_{\rm eff}$ is Hermitian, non-negative, Markovian, and has unit trace.

In the muon decay example, the anti-Hermitian part of the effective Hamiltonian is given by Eq. (8b). There is a single Lindblad operator at leading order, and the Lindblad equation reduces to

$$i\frac{d}{dt}\hat{\rho}_{\rm eff} = \left[H_{\rm eff}, \hat{\rho}_{\rm eff}\right] - \frac{i}{2}\Gamma_\mu\left\{\hat{N}_\mu, \hat{\rho}_{\rm eff}\right\} + i\Gamma_\mu\int d^3\boldsymbol{r}\,\psi_\mu(\boldsymbol{r})\,\hat{\rho}_{\rm eff}\,\psi_\mu^\dagger(\boldsymbol{r}). \tag{14}$$

The role of the Lindblad term is easily understood if we use the evolution equation to calculate the rate of change of the probability $P_n(t)$ for finding $n$ muons in the system. This probability equals the partial trace of $\hat{\rho}_{\rm eff}$ over all states $|X_n\rangle$ that contain $n$ muons:

$$P_n(t) \equiv \sum_{X_n}\langle X_n|\hat{\rho}_{\rm eff}(t)|X_n\rangle. \tag{15}$$

The partial trace of the evolution equation in Eq. (14) gives

$$\frac{d}{dt}P_n(t) = -n\Gamma_\mu P_n(t) + (n+1)\Gamma_\mu P_{n+1}(t). \tag{16}$$

The commutator term in Eq. (14) does not contribute to the partial trace. The anticommutator term gives $-n\Gamma_\mu P_n$, which is the rate at which probability leaves the $n$-muon sector because of the decay of a muon. The Lindblad term gives $+(n+1)\Gamma_\mu P_{n+1}$, which is the rate at which probability enters the $n$-muon sector from the decay of muons in the $(n+1)$-muon sector.[2]

The Lindblad term in Eq. (14) is essential to get the correct physical behavior for the time evolution of the total number of muons. The expectation value of the muon number, for example, is

$$N_\mu(t) \equiv \mathrm{Tr}\big(\hat{N}_\mu\,\hat{\rho}_{\rm eff}(t)\big) = \sum_n nP_n(t). \tag{17}$$

We can use Eq. (16) to determine the time dependence of $N_\mu(t)$:

$$\frac{d}{dt}N_\mu(t) = -\Gamma_\mu\Big[\sum_n n^2P_n(t) - \sum_n n(n+1)P_{n+1}(t)\Big]. \tag{18}$$

[2] This result follows because $\psi^\dagger(\mathbf{r})$ acting on an $n$-muon basis state gives $\sqrt{n+1}$ times an $(n+1)$-muon state.

After shifting the index of the second term on the right side, we obtain $(d/dt)N_\mu = -\Gamma_\mu N_\mu$, which implies that $N_\mu(t) = N_0\exp(-\Gamma_\mu t)$, as expected.

In order to obtain the Lindblad equation in Eq. (13), it is essential that $K_{\rm deep}$ have the structure shown in Eq. (12). This is generally the case in a nonrelativistic effective field theory. In the muon decay example, the operator $K_{\rm deep}$ in Eq. (8b) can be put into the canonical form in Eq. (12) by expressing the expansion inside the braces as the square of the expansion of its square root. The corresponding corrections to the Lindblad term in Eq. (14) can be obtained by making the substitution

$$\psi_\mu(\boldsymbol{r}) \longrightarrow \left\{1 + c_2\frac{\boldsymbol{D}^2}{2m_\mu^2} + \cdots\right\}\psi_\mu(\boldsymbol{r}). \tag{19}$$

More complicated operators in $K_{\rm deep}$, like those that come from the electron-muon terms in Eq. (7), can also be rewritten in the required form. Such operators have the generic form

$$K_{\rm deep} = \int d^3\boldsymbol{r}\sum_{nm} c_{nm}L_n^\dagger(\boldsymbol{r})L_m(\boldsymbol{r}), \tag{20}$$

where the $L_m(\boldsymbol{r})$ are local operators made of low-energy fields and where $c_{nm}$ is a Hermitian matrix, because $K_{\rm deep}$ is Hermitian by definition. It is also guaranteed to be a positive matrix by the optical theorem: $-i(T - T^\dagger) = T^\dagger T$. The double sum is easily rewritten in the canonical form of Eq. (12) by expanding $c_{nm}$ in terms of outer products of its eigenvectors.

