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Neutrino Quantum Kinetic Equations: The Collision Term

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We derive the collision term relevant for neutrino quantum kinetic equations in the early universe and compact astrophysical objects, displaying its full matrix structure in both flavor and spin degrees of freedom. We include in our analysis neutrino-neutrino processes, scattering and annihilation with electrons and positrons, and neutrino scattering off nucleons (the latter in the low-density limit). After presenting the general structure of the collision terms, we take two instructive limiting cases. The one-flavor limit highlights the structure in helicity space and allows for a straightforward interpretation of the off-diagonal entries in terms of the product of scattering amplitudes of the two helicity states. The isotropic limit is relevant for studies of the early universe: in this case the terms involving spin coherence vanish and the collision term can be expressed in terms of two-dimensional integrals, suitable for computational implementation.

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I. INTRODUCTION

The evolution of an ensemble of neutrinos in hot and dense media is described by an appropriate set of quantum kinetic equations (QKEs), accounting for kinetic, flavor, and spin degrees of freedom [1–14]. QKEs are central to obtain a complete description of neutrino transport in the early universe, core collapse supernovae, and compact object mergers, valid before, during, and after the neutrino decoupling epoch (region). A self-consistent treatment of neutrino transport is highly relevant because in such environments neutrinos carry a significant fraction of the energy and entropy, and through their flavor- and energy-dependent weak interactions play a key role in setting the neutron-to-proton ratio, a critical input for the nucleosynthesis process.

In Ref. [9] the QKEs describing the evolution of Majorana neutrinos were derived using field-theoretic methods (see [15–19] for an introduction to non-equilibrium QFT). These QKEs include spin degrees of freedom and encompass effects up to second order in small ratios of scales characterizing the neutrino environments we are interested in. Specifically, we treat neutrino masses, mass-splitting, and matter potentials induced by forward scattering, as well as external gradients as much smaller than the typical neutrino energy scale E, set by the temperature or chemical potential: namely $m_\nu/E \sim \Delta m_\nu/E \sim \Sigma_{\rm forward}/E \sim \partial_X/E \sim O(\epsilon)$ [20]. The inelastic scattering can also be characterized by a potential $\Sigma_{\rm inelastic} \sim \Sigma_{\rm forward} \times G_F E^2$ which we therefore power-count as $\Sigma_{\rm inelastic}/E \sim O(\epsilon^2)$. This power-counting is tantamount to the statement that physical quantities vary slowly on the scale of the neutrino de Broglie wavelength.

In this paper we elaborate on the terms of the QKEs describing inelastic collisions or production and absorption in the medium. These terms are essential for a correct description of neutrinos in the decoupling epoch (region), in which the neutrino spectra and flavor composition are determined [21–25]. While Ref. [9] only included a discussion of neutrino-neutrino scattering in isotropic environment, here we compute the collision terms induced by neutrino-(anti)neutrino processes, neutrino scattering and annihilation with electrons and positrons, and neutrino scattering off nucleons. Our expressions for processes involving nucleons are valid in the low-density limit, i.e. do not take into account nucleon interactions. However, the effects of strong interactions in dense matter — relevant for supernovae environments — can be included by appropriately modifying the medium response functions (see for example [26–28]).

The paper is organized as follows: in Sect. II we review the Quantum Kinetic Equations (QKEs) in the field-theoretic approach, for both Dirac and Majorana neutrinos. In Sect. III we provide a derivation of the generalized collision term for Majorana neutrinos, and present general expressions involving coherence terms both in flavor and spin space, valid for any geometry, relegating some lengthy results to Appendix C. The collision terms for Dirac neutrinos are discussed in Section IV. After presenting general results, we discuss two limiting cases. In Sect. V we take the one-flavor limit and illustrate the structure of the collision term for the two spin degrees of freedom of a Majorana neutrino (i.e neutrino and antineutrino). In Sect. VI we consider the isotropic limit relevant for the description of neutrinos in the early universe, with some details reported in Appendix D. Finally, we present our concluding remarks in Sect. VII. To keep the paper self-contained, we include a number of appendices with technical details and lengthy results.

II. REVIEW OF QUANTUM KINETIC EQUATIONS (QKES)

In this section we review the field-theoretic approach to neutrino QKEs, following Refs. [9, 11], with the dual purpose of having a self-contained presentation and setting

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the notation for the following sections. After a brief discussion of neutrino interactions in the Standard Model (SM) at energy scales much smaller than the W and Z boson masses, we present the Green's function approach to neutrino propagation in hot and dense media, we describe the structure of the QKEs, and we finally review the content of the "coherent" (collisionless) QKEs. Throughout, we use four-component spinors to describe both Dirac and Majorana neutrinos, providing an alternative description to the one of Ref. [9], that employs two-dimensional Weyl spinors. In this section we present results for both Majorana and Dirac neutrinos.

A. Neutrino interactions

In this work we describe neutrino fields (Dirac or Majorana) in terms of 4-component spinors ν_α, where α is a flavor or family index. In the Majorana case the fields satisfy the Majorana condition $\nu^c = \nu$, with $\nu^c \equiv C\bar{\nu}^T$, where $C = i\gamma_0\gamma_2$ is the charge-conjugation matrix. In the Majorana case, the kinetic Lagrangian can be written as

$$\mathcal{L}_{\rm Kin} = \frac{i}{2}\,\bar{\nu}\overleftrightarrow{\partial\!\!\!/}\nu \;-\; \frac{1}{2}\bar{\nu}\,m\,\nu\,, \qquad\qquad \nu = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}\,, \tag{1}$$

where $m = m^T$ is the Majorana mass matrix (a complex symmetric matrix). [29]

In situations of physical interest, such as neutrino decoupling in the early universe and neutrino propagation in compact astrophysical objects, the typical neutrino energy is well below the electroweak scale ($\sim$ 100 GeV). Therefore, in computing the collision integrals it is safe to use the contact-interaction limit of the full Standard Model, and to replace the quark degrees of freedom with nucleon degrees of freedom.

After integrating out W and Z bosons, the part of the Standard Model effective Lagrangian controlling neutrino interactions can be written in the following current-current form (in terms of 4-dimensional spinors):

$$\mathcal{L}_{\nu\nu} = -\frac{G_F}{\sqrt{2}}\;\bar{\nu}\gamma_\mu P_L\nu\;\;\bar{\nu}\gamma^\mu P_L\nu\,, \tag{2a}$$

$$\mathcal{L}_{\nu e} = -2\sqrt{2}G_F\;\Big(\bar{\nu}\gamma_\mu P_L Y_L\nu\;\;\bar{e}\gamma^\mu P_L e \;+\; \bar{\nu}\gamma_\mu P_L Y_R\nu\;\;\bar{e}\gamma^\mu P_R e\Big)\,, \tag{2b}$$

$$\mathcal{L}_{\nu N} = -\sqrt{2}G_F\sum_{N=p,n}\bar{\nu}\gamma_\mu P_L\nu\;\;\bar{N}\gamma^\mu \times\Big(C_V^{(N)} - C_A^{(N)}\gamma_5\Big)\,N\,, \tag{2c}$$

$$\mathcal{L}_{CC} = -\sqrt{2}G_F\;\bar{e}\gamma_\mu P_L\nu_e\;\bar{p}\,\gamma^\mu\,(1 - g_A\gamma_5)\,n \;+\;\text{h.c.}\,, \tag{2d}$$

where G_F is the Fermi constant, $P_{L,R} = (\mathbb{1} \mp \gamma_5)/2$, and

$$Y_L = \begin{pmatrix} \frac{1}{2} + \sin^2\theta_W & 0 & 0 \\ 0 & -\frac{1}{2} + \sin^2\theta_W & 0 \\ 0 & 0 & -\frac{1}{2} + \sin^2\theta_W \end{pmatrix},$$
$$Y_R = \sin^2\theta_W \times \mathbb{1}\;. \tag{3}$$

The nucleon couplings are given in terms of $g_A \simeq 1.27$ by

$$C_V^{(p)} = \frac{1}{2} - 2\sin^2\theta_W\,, \qquad C_V^{(n)} = -\frac{1}{2}\,,$$
$$C_A^{(p)} = \frac{g_A}{2}\,, \qquad C_A^{(n)} = -\frac{g_A}{2}\,. \tag{4}$$

B. Neutrinos in hot and dense media

QKEs are the evolution equations for suitably defined dynamical quantities that characterize a neutrino ensemble, which we will refer to as neutrino density matrices. In the most general terms a neutrino ensemble is described by the set of all $2n$-field Green's functions, encoding n-particle correlations. These obey coupled integro-differential equations, equivalent to the BBGKY equations [18]. As discussed in Refs. [1, 9], for weakly interacting neutrinos ($\Sigma/E \sim O(\epsilon, \epsilon^2)$) the set of coupled equations can be truncated by using perturbation theory to express all higher order Green's functions in terms of the two-point functions. In this case the neutrino ensemble is characterized by one-particle correlations.[30]

One-particle states of massive neutrinos and antineutrinos are specified by the three-momentum $\vec{p}$, the helicity $h \in \{L, R\}$, and the family label i (for eigenstates of mass m_i), with corresponding annihilation operators $a_{i,\vec{p},h}$ and $b_{j,\vec{p},h}$ satisfying the canonical anti-commutation relations $\{a_{i,\vec{p},h}, a^\dagger_{j,\vec{p}',h'}\} = (2\pi)^3\,2\,\omega_i(\vec{p})\,\delta_{hh'}\,\delta_{ij}\,\delta^{(3)}(\vec{p} - \vec{p}')$, *etc.*, where $\omega_i(\vec{p}) = \sqrt{\vec{p}^2 + m_i^2}$. Then, the neutrino state is specified by the matrices $f^{ij}_{hh'}(\vec{p})$ and $\bar{f}^{ij}_{hh'}(\vec{p})$ (which we call density matrices, with slight abuse of language)

$$\langle a^\dagger_{j,\vec{p}',h'}\,a_{i,\vec{p},h}\rangle = (2\pi)^3\,2n(\vec{p})\,\delta^{(3)}(\vec{p} - \vec{p}')\;\,f^{ij}_{hh'}(\vec{p})\;, \tag{5a}$$

$$\langle b^\dagger_{i,\vec{p}',h'}\,b_{j,\vec{p},h}\rangle = (2\pi)^3\,2n(\vec{p})\,\delta^{(3)}(\vec{p} - \vec{p}')\;\,\bar{f}^{ij}_{hh'}(\vec{p})\;, \tag{5b}$$

where $\langle\ldots\rangle$ denotes the ensemble average and $n(\vec{p})$ is a normalization factor. [31] For inhomogeneous backgrounds, the density matrices depend also on the space-time label, denoted by x in what follows.

The physical meaning of the generalized density matrices $f^{ij}_{hh'}(\vec{p})$ and $\bar{f}^{ij}_{hh'}(\vec{p})$ is dictated by simple quantum mechanical considerations: the diagonal entries $f^{ii}_{hh}(\vec{p})$ represent the occupation numbers of neutrinos of mass m_i, momentum $\vec{p}$, and helicity h; the off diagonal elements $f^{ij}_{hh}(\vec{p})$ represent quantum coherence of states of same helicity and different mass (familiar in the context of neutrino oscillations); $f^{ii}_{hh'}(\vec{p})$ represent coherence of states of different helicity and same mass, and finally $f^{ij}_{hh'}(\vec{p})$ represent coherence between states of different helicity and mass.

In summary, the basic dynamical objects describing ensembles of neutrinos and antineutrinos are the $2n_f \times 2n_f$ matrices,

$$F(\vec{p},x) = \left(\begin{array}{cc} f_{LL} & f_{LR} \\ f_{RL} & f_{RR} \end{array} \right),$$
$$\bar{F}(\vec{p},x) = \left(\begin{array}{cc} \bar{f}_{RR} & \bar{f}_{RL} \\ \bar{f}_{LR} & \bar{f}_{LL} \end{array} \right), \tag{6}$$

where we have suppressed the generation indices (each block $f_{hh'}$ is a square $n_f \times n_f$ matrix). QKEs are the evolution equations for F and $\bar{F}$. Before sketching their derivation in the following subsections, we discuss how this formalism allows one to describe both Dirac and Majorana neutrinos:

- For Dirac neutrinos, one needs both F and $\bar{F}$, with f_{LL} and $\bar{f}_{RR}$ denoting the occupation numbers of active states, left-handed neutrinos and right-handed antineutrinos, respectively. Similarly, f_{RR} and $\bar{f}_{LL}$ describe the occupation number of wrong-helicity sterile states.

- For Majorana neutrinos, one can choose the phases so that $a_i(\vec{p},h) = b_i(\vec{p},h)$ and therefore $f_{hh'} = \bar{f}^T_{hh'}$ (transposition acts on flavor indices). Therefore the ensemble is described by just the matrix $F(\vec{p},x)$. With the definitions $f \equiv f_{LL}$, $\bar{f} \equiv \bar{f}_{RR} = f^T_{RR}$, and $\phi \equiv f_{LR}$, one needs evolution equations only for the matrix $\mathcal{F}$ introduced in Ref. [9]:

$$F \to \mathcal{F} = \left(\begin{array}{cc} f & \phi \\ \phi^\dagger & \bar{f}^T \end{array} \right) . \tag{7}$$

  Here f and $\bar{f}$ are $n_f \times n_f$ matrices describing the occupation and flavor coherence of neutrinos and antineutrinos, respectively. The $n_f \times n_f$ "spin coherence" matrix ϕ describes the degree to which the ensemble contains coherent superpositions of neutrinos and antineutrinos of any flavor.

