The polynomial form of the scattering equations is an H-basis

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The Polynomial Form of the Scattering Equations is an H-Basis

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We prove that the polynomial form of the scattering equations is a Macaulay H-basis. We demonstrate that this H-basis facilitates integrand reduction and global residue computations in a way very similar to using a Gröbner basis, but circumvents the heavy computation of the latter. As an example, we apply the H-basis to prove the conjecture that the dual basis of the polynomial scattering equations must contain one constant term.

I. INTRODUCTION

The Cachazo, He and Yuan (CHY) formulation [1–5] of the perturbative S-matrix relies on a collection of rational maps $\{f_a\}$ from the space of massless kinematic configurations to the moduli space $\mathcal{M}_{0,n}$ of Riemann spheres with n marked points $z_a \in \mathbb{CP}^1$ associated with each of the external particles. Particularly, the set $\mathcal{S}$ of the $(n-3)!$ solutions to simultaneous constraints,

$$f_a(z, k) = \sum_{b \neq a} \frac{k_a \cdot k_b}{z_a - z_b} = 0 \,, \quad \forall a \in A = \{1, 2, \ldots, n\} \,. \tag{1}$$

dubbed as the *scattering equations*, provides a very intriguing basis for decomposing massless scattering processes in generic quantum field theories. The external particles have momenta k_a and polarizations ϵ_a. Scattering amplitudes in arbitrary dimension are expressed as a multidimensional integral of a certain rational function on $\mathcal{M}_{0,n}$ [1–5]. At the heart of the formalism lies the principle that the integration is localized on the support of the scattering equations. We write amplitudes as

$$\mathcal{A}_n^{\text{tree}}(\{k_a, \epsilon_a\}) = \int d\Omega_{\text{CHY}} \mathcal{I}(\{z_a\}, \{k_a, \epsilon_a\}) \,, \tag{2}$$

where $d\Omega_{\text{CHY}}$ is the integration measure,

$$d\Omega_{\text{CHY}} \equiv \frac{d^n z}{\text{vol} \, \text{SL}(2, \mathbb{C})} {\prod_a}' \delta(f_a) \,, \tag{3}$$

and $\mathcal{I}$ is referred to as the CHY integrand. The latter is a rational function of the marked points z_a and $d\Omega_{\text{CHY}}$ is constructed from the f_a's. The philosophy is that for a given theory in consideration, for example Yang-Mills theory, there exists a compact integrand $\mathcal{I}$ such that Eq. (2) reproduces the correct S-matrix.

In order to briefly explain the notation in Eq. (3), we remark that $\text{SL}(2, \mathbb{C})$ invariance implies that imposition of merely any $(n-3)$ of the scattering equations suffices to restrict the solution to Eq. (1). Evidently,

$${\prod_a}' \delta(f_a) \equiv z_{ij} z_{jk} z_{ki} \prod_{a \in A \setminus \{i,j,k\}} \delta(f_a) \,, \tag{4}$$

where the labels i, j, k specify the arbitrary choice of the the three extraneous scattering equations to be disregarded. Throughout this paper $z_{ab} \equiv z_a - z_b$. Moreover, the $\text{SL}(2, \mathbb{C})$ redundancy is explicitly quotiented out by fixing the values of, say, z_r, z_s and z_t.

For the purpose of investigating aspects of the CHY formalism via algebraic geometry, it is essential to interpret Eq. (2) as a multivariate *global residue* with respect to the polynomial form of the scattering equations derived by Dolan and Goddard [6]. Let h_m be the multilinear homogeneous polynomial of degree m defined by

$$h_m = \frac{1}{m!} \sum_{\substack{a_1, a_2, \ldots, a_m \in A' \\ a_i \neq a_j}} \sigma_{a_1 a_2 \cdots a_m} z_{a_1} z_{a_2} \cdots z_{a_m} \,, \tag{5}$$

where, in our notation, we take $A' = \{1, \ldots, n-2\}$ and $\sigma_{a_1 a_2 \cdots a_m} \equiv (k_{a_1} + \cdots + k_{a_m} + k_n)^2$. Then Eqs. (1) subject to the partial gauge fixing $z_{n-1} \to 0$ and $z_n \to \infty$ are equivalent to the polynomial equations,