## III. A SIMPLE EXAMPLE

In this section, we illustrate how the Lindblad equation for an effective density matrix emerges naturally from a simple model in perturbation theory. We consider a theory that describes a nonrelativistic particle of mass $M$ (with field $\psi$) that can decay into two massless particles (with field $\phi$). The Hamiltonian for the full theory is

$$H = H_0^\psi + H_0^\phi + H_{\rm int}, \tag{21}$$

where

$$H_0^\psi = \int_{\boldsymbol{r}}\psi^\dagger(\boldsymbol{r})\left(M - \frac{\nabla^2}{2M}\right)\psi(\boldsymbol{r}), \tag{22a}$$

$$H_0^\phi = \int_{\boldsymbol{r}}\tfrac{1}{2}\left(\dot{\phi}^2 + (\nabla\phi)^2\right), \tag{22b}$$

$$H_{\rm int} = \tfrac{1}{2}g\int_{\boldsymbol{r}}\Big(\psi^\dagger(\boldsymbol{r})\phi^2(\boldsymbol{r}) + \phi^2(\boldsymbol{r})\psi(\boldsymbol{r})\Big). \tag{22c}$$

To reduce visual clutter, we have introduced the compact notation

$$\int_{\boldsymbol{r}} \equiv \int d^3\boldsymbol{r}. \tag{23}$$

The free Hamiltonians describe particles with on-shell energies

$$E_{\boldsymbol{p}} = M + \frac{\boldsymbol{p}^2}{2M}, \tag{24a}$$

$$\omega_{\boldsymbol{q}} = |\boldsymbol{q}|, \tag{24b}$$

respectively. The interaction Hamiltonian allows a $\psi$ particle to decay into a pair of $\phi$ particles. Our analysis below is simplified if we isolate the part of the interaction Hamiltonian in Eq. (22c) that causes the deeply inelastic decay:

$$H_{\rm deep} = \tfrac{1}{2} g \int_{\boldsymbol{r}} \Big( \psi^\dagger(\boldsymbol{r}) \tilde{\phi}^2(\boldsymbol{r}) + \tilde{\phi}^{\dagger 2}(\boldsymbol{r}) \psi(\boldsymbol{r}) \Big), \tag{25}$$

where $\phi = \tilde{\phi} + \tilde{\phi}^\dagger$ and $\tilde{\phi}$ is the annihilation part of $\phi$.

We are interested in systems consisting of nonrelativistic $\psi$ particles, whose energies and momenta satisfy

$$E_{\boldsymbol{p}} \approx M \qquad |\boldsymbol{p}| \ll M. \tag{26}$$

The decay products in $\psi \to \phi\phi$ therefore have momenta that are approximately $\pm\boldsymbol{q}$, where $\boldsymbol{q}$ is much larger than the $\psi$'s momentum $\boldsymbol{p}$:

$$|\boldsymbol{q}| \approx M/2 \gg |\boldsymbol{p}|. \tag{27}$$

We show below how the entire decay process in this limit is effectively local and instantaneous: it takes place over a spatial region of size $\Delta x \sim 1/M$, which is much larger than the typical length scale $1/|\boldsymbol{p}|$ associated with a nonrelativistic $\psi$ particle, and during a time interval $\Delta t \sim 1/M$, which is much larger than the typical time scale $1/(\boldsymbol{p}^2/M)$. We use this locality to remove all $\phi$ particles from the theory, creating an effective theory of unstable $\psi$ particles. We do this first for a single $\psi$ particle, and then for a system containing multiple $\psi$ particles. Finally, we show how to adapt these results to a different model in which $\psi$ particles are lost through $\psi\psi$ collisions rather than $\psi$ decays. For simplicity, we assume the coupling $g$ is small, and we work to leading order in $g^2$.