The above discussion in terms of creation and annihilation operators has been presented in the mass eigenstate basis [32]. One can define "flavor basis" density matrices $f_{\alpha\beta}$ in terms of the mass-basis f_{ij} as $f_{\alpha\beta} = U_{\alpha i} f_{ij} U^*_{\beta j}$, where U is the unitary transformation $\nu_\alpha = U_{\alpha i}\nu_i$ that puts the inverse neutrino propagator in diagonal form. While the QKEs can be written in any basis, we give our results below in the "flavor" basis.

C. Green's function approach to the QKEs

1. Generalities

The description in terms of creation and annihilation operators presented so far has a simple counterpart in the QFT approach of Ref. [9]. In that approach, the basic dynamical objects are the neutrino two-point functions (a and b denote flavor indices, and we suppress spinor indices)

$$\left(G^{(\nu)}_{ab}\right)^+(x,y) \equiv \langle \nu_a(x)\bar{\nu}_b(y)\rangle , \tag{8a}$$
$$\left(G^{(\nu)}_{ab}\right)^-(x,y) \equiv \langle \bar{\nu}_b(y)\nu_a(x)\rangle , \tag{8b}$$

from which one can construct the statistical (F) and spectral (ρ) functions,

$$\begin{aligned} F^{(\nu)}_{ab}(x,y) &\equiv \frac{1}{2}\langle\, [\nu_a(x), \bar{\nu}_b(y)]\,\rangle \\ &= \frac{1}{2}\left(\left(G^{(\nu)}_{ab}\right)^+(x,y) - \left(G^{(\nu)}_{ab}\right)^-(x,y)\right) , \end{aligned} \tag{9a}$$

$$\begin{aligned} \rho^{(\nu)}_{ab}(x,y) &\equiv i\,\langle \{\nu_a(x), \bar{\nu}_b(y)\}\rangle \\ &= i\left(\left(G^{(\nu)}_{ab}\right)^+(x,y) + \left(G^{(\nu)}_{ab}\right)^-(x,y)\right) , \end{aligned} \tag{9b}$$

and the time-ordered propagator

$$\begin{aligned} G^{(\nu)}_{ab}(x,y) &\equiv \langle T\,(\nu_a(x)\bar{\nu}_b(y))\rangle \\ &= \theta(x^0-y^0)\left(G^{(\nu)}_{ab}\right)^+(x,y) \\ &\quad - \theta(y^0-x^0)\left(G^{(\nu)}_{ab}\right)^-(x,y) . \end{aligned} \tag{10}$$

The statistical and spectral function have a simple physical interpretation (see for example [19]): roughly speaking the spectral function encodes information on the spectrum of the theory, i.e. the states that are available, while the statistical function gives information about the occupation numbers and quantum coherence for the available states. As we will show below, the Wigner Transform (i.e. Fourier transform with respect to the relative coordinate) of the statistical function

$$F^{(\nu)}_{ab}(k,x) = \int d^4r\; e^{ik\cdot r}\; F^{(\nu)}_{ab}(x+r/2, x-r/2) \tag{11}$$

contains all the information about the density matrices introduced in Eqs (5) and (6). Below we sketch the various steps leading to the QKEs.

2. Equations of motion

Starting point is the equation of motion for the two point function $G^{(\nu)}_{ab}(x,y)$, equivalent to the Dyson-Schwinger equation

$$\left[G^{(\nu)}(x,y)\right]^{-1} = \left[G^{(\nu)}_0(x,y)\right]^{-1} - \tilde{\Sigma}(x,y) , \tag{12}$$

where we have suppressed for simplicity the flavor indices. $G^{(\nu)}_0(x,y)$ is the tree-level two point function and $\tilde{\Sigma}(x,y)$

is the neutrino self-energy, i.e. the sum of all amputated one-particle-irreducible (1PI) diagrams with two external neutrino lines. $\tilde{\Sigma}(x,y)$ is itself a functional of the two point function $G^{(\nu)}$ and admits the decomposition into a local term, and $^{\pm}$ components:

$$\tilde{\Sigma}(x,y) = -i\Sigma(x)\delta^{(4)}(x-y) + \theta(x^0-y^0)\tilde{\Pi}^+(x,y) - \theta(y^0-x^0)\tilde{\Pi}^-(x,y)\ . \tag{13}$$

With the interactions given in Sect. II A, one can show that $\Sigma(x)$ receives contributions starting at one loop (Fig. 1), i.e. first order in G_F, while $\Pi^{\pm}(x,y)$ receive contributions starting at two loops (Fig. 2), and are thus of second order in G_F.

Wigner-transforming the equation of motion for the two-point function and keeping terms up to $O(\epsilon^2)$ in the small ratios discussed in Sect. I, namely,

$$\frac{\partial_x, m, \Sigma}{E_\nu} = O(\epsilon)\,, \qquad \frac{\tilde{\Pi}^{\pm}}{E} = O(\epsilon^2)\ , \tag{14}$$

one arrives at [9]

$$\hat{\Omega}\, F^{(\nu)}(k,x) = -\frac{i}{2}\Big(\tilde{\Pi}^+(k,x)G^{(\nu)-}(k,x) - \tilde{\Pi}^-(k,x)G^{(\nu)+}(k,x)\Big)\,, \tag{15a}$$

$$\hat{\Omega} = \not{k} + \frac{i}{2}\not{\partial} - \Sigma(x) - m + \frac{i}{2}\frac{\partial\Sigma}{\partial x^\mu}\frac{\partial}{\partial k_\mu}\ . \tag{15b}$$

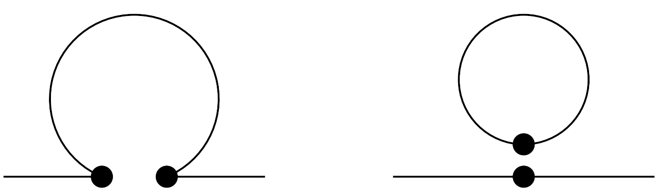

FIG. 1. Feynman graphs contributing to $\Sigma(x)$. External lines represent neutrinos. Internal lines represent ν, e, n, p propagators. We represent each 4-fermion interaction vertex from Eqs. (2) in terms of two displaced fermionic current vertices.

3. Decomposition in spinor components

The Wigner transform of the statistical function $F^{(\nu)}_{ab}(k,x)$ (and any other two-point function) has sixteen spinor components (scalar, pseudoscalar, vector, axial-vector, tensor),

$$F^{(\nu)} = \left[F_S + \left(F_V^R\right)^\mu\gamma_\mu - \frac{i}{4}\left(F_T^L\right)^{\mu\nu}\sigma_{\mu\nu}\right]P_L + \left[F_S^\dagger + \left(F_V^L\right)^\mu\gamma_\mu + \frac{i}{4}\left(F_T^R\right)^{\mu\nu}\sigma_{\mu\nu}\right]P_R\,, \tag{16}$$

where $P_{L,R} \equiv (1\mp\gamma_5)/2$ and $\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu]$. The various components satisfy the hermiticity conditions $F_V^{L\dagger} = F_V^L$, $F_V^{R\dagger} = F_V^R$, and $F_T^{L\dagger} = F_T^R$. The forward scattering potential $\Sigma(x)$ and the inelastic collision self-energies $\tilde{\Pi}^{\pm}(k,x)$ admit a similar decomposition in spinor components (we give here only the decomposition for $\tilde{\Pi}$, a completely analogous one exists for Σ):

$$\tilde{\Pi} = \left[\Pi_S + \Pi_R^\mu\gamma_\mu - \frac{i}{4}\left(\Pi_T^L\right)^{\mu\nu}\sigma_{\mu\nu}\right]P_L + \left[\Pi_S^\dagger + \Pi_L^\mu\gamma_\mu + \frac{i}{4}\left(\Pi_T^R\right)^{\mu\nu}\sigma_{\mu\nu}\right]P_R\ , \tag{17}$$

where we suppress the $^{\pm}$ superscripts.

For ultra-relativistic neutrinos of three-momentum $\vec{k}$ (characterized by polar angle θ and azimuthal angle φ), it is convenient to express all Lorentz tensors and components of the two-point functions (such as the $(F_V^{L,R})^\mu$ and $(F_T^{L,R})^{\mu\nu}$) in terms of a basis formed by two light-like four-vectors $\hat{\kappa}^\mu(k) = (\mathrm{sgn}(k^0), \hat{k})$ and $\hat{\kappa}'^\mu(k) = (\mathrm{sgn}(k^0), -\hat{k})$ ($\hat{\kappa}\cdot\hat{\kappa} = \hat{\kappa}'\cdot\hat{\kappa}' = 0$, $\hat{\kappa}\cdot\hat{\kappa}' = 2$) and two transverse four vectors $\hat{x}_{1,2}(k)$ such that $\hat{\kappa}\cdot\hat{x}_i = \hat{\kappa}'\cdot\hat{x}_i = 0$ and $\hat{x}_i\cdot\hat{x}_j = -\delta_{ij}$, or equivalently $\hat{x}^{\pm} \equiv \hat{x}_1 \pm i\hat{x}_2$, with $\hat{x}^+\cdot\hat{x}^- = -2$ (see Appendix A for additional details).

The components of the self-energy entering the QKEs are obtained by the projections (see Appendix A)

$$\Pi^\kappa_{L,R}(k,x) = \frac{1}{2}\hat{\kappa}_\mu\,\mathrm{Tr}\left[\tilde{\Pi}(k,x)\,\gamma^\mu P_{L,R}\right]\ , \tag{18a}$$

$$P_T(k,x) = \frac{ie^{i\varphi}}{16}\left(\hat{\kappa}\wedge\hat{x}^+\right)^{\mu\nu}\ \mathrm{Tr}\left[\tilde{\Pi}(k,x)\,\sigma_{\mu\nu}P_R\right]\ , \quad (a\wedge b)^{\mu\nu} = a^\mu b^\nu - a^\nu b^\mu\,, \tag{18b}$$

$$\Sigma^\mu_{L,R}(x) = \frac{1}{2}\mathrm{Tr}\Big[\Sigma(x)\gamma^\mu P_{L,R}\Big]\,, \tag{18c}$$

which can be arranged in the $2n_f\times 2n_f$ structures:

$$\Sigma^\mu(x) = \begin{pmatrix} \Sigma_R^\mu & 0 \\ 0 & \Sigma_L^\mu \end{pmatrix}\,, \qquad \hat{\Pi}^{\pm}(k,x) = \begin{pmatrix} \Pi_R^{\kappa\pm} & 2P_T^{\pm} \\ 2P_T^{\pm\dagger} & \Pi_L^{\kappa\pm} \end{pmatrix}\ . \tag{19}$$

4. Leading order analysis

The equations of motion (15) impose relations between the sixteen components of $F^{(\nu)}_{ab}(k,x)$. Solving (15) to $O(\epsilon^0)$, only four components survive (L- and R- vector components and two tensor components), parameterized by the real functions $F_{L,R}(k,x)$ and the complex function

$\Phi(k,x)$:

$$\left(F_V^{L,R}\right)^\mu(k,x) = \hat{\kappa}^\mu(k)\, F_{L,R}(k,x) \ , \tag{20a}$$
$$\left(F_T^L\right)_{\mu\nu}(k,x) = e^{-i\varphi(k)}(\hat{\kappa}(k)\wedge\hat{x}^-(k))_{\mu\nu}\, \Phi(k,x) \ , \tag{20b}$$
$$\left(F_T^R\right)_{\mu\nu}(k,x) = e^{i\varphi(k)}(\hat{\kappa}(k)\wedge\hat{x}^+(k))_{\mu\nu}\, \Phi^\dagger(k,x) \ , \tag{20c}$$

with all other spinor components vanishing.

Beyond $O(\epsilon^0)$, the four independent spinor components of $F_{ab}^{(\nu)}(k,x)$ can be conveniently chosen to coincide with the ones non-vanishing to $O(\epsilon^0)$ [9, 11]. They can be isolated by the following projections:

$$F_{L,R}(k,x) \equiv \frac{1}{4}\mathrm{Tr}\Big(\gamma_\mu P_{L,R}\ F^{(\nu)}(k,x)\Big)\, \hat{\kappa}'^\mu(k) \ , \tag{21a}$$
$$\Phi^{(\dagger)}(k,x) \equiv \mp\frac{i}{16}\mathrm{Tr}\Big(\sigma_{\mu\nu} P_{L/R}\ F^{(\nu)}(k,x)\Big) \times(\hat{\kappa}'(k)\wedge\hat{x}^\pm(k))^{\mu\nu} e^{\pm i\varphi(k)} \ , \tag{21b}$$

where the upper (lower) signs and indices refer to Φ ($\Phi^\dagger$) [33]. These components can be collected in a $2n_f\times 2n_f$ matrix [34]

$$\hat{F} = \begin{pmatrix} F_L & \Phi \\ \Phi^\dagger & F_R \end{pmatrix} \ . \tag{22}$$

In the free theory, the positive and negative frequency integrals of $\hat{F}$ give (up to a constant) the particle and antiparticle density matrices of Eq. (6) defined in terms of ensemble averages of creation and annihilation operators (see Eqs (5)):

$$-2\int_0^\infty \frac{dk^0}{2\pi}\hat{F}(k,x) = F(\vec{k},x) - \frac{1}{2}\mathbb{1} \ , \tag{23a}$$
$$-2\int_{-\infty}^0 \frac{dk^0}{2\pi}\hat{F}(k,x) = \bar{F}(-\vec{k},x) - \frac{1}{2}\mathbb{1} \ . \tag{23b}$$

In the interacting theory, we take the above equations as definitions of neutrino and antineutrino number densities (generalized to include the off-diagonal coherence terms). These correspond to spectrally-dressed densities in the language adopted in Refs. [35, 36], with quasi-particle spectra dictated by Eq. (24b) below.