$$h_m = 0 \,, \quad 1 \leq m \leq n - 3 \,. \tag{6}$$

In terms of the h_m's, Eq. (2) can be rewritten as a $(n-3)$-fold integral over a contour $\mathcal{O}$ encircling all points in $\mathcal{S}$, with the replacements $\mathcal{I} \to \tilde{\mathcal{I}}$ and $d\Omega_{\text{CHY}} \to d\tilde{\Omega}_{\text{CHY}}$,

$$\tilde{\mathcal{I}} \equiv \mathcal{I} \prod_{a \in A} (z_a - z_{a+1})^2 \,, \tag{7}$$

$$d\tilde{\Omega}_{\text{CHY}} \equiv \frac{z_1}{z_{n-2}} \prod_{m=1}^{n-3} \frac{1}{h_m} \tag{8}$$

$$\times \prod_{1 \leq a < b \leq n-2} (z_a - z_b) \prod_{a=1}^{n-3} \frac{z_a dz_{a+1}}{(z_a - z_{a+1})^2} \,.$$

The CHY literature is by now fairly extensive. We suggest a partial list [6–15] of recent developments. The loop-level generalization is addressed in ref. [16–23]. Although Eqs. (1) look very simple, it is a formidable task to solve them to actually compute amplitudes. This problem has received considerable attention recently, and it is now clear that the explicit solutions can be bypassed completely. We mention the integration rules [24–28] and various other approaches [29–32]. See also refs. [33, 34]

for related progress. In ref. [35] two of the present authors proposed an algebraic resolution offered by the Bezoutian matrix method. We will revisit this part later.

The main result of this paper is that the polynomial scattering equations (5) (strictly speaking, the polynomials, not the equations) automatically form an H-basis for the zero-dimensional ideal $I = \langle h_1, \ldots, h_{n-3} \rangle$.

II. H AND G(RÖBNER) BASES

We consider the ring $R = \mathbb{K}[z_1, \ldots, z_n]$ of polynomials in n variables $z_1, \ldots, z_n$ over a field $\mathbb{K}$. Typically, $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{Q}$. Here we follow refs. [40, 41].

Let P_d be the subset of all polynomials of degree d or less, and S_d be the subset of homogeneous polynomials of degree d. We have the direct sum decomposition,

$$P_d = \bigoplus_{i=0}^{d} S_d . \qquad (9)$$

Consider an ideal I generated by polynomials $f_1, \ldots, f_k$, $I = \langle f_1, \ldots, f_k \rangle$. So for any $f \in I$,

$$f = \sum_{i=1}^{k} q_i f_i , \quad q_i \in R . \qquad (10)$$

In practice, it is much easier to carry out polynomial reduction if

$$\max_{i=1}^{k} \deg(q_i f_i) = \deg f , \qquad (11)$$

since the quotients q_i's degrees are under control. However, the condition (11) in general may not be satisfied for any set of generators for I. Hence F. Macaulay [42] defined the H-basis of an ideal:

Definition 1. *We say that* $\{f_1, \ldots, f_k\} \subset I$ *is an* H-*basis of an ideal* $I \subseteq R$, *if* $\forall f \in I$, $\exists q_1, \ldots, q_k \in R$ *such that* $f = \sum_{i=1}^{k} q_i f_i$, *and*

$$\max_{i=1}^{k} \deg(q_i f_i) = \deg f . \qquad (12)$$

By this definition, the condition of being an H-basis is equivalent to that $P_d \cap I$ is generated as

$$P_d \cap I = \mathrm{span}_{\mathbb{K}}\{a f_i\} , \quad \forall a \in R , \ \deg a \leq d - \deg f_i . \qquad (13)$$

For any polynomial $f \in R$, define the *initial form* of f, $\mathrm{in}(f)$, as the homogeneous part of f, with the degree $\deg f$. The condition of being an H-basis can be further reformulated as [40],

$$\langle \mathrm{in}(I) \rangle = \langle \mathrm{in}(f_1), \ldots, \mathrm{in}(f_k) \rangle , \qquad (14)$$

where $\mathrm{in}(I)$ is the collection of initial forms of all polynomials in I.

When the number of generators equals the number of variables, there is a simple way to check if the generator set is an H-basis [40, 41]:

Theorem 2. $\{f_1, \ldots, f_k\}$ *is an* H-*basis for the ideal* $I = \langle f_1, \ldots, f_k \rangle$, *provided that* $(0, \ldots, 0)$ *is the only simultaneous zero of the initial forms* $\mathrm{in}(f_1), \ldots, \mathrm{in}(f_k)$.