### A. Locality

The leading decay contribution to the $\psi$ self-energy comes from the diagram for $\psi \to \phi\phi \to \psi$ in Fig. 4:

$$\tilde{\Pi}(E, \boldsymbol{p}) = g^2 \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{4\omega_{\boldsymbol{q}}\omega_{\boldsymbol{p}-\boldsymbol{q}}} \frac{1}{E - \omega_{\boldsymbol{p}} - \omega_{\boldsymbol{p}-\boldsymbol{q}} + i\epsilon}, \tag{28}$$

where we have used standard time-dependent perturbation theory to calculate the contribution from $H_{\rm deep}$ in Eq. (25) [15]. The integral over $q$ is dominated by momentum scales of order $M$ or larger. This is true as well of its imaginary part evaluated on-shell at $E = E_{\boldsymbol{p}}$:

$$\operatorname{Im} \tilde{\Pi}(E_{\boldsymbol{p}}, \boldsymbol{p}) = -\tfrac{1}{2} g^2 \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{4\omega_{\boldsymbol{q}}\omega_{\boldsymbol{p}-\boldsymbol{q}}} 2\pi\delta(E_{\boldsymbol{p}} - \omega_{\boldsymbol{q}} - \omega_{\boldsymbol{p}-\boldsymbol{q}}), \tag{29}$$

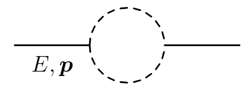


FIG. 4. Self-energy diagram of order $g^2$ for a $\psi$ particle. The propagators for $\psi$ and $\phi$ are represented by solid lines and dashed lines, respectively.

since the delta function forces $|\boldsymbol{q}| \approx M/2 \gg |\boldsymbol{p}|$. As a result, the distance and time scales that dominate the Fourier transform of $\tilde{\Pi}(E, \boldsymbol{p})$ are of order $1/M$, and, therefore, the decay process is local so far as the external (nonrelativistic) $\psi$ particle is concerned. This also means that we can expand $\tilde{\Pi}(E_{\boldsymbol{p}}, \boldsymbol{p})$ in powers of $\boldsymbol{p}^2$:

$$\tilde{\Pi}(E_{\boldsymbol{p}}, \boldsymbol{p}) = \tilde{\Pi}(M, 0) \left( 1 + \sum_{n=1}^{\infty} c_n \left( \frac{\boldsymbol{p}^2}{M^2} \right)^n \right), \tag{30}$$

where the coefficients $c_n$ are independent of $\boldsymbol{p}$. We are interested here in the leading term ($n = 0$) in the power series expansion of $\tilde{\Pi}(E_{\boldsymbol{p}}, \boldsymbol{p})$. We ignore the remaining terms in what follows. They are easily included as higher-order corrections to the effective Hamiltonian, as we discussed in Section II.

The leading effect of $\tilde{\Pi}(E, \boldsymbol{p})$ on a single-$\psi$ state is to renormalize its free Hamiltonian:

$$H_0^\psi \longrightarrow H_{\rm eff} - iK_{\rm deep}, \tag{31}$$

where

$$H_{\rm eff} = \int_{\boldsymbol{r}} \psi^\dagger(\boldsymbol{r}) \Big( M - \frac{\nabla^2}{2M} + \operatorname{Re} \tilde{\Pi}(M, 0) \Big) \psi(\boldsymbol{r}), \tag{32a}$$

$$K_{\rm deep} = \frac{1}{2} \Gamma \int_{\boldsymbol{r}} \psi^\dagger(\boldsymbol{r}) \psi(\boldsymbol{r}). \tag{32b}$$