5. *Kinetic equations and shell conditions beyond leading order*

Beyond $O(\epsilon^0)$, the dynamics of $\hat{F}(k,x)$ is still controlled by Eq. (15). Projecting out all the spinor components of (15) one finds [9]: (i) constraint relations that express "small components" of $F^{(\nu)}$ in terms of $F_{L,R}$ and Φ; (ii) evolution equations (i.e. first order in space-time derivatives) for $\hat{F}(k,x)$; (iii) constraint relations on the components $\hat{F}(k,x)$, which determine the shell structure of the solutions. Defining $\partial^\kappa \equiv \hat{\kappa}(k)\cdot\partial$, $\partial^i \equiv \hat{x}^i(k)\cdot\partial$, the kinetic and constraint equations (items (ii) and (iii) above) for the matrix $\hat{F}(k,x)$ are

$$\partial^\kappa\hat{F} + \frac{1}{2|\vec{k}|}\left\{\Sigma^i, \partial^i\hat{F}\right\} + \frac{1}{2}\left\{\frac{\partial\Sigma^\kappa}{\partial x^\mu}, \frac{\partial\hat{F}}{\partial k_\mu}\right\} = -i\left[H,\hat{F}\right] + \hat{C}, \tag{24a}$$
$$\left\{\hat{\kappa}(k)\cdot k - \Sigma^\kappa\ ,\ \hat{F}\right\} = 0 \ . \tag{24b}$$

Using $k^\mu = |\vec{k}|\,\hat{\kappa}^\mu(k) + O(\epsilon^2)$ the constraint equation (24b) can be written in the more familiar form of a shell-condition:

$$\left\{k^2 - |\vec{k}|\,\Sigma^\kappa(x)\ ,\ \hat{F}(k,x)\right\} = 0 \ . \tag{25}$$

The $2n_f\times 2n_f$ potential $\Sigma(x)$ is defined in (19) and its projections are $\Sigma^\kappa = \hat{\kappa}(k)\cdot\Sigma$ and $\Sigma^i = \hat{x}_i\cdot\Sigma$.

The Hamiltonian-like operator controlling the coherent evolution in Eq. (24a) is given by

$$H = \begin{pmatrix} H_R & H_{LR} \\ H_{LR}^\dagger & H_L \end{pmatrix} \tag{26}$$

with

$$H_R = \Sigma_R^\kappa + \frac{1}{2|\vec{k}|}\Big(m^\dagger m - \epsilon^{ij}\partial^i\Sigma_R^j + 4\Sigma_R^+\Sigma_R^-\Big) \ , \tag{27a}$$
$$H_L = \Sigma_L^\kappa + \frac{1}{2|\vec{k}|}\Big(m m^\dagger + \epsilon^{ij}\partial^i\Sigma_L^j + 4\Sigma_L^-\Sigma_L^+\Big) \ , \tag{27b}$$
$$H_{LR} = -\frac{1}{|\vec{k}|}\left(\Sigma_R^+ m^\dagger - m^\dagger\,\Sigma_L^+\right) \ , \tag{27c}$$

where $\Sigma_{L,R}^\pm \equiv (1/2)\, e^{\pm i\varphi}\,(x_1 \pm i x_2)_\mu\, \Sigma_{L,R}^\mu$ and ϵ^{ij} is the two-dimensional Levi-Civita symbol ($\epsilon^{12}=1$).

Finally, the collision term in Eq. (24a) reads

$$\hat{C} = -\frac{1}{2}\left\{\hat{\Pi}^+, \hat{G}^-\right\} + \frac{1}{2}\left\{\hat{\Pi}^-, \hat{G}^+\right\} \ , \tag{28}$$

where the $2n_f\times 2n_f$ gain and loss potentials $\hat{\Pi}^\pm(k)$ are given in Eq. (19) in terms of spinor components of the self-energy, extracted from a calculation of the two-loop diagrams of Fig. 2, and

$$\hat{G}^\pm = -\frac{i}{2}\hat{\rho}\,\mathbb{1} \pm \hat{F} \ . \tag{29}$$

In order to obtain the collision terms to $O(\epsilon^2)$ we will need only the $O(\epsilon^0)$ expression for the vector component of the spectral function, namely $\hat{\rho}(k) = 2i\pi|\vec{k}|\delta(k^2)\mathrm{sgn}(k^0)$ (see Appendix B).

6. *Integration over frequencies: QKEs for Dirac and Majorana neutrinos*

The final step to obtain the QKEs requires integrating (24a) over positive and negative frequencies, taking into account the $O(\epsilon)$ shell corrections from (24b), whenever required in order to keep terms up to $O(\epsilon^2)$ in power-counting. Recalling the definitions (23a), the integrations over positive and negative frequency lead to:

$$\partial^\kappa F + \frac{1}{2|\vec{k}|}\left\{\Sigma^i, \partial^i F\right\} - \frac{1}{2}\left\{\frac{\partial \Sigma^\kappa}{\partial \vec{x}}, \frac{\partial F}{\partial \vec{k}}\right\} = -i[H, F] + \mathcal{C} \,, \tag{30a}$$

$$\partial^\kappa \bar{F} - \frac{1}{2|\vec{k}|}\left\{\Sigma^i, \partial^i \bar{F}\right\} + \frac{1}{2}\left\{\frac{\partial \Sigma^\kappa}{\partial \vec{x}}, \frac{\partial \bar{F}}{\partial \vec{k}}\right\} = -i[\bar{H}, \bar{F}] + \bar{\mathcal{C}} \,. \tag{30b}$$

The differential operator on the left-hand side generalizes the “Vlasov” term. The first term on the right-hand side controls coherent evolution due to mass and forward scattering and generalizes the standard Mikheyev-Smirnov-Wolfenstein (MSW) effect [37–39]. Finally, the second term on the right hand side encodes inelastic collisions and generalizes the standard Boltzmann collision term [40–48]. Let us now discuss in greater detail each term.

The physical meaning of the differential operators on the LHS of (30) becomes more transparent by noting that they can be re-written as

$$\partial_t + \frac{1}{2}\{\partial_{\vec{k}}\omega_\pm, \partial_{\vec{x}}\} - \frac{1}{2}\{\partial_{\vec{x}}\omega_\pm, \partial_{\vec{k}}\}, \tag{31}$$

with $\omega_+ = |\vec{k}| + \Sigma^\kappa$ for neutrinos and $\omega_- = |\vec{k}| - \Sigma^\kappa$ for antineutrinos. Recalling that $\omega_\pm(\vec{k}) = |\vec{k}| \pm \Sigma^\kappa$ are the $O(\epsilon)$ neutrino (+) and antineutrino (−) Hamiltonian operators, one sees that the differential operators on the LHS of (30) generalize the total time-derivative operator $d_t = \partial_t + \dot{\vec{x}}\,\partial_{\vec{x}} + \dot{\vec{k}}\,\partial_{\vec{k}}$, with $\dot{\vec{k}} = -\partial_{\vec{x}}\,\omega$ and $\dot{\vec{x}} = \partial_{\vec{k}}\,\omega$, thus encoding the familiar drift and force terms.

In terms of the mass matrix m and the potentials $\Sigma^\mu_{L,R}$, the Hamiltonian-like operators controlling the coherent evolution are given by

$$H = \begin{pmatrix} H_R & H_{LR} \\ H_{LR}^\dagger & H_L \end{pmatrix} \,, \quad \bar{H} = \begin{pmatrix} \bar{H}_R & H_{LR} \\ H_{LR}^\dagger & \bar{H}_L \end{pmatrix} \,, \tag{32}$$

with H_L, H_R, H_{LR} given in Eqs. (27). The antineutrino operators $\bar{H}_{L,R}$ can be obtained from $H_{L,R}$ by flipping the sign of the entire term multiplying $1/(2|\vec{k}|)$. The first two terms in $H_{L,R}$ are included in all analyses of neutrino oscillations in medium. $\Sigma^\kappa_{L,R}$ include the usual forward scattering off matter and neutrinos, and are functions of $F, \bar{F}$ thereby introducing non-linear effects in the coherent evolution. The $m^\dagger m/|\vec{k}|$ term encodes vacuum oscillations. The additional terms in $H_{L,R}$ and the spin-flip term H_{LR}, discussed in detail in Refs. [11, 49], complete the set of contributions to $O(\epsilon^2)$.

Finally, the collision terms on the RHS of Eqs. (30) are

$$\mathcal{C} = \frac{1}{2}\left\{\Pi^+, F\right\} - \frac{1}{2}\left\{\Pi^-, \mathbb{1} - F\right\} \,, \tag{33a}$$

$$\bar{\mathcal{C}} = \frac{1}{2}\left\{\bar{\Pi}^+, \bar{F}\right\} - \frac{1}{2}\left\{\bar{\Pi}^-, \mathbb{1} - \bar{F}\right\} \,, \tag{33b}$$

where [50]

$$\Pi^\pm(\vec{k}) = \int_0^\infty dk^0\ \hat{\Pi}^\pm(k^0, \vec{k})\,\delta(k^0 - |\vec{k}|)\,,$$
$$\bar{\Pi}^\pm(\vec{k}) = -\int_{-\infty}^0 dk^0\ \hat{\Pi}^\mp(k^0, -\vec{k})\,\delta(k^0 + |\vec{k}|)\,. \tag{34}$$

and the $2n_f \times 2n_f$ gain and loss potentials $\hat{\Pi}^\pm$ are given in Eq. (19) in terms of spinor components of the self-energy.

The above discussion directly applies to Dirac neutrinos, with Eqs. (30) representing QKEs for neutrino ($F(k,x)$) and antineutrino ($\bar{F}(k,x)$) density matrices. In the Majorana case $\bar{F}(k,x)$ contains no additional information compared to $F(k,x)$. So one can get QKEs for Majorana neutrino exclusively from the positive frequency integral of (24a). To avoid confusion, in the Majorana case we denote the positive frequency integral of $\hat{F}(k,x)$ by $\mathcal{F}(k,x)$ (see Eq. (7) for a discussion of its physical content). The Majorana QKE is formally identical to the first one of (30):

$$\partial^\kappa \mathcal{F} + \frac{1}{2|\vec{k}|}\left\{\Sigma^i, \partial^i \mathcal{F}\right\} - \frac{1}{2}\left\{\frac{\partial \Sigma^\kappa}{\partial \vec{x}}, \frac{\partial \mathcal{F}}{\partial \vec{k}}\right\} = -i[H, \mathcal{F}] + \mathcal{C}_M \,, \tag{35}$$

with

$$\mathcal{C}_M = \frac{1}{2}\left\{\Pi^+, \mathcal{F}\right\} - \frac{1}{2}\left\{\Pi^-, \mathbb{1} - \mathcal{F}\right\} \,, \tag{36}$$

and $\Pi^\pm$ formally given in (34). The analogy, however, is only superficial because the potentials Σ_μ and $\Pi^\pm$ have a very different structure in the Dirac and Majorana cases. Anticipating results to be described later, we note that:

- Concerning the potentials induced by forward scattering, $\Sigma_{R,L}$, for Dirac neutrinos $\Sigma_R \neq 0$ while $\Sigma_L \propto G_F m^2 \sim O(\epsilon^3)$ (massless right-handed neutrinos do not interact). On the other hand, in the Majorana case one has $\Sigma_L = -\Sigma_R^T$, with transposition acting on flavor indices: right-handed antineutrinos do interact even in the massless limit.

- Concerning the inelastic potentials $\hat{\Pi}^\pm$ given in (19), in the Majorana case all four $n_f \times n_f$ blocks are non-vanishing, i.e. $\Pi^{\kappa\pm}_{R,L} \neq 0$ and $P^\pm_T \neq 0$. In the Dirac case, on the other hand, in the massless limit only the upper diagonal block is non-vanishing, i.e. $\Pi^{\kappa\pm}_R \neq 0$ and $\Pi^{\kappa\pm}_L = P^\pm_T = 0$. This again corresponds to the fact that massless right-handed neutrinos and left-handed antineutrinos have no interactions in the Standard Model.

D. Refractive effects

Before discussing in detail the collision terms in the next section, for completeness we briefly describe refractive effects in the coherent evolution, controlled by the potential Σ. Σ_R and Σ_L (see Eq. 19) are the 4-vector potentials induced by forward scattering for left-handed and right-handed neutrinos, respectively. For Dirac neutrinos $\Sigma_R \neq 0$ and $\Sigma_L \propto G_F m^2 \sim O(\epsilon^3)$ while for Majorana neutrinos $\Sigma_L = -\Sigma_R^T$.