With an H-basis, many problems in commutative algebra can be translated to linear algebra problems by Eq. (13). We remark that the H-basis, in many aspects, resembles the Gröbner basis (G-basis). They both rely on the order of monomials. An H-basis sorts monomials by the degree, while a G-basis sorts monomials by a total monomial order $\succ$. The definition of a G-basis can be rephrased in a similar form of Def. 1: $\{g_1, \ldots, g_k\} \subset I$ is a G-basis of an ideal $I \subseteq R$ with respect to $\succ$, if $\forall f \in I$, $\exists q_1, \ldots, q_k \in R$ such that $f = \sum_{i=1}^{k} q_i g_i$, and

$$\max_{i=1}^{k} \mathrm{LT}(q_i g_i) = \mathrm{LT}(f) , \qquad (15)$$

where LT stands for the highest monomial with respect to $\succ$, and "max" and "=" are to be understood in the context of this monomial order.

The G-basis concept can be considered as a refined version of the H-basis, since roughly speaking, it converts commutative algebra problems to one-dimensional linear algebra problems. On the other hand, in many cases, like the high-point scattering equations, the computation to obtain a G-basis is heavy. Furthermore, for symmetric ideals, the G-basis introduces an artificial order between the variables which explicitly breaks the symmetry.

III. THE POLYNOMIAL FORM OF THE SCATTERING EQUATIONS IS AN H-BASIS

We are now ready to prove our principal result.

Theorem 3. *The polynomial scattering equations* h_m, $1 \leq m \leq n-3$, *form an* H-*basis*.

Proof. We show that $\mathcal{Z}(J) \equiv \mathcal{Z}(\langle \mathrm{in}(h_1), \ldots, \mathrm{in}(h_{n-3}) \rangle)$, the zero locus of the initial forms of the polynomial scattering equations, consists of only the point $(0, \ldots, 0)$, after which the result follows from Theorem 2. Note that since our field of interest $\mathbb{K} = \mathbb{C}$ is algebraically closed, $\mathcal{Z}(J) = \mathcal{Z}(\sqrt{J})$, so it suffices to consider the *radical ideal* $\sqrt{J} \supset J$. From inspection of Eq. (5), after applying the gauge fixing $z_1 \to 1$, we have

$$\mathrm{in}(h_m) = \frac{1}{m!} \sum_{\substack{a_1, \ldots, a_m \in A'' \\ a_i \neq a_j}} \sigma_{a_1 \cdots a_m} z_{a_1} \cdots z_{a_m} , \qquad (16)$$

where $A'' = \{2, \ldots, n-2\}$. By iteratively considering

$$z_2 \cdots z_{n-2-i} \, \mathrm{in}(h_{n-3-i}) \in \sqrt{J} \qquad (17)$$

for $i = 0, \ldots, n-4$, we find that, after each step, $z_2^2 \cdots z_{n-2-i}^2 \in \sqrt{J}$ and hence $z_2 \cdots z_{n-2-i} \in \sqrt{J}$. As the same works for permutations of the indices, we realize that $z_2, \ldots, z_{n-2} \in \sqrt{J}$ and thus

$$\mathcal{Z}(\sqrt{J}) = \mathcal{Z}(\langle z_2, \ldots, z_{n-2} \rangle) = \{(0, \ldots, 0)\} . \qquad (18)$$

$\square$

IV. INTEGRAND REDUCTION

One of the crucial features of an H-basis is that we can perform polynomial reduction towards it, in a similar way as the G-basis. Here we do not need the reduction algorithm for a generic H-basis [40], but only the case of our interest, i.e. the polynomial scattering equations as an H-basis. The goal is to reduce an arbitrary polynomial to a polynomial with degree low enough.