In the Hermitian part of the effective Hamiltonian, $\operatorname{Re} \tilde{\Pi}(M, 0)$ is absorbed into a renormalization of the mass $M$. In the anti-Hermitian part, $\Gamma$ is the decay rate of a $\psi$ particle:

$$\Gamma = -2 \operatorname{Im} \tilde{\Pi}(M, 0). \tag{33}$$

The locality of the decay process for nonrelativistic momenta, which implies $\tilde{\Pi}(E, \boldsymbol{p}) \approx \tilde{\Pi}(M, 0)$, allows us to simplify correlators that involve $\tilde{\phi}$ fields and $\psi^\dagger$ fields. For example, in the absence of interactions, the correlator $\langle 0 | \tilde{\phi}^2(\boldsymbol{r}, t)\, \psi^\dagger(0, 0) | 0 \rangle$ vanishes. With interactions, we can use locality to replace the $\tilde{\phi}$ fields by a $\psi$ field, as illustrated in Fig. 5:

$$\tfrac{1}{2} g \tilde{\phi}^2(\boldsymbol{r}) \longrightarrow \tilde{\Pi}(M, 0)\, \psi(\boldsymbol{r}, t) \tag{34}$$

when the operators are acting to the right on a state in the Fock space of $\psi$. Similarly, we can replace

$$\tfrac{1}{2} g \tilde{\phi}^{\dagger 2}(\boldsymbol{r}) \longrightarrow \tilde{\Pi}^*(M, 0)\, \psi^\dagger(\boldsymbol{r}) \tag{35}$$

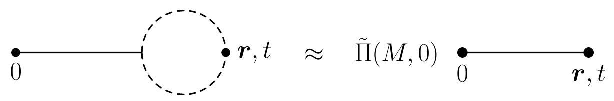


FIG. 5. The operator $\frac{1}{2}g\tilde{\phi}^2(\boldsymbol{r},t)$ can be replaced by $\tilde{\Pi}(M,0)\psi(\boldsymbol{r},t)$ in a nonrelativistic correlator.

when the operators are acting to the left on a state in the Fock space of $\psi$. We use these substitutions in the next section.

### B. Emergence of the Lindblad equation

Replacing the free Hamiltonian $H_0^\psi$ by the effective Hamiltonian $H_{\rm eff} - iK_{\rm dir}$ is all that is needed to analyze the impact of the high-momentum decay on single-$\psi$ states. Analyzing multi-$\psi$ states is more complicated, however, as we discussed in Section II: a system that is described initially by a state with $n$ $\psi$ particles evolves into a mixture of states with $n, n{-}1, n{-}2, \ldots$ $\psi$ particles. The single-$\psi$ state also evolves into a mixture, but there are only two states, $n=1$ and $n=0$, and we don't care about the second one. For $n>1$, we need the density matrix $\hat{\rho}(t)$ to track the superposition of states containing different numbers of $\psi$ particles over time. Following Section II, we replace this density matrix by an effective density matrix obtained by tracing over the Hilbert space of the decay products:

$$\hat{\rho}_{\rm eff}(t) \equiv {\rm Tr}_\phi\, \hat{\rho}(t). \tag{36}$$

Our goal is to derive an evolution equation for $\hat{\rho}_{\rm eff}$.

The temporal evolution of the density matrix is given by

$$i\frac{d}{dt}\hat{\rho} = \big[H, \hat{\rho}\big], \tag{37}$$

where $H$ is the Hamiltonian for the full theory in Eq. (21). We obtain an evolution equation for the effective density matrix by tracing both sides of this equation over all $\phi$ states:

$$i\frac{d}{dt}\hat{\rho}_{\rm eff} = {\rm Tr}_\phi\big[H, \hat{\rho}\big]. \tag{38}$$

The contributions to this equation from the kinetic terms are simple since

$$\begin{aligned} {\rm Tr}_\phi\big[H_0^\psi, \hat{\rho}\big] &= \big[H_0^\psi, \hat{\rho}_{\rm eff}\big], &\text{(39a)}\\ {\rm Tr}_\phi\big[H_0^\phi, \hat{\rho}\big] &= 0, &\text{(39b)}\end{aligned}$$

The first equation holds because $H_0^\psi$ does not act on $\phi$ states. The second equation holds because $H_0^\phi$ depends only on $\phi$ fields.[3] The evolution equation (38) reduces to

$$i\frac{d}{dt}\hat{\rho}_{\rm eff} = \big[H_0^\psi, \hat{\rho}_{\rm eff}\big] + {\rm Tr}_\phi\big[H_{\rm int}, \hat{\rho}\big]. \tag{40}$$