The potential induced by a background of electrons and positrons is given for any geometry by the following expressions:

$$\left[\Sigma_R^\mu \Big|_e\right]_{ab} = 2\sqrt{2}G_F \times \Big[\left(\delta_{ea}\delta_{eb} + \delta_{ab}\left(\sin^2\theta_W - \frac{1}{2}\right)\right) J^\mu_{(e_L)} + \delta_{ab}\,\sin^2\theta_W\, J^\mu_{(e_R)}\Big] , \tag{37a}$$

$$J^\mu_{(e_L)}(x) = \int \frac{d^3q}{(2\pi)^3}\, v^\mu_{(e)}(q)\, \left(f_{e_L}(\vec{q},x) - \bar{f}_{e_R}(\vec{q},x)\right) , \tag{37b}$$

$$J^\mu_{(e_R)}(x) = \int \frac{d^3q}{(2\pi)^3}\, v^\mu_{(e)}(q)\, \left(f_{e_R}(\vec{q},x) - \bar{f}_{e_L}(\vec{q},x)\right) , \tag{37c}$$

where a, b are flavor indices, $v^\mu_{(e)} = (1, \vec{q}/\sqrt{m_e^2+q^2}\,)$, and we use the notation $f_{e_L}(\vec{q},x)$ ($\bar{f}_{e_L}(\vec{q},x)$) for the distribution function of L-handed electrons (positrons), etc.

The nucleon-induced potentials have similar expressions, with appropriate replacements of the L- and R-handed couplings to the Z and the distribution functions $f_{e_L} \to f_{N_L}$, etc. For unpolarized electron and nucleon backgrounds of course one has $f_{e_L} = f_{e_R}$, etc., and the nucleon contribution to the potential is:

$$\left[\Sigma_R^\mu \Big|_N\right]_{ab} = \sqrt{2}G_F\, C_V^{(N)}\, J^\mu_{(N)}\, \delta_{ab} , \tag{38}$$

with $C_V^{(n,p)}$ given in Eq. (4).

Finally, the neutrino-induced potentials are given by

$$\left[\Sigma_R^\mu \Big|_\nu\right]_{ab} = \sqrt{2}G_F \left(\left[J^\mu_{(\nu)}\right]_{ab} + \delta_{ab}\,\mathrm{Tr}J^\mu_{(\nu)}\right) , \tag{39a}$$

$$J^\mu_{(\nu)}(x) = \int \frac{d^3q}{(2\pi)^3} n^\mu(q)\, \left(f_{LL}(\vec{q},x) - \bar{f}_{RR}(\vec{q},x)\right) , \tag{39b}$$

with $n^\mu(q) = (1,\hat{q})$. For a test-neutrino of three-momentum $\vec{k}$, these potentials can be further projected along the basis vectors: with light-like component $\Sigma^\kappa \equiv n(k)\cdot\Sigma$ along the neutrino trajectory (in the massless limit); and space-like component $\Sigma^i \equiv x^i(k)\cdot\Sigma$, transverse to the neutrino trajectory. In particular, for the neutrino-induced contribution we find $\Sigma^\kappa(x) \propto \int d^3q\,(1-\cos\theta_{kq})\cdot(f_{LL}(\vec{q},x) - \bar{f}_{RR}(\vec{q},x))$, consistently with the familiar results in the literature (see [51] and references therein).

Having summarized the structure of the neutrino QKEs, we next discuss the main new results of this paper, namely the calculation of the collision terms.

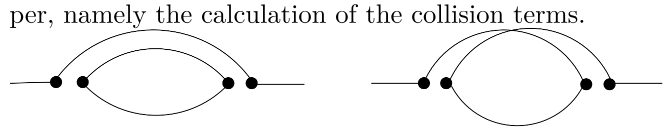

FIG. 2. Feynman graphs contributing to $\Pi(k,x)$. We represent each 4-fermion interaction vertex from Eqs. (2) in terms of two displaced fermionic current vertices. External lines represent neutrinos. Internal lines represent ν, e, n, p propagators (left diagram) and ν propagators (right diagram).

III. MAJORANA COLLISION TERM: DERIVATION AND GENERAL RESULTS

Direction-changing scattering and inelastic processes such as neutrino emission, absorption, pair production, or pair annihilation are encoded in $\mathcal{C}_M$ on the RHS of (35). In this section we derive the structure and detailed expressions for $\mathcal{C}_M$, providing, for the sake of completeness, several intermediate steps in the derivation. All the results presented in this section pertain to Majorana neutrinos. The collision term for Dirac neutrinos is discussed in Section IV.

We recall that the Majorana collision term is given by

$$\mathcal{C}_M = \frac{1}{2}\left\{\Pi^+, \mathcal{F}\right\} - \frac{1}{2}\left\{\Pi^-, \mathbb{1} - \mathcal{F}\right\} , \tag{40a}$$

$$\Pi^\pm(\vec{k}) = \int_0^\infty dk^0\, \hat{\Pi}^\pm(k^0,\vec{k})\,\delta(k^0 - |\vec{k}|) , \tag{40b}$$

with the $2n_f \times 2n_f$ gain and loss potentials $\Pi^\pm$ given in Eq. (19) in terms of spinor components of the self-energy, extracted from a calculation of the two-loop self-energies of Fig. 2. The $2n_f \times 2n_f$ collision term matrix is thus given by

$$\mathcal{C}_M = \begin{pmatrix} C & C_\phi \\ C_\phi^\dagger & \bar{C}^T \end{pmatrix} \tag{41}$$

with

$$C(\vec{k}) = \frac{1}{2}\left(\{(\Pi_R^\kappa)^+, f\} - \{(\Pi_R^\kappa)^-, (1-f)\}\right) + (P_T^+ + P_T^-)\phi^\dagger + \phi(P_T^+ + P_T^-)^\dagger \,,$$
$$\bar{C}^T(\vec{k}) = \frac{1}{2}\left(\{(\Pi_L^\kappa)^+, \bar{f}^T\} - \{(\Pi_L^\kappa)^-, (1-\bar{f}^T)\}\right) + (P_T^+ + P_T^-)^\dagger\phi + \phi^\dagger(P_T^+ + P_T^-)\,,$$
$$C_\phi(\vec{k}) = \frac{1}{2}\Big(\big((\Pi_R^\kappa)^+ + (\Pi_R^\kappa)^-\big)\phi + \phi\big((\Pi_L^\kappa)^+ + (\Pi_L^\kappa)^-\big)\Big) + f\,(P_T^+ + P_T^-) + (P_T^+ + P_T^-)\,\bar{f}^T - 2P_T^- \,. \tag{42}$$

Note that the collision term has a non-diagonal matrix structure in both flavor [1, 2] and spin space [9]. The matrix components of $\Pi^\pm$ can be expressed in terms of neutrino density matrices and distribution functions of the medium particles (electrons, etc.). To the order we are working, we need only the $O(\epsilon^0)$ expressions for the neutrino and matter Green's functions in the collision term. These expressions are collected in Appendix B.

The contribution to $\nu - \nu$ scattering neglecting spin coherence is given in Ref. [9]. We present below the full analysis including scattering off neutrinos, electrons, and low-density nucleons. Deriving the collision term requires the following steps:

- Calculation of the self-energy diagrams in Fig. 2.
- Identification of the components of $\hat{\Pi}^\pm$ (see (19)) by projecting the self-energy diagrams on appropriate spinor and Lorentz components via Eq.(18).
- Integration over positive frequencies according to Eq. (40b) to obtain $\Pi^\pm(\vec{k})$.
- Matrix multiplications to obtain the various components of $\mathcal{C}_M$ in (42).

In the following subsections we present results on each of the above items.

A. Self-energies to two loops

The Standard Model interactions allow for neutrino-neutrino processes and neutrino interactions with charged leptons and nucleons, cf. Eqn. (2). We report below the various contributions.

- Neutrino-neutrino processes receive contributions from both topologies in Fig. 2.
  Left diagram:

$$\tilde{\Pi}^\pm_{ab}(k) = -G_F^2 \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8}\delta^{(4)}(k - q_3 - q_1 + q_2) \times \gamma_\mu(P_L - P_R)\, G^{(\nu)\pm}_{ab}(q_3)\,\gamma_\nu(P_L - P_R) \times \mathrm{Tr}\Big[\gamma^\nu(P_L - P_R)\, G^{(\nu)\mp}_{cd}(q_2) \times \gamma^\mu(P_L - P_R)\; G^{(\nu)\pm}_{dc}(q_1)\Big] \tag{43}$$

  Right diagram:

$$\tilde{\Pi}^\pm_{ab}(k) = 2G_F^2 \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8}\delta^{(4)}(k - q_3 - q_1 + q_2) \times \gamma_\mu(P_L - P_R)\, G^{(\nu)\pm}_{aa'}(q_1)\,\gamma_\nu(P_L - P_R) \times G^{(\nu)\mp}_{a'b'}(q_2)\,\gamma^\mu(P_L - P_R) \times G^{(\nu)\pm}_{b'b}(q_3)\,\gamma^\nu(P_L - P_R) \tag{44}$$

  where a, b, c, d are flavor indices.

- Neutrino-electron processes receive contributions only from the first topology in Fig. 2:

$$\tilde{\Pi}^\pm_{ab}(k) = -8G_F^2 \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8}\delta^{(4)}(k - q_3 - q_1 + q_2) \times \sum_{A,B=L,R}\Big\{\gamma_\mu(P_L - P_R) \times \Big[Y_A G^{(\nu)\pm}(q_3) Y_B\Big]_{ab}\gamma_\nu(P_L - P_R) \times \mathrm{Tr}\Big[\gamma^\nu P_B\, G^{(e)\mp}(q_2)\,\gamma^\mu P_A\; G^{(e)\pm}(q_1)\Big]\Big\} \tag{45}$$

- Neutrino-nucleon processes receive contributions only from the first topology in Fig. 2. There are two contributions, scattering and absorption.
  Scattering:

$$\tilde{\Pi}^\pm_{ab}(k) = -2G_F^2 \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8}\delta^{(4)}(k - q_3 - q_1 + q_2) \times \sum_{N=n,p}\Big\{\gamma_\mu(P_L - P_R) G^{(\nu)\pm}_{ab}(q_3)\gamma_\nu(P_L - P_R) \times \mathrm{Tr}\Big[\Gamma_N^\nu\, G^{(N)\mp}(q_2)\,\Gamma_N^\mu\; G^{(N)\pm}(q_1)\Big]\Big\}\,,$$
$$\Gamma_N^\mu = \gamma^\mu\left(C_V^{(N)} - C_A^{(N)}\gamma_5\right)\,. \tag{46}$$

Neutrino absorption and emission:

$$\tilde{\Pi}^{\pm}_{ab}(k) = -2G_F^2 \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8} \delta^{(4)}(k - q_3 - q_1 + q_2)$$
$$\times \gamma_\mu P_L \; [I_e]_{ab} \; G^{(e)\pm}(q_3) \; \gamma_\nu P_L$$
$$\times \mathrm{Tr}\Big[\Gamma^\mu \, G^{(p)\pm}(q_1) \, \Gamma^\nu \; G^{(n)\mp}(q_2)\Big]$$
$$- 2G_F^2 \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8} \delta^{(4)}(k + q_3 - q_1 + q_2)$$
$$\times \gamma_\mu P_R \; [I_e]_{ab} \, G^{(e)\mp}(q_3) \; \gamma_\nu P_R$$
$$\times \mathrm{Tr}\Big[\Gamma^\mu \, G^{(n)\pm}(q_1) \, \Gamma^\nu \; G^{(p)\mp}(q_2)\Big] ,$$
$$\Gamma^\mu = \gamma^\mu \left(1 - g_A \gamma_5\right) . \tag{47}$$

In the above expression we have introduced the projector on the electron flavor,

$$[I_e]_{ab} \equiv \; \delta_{ae}\delta_{be} \tag{48}$$

and we are neglecting contributions from $\mu^\pm$ and $\tau^\pm$, which are kinematically suppressed at the energies and temperatures of interest. Their contributions are formally identical to the electron one, with the replacements $I_e \to I_\mu, I_\tau$ and $G^{(e)} \to G^{(\mu)}, G^{(\tau)}$.

B. Projections on Lorentz structures

Using the identities collected in Appendix A 2 and A 3, one can perform all the needed projections in a straightforward way (at most four gamma matrices and a γ_5 appear in the traces). In the following, we will suppress flavor indices. The vector and tensor components of the self-energies (defined in Eqn. (18)) for all processes considered in this work are:

Neutrino-nucleon scattering processes

$$(\Pi_R^\kappa)^\pm(k) = -\frac{1}{|\vec{k}|} \int \frac{d^4q_3}{(2\pi)^4} \left(R_N^\pm(k,q_3)\right)^{\mu\nu}$$
$$\times \left[\bar{G}_V^L(q_3)\right]^\pm \; (T_R)_{\mu\nu}(k,q_3) , \tag{49a}$$
$$(\Pi_L^\kappa)^\pm(k) = -\frac{1}{|\vec{k}|} \int \frac{d^4q_3}{(2\pi)^4} \left(R_N^\pm(k,q_3)\right)^{\mu\nu}$$
$$\times \left[\bar{G}_V^R(q_3)\right]^\pm \; (T_L)_{\mu\nu}(k,q_3) , \tag{49b}$$
$$P_T^\pm(k) = \mp\frac{1}{2} \int \frac{d^4q_3}{(2\pi)^4} \left(R_N^\pm(k,q_3)\right)^{\mu\nu}$$
$$\times \left[\Phi(q_3)\right] \; (T_T)_{\mu\nu}(k,q_3) . \tag{49c}$$

The various tensors are given by:

$$(T_{R,L})_{\mu\nu}(k,k') = k_\mu k'_\nu + k_\nu k'_\mu - k\cdot k' \eta_{\mu\nu}$$
$$\mp i\epsilon_{\mu\nu\alpha\beta} k^\alpha k'^\beta , \tag{50a}$$
$$(T_T)_{\mu\nu}(k,k') = [\hat{\kappa}(k') \wedge \hat{x}^-(k')]_{\mu\alpha} \; [\hat{\kappa}(k) \wedge \hat{x}^+(k)]_\nu{}^\alpha$$
$$\times e^{i\left(\varphi(k) - \varphi(k')\right)} . \tag{50b}$$

In the last expression we have explicitly indicated the dependence of the basis vectors $\hat{\kappa}, \hat{\kappa}', \hat{x}_\pm$ on the four-momenta k and k'. Finally, the nucleon response function is given by:

$$\left(R_N^\pm(k,q_3)\right)^{\mu\nu} = 2G_F^2 \int \frac{d^4q_1 d^4q_2}{(2\pi)^8} \delta^{(4)}(k - q_3 - q_1 + q_2)$$
$$\times (2\pi)^4 \mathrm{Tr}\Big[\Gamma_N^\mu \, G^{(N)\pm}(q_1) \Gamma_N^\nu G^{(N)\mp}(q_2)\Big] . \tag{51}$$

We give the explicit expressions for the neutrino and nucleon Green's functions $\bar{G}_V^{L,R}$, Φ, and $G^{(N)}$ to $O(\epsilon^0)$ in Appendix B.