Let $\{f_1, \ldots f_n\}$ be an H-basis in n variables $z_1, \ldots, z_n$ and let $I = \langle f_1, \ldots, f_n \rangle$, so $\text{in}(I)$ is the ideal of the initial forms. As $\mathcal{Z}(\text{in}(I)) = \{(0, \ldots, 0)\}$, by Hilbert's Nullstellensatz [45], there exists a (minimal) positive integer d^* such that

$$S_d \subset \text{in}(I), \quad \forall d > d^* . \tag{19}$$

Now we estimate d^*, referred to as the *critical degree*, for the scattering equations:

Theorem 4. *For the n-point scattering equations, define $d^* = (n-3)(n-4)/2$. Then $S_d \subset \text{in}(I)$, $\forall d > d^*$.*

Proof. Homogeneity of the initial forms implies that $R/\langle \text{in}(I) \rangle = \bigoplus_{i=0}^{\infty} A_d$ is a graded algebra. Then,

$$\dim_{\mathbb{K}} A_d = 0 , \quad \text{if } d > \sum_{i=1}^{k} \deg f_i - n , \tag{20}$$

when $V(\text{in}(I)) = \{(0, \ldots, 0)\}$ [46]. The theorem follows immediately from the degree counting of the scattering equations. $\square$

The upshot of the above discussion is that we can reduce any desired polynomial with degree larger than $d^* = (n-3)(n-4)/2$ to a polynomial with degree equal or less than d^* towards an H-basis. The algorithm is recursive: given a polynomial f with $\deg f = d > d^*$, since $\text{in}(f) \in \langle \text{in}(I) \rangle$,

$$\text{in}(f) = \sum_{i=1}^{n-3} q_i^{(d)} \text{in}(f_i) , \tag{21}$$

where $\deg q_i^{(d)} = d - \deg f_i$. Since the degrees are confined by virtue of the H-basis, the $q_i^{(d)}$'s can be obtained by just solving a finite system of linear equations. Define

$$\tilde{f} = f - \sum_{i=1}^{n-3} q_i^{(d)} f_i . \tag{22}$$

By construction, $\tilde{f}$ is polynomial with degree less than d, because we subtracted off the leading part. Repeat at most $d - d^*$ times and collect the intermediate coefficients,

$$f = \sum_{i=1}^{n-3} q_i f_i + r , \tag{23}$$

where r, the *remainder*, is a polynomial with degree less than or equal to $d^* = (n-3)(n-4)/2$. The reduction result r is not unique, however, different results only differ by polynomials in I, hence will give the same residue, in the following computations.

V. GLOBAL RESIDUES AND THE BEZOUTIAN

The H-basis property of the polynomial scattering equations allows us to prove an exciting empirical observation [35]: the CHY formula produces a global residue proportional to merely a single monomial coefficient, and the constant of proportionality is universal.

For the benefit of the reader, we recall the notion of a global residue for polynomials. We refer to the text books [43, 44] and related applications [35] (see also refs. [36–39]). An individual (local) residue may require algebraic extensions such as $\sqrt{2}$. On the contrary, a global residue is a manifestly rational quantity of the monomial coefficients. Let $I = \langle f_1, \ldots, f_n \rangle$ be a zero-dimensional polynomial ideal, so R/I is a finite-dimensional $\mathbb{C}$-linear space. The global residue is a linear map $R/I \to \mathbb{C}$ that computes the total sum of local residues,

$$\text{Res}_{\{f_1, \ldots, f_n\}}(N) \equiv \sum_{\xi_i \in \mathcal{Z}(I)} \text{Res}_{\{f_1, \ldots, f_n\}, \xi_i}(N) . \tag{24}$$

Keep in mind that the residue only depends on the equivalence class $[N]$ of N in R/I. We can make connection with the CHY formalism by noting that

$$N(z) \equiv \frac{z_2}{z_{n-1}} \prod_{2 \le a < b \le n-1} (z_a - z_b) \prod_{a=2}^{n-2} \frac{z_a}{(z_a - z_{a+1})^2} \times \tilde{\mathcal{I}} , \tag{25}$$

and therefore, Eq. (2) equals $\text{Res}_{\{h_1, \ldots, h_{n-3}\}}(N)$. In particular, if N is not a polynomial, but a rational function $N = N_1/N_2$ and N_2 is nonvanishing on $\mathcal{Z}(I)$, then [35]

$$\text{Res}_{\{f_1, \ldots, f_n\}}(N_1/N_2) \equiv \text{Res}_{\{f_1, \ldots, f_n\}}(N_1 G_2) . \tag{26}$$

Here, G_2 is the *polynomial inverse* of N_2 modulo I.

Efficient algebraic evaluation of global residues without computing the individual residues is facilitated by the following theorem [43].