Again we focus on the interaction term in Eq. (25) that causes decays. The other parts of the interaction term generate additional contributions to the Hermitian part of the effective Hamiltonian ($H_{\rm eff}$), which we ignore because they are irrelevant to decays. The decay interaction contributes four terms to the right side of the evolution equation (40):

$$\begin{aligned} &{\rm Tr}_\phi\big[H_{\rm deep}, \hat{\rho}\big] \\ &\quad = \tfrac{1}{2}g\int_{\boldsymbol{r}} {\rm Tr}_\phi\Big(\psi^\dagger(\boldsymbol{r})\tilde{\phi}^2(\boldsymbol{r})\,\hat{\rho} + \tilde{\phi}^{\dagger 2}(\boldsymbol{r})\psi(\boldsymbol{r})\,\hat{\rho}\Big) \\ &\quad - \tfrac{1}{2}g\int_{\boldsymbol{r}} {\rm Tr}_\phi\Big(\hat{\rho}\,\psi^\dagger(\boldsymbol{r})\tilde{\phi}^2(\boldsymbol{r}) + \hat{\rho}\,\tilde{\phi}^{\dagger 2}(\boldsymbol{r})\psi(\boldsymbol{r})\Big). \end{aligned} \tag{41}$$

For simplicity, we consider the initial time when the system consists only of $\psi$ particles. The action of $\rho(t)$ at this time includes a projection onto the Fock space of $\psi$. We proceed to examine each of the four terms in Eq. (41) in turn. We can use the substitution in Eq. (34) to rewrite the first trace in Eq. (41) as:

$$\tfrac{1}{2}g{\rm Tr}_\phi\Big(\psi^\dagger(\boldsymbol{r})\tilde{\phi}^2(\boldsymbol{r})\,\hat{\rho}\Big) \approx \tilde{\Pi}(M,0)\,\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\hat{\rho}_{\rm eff}. \tag{42}$$

Similarly we can use Eq. (35) to rewrite the trace in the second decay term as:

$$\begin{aligned} \tfrac{1}{2}g{\rm Tr}_\phi\Big(\tilde{\phi}^{\dagger 2}(\boldsymbol{r})\psi(\boldsymbol{r})\,\hat{\rho}\Big) &= \tfrac{1}{2}g{\rm Tr}_\phi\Big(\psi(\boldsymbol{r})\,\hat{\rho}\,\tilde{\phi}^{\dagger 2}(\boldsymbol{r})\Big) \\ &\approx \tilde{\Pi}^*(M,0)\,\psi(\boldsymbol{r})\,\hat{\rho}_{\rm eff}\,\psi^\dagger(\boldsymbol{r}). \end{aligned} \tag{43}$$

The traces in the remaining two terms follow the same patterns:

$$\begin{aligned} \tfrac{1}{2}g{\rm Tr}_\phi\big(\hat{\rho}\psi^\dagger(\boldsymbol{r})\tilde{\phi}^2(\boldsymbol{r})\big) &\approx \tilde{\Pi}(M,0)\psi(\boldsymbol{r})\hat{\rho}_{\rm eff}\psi^\dagger(\boldsymbol{r}), &\text{(44a)}\\ \tfrac{1}{2}g{\rm Tr}_\phi\big(\hat{\rho}\tilde{\phi}^{\dagger 2}(\boldsymbol{r})\psi(\boldsymbol{r})\big) &\approx \tilde{\Pi}^*(M,0)\hat{\rho}_{\rm eff}\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}) &\text{(44b)}\end{aligned}$$

Inserting these traces into Eq. (41), we obtain our final result for the evolution equation in Eq. (40):

$$\begin{aligned} i\frac{d}{dt}\hat{\rho}_{\rm eff} = \big[H_{\rm eff}, \hat{\rho}_{\rm eff}\big] - \frac{i}{2}\Gamma\int_{\boldsymbol{r}}\Big(\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})\,\hat{\rho}_{\rm eff} \\ +\hat{\rho}_{\rm eff}\,\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}) - 2\psi(\boldsymbol{r})\,\hat{\rho}_{\rm eff}\,\psi^\dagger(\boldsymbol{r})\Big), \end{aligned} \tag{45}$$

where $H_{\rm eff}$ and $\Gamma$ are defined in Eqs. (32a) and (33). This equation has the standard Lindbladian form. The last term removes $\psi$ particles one at a time to account

[3] The identity ${\rm Tr}_\phi(\hat{A}\hat{B}) = {\rm Tr}_\phi(\hat{B}\hat{A})$ holds for any operator $\hat{A}$ constructed out of the field $\phi$ and any operator $\hat{B}$. This can be verified by expressing the partial trace as a sum over a complete set of $\phi$ states and inserting a complete set of $\phi$ states between $\hat{A}$ and $\hat{B}$.