Neutrino-electron processes

$$(\Pi_R^\kappa)^\pm(k) = -\sum_{A,B=L,R} \frac{1}{|\vec{k}|} \int \frac{d^4q_3}{(2\pi)^4} \left(R_{BA}^\pm(k,q_3)\right)^{\mu\nu}$$
$$\times \left[Y_A \; \left(\bar{G}_V^L(q_3)\right)^\pm Y_B\right] \; (T_R)_{\mu\nu}(k,q_3) , \tag{52a}$$

$$(\Pi_L^\kappa)^\pm(k) = -\sum_{A,B=L,R} \frac{1}{|\vec{k}|} \int \frac{d^4q_3}{(2\pi)^4} \left(R_{BA}^\pm(k,q_3)\right)^{\mu\nu}$$
$$\times \left[Y_A \; \left(\bar{G}_V^R(q_3)\right)^\pm Y_B\right] \; (T_L)_{\mu\nu}(k,q_3) , \tag{52b}$$

$$P_T^\pm(k) = \mp\frac{1}{2} \sum_{A,B=L,R} \int \frac{d^4q_3}{(2\pi)^4} \left(R_{BA}^\pm(k,q_3)\right)^{\mu\nu}$$
$$\times \left[Y_A \Phi(q_3) Y_B\right] \; (T_T)_{\mu\nu}(k,q_3) . \tag{52c}$$

The electron response functions are given by (recall $A, B \in \{L,R\}$ and $P_{A,B} = P_{L,R}$):

$$\left(R_{BA}^\pm(k,q_3)\right)^{\mu\nu} = 8G_F^2 \int \frac{d^4q_1 d^4q_2}{(2\pi)^4} \delta^{(4)}(k - q_3 - q_1 + q_2)$$
$$\times \mathrm{Tr}\Big[\gamma^\mu P_A G^{(e)\pm}(q_1) \gamma^\nu P_B G^{(e)\mp}(q_2)\Big] . \tag{53}$$

Expressions for the electron Green's functions $G^{(e)}$ to $O(\epsilon^0)$ are given in Appendix B.

Charged-current processes

$$(\Pi^\kappa_R)^\pm_{ab}(k) = -\frac{1}{|\vec{k}|}\int\frac{d^4q_3}{(2\pi)^4}\left(R^\pm_{CC}(k,q_3)\right)^{\mu\nu} \times\left[[I_e]_{ab}\ \bar{G}^{(e)\pm}(q_3)\right]\ (T_R)_{\mu\nu}(k,q_3)\,, \quad (54a)$$

$$(\Pi^\kappa_L)^\pm_{ab}(k) = -\frac{1}{|\vec{k}|}\int\frac{d^4q_3}{(2\pi)^4}\left(\tilde{R}^\pm_{CC}(k,q_3)\right)^{\mu\nu} \times\left[[I_e]_{ab}\ \bar{G}^{(e)\mp}(q_3)\right]\ (T_L)_{\mu\nu}(k,q_3)\,. \quad (54b)$$

There is no tensor projection from these processes. The charged-current response is given by:

$$\left(R^\pm_{CC}(k,q_3)\right)^{\mu\nu} = 2G_F^2\int\frac{d^4q_1d^4q_2}{(2\pi)^4}\delta^{(4)}(k-q_3-q_1+q_2) \times\mathrm{Tr}\left[\Gamma^\mu G^{(p)\pm}(q_1)\Gamma^\nu G^{(n)\mp}(q_2)\right], \quad (55a)$$

$$\left(\tilde{R}^\pm_{CC}(k,q_3)\right)^{\mu\nu} = 2G_F^2\int\frac{d^4q_1d^4q_2}{(2\pi)^4}\delta^{(4)}(k+q_3-q_1+q_2) \times\mathrm{Tr}\left[\Gamma^\mu G^{(n)\pm}(q_1)\Gamma^\nu G^{(p)\mp}(q_2)\right]\ . \quad (55b)$$

Neutrino-neutrino processes

The diagram in the left panel of Fig. 2 induces:

$$(\Pi^\kappa_R)^\pm(k) = -\frac{1}{|\vec{k}|}\int\frac{d^4q_3}{(2\pi)^4}\left(R^\pm_{(\nu)}(k,q_3)\right)^{\mu\nu} \times\left[\bar{G}^L_V(q_3)\right]^\pm(T_R)_{\mu\nu}(k,q_3)\,, \quad (56a)$$

$$(\Pi^\kappa_L)^\pm(k) = -\frac{1}{|\vec{k}|}\int\frac{d^4q_3}{(2\pi)^4}\left(R^\pm_{(\nu)}(k,q_3)\right)^{\mu\nu} \times\left[\bar{G}^R_V(q_3)\right]^\pm(T_L)_{\mu\nu}(k,q_3)\,, \quad (56b)$$

$$P^\pm_T(k) = \mp\frac{1}{2}\int\frac{d^4q_3}{(2\pi)^4}\left(R^\pm_{(\nu)}(k,q_3)\right)^{\mu\nu} \times\left[\Phi(q_3)\right](T_T)_{\mu\nu}(k,q_3)\ . \quad (56c)$$

$$\left(R^\pm_{(\nu)}(k,q_3)\right)_{\mu\nu} = 2\,G_F^2\int\frac{d^4q_1d^4q_2}{(2\pi)^4}\,\delta^{(4)}(k-q_3-q_1+q_2) \times\mathrm{Tr}\Bigg[\left[\bar{G}^L_V(q_2)\right]^\mp\left[\bar{G}^L_V(q_1)\right]^\pm(T_R)_{\mu\nu}(q_2,q_1)+\left[\bar{G}^R_V(q_2)\right]^\mp\left[(\bar{G}^R_V(q_1)\right]^\pm(T_L)_{\mu\nu}(q_2,q_1) - \Phi^\dagger(q_2)\Phi(q_1)(T_T)_{\mu\nu}(q_2,q_1)-\Phi(q_2)\Phi^\dagger(q_1)(T_T)^*_{\mu\nu}(q_2,q_1)\Bigg]. \quad (56d)$$

The diagram in the right panel of Fig. 2 induces

$$
\begin{aligned}
(\Pi_R^\kappa)^\pm(k) = &-\frac{8\,G_F^2}{|\vec{k}|}\int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8}\,\delta^{(4)}(k-q_3-q_1+q_2)\\
&\times\Big\{k^\alpha q_2^\beta\,(T_T)_{\alpha\beta}(q_3,q_1)\;\Phi(q_1)\;\left[\bar{G}_V^R(q_2)\right]^\mp\;\Phi^\dagger(q_3)\\
&\quad - k^\alpha q_3^\beta\,(T_T)_{\alpha\beta}(q_2,q_1)\;\Phi(q_1)\,\Phi^\dagger(q_2)\;\left[\bar{G}_V^L(q_3)\right]^\pm\\
&\quad - k^\alpha q_1^\beta\,(T_T)_{\beta\alpha}(q_3,q_2)\;\left[\bar{G}_V^L(q_1)\right]^\pm\;\Phi(q_2)\,\Phi^\dagger(q_3)\\
&\quad + 2(kq_2)(q_1q_3)\left[\bar{G}_L(q_1)\right]^\pm\left[\bar{G}_L(q_2)\right]^\mp\left[\bar{G}_L(q_3)\right]^\pm\Big\}\,,
\end{aligned}
\tag{57a}
$$

$$
\begin{aligned}
(\Pi_L^\kappa)^\pm(k) = &-\frac{8\,G_F^2}{|\vec{k}|}\int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8}\,\delta^{(4)}(k-q_3-q_1+q_2)\\
&\times\Big\{k^\alpha q_2^\beta\,(T_T)^*_{\alpha\beta}(q_3,q_1)\;\Phi^\dagger(q_1)\;\left[\bar{G}_V^L(q_2)\right]^\mp\;\Phi(q_3)\\
&\quad - k^\alpha q_3^\beta\,(T_T)^*_{\alpha\beta}(q_2,q_1)\;\Phi^\dagger(q_1)\,\Phi(q_2)\;\left[\bar{G}_V^R(q_3)\right]^\pm\\
&\quad - k^\alpha q_1^\beta\,(T_T)^*_{\beta\alpha}(q_3,q_2)\;\left[\bar{G}_V^R(q_1)\right]^\pm\;\Phi^\dagger(q_2)\,\Phi(q_3)\\
&\quad + 2(kq_2)(q_1q_3)\left[\bar{G}_V^R(q_1)\right]^\pm\left[\bar{G}_V^R(q_2)\right]^\mp\left[\bar{G}_V^R(q_3)\right]^\pm\Big\}\,,
\end{aligned}
\tag{57b}
$$

and

$$
\begin{aligned}
P_T^\pm(k) = &-4\,G_F^2\int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8}\,\delta^{(4)}(k-q_3-q_1+q_2)\\
&\times\Big\{\pm\frac{1}{2}\Phi(q_1)\Phi^\dagger(q_2)\Phi(q_3)\,(T_T)_{\mu\nu}(q_2,q_1)\,(T_T)^{\mu\nu}(k,q_3)\\
&\quad \pm\Phi(q_1)\left[\bar{G}_V^R(q_2)\right]^\mp\left[\bar{G}_V^R(q_3)\right]^\pm\;q_2^\alpha q_3^\beta\,(T_T)_{\alpha\beta}(k,q_1)\\
&\quad \mp\left[\bar{G}_V^L(q_1)\right]^\pm\Phi(q_2)\left[\bar{G}_V^R(q_3)\right]^\pm\;q_1^\alpha q_3^\beta\,(T_T)_{\alpha\beta}(k,q_2)\\
&\quad \pm\left[\bar{G}_V^L(q_1)\right]^\pm\left[\bar{G}_V^L(q_2)\right]^\mp\Phi(q_3)\;q_1^\alpha q_2^\beta\,(T_T)_{\beta\alpha}(k,q_3)\Big\}\,.
\end{aligned}
\tag{57c}
$$

C. Frequency projections: general results for loss and gain potentials

To obtain the Majorana collision term $\mathcal{C}_M$, we need the positive-frequency ($k^0>0$) integrals of $\hat{\Pi}^\pm(k^0,\vec{k})$ defined in Eq. (40). Furthermore, we also integrate over $q^0_{1,2,3}$ using the δ-functions present in all Green functions, see (B4) and (B6).

In the following, we will use abbreviations for the various density matrices, i.e. $f_i\equiv f(\vec{q}_i)$, $\bar{f}_i\equiv\bar{f}(\vec{q}_i)$, $\bar{f}_i^T\equiv\bar{f}^T(\vec{q}_i)$, $f_i^T\equiv f^T(\vec{q}_i)$, and $\phi_i\equiv\phi(\vec{q}_i)$, $\phi_i^T\equiv\phi^T(\vec{q}_i)$, $\phi_i^\dagger\equiv\phi^\dagger(\vec{q}_i)$, $\phi_i^*\equiv\phi^*(\vec{q}_i)$. We will, however, omit subscripts $_k$: $f\equiv f(\vec{k})$, $\phi\equiv\phi(\vec{k})$. Note that all density matrices and distribution functions appear with argument "$+\vec{q}_i$". i.e. $f(\vec{q}_i)$ and not $f(-\vec{q}_i)$, something we achieve via variable substitution under the integrals. Furthermore, f, $\bar{f}$ with subscripts $_{(N)},_{(e)}$ indicate them being nucleon and electron (anti)particle distributions (and thus scalars in flavor space) rather than neutrino distributions.

Finally, we write

$$
\int\widetilde{dq_i}\equiv\int\frac{d^3\vec{q}_i}{2E_i(2\pi)^3}\ ,
\tag{58}
$$

where the energy is $E_i\approx\sqrt{(\vec{q}_i)^2}$ for neutrinos (since their masses would give $O(\epsilon^3)$ contributions in the collision term), and $E_i=\sqrt{(\vec{q}_i)^2+M^2}$ for electrons and nucleons with $M=M_e$ and $M=M_N$, respectively.

Below, we give the expressions for the loss potentials $\Pi_R^{\kappa+}(\vec{k})$, $P_T^+(\vec{k})$ corresponding to each class of processes in the medium. From these, the gain potentials $(\Pi_R^\kappa)^-(\vec{k})$ and $P_T^-(\vec{k})$ can be obtained as follows,

$$
\begin{aligned}
(\Pi_R^\kappa)^-(\vec{k}) &= (\Pi_R^\kappa)^+(\vec{k})\big|_{f_i\to1-f_i,\ \phi_j\to-\phi_j}\ ,\\
P_T^-(\vec{k}) &= P_T^+(\vec{k})\big|_{f_i\to1-f_i,\ \phi_j\to-\phi_j}
\end{aligned}
\tag{59}
$$

for all f_i (all particle species, including barred ones) and ϕ_j. For each class of processes, we also give below the recipe to obtain the antineutrino potentials $\Pi_L^{\kappa\pm}(\vec{k})$ from the neutrino potentials $\Pi_R^{\kappa\pm}(\vec{k})$.