Theorem 5 (Global Duality). $\langle \bullet, \bullet \rangle : R/I \otimes R/I \to \mathbb{C}$ *defined by*

$$\langle N_1, N_2 \rangle \equiv \text{Res}(N_1 \cdot N_2) \tag{27}$$

is a nondegenerate inner product.

Let $\{e_i\}$ be a basis for R/I. The strength of this theorem is that it requires the existence of a *dual basis* $\{\Delta_i\}$ in R/I, defined by the orthonormality conditions,

$$\langle e_i, \Delta_j \rangle = \delta_{ij} . \tag{28}$$

Indeed, if we decompose $[N] = \sum_i \lambda_i e_i$ and $1 = \sum_i \mu_i \Delta_i$ with $\lambda_i, \mu_i \in \mathbb{C}$, a tractable expression emerges,

$$\text{Res}_{\{f_1, \ldots, f_n\}}(N) = \sum_{i,j} \lambda_i \mu_j \langle e_i, \Delta_j \rangle = \sum_i \lambda_i \mu_i . \tag{29}$$

The dual and canonical bases are obtained algorithmically by means of the Gröbner basis method and the Bezoutian matrix [35]. This method boils down the problem to taking linear combinations of monomial coefficients of the numerator in question. Henceforth we will restrict attention to a special case. If we randomly write down a zero-dimensional ideal, none of the entries of the dual basis may be constant. But if Δ_1 is a scalar, the decomposition of unity over the dual basis becomes trivial and the global residue (29) truncates to a single term,

$$\mathrm{Res}_{\{f_1,\ldots,f_n\}}(N) = \lambda_1/\Delta_1 \ . \qquad (30)$$

We will momentarily show that an H-basis gives rise to this particular result. Our starting point is the Euler-Jacobi vanishing theorem.

Theorem 6 (Euler-Jacobi). *Suppose $I = \langle f_1,\ldots,f_n\rangle$ is a zero-dimensional ideal whose generators form an H-basis. Then, for any $N \in R$,*

$$\mathrm{Res}_{\{f_1,\ldots,f_n\}}(N) = 0 \ , \quad \text{if } \deg N < d^* \ , \qquad (31)$$

where $d^* \equiv \sum_{i=1}^{n} \deg f_i - n$ is the critical degree.

Accordingly, for the n-point tree-level polynomial scattering equations, if the degree of the numerator is strictly less than $d^* = (n-3)(n-4)/2$, the global residue vanishes identically. This observation leads us the following theorem and corollary.

Theorem 7. *Let $I = \langle h_1,\ldots,h_{n-3}\rangle$ be the ideal generated by the polynomial scattering equations. The canonical linear basis for R/I must contain a monomial of degree at least d^*.*

Proof. If not, then all monomials in the canonical linear basis have degree strictly less than d^*. By Euler-Jacobi's theorem, $\langle 1, m\rangle = 0$, for every monomial m in the canonical basis. This is a contradiction of the non-degenerate property of the inner product. $\square$

Corollary 7.1. *Let $I = \langle h_1,\ldots,h_{n-3}\rangle$ be the ideal generated by the polynomial scattering equations. Then, the dual basis of R/I must contain a constant.*

Proof. The kth row of the $(n-3) \times (n-3)$ Bezoutian matrix B associated with the h_m's has degree $k-1$, cf. ref. [35]. Ergo we have the bound $\deg(\det B) \leq d^*$. The rest follows immediately from Theorem 7. $\square$

We have thus confirmed that the global residue with respect to an H-basis of any polynomial $N \in R$ is always dictated entirely by the leading term of the Gröbner basis normal form of N. More specifically, Corollary 7.1 implies that

$$\mathrm{Res}_{\{h_1,\ldots,h_{n-3}\}}(N) = [N]_{z_{n-1}^{d^*}}/\Delta_1 \ , \qquad (32)$$

where the subscript indicates that only a single coefficient is extracted.