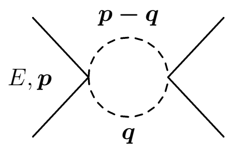


FIG. 6. Diagram of order $g^2$ for $\psi\psi \to \psi\psi$ through an intermediate state with two $\phi$ particles.

for their disappearance due to decays into pairs of high-momentum $\phi$ particles.

Note that we are making a nontrivial physical assumption about $\hat{\rho}$ when we use, for example, Eq. (34) to remove $\tilde{\phi}$ fields from the effective evolution equation, as in Eq. (42). This substitution is valid provided $\tilde{\phi}^2$ annihilates $\phi$ particles coming from the $\psi$ sector of the density matrix (that is from $\psi$ decays). In principle, it is also possible for $\tilde{\phi}^2(\boldsymbol{r})$ to annihilate $\phi$ particles from the $\phi$-sector of $\hat{\rho}$. We assume that such contributions can be ignored because the probability for finding two $\phi$ particles at the same space-time point is vanishingly small (and therefore the probability of an inverse decay, $\phi\phi \to \psi$, is negligible). This is the case if $\hat{\rho}$ describes a situation in which all $\phi$ particles are produced by $\psi$ decays and, once produced, they escape from the system or otherwise decouple.

### C. Inelastic scattering

A variation on our simple model is to replace the decay process $\psi \to \phi\phi$ by a deeply inelastic scattering process $\psi\psi \to \phi\phi$ as the mechanism by which probability leaks from the $\psi$ sector. We replace the interaction Hamiltonian in Eq. (22c) by

$$H_{\rm int} = \tfrac{1}{4} g \int_{\boldsymbol{r}} \Big( \psi^{\dagger 2}(\boldsymbol{r})\phi^2(\boldsymbol{r}) + \phi^2(\boldsymbol{r})\psi^2(\boldsymbol{r}) \Big). \quad (46)$$

This interaction term allows the inelastic scattering reaction $\psi\psi \to \phi\phi$, where now the decay products have approximate momenta $\pm\boldsymbol{q}$ with $|\boldsymbol{q}| \approx M$. The leading contribution to the transition amplitude for $\psi\psi \to \psi\psi$ comes from the diagram in Fig. 6.

The analysis of the effective density matrix for this model, where the $\phi$ states are traced out, is almost identical to our decay model analysis above. Here, in place of Eqs. (34) and (35), we have substitutions

$$\tfrac{1}{2} g\, \tilde{\phi}^2(\boldsymbol{r},t) \longrightarrow \tfrac{1}{2}\tilde{\Pi}(2M,0)\, \psi^2(\boldsymbol{r},t), \quad (47a)$$

$$\tfrac{1}{2} g\, \tilde{\phi}^{\dagger 2}(\boldsymbol{r},t) \longrightarrow \tfrac{1}{2}\tilde{\Pi}^*(2M,0)\, \psi^{\dagger 2}(\boldsymbol{r},t), \quad (47b)$$

where $\tilde{\Pi}(E,\boldsymbol{p})$ is the same function defined in Eq. (28). The new term in the effective Hamiltonian is an interaction term instead of a mass term:

$$H_{\rm eff} - iK_{\rm deep} = H_0^{\psi} + \tfrac{1}{4}\tilde{\Pi}(2M,0) \int_{\boldsymbol{r}} \psi^{\dagger 2}(\boldsymbol{r})\psi^2(\boldsymbol{r}). \quad (48)$$