Neutrino-nucleon scattering processes

Neutrino-nucleon scattering $\nu(k)N(q_2) \to \nu(q_3)N(q_1)$ induces the following contributions to the loss potentials $\Pi_R^{\kappa+}(\vec{k})$ and $P_T^+(\vec{k})$:

$$\Pi_R^{\kappa+}(\vec{k}) = -\frac{4G_F^2}{|\vec{k}|}\int \widetilde{dq}_1\widetilde{dq}_2\widetilde{dq}_3\,(2\pi)^4\mathcal{M}_R(q_1,q_2,q_3,k)(1-f_{(N),1})f_{(N),2}\,(1-f_3)\;, \tag{60a}$$

$$P_T^+(\vec{k}) = \frac{8G_F^2}{|\vec{k}|}\int \widetilde{dq}_1\widetilde{dq}_2\widetilde{dq}_3\,(2\pi)^4(C_V^2+C_A^2)\mathcal{M}_T(q_1,q_2,q_3,k)f_{(N),2}(1-f_{(N),1})\phi_3\,, \tag{60b}$$

with

$$\begin{aligned}\mathcal{M}_{R,L}(q_1,q_2,q_3,k) &= \delta^{(4)}(k-q_3-q_1+q_2)\,4\Big((C_V^2+C_A^2)\big((q_1q_3)(kq_2)+(q_1k)(q_2q_3)\big)\\ &\qquad -(C_V^2-C_A^2)M_N^2(q_3k)\pm 2C_VC_A\big((q_1q_3)(kq_2)-(q_2q_3)(kq_1)\big)\Big)\,,\\ \mathcal{M}_T(q_1,q_2,q_3,k) &= \delta^{(4)}(k-q_3-q_1+q_2)\,|\vec{k}||\vec{q}_3|\,q_1^\mu q_2^\nu\,(T_T)_{\mu\nu}(k,q_3)\;,\end{aligned} \tag{61}$$

where $(T_T)_{\mu\nu}(k,k')$ is defined in (50), we suppressed the superscripts $^{(N)}$ on the couplings $C_{V,A}$ and all four-momenta are on-shell, i.e. $q_i^0=E_i$ and thus $\delta^{(4)}(k-q_3-q_1+q_2)=\delta(E_k-E_3-E_1+E_2)\delta^{(3)}(\vec{k}-\vec{q}_3-\vec{q}_1+\vec{q}_2)$. The antineutrino potentials are obtained by the relation

$$(\Pi_L^\kappa)^\pm(\vec{k}) = (\Pi_R^\kappa)^\pm(\vec{k})\big|_{f_i\to\bar{f}_i^T,\ \mathcal{M}_R\to\mathcal{M}_L}\;, \tag{62}$$

where $\mathcal{M}_R\to\mathcal{M}_L$ amounts to a change of sign in the axial coupling C_A.

Charged-current processes

The loss potential term from charged-current neutrino absorption $\nu(k)n(q_2)\to e^-(q_3)p(q_1)$ is

$$\begin{aligned}\Pi_R^{\kappa+}(\vec{k}) = -\frac{4G_F^2}{|\vec{k}|}&\int \widetilde{dq}_1\widetilde{dq}_2\widetilde{dq}_3\mathcal{M}_R^{CC}(q_1,q_2,q_3,k)\\ &\times(2\pi)^4(1-f_{(p),1})f_{(n),2}(1-f_{(e),3})I_e\;,\end{aligned} \tag{63}$$

where the flavor projector I_e is defined in Eq. (48) and

$$\begin{aligned}\mathcal{M}_{R,L}^{CC}(q_1,q_2,q_3,k) = 4\Big(&(1+g_A^2)\big((q_3q_1)(kq_2)+(q_3q_2)(kq_1)\big)-M_pM_n(1-g_A^2)(kq_3)\\ &\pm 2g_A\big((q_3q_1)(kq_2)-(kq_1)(q_3q_2)\big)\Big)\delta^{(4)}(k-q_3-q_1+q_2)\,.\end{aligned} \tag{64}$$

Neutrino absorption and emission does not induce $P_T^\pm(\vec{k})$. The antineutrino potentials are obtained by the relation

$$(\Pi_L^\kappa)^\pm(\vec{k}) = (\Pi_R^\kappa)^\pm(\vec{k})\big|_{f_n\leftrightarrow f_p,\ f_{(e)}\to\bar{f}_{(e)},\ \mathcal{M}_R^{CC}\to\mathcal{M}_L^{CC}}\;, \tag{65}$$

where again $\mathcal{M}_R^{CC}\to\mathcal{M}_L^{CC}$ amounts to a change of sign in the axial coupling g_A.

Neutrino-electron processes

Neutrino electron processes contribute to the loss potentials as follows,

$$
\begin{aligned}
\Pi_R^{\kappa +}(\vec{k}) = &-\frac{32G_F^2}{|\vec{k}|}\int \widetilde{dq}_1\widetilde{dq}_2\widetilde{dq}_3\,(2\pi)^4 \qquad (66a)\\
&\times \sum_{I=L,R}\Bigg[(1-f_{(e),1})f_{(e),2}Y_I(1-f_3)\Big(2Y_I\mathcal{M}_I^R(q_1,q_2,q_3,k)-Y_{J\neq I}\mathcal{M}_m(q_1,q_2,q_3,k)\Big)\\
&+\bar{f}_{(e),1}(1-\bar{f}_{(e),2})Y_I(1-f_3)\Big(2Y_I\mathcal{M}_I^R(-q_1,-q_2,q_3,k)-Y_{J\neq I}\mathcal{M}_m(-q_1,-q_2,q_3,k)\Big)\\
&+(1-f_{(e),1})(1-\bar{f}_{(e),2})Y_I\bar{f}_3\Big(2Y_I\mathcal{M}_I^R(q_1,-q_2,-q_3,k)-Y_{J\neq I}\mathcal{M}_m(q_1,-q_2,-q_3,k)\Big)\Bigg],\\
P_T^{+}(\vec{k}) = &-\frac{32G_F^2}{|\vec{k}|}\int \widetilde{dq}_1\widetilde{dq}_2\widetilde{dq}_3\,(2\pi)^4\\
&\times \sum_{I=L,R}\Bigg[(1-f_{(e),1})f_{(e),2}Y_I(-\phi_3)Y_I\mathcal{M}_T(q_1,q_2,q_3,k)\\
&+\bar{f}_{(e),1}(1-\bar{f}_{(e),2})Y_I(-\phi_3)Y_I\mathcal{M}_T(-q_1,-q_2,q_3,k)\\
&+(1-f_{(e),1})(1-\bar{f}_{(e),2})Y_I\,\phi_3^T\,Y_I\mathcal{M}_T(q_1,-q_2,-q_3,k)\Bigg], \qquad (66b)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{M}_I^L(q_1,q_2,q_3,k) &= \big(\delta_I^R(q_3q_1)(kq_2)+\delta_I^L(q_3q_2)(kq_1)\big)\ \delta^{(4)}(k-q_3-q_1+q_2)\,,\\
\mathcal{M}_I^R(q_1,q_2,q_3,k) &= \big(\delta_I^L(q_3q_1)(kq_2)+\delta_I^R(q_3q_2)(kq_1)\big)\ \delta^{(4)}(k-q_3-q_1+q_2)\,,\\
\mathcal{M}_m(q_1,q_2,q_3,k) &= m_e^2\,(kq_3)\,\delta^{(4)}(k-q_3-q_1+q_2) \qquad (67)
\end{aligned}
$$

and $\mathcal{M}_T(q_1,q_2,q_3,k)$ is defined in (61). The first term of the sum in Eq. (66a) stems from neutrino scattering off electrons ($\nu(k)e^-(q_2)\to\nu(q_3)e^-(q_1)$), the second from neutrino scattering off positrons ($\nu(k)e^+(q_1)\to\nu(q_3)e^+(q_2)$) , and the third ones from neutrino-antineutrino annihilation into electron-positron pairs ($\nu(k)\bar{\nu}(q_3)\to e^+(q_2)e^-(q_1)$). The antineutrino potentials are obtained by the relation

$$
(\Pi_L^\kappa)^\pm(\vec{k}) = (\Pi_R^\kappa)^\pm(\vec{k})\big|_{f_i\to\bar{f}_i^T,\ \bar{f}_i\to f_i^T,\ Y_R\leftrightarrow Y_L}\,, \qquad (68)
$$

where $Y_L\leftrightarrow Y_R$ is equivalent to the replacement $\mathcal{M}_I^R\to\mathcal{M}_I^L$.

Neutrino-neutrino processes

Neutrino-neutrino scattering $\nu\nu\to\nu\nu$ and neutrino-antineutrino scattering $\nu\bar{\nu}\to\nu\bar{\nu}$ contribute to the neutrino loss potentials as follows,

$$
\begin{aligned}
(\Pi_R^\kappa)^+(\vec{k}) = -4\frac{G_F^2}{|\vec{k}|}\int \widetilde{dq}_1\widetilde{dq}_2\widetilde{dq}_3(2\pi)^4 \Big(& \Big((1-f_1)f_2 + \mathrm{tr}\left((1-f_1)f_2\right)\Big)(1-f_3)\mathcal{M}(q_1,q_2,q_3,k) \\
& - \Big(2\phi_1\phi_2^\dagger + \mathrm{tr}\big(\phi_1\phi_2^\dagger\big)\Big)(1-f_3)\mathcal{M}_T(q_3,k,q_1,q_2) \\
& - (1-f_1)\Big(2\phi_2\phi_3^\dagger + \mathrm{tr}\big(\phi_2\phi_3^\dagger\big)\Big)\mathcal{M}_T(q_1,k,q_2,q_3) \\
& + 2\phi_1\bar{f}_2^T\phi_3^\dagger\mathcal{M}_T(q_2,k,q_1,q_3) \\
+ \Big\{ & q_{2,3}\to -q_{2,3}, \quad f_{2,3}\to(1-\bar{f}_{2,3}), \quad \bar{f}_{2,3}^T\to(1-f_{2,3}^T), \quad \phi_{2,3}\to-\phi_{2,3}^T, \quad \phi_{2,3}^\dagger\to-\phi_{2,3}^*\Big\} \\
+ \Big\{ & q_{1,2}\to -q_{1,2}, \quad f_{1,2}\to(1-\bar{f}_{1,2}), \quad \bar{f}_{1,2}^T\to(1-f_{1,2}^T), \quad \phi_{1,2}\to-\phi_{1,2}^T, \quad \phi_{1,2}^\dagger\to-\phi_{1,2}^*\Big\}\,,
\end{aligned} \tag{69}
$$

with $\mathcal{M}_T(q_1,q_2,q_3,k)$ defined in (61) and

$$
\mathcal{M}(q_1,q_2,q_3,k) = 4(q_1q_3)(q_2k)\delta^{(4)}(k-q_3-q_1+q_2)\;. \tag{70}
$$

In absence of spin-coherence ($\phi_i\to 0$) the first term in (69) encodes loss terms due to $\nu_k\nu_2\to\nu_1\nu_3$, while the terms in the last two lines in (69) encode the effects of $\nu\bar{\nu}\to\nu\bar{\nu}$ processes. All the remaining terms, involving ϕ_i, arise due to the fact that target neutrinos in the thermal bath can be in coherent linear superpositions of the two helicity states (see Section V for a discussion of this point). The corresponding antineutrino potentials are obtained by the relation

$$
(\Pi_L^\kappa)^\pm(\vec{k}) = (\Pi_R^\kappa)^\pm(\vec{k})\big|_{f_i\to\bar{f}_i^T,\ \bar{f}_i\to f_i^T,\ \phi_j\leftrightarrow\phi_j^\dagger,\ \mathcal{M}_T\to\mathcal{M}_T^*}\,. \tag{71}
$$

and the contributions of neutrino-neutrino processes to the helicity off-diagonal loss potentials read

$$
\begin{aligned}
P_T^+(\vec{k}) = 4\frac{G_F^2}{|\vec{k}|}\int & \widetilde{dq}_1\widetilde{dq}_2\widetilde{dq}_3(2\pi)^4 \\
\times\Bigg(& \Big(\phi_1\phi_2^\dagger - \tfrac{1}{2}\mathrm{tr}\big(\phi_1\phi_2^\dagger\big)\Big)\phi_3\mathcal{M}_{TT}(q_1,q_2,q_3,k) - \tfrac{1}{2}\mathrm{tr}\big(\phi_1^\dagger\phi_2\big)\phi_3\widetilde{\mathcal{M}}_{TT}(q_1,q_2,q_3,k) \\
& + \Big((1-f_1)f_2 + \tfrac{1}{2}\mathrm{tr}\left((1-f_1)f_2\right)\Big)\phi_3\mathcal{M}_T(q_1,q_2,q_3,k) \\
& + \phi_1\Big(\bar{f}_2^T(1-\bar{f}_3^T) + \tfrac{1}{2}\mathrm{tr}\left(\bar{f}_2^T(1-\bar{f}_3^T)\right)\Big)\mathcal{M}_T(q_3,q_2,q_1,k) \\
& - (1-f_1)\phi_2(1-\bar{f}_3^T)\mathcal{M}_T(q_1,q_3,q_2,k) \\
+ \Big\{ & q_{2,3}\to -q_{2,3}, \quad f_{2,3}\to(1-\bar{f}_{2,3}), \quad \bar{f}_{2,3}^T\to(1-f_{2,3}^T), \quad \phi_{2,3}\to-\phi_{2,3}^T, \quad \phi_{2,3}^\dagger\to-\phi_{2,3}^*\Big\} \\
+ \Big\{ & q_{1,2}\to -q_{1,2}, \quad f_{1,2}\to(1-\bar{f}_{1,2}), \quad \bar{f}_{1,2}^T\to(1-f_{1,2}^T), \quad \phi_{1,2}\to-\phi_{1,2}^T, \quad \phi_{1,2}^\dagger\to-\phi_{1,2}^*\Big\}\,,
\end{aligned} \tag{72}
$$

with $\mathcal{M}_T(q_1,q_2,q_3,k)$ defined in (61) and

$$
\begin{aligned}
\mathcal{M}_{TT}(q_1,q_2,q_3,k) &= \frac{1}{2}|\vec{q_1}||\vec{q_2}||\vec{q_3}||\vec{k}|\,(T_T)_{\mu\nu}(k,q_1)\,(T_T)^{\mu\nu}(q_2,q_3)\;\delta^{(4)}(k-q_3-q_1+q_2)\,, \\
\widetilde{\mathcal{M}}_{TT}(q_1,q_2,q_3,k) &= \frac{1}{2}|\vec{q_1}||\vec{q_2}||\vec{q_3}||\vec{k}|\;(T_T)_{\mu\nu}(k,q_2)\,(T_T)^{\mu\nu}(q_1,q_3)\;\delta^{(4)}(k-q_3-q_1+q_2)\,.
\end{aligned} \tag{73}
$$

With the gain and loss potentials at hand, the collision terms C and C_ϕ are then assembled according to Eqn. (42). We present the lengthy results for (some of) the assembled collision terms in Appendix C.