VI. GLOBAL RESIDUES AND H-BASIS

Alternatively, besides the Bezoutian matrix and Gröbner basis approach, we can also use the H-basis to calculate global residues with respect to the scattering equations for polynomial numerators. Given a degree d polynomial N, if $d > d^* = (n-3)(n-4)/2$, using the integrand reduction algorithm,

$$N = \sum_{i=1}^{n-3} q_i h_i + \tilde{N} \ , \qquad (33)$$

where $\tilde{N}$ is a polynomial with degree at most d^*. From the preceding H-basis discussion,

$$\mathrm{Res}_{\{h_1,\ldots,h_{n-3}\}}(N) = \mathrm{Res}_{\{h_1,\ldots,h_{n-3}\}}(\mathrm{in}(\tilde{N})) \ , \qquad (34)$$

and $\mathrm{in}(N)$ consists of monomials with degree d^*. From the proper map theorem [41] for H-bases,

$$\mathrm{Res}_{\{h_1,\ldots,h_{n-3}\}}(N) = \mathrm{Res}_{\{\mathrm{in}(h_1),\ldots,\mathrm{in}(h_{n-3})\}}(\mathrm{in}(\tilde{N})) \ . \qquad (35)$$

Note that the $\mathrm{in}(h_i)$'s have only one common zero, namely at $(0,\ldots,0)$. Hence we just need to evaluate the residue at *one point*. Furthermore from the H-basis graded algebra [46],

$$\dim_{\mathbb{C}} S_{d^*} - \dim_{\mathbb{C}}(S_{d^*} \cap \langle \mathrm{in}(I)\rangle) = 1 \ . \qquad (36)$$

Consequently, if a degree-d^* monomial's residue is obtained and nonzero, all other degree-d^* monomial's residues are obtained from linear relations. Such a residue can be found using the transformation law from algebraic geometry.

Proposition 8. *For the n-point scattering equations in polynomial form, with the gauge fixing $z_1 \to 1$, $z_{n-1} \to 0$ and $z_n \to \infty$,*

$$\mathrm{Res}_{\{h_1,\ldots,h_{n-3}\}}(z_3 z_4^2 \ldots z_{n-2}^{n-4}) = \frac{(-1)^{(n-3)(n-4)/2}}{\prod_{j=2}^{n-2} \sigma_{j,j+1,\ldots,n-2}} \ . \qquad (37)$$

Proof. Since the polynomial scattering equations form an H-basis, $\mathrm{in}(h_i) \in \langle z_2,\ldots,z_{n-2}\rangle$, $1 \leq i \leq n-3$. That is,

$$\mathrm{in}(h_i) = \sum_{j=2}^{n-2} a_{ij} z_j \ . \qquad (38)$$

Choosing the matrix $A = (a_{ij})$ to be upper triangular, the determinant becomes,

$$\det A = (-1)^{\frac{(n-3)(n-4)}{2}} z_3 z_4^2 \ldots z_{n-2}^{n-4} \prod_{j=2}^{n-2} \sigma_{j,j+1,\ldots,n-2} \ . \qquad (39)$$

Hence, Eq. (37) follows from the transformation law [43]. $\square$

Using this straightforward approach, we are able to get the residue of any polynomial numerator in analytic form using the H-basis. For example, for $n = 8$ we *analytically* find the residues of all 1001 monomials with the critical degree, in 3 minutes, on a laptop with 2.5 GHz Intel Core i7 processor and 16 GB RAM.

VII. CONCLUSION

We have uncovered and proved that the polynomial form of the scattering equations is an H-basis. We have explored and emphasized several compelling implications of this observation, and briefly compared with the presumably more familiar Gröbner basis, which can be computationally expensive to obtain.

In particular, the H-basis enables us to perform reductions of high-degree multivariate polynomials without the need for a Gröbner basis. More concretely, in connection with the scattering equations we have shown that any monomial with degree greater than $d^* = (n-3)(n-4)/2$ can always be reduced to a polynomial of degree at most d^*, modulo the H-basis. This procedure only involves linear algebra. The H-basis greatly enhances our ability to compute global residues and thus calculate scattering amplitudes in the CHY framework.

In this direction we have also proved a conjecture recently made in ref. [35], namely that the dual basis associated with the polynomial scattering equations always contains a constant and that hence any global residue is just one rational monomial coefficient, multiplied by a universal factor. In a forthcoming paper [47] we expect to tabulate analytic expressions for many of the CHY global residues, also at loop level.

It remains intriguing to gain a complete insight into the algebraic geometry underlying the whole CHY formalism. We anticipate that the explicit identification of the polynomial scattering equations as an H-basis paves the way for new exciting advances in this direction.

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