The anti-Hermitian part of the effective Hamiltonian comes from the imaginary part of $\tilde{\Pi}(2M,0)$, which we denote by $-\Gamma/2$. The final evolution equation for the effective density matrix is

$$i\frac{d}{dt}\hat{\rho}_{\rm eff} = \big[H_{\rm eff}, \hat{\rho}_{\rm eff}\big] - \frac{i}{2}\Gamma \int_{\boldsymbol{r}} \Big( \psi^{\dagger 2}(\boldsymbol{r})\psi^2(\boldsymbol{r})\,\hat{\rho}_{\rm eff} + \hat{\rho}_{\rm eff}\,\psi^{\dagger 2}(\boldsymbol{r})\psi^2(\boldsymbol{r}) - 2\psi^2(\boldsymbol{r})\,\hat{\rho}_{\rm eff}\,\psi^{\dagger 2}(\boldsymbol{r}) \Big). \quad (49)$$

This equation again has the standard Lindbladian form. The last term removes $\psi$ particles two at a time to account for their disappearance due to inelastic scattering into pairs of high-momentum $\phi$ particles.

## IV. DISCUSSION

The effective Hamiltonian for an effective field theory obtained by integrating out high-momentum particles produced by deeply inelastic reactions is local but non-Hermitian. We have pointed out that states consisting of low-energy particles are naturally described by an effective density matrix obtained by tracing over states containing high-momentum particles, as in Eq. (10). The time evolution of the effective density matrix is given by the Lindblad equation in Eq. (13). The Lindblad operators $L_n(\boldsymbol{r})$ are local, and they can be deduced from the anti-Hermitian terms in the effective Hamiltonian density, which can be expressed in the form in Eq. (12). The Lindblad terms in the evolution equation are essential to get the correct behavior for the time evolution of multiparticle observables, such as the number of low-energy particles.

The Lindblad equation is familiar in quantum information theory [13]. An *open quantum system* consists of a *subsystem* of interest together with its *environment*. A time evolution equation for the density matrix of the subsystem is called a *master equation*. Under special conditions, the master equation has the form of the Lindblad equation [11, 12]. These conditions ensure that the autocorrelation function of the interaction Hamiltonian that connects the subsystem and the environment decreases to 0 at large times.

An *open effective field theory* is an open quantum system in which the subsystem of interest is an effective field theory [16, 17]. Grozdanov and Polonyi have proposed an open effective field theory for the hydrodynamic modes of a quantum field theory as a framework for deriving dissipative hydrodynamics [16]. Burgess, Holman, Tasinato, and Williams have applied open effective field theory to the super-Hubble modes of primordial quantum fluctuations in the early universe [17, 18]. In the stochastic inflation framework, the master equation is the Lindblad equation. We have shown that an effective field theory in which deeply inelastic reaction products have been integrated out is an open effective field theory. In this case, the environment consists of the high-momentum particles produced by the deeply inelastic reactions.

A heavy quark and heavy antiquark in the quark-gluon plasma can be regarded as an open quantum system in which the heavy quark-antiquark pair is the subsystem of interest and the quark-gluon plasma is the environment. The quark-gluon plasma can cause the decoherence of the heavy quark-antiquark pair and the dissociation of heavy-quarkonium bound states. A master equation for the heavy quark-antiquark subsystem that has the Lindblad form has been derived [19]. This problem could perhaps be formulated in terms of an open effective field theory using potential NRQCD [20].

Ultracold atoms can be described by a local nonrelativistic effective field theory for which the coupling constant is the scattering length [21]. Many loss processes for ultracold atoms involve deeply inelastic reactions. An important example is three-body recombination, in which a collision of three low-energy atoms results in the binding of two of the atoms into a diatomic molecule with a large binding energy. The Lindblad equation is useful for deriving universal relations for the loss rate of ultracold atoms [22].

Open effective field theories from integrating out deeply inelastic reactions may have other applications in high energy physics. One particularly interesting application is dark matter. The deeply inelastic reactions are annihilation collisions of pairs of dark matter particles, which produce Standard Model particles that may be observed in indirect detection experiments. The Lindblad equation could prove to be especially useful if dark matter particles have strong self-interactions or if they are in a Bose-Einstein condensate.

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