IV. DIRAC COLLISION TERM

In this section we discuss the structure of the collision terms $\mathcal{C}$ and $\bar{\mathcal{C}}$ (33) appearing in the QKEs (30) for Dirac neutrinos and antineutrinos. We do not repeat all the steps reported in Section III, but simply outline how to map the Majorana expressions into the ones relevant for Dirac neutrinos.

First, note that the self-energy diagrams in Fig. 2 for Dirac neutrinos are obtained from the ones in Sect. III A, that refer to the Majorana case, with the following simple changes:

(i) in the weak vertices one should make the replacement $\gamma_\mu(P_L - P_R) \to \gamma_\mu P_L$;

(ii) in Eq. (43) the trace should be multiplied by a factor of 2;

(iii) in Eq. (47) the second term (with $\gamma_\mu P_R$ in the vertices) should be dropped.

As a consequence of the different structure of the vertices, the projections over various spinor and Lorentz components of $\hat{\Pi}^{\pm}$ (see (19)) simplify greatly. Using Eq.(18) one sees that in the Dirac case Π_L and P_T vanish. Moreover, in the expressions of Π_R only the terms proportional to $\bar{G}_V^L$ survive. The above simplifications simply reflect the sterile nature of R-handed neutrinos and L-handed antineutrinos.

Results for the neutrino ($\Pi_R^{\kappa\pm}$) and antineutrino ($\bar{\Pi}_R^{\kappa\pm}$) gain and loss potentials are obtained integrating over positive and negative frequencies according to Eq. (34). The positive frequency integral is fairly similar to the Majorana one. The negative frequency integral can be cast in a simpler form by performing the change of variables $k^0 \to -k^0$, leading to

$$\Pi_R^{\kappa\pm}(\vec{k}) = \int_0^\infty dk^0 \; \Pi_R^{\kappa\pm}(k^0, \vec{k})\, \delta(k^0 - |\vec{k}|) \,, \tag{74a}$$

$$\bar{\Pi}_R^{\kappa\pm}(\vec{k}) = -\int_0^\infty dk^0 \; \Pi_R^{\kappa\mp}(-k^0, -\vec{k})\, \delta(k^0 - |\vec{k}|) \,. \tag{74b}$$

Finally, performing the matrix multiplications to obtain the various components of $\mathcal{C}$ and $\bar{\mathcal{C}}$ we obtain the following form in terms of $n_f \times n_f$ blocks:

$$\mathcal{C} = \begin{pmatrix} C_{LL} & C_{LR} \\ C_{LR}^\dagger & C_{RR} \end{pmatrix} , \qquad \bar{\mathcal{C}} = \begin{pmatrix} \bar{C}_{RR} & \bar{C}_{RL} \\ \bar{C}_{RL}^\dagger & \bar{C}_{LL} \end{pmatrix} , \tag{75}$$

with

$$C_{LL} = \frac{1}{2}\{\Pi_R^{\kappa+}, f_{LL}\} - \frac{1}{2}\{\Pi_R^{\kappa-}, 1 - f_{LL}\} \,, \tag{76a}$$

$$C_{RR} = 0 \,, \tag{76b}$$

$$C_{LR} = \frac{1}{2}\left(\Pi_R^{\kappa+} + \Pi_R^{\kappa-}\right) f_{LR} \,, \tag{76c}$$

and

$$\bar{C}_{RR} = \frac{1}{2}\{\bar{\Pi}_R^{\kappa+}, \bar{f}_{RR}\} - \frac{1}{2}\{\bar{\Pi}_R^{\kappa-}, 1 - \bar{f}_{RR}\} \,, \tag{77a}$$

$$\bar{C}_{LL} = 0 \,, \tag{77b}$$

$$\bar{C}_{RL} = \frac{1}{2}\left(\bar{\Pi}_R^{\kappa+} + \bar{\Pi}_R^{\kappa-}\right) \bar{f}_{LR} \,. \tag{77c}$$

The collision terms C_{RR} and $\bar{C}_{LL}$ vanish because R-handed neutrinos and L-handed antineutrinos do not interact in the massless limit that we adopt here (mass effects in the collision term are higher order in the ϵ counting). The gain and loss potentials $\Pi_R^{\kappa\pm}$ and $\bar{\Pi}_R^{\kappa\pm}$ can be expressed in terms of neutrino density matrices and distribution functions of the medium particles (electrons, etc.), as in the Majorana case. In fact, the expressions for the Dirac case can be obtained from the ones in the Majorana case with the following mapping, which we have checked with explicit calculations:

- The Dirac neutrino potentials $\Pi_R^{\kappa\pm}$ are obtained from the Majorana ones by replacing $f \to f_{LL}$ and $\bar{f} \to \bar{f}_{RR}$ everywhere.

- The Dirac antineutrino potentials $\bar{\Pi}_R^{\kappa\pm}$ are in one-to-one correspondence to the Majorana potentials $(\Pi_L^{\kappa\pm})^T$. Their expressions are simply obtained from the Dirac neutrino potentials $\Pi_R^{\kappa\pm}$ with the following simple changes: (i) in the νN, νe, and $\nu\nu$ processes replace $f_{LL} \to \bar{f}_{RR}$ and $\bar{f}_{RR} \to f_{LL}$ everywhere, and flip the signs of the axial couplings ($C_A \to -C_A$ in νN terms and $\mathcal{M}_R \to \mathcal{M}_L$ in νe terms). (ii) In the CC processes, make the replacements $f_e \to \bar{f}_e$, $f_n \leftrightarrow f_p$, and flip the sign of the axial coupling ($g_A \to -g_A$).

V. ONE-FLAVOR LIMIT AND INTERPRETATION OF OFF-DIAGONAL ENTRIES

We now specialize to the one-flavor limit and illustrate the structure of the collision term for the two spin degrees of freedom corresponding to Majorana neutrino and antineutrino. We will discuss explicitly only the simplest process, namely neutrino-nucleon scattering. We provide a simple form for the various components of the gain and loss potentials $\Pi^{\pm}$ in terms of scattering amplitudes of the two spin states (neutrino and antineutrino) off nucleons. We also provide a heuristic interpretation of the results for $\Pi^{\pm}$ in terms of changes in occupation numbers and quantum coherence due to scattering processes in the medium. Finally, in the limiting case of nearly-forward scattering we are able to recover earlier results by Stodolsky and collaborators [2, 52, 53]. Note that while we work with spin degrees of freedom, the discussion applies to the case of any internal degree of freedom.

A. Scattering amplitudes

In the collision terms calculated in the previous sections we have set the neutrino mass to zero, as terms proportional to the neutrino mass in the collision term would be $O(\epsilon^3)$ in our counting. Therefore, when computing neutrino-nucleon scattering from the interaction

Lagrangian $\mathcal{L}_{\nu N}$ of (2), we need to use the massless Majorana neutrino fields, ν_L, which can be expressed as follows (with $\widetilde{dk}$ defined in (58))

$$\nu_L(x) = P_L\,\nu(x) = \int \widetilde{dk}\,\Big(u(k,-)a(k,-)e^{-ikx} + v(k,+)a^\dagger(k,+)e^{ikx}\Big)\,, \qquad (78)$$

in terms of spinors $v(k,\pm) = u(k,\mp)$ and creation / annihilation operators $a^\dagger(k,\mp)$ and $a(k,\mp)$. The $\mp$ label refers to helicity: negative (L-handed) helicity corresponds to the neutrino (ν_-), while positive helicity (R-handed) to the antineutrino (ν_+). The spinors satisfy the following relations,

$$\begin{aligned} u(k,\pm)\bar{u}(k,\pm) &= \not{k}P_{L/R}\,,\\ u(k,\pm)\bar{u}(k,\mp) &= \pm|\vec{k}|\frac{i}{4}e^{\pm i\varphi}(\hat{\kappa}\wedge\hat{x}^\pm)_{\mu\nu}\sigma^{\mu\nu}\,P_{R/L}\,, \end{aligned} \qquad (79)$$

in terms of the basis vectors $\hat{\kappa}(k)$ and $\hat{x}^\pm(k)$ (see Appendix A).

The gain and loss terms can be expressed in terms of the following neutrino and antineutrino scattering amplitudes (and their conjugates):

$$A_\mp(k) \equiv A\Big(\nu_\mp(k)N(p)\to\nu_\mp(k')N(p')\Big)\,, \qquad (80a)$$

$$\bar{A}_\mp(k) \equiv A\Big(\nu_\mp(k')N(p')\to\nu_\mp(k)N(p)\Big) = A_\mp(k)^*\,. \qquad (80b)$$

The amplitudes $A_\mp(k)$ depend also on k', p, p', but to avoid notational clutter we do not write this down explicitly. From the interaction Lagrangian $\mathcal{L}_{\nu N}$ of (2), recalling $\Gamma^\mu_N = \gamma^\mu(C_V^{(N)} - C_A^{(N)}\gamma_5)$, one finds:

$$A_-(k) = -\sqrt{2}G_F\,\bar{u}(k',-)\gamma^\mu u(k,-)\;\bar{u}_N(p')\Gamma^\mu_N u_N(p)\,, \qquad (81a)$$

$$A_+(k) = \sqrt{2}G_F\,\bar{u}(k,-)\gamma^\mu u(k',-)\;\bar{u}_N(p')\Gamma^\mu_N u_N(p)\,. \qquad (81b)$$

Note that the scattering processes do not flip the neutrino spin (this effect enters to $O(m_\nu/E_\nu)$). In other words, we consider the case in which collisions do not change the internal quantum number. In the Standard Model this applies to both spin (neglecting neutrino mass) and flavor.

Taking the average over the initial and sum over final nucleon polarizations, i.e.

$$\langle A^*_\alpha A_\beta\rangle = \frac{1}{2}\sum_{N\ \mathrm{pol}} A^*_\alpha A_\beta \qquad \forall\ \alpha,\beta\,, \qquad (82)$$

using $\bar{u}(p,-)\gamma^\mu u(q,-) = \bar{u}(q,+)\gamma^\mu u(p,+)$, the relations (79), and the trace identities for gamma matrices, one can show that:

$$\langle|A_\mp(k)|^2\rangle \propto M_{R/L}(p',p,k',k), \qquad (83a)$$

$$\langle A^*_-(k)A_+(k)\rangle \propto M_T(p',p,k',k), \qquad (83b)$$

where $M_{R,L,T}$ are given in (61) (note the proportionality holds modulo the 4-momentum conservation δ-function in (61)). These results imply that $\Pi^\kappa_{R,L}$ and P_T can be expressed in terms of $\langle A^*_-A_-\rangle$, $\langle A^*_+A_+\rangle$, and $\langle A^*_-A_+\rangle$, respectively.

B. Gain and loss potentials in terms of ν and $\bar{\nu}$ scattering amplitudes

Keeping track of all factors, we find that the gain and loss potentials $\hat{\Pi}^\pm$ (as per (19)) in the one flavor limit can be written in terms of $\langle A^*_\alpha A_\beta\rangle$ as follows. The gain term is given by:

$$\hat{\Pi}^-(\vec{k}) = -\frac{1}{|\vec{k}|}\int \widetilde{dk'}\widetilde{dp}\widetilde{dp'}\;(2\pi)^4\delta^{(4)}(k+p-k'-p')\Big(1-f_N(p)\Big)f_N(p') \times\begin{pmatrix} \langle|\bar{A}_-(k)|^2\rangle\;f(\vec{k}') & \langle\bar{A}_-(k)\bar{A}^*_+(k)\rangle\phi(\vec{k}') \\ \langle\bar{A}^*_-(k)\bar{A}_+(k)\rangle\phi^*(\vec{k}') & \langle|\bar{A}_+(k)|^2\rangle\;\bar{f}(\vec{k}') \end{pmatrix}\,, \qquad (84)$$

while the loss term reads:

$$\hat{\Pi}^+(\vec{k}) = -\frac{1}{|\vec{k}|}\int \widetilde{dk'}\widetilde{dp}\widetilde{dp'}\;(2\pi)^4\delta^{(4)}(k+p-k'-p')\,f_N(p)\Big(1-f_N(p')\Big) \times\begin{pmatrix} \langle|A_-(k)|^2\rangle\;\Big(1-f(\vec{k}')\Big) & \langle A^*_-(k)A_+(k)\rangle\;\Big(-\phi(\vec{k}')\Big) \\ \langle A_-(k)A^*_+(k)\rangle\;\Big(-\phi^*(\vec{k}')\Big) & \langle|A_+(k)|^2\rangle\;\Big(1-\bar{f}(\vec{k}')\Big) \end{pmatrix}\,. \qquad (85)$$

Using (19) one can easily identify $\Pi^{\kappa\,\pm}_{R/L}(k)$ and $P^{\pm}_{T}(k)$ as the diagonal and off-diagonal entries in the above equations. Moreover, one can check that the positive-frequency integrals of (49) in the one-flavor case reduce to the matrix entries in (84) and (85).

The diagonal entries in the above expressions correspond to the familiar gain and loss terms for neutrino and antineutrinos ($\nu_{\mp}(k)$), that one could have guessed without the field-theoretic derivation: they are proportional to the square moduli of the scattering amplitudes of each state ($|A_{\mp}(k)|^2$). The off-diagonal entries, however, are proportional to the products $A^*_{-}(k)A_{+}(k)\,\phi(\vec{k}')$ and thus are related to interference effects that arise when initial and final states in a scattering process are given by coherent linear combinations of $\nu_+(k)$, $\nu_-(k)$ and $\nu_+(k')$, $\nu_-(k')$ ($\phi(\vec{k}')\neq 0$).

While so far we have phrased our discussion in terms of neutrinos and antineutrinos of the same flavor, the results generalize to a system with any internal degree of freedom, such as flavor, denoted by labels a, b. Assuming that scattering processes do not change the internal degree of freedom, i.e. $A(\nu_a N \to \nu_b N) \propto \delta_{ab} A_a$, one gets the general structures:

$$\Pi^{-}_{ab}(k) = -\frac{1}{|\vec{k}|}\int \widetilde{dk'}\widetilde{dp}\widetilde{dp'}\,(2\pi)^4\delta^{(4)}(k+p-k'-p') \times \Big(1-f_N(p)\Big) f_N(p')\,\bar{A}_a(k) f_{ab}(k') A_b(k)\,, \tag{86a}$$

$$\Pi^{+}_{ab}(k) = -\frac{1}{|\vec{k}|}\int \widetilde{dk'}\widetilde{dp}\widetilde{dp'}\,(2\pi)^4\delta^{(4)}(k+p-k'-p') \times f_N(p)\Big(1-f_N(p')\Big)\,\bar{A}_a(k) \times (\mathbb{1}-f(k'))_{ab}A_b(k)\;, \tag{86b}$$

in agreement with earlier work on collisional terms for particles with internal degrees of freedom, such as color, flavor, and/or spin [35, 54, 55].

The above results for $\hat{\Pi}^{\pm}(k)$ are derived in the field theoretic context with a well defined set of truncations, dictated by our power-counting in ϵ's. In addition, heuristic arguments can help explaining the structure of the gain and loss potentials. Let us discuss $\Pi^{-}_{ab}(k)$, i.e. the "gain term". As we already mentioned, the off-diagonal terms must be related to interference effects in the scattering, arising when the thermal bath contains states that are coherent superpositions of $|\vec{k},a\rangle$ and $|\vec{k},b\rangle$, i.e. states with same momentum but different internal quantum number. So let us consider the evolution of an initial state $|i\rangle = c_a(k')|\vec{k}',a\rangle + c_b(k')|\vec{k}',b\rangle$, where $\vec{k}'$ represents a generic momentum other than the momentum $\vec{k}$ of our "test" neutrino. Modulo normalizations, the density matrix associated with this (pure) state reads $f_{ab}(k') \propto c_a(k')c^*_b(k')$. Under S-matrix evolution the state $|i\rangle$ evolves into $|f\rangle = S|i\rangle \propto c_a(k')\bar{A}_a(k)|\vec{k},a\rangle + c_b(k')\bar{A}_b(k)|\vec{k},b\rangle + \dots$, where we used $\langle\vec{k},a|S|\vec{k}',a\rangle \propto \bar{A}_a(k)$ (see (80)) and the dots represent states with $\vec{p}\neq\vec{k}$ onto which the final state $S|i\rangle$ can project. So as a net result of evolving the state $|i\rangle$, a linear superposition of internal states with momentum $\vec{k}$ is generated. The change in the density matrix for momentum $\vec{k}$ reads $\Delta f_{ab}(k) \propto f_{ab}(k')\bar{A}_a(k)\bar{A}^*_b(k)$, which has the same structure of (86a). So we see that $\Pi^{-}_{ab}(k) \propto \Delta f_{ab}(k)$, i.e. the gain potential is related to the change in occupation number ($a=b$) or coherence ($a\neq b$) in the momentum state $\vec{k}$ resulting from scattering from all bins $\vec{k}'$ into the bin $\vec{k}$.

C. Coherence damping

The coherence damping rate has been estimated in Refs. [2, 52, 53] in the special case of nearly forward scattering, namely $k'\sim k$, and our expression can reproduce their result. In fact, for $k'\sim k$ one has $f_N(p)(1-f_N(p'))\sim f_N(p')(1-f_N(p))$. Using this result in (84) and (85) one obtains $P^{-}_{T}(k) = -P^{+}_{T}(k)$ (the latter relation holds under the weaker condition $|\vec{k}|\sim|\vec{k}'|$). Inspection of the collision terms C, $\bar{C}$, and C_ϕ of Eqs. (41) and (42) shows that in this limit $C(k)=\bar{C}(k)\simeq 0$. On the other hand, using $\phi(k')\sim\phi(k)$, one finds $C_\phi(k)$ in (42) to be proportional to $|A_-(k)|^2+|A_+(k)|^2-2A^*_-(k)A_+(k)$. Recalling that for $k'\sim k$ then $A^*_-A_+$ becomes real, one arrives at the result

$$C_\phi(\vec{k}) = -\Gamma_\phi(\vec{k})\,\phi(\vec{k})\,,$$
$$\Gamma_\phi(\vec{k}) = -\frac{1}{|\vec{k}|}\int \widetilde{dk'}\widetilde{dp}\widetilde{dp'}\,(2\pi)^4\delta^{(4)}(k+p-k'-p') \times f_N(p)\Big(1-f_N(p')\Big)\,\frac{1}{2}\langle|A_-(k)-A_+(k)|^2\rangle\,, \tag{87}$$

which agrees qualitatively with [2, 52, 53, 56]: the damping rate for the coherence $\phi(\vec{k})$ is proportional to a statistical average of the square of the difference of the scattering amplitudes of the two states. In the case of neutrinos and antineutrinos, since weak interactions are spin-dependent, $A_- - A_+ \neq 0$ and we expect damping of spin coherence with a typical weak-interaction time scale. On the other hand, neutrino-nucleon scattering is flavor blind and therefore does not contribute to damping of flavor coherence in the case of nearly forward scattering.

"Flavor blind" scattering (i.e. $A_+ = A_-$) can still cause coherence damping, as long as the collisions involve energy transfer. Assuming for simplicity thermal equilibrium for the "scatterers" (the nucleons in our example, so that $f_N(p) = 1/(e^{E_p/T}+1)$), we find

$$C_\phi(\vec{k}) = -\frac{1}{|\vec{k}|}\int \widetilde{dk'}\widetilde{dp}\widetilde{dp'}\,(2\pi)^4\delta^{(4)}(k+p-k'-p')\;|A_-(k)|^2\;f_N(p)\Big(1-f_N(p')\Big)$$
$$\times\Bigg\{\phi(\vec{k}) - e^{(E_p-E_{p'})/T}\phi(\vec{k}') + \frac{1}{2}\left(e^{(E_p-E_{p'})/T}-1\right)\Big[\phi(\vec{k})\left(f(\vec{k}')+\bar{f}(\vec{k}')\right)+\phi(\vec{k}')\left(f(\vec{k})+\bar{f}(\vec{k})\right)\Big]\Bigg\}. \tag{88}$$

The vanishing of $\int d^3k C_\phi(\vec{k}) = 0$ (in agreement with Ref. [2]) signals that coherence at the level of the "integrated" density matrix is not damped for flavor blind interactions. On the other hand, the fact that the individual $C_\phi(\vec{k}) \neq 0$ signals that flavor-blind collisions "shuffle" or transfer coherence between momentum modes.

For the "flavor diagonal" collision term $C(\vec{k})$, in the same limit we obtain

$$\begin{aligned}C(\vec{k}) = &-\frac{1}{|\vec{k}|}\int \widetilde{dk'}\widetilde{dp}\widetilde{dp'}(2\pi)^4\delta^{(4)}(k+p-k'-p')\\ &\times |A_-(k)|^2\; f_N(p)\,(1-f_N(p'))\\ &\times\Big\{f(\vec{k})(1-f(\vec{k}')) - e^{(E_p-E_{p'})/T}f(\vec{k}')(1-f(\vec{k}))\\ &\quad+\left(e^{(E_p-E_{p'})/T}-1\right)\mathrm{Re}\left(\phi(\vec{k})\phi^*(\vec{k}')\right)\Big\}\,.\end{aligned}$$

VI. ISOTROPIC LIMIT AND THE EARLY UNIVERSE

In this section we revert to the full three-flavor analysis and consider the limiting case of our expressions corresponding to isotropic space, that allows us to further evaluate analytically the expressions derived earlier in this work. The isotropic limit is of considerable physical interest, as it applies to the description of the early universe (see earlier works [21–25]). In this setup we may assume that all f's depend only on the absolute values of the momenta (not the angles) and additionally all ϕ (for Majorana neutrinos) and f_{LR} (for Dirac neutrinos) vanish, thus greatly simplifying our collision terms. In particular, all collision terms relating to spin-coherence vanish, i.e. $C_\phi = 0$ in the Majorana case and $C_{LR} = \bar{C}_{RL} = 0$ in the Dirac case.

In the Majorana QKEs, the non-vanishing $n_f \times n_f$ blocks of $\mathcal{C}_M$ in Eq. (41) are given by:

$$C = \frac{1}{2}\left(\{\Pi_R^{+,\kappa}, f\} - \{\Pi_R^{-,\kappa},(1-f)\}\right)\,, \tag{89a}$$
$$\bar{C}^T = \frac{1}{2}\left(\{(\Pi_L^\kappa)^+, \bar{f}^T\} - \{(\Pi_L^\kappa)^-,(1-\bar{f}^T)\}\right)\,. \tag{89b}$$

In the Dirac QKEs, the non-vanishing $n_f \times n_f$ blocks of $\mathcal{C}$ and $\bar{\mathcal{C}}$ in Eq. (75) are obtained from the Majorana results as follows:

$$\begin{aligned}C_{LL} &= C\Big|_{f\to f_{LL},\bar{f}\to\bar{f}_{RR}}\,,\\ \bar{C}_{RR} &= \bar{C}\Big|_{f\to f_{LL},\bar{f}\to\bar{f}_{RR}}\,.\end{aligned} \tag{90}$$

The key trick [22, 57] leading to closed expressions for the collision integrals is to write the momentum conserving δ-function in terms of its Fourier representation

$$\begin{aligned}\delta^{(3)}(\vec{k}-\vec{q}_3-\vec{q}_1+\vec{q}_2) &= \int\frac{d^3\lambda}{(2\pi)^3}e^{i\vec{\lambda}(\vec{k}-\vec{q}_3-\vec{q}_1+\vec{q}_2)}\\ &= \int\frac{r_\lambda^2 dr_\lambda d(\cos\theta_\lambda)d\varphi_\lambda}{(2\pi)^3}e^{i\vec{\lambda}(\vec{k}-\vec{q}_3-\vec{q}_1+\vec{q}_2)}\,.\end{aligned} \tag{91}$$

With this result, one can integrate out all angles ultimately arriving at an expression with only two integrals left [22, 57]. The types of integrals appearing for the $x = \cos\theta_i$ are

$$\begin{aligned}\int_{-1}^{1} dx e^{iAx} &= \frac{2}{A}\sin A\,,\\ \int_{-1}^{1} dx x e^{iAx} &= \frac{-2i}{A}\left(\cos A - \frac{\sin A}{A}\right)\,.\end{aligned} \tag{92}$$

The integrals over the φ_i are trivial when aligning the $\hat{z}$-axis with $\vec{\lambda}$.

Below, we report our results for the collision term in isotropic environments. Each contribution to the collision term has a factorized structure, in terms of weak matrix elements and distribution functions of the "scatterers", multiplied by a matrix structure involving the neutrino and antineutrino density matrices $f, f_{1,2,3}$ and $\bar{f}, \bar{f}_{1,2,3}$.

Neutrino-nucleon scattering processes

Neglecting contributions from antinucleons (irrelevant in the early universe in the interesting decoupling region, $T < 20$ MeV) the scattering processes $\nu(k)N(q_2) \leftrightarrow \nu(q_3)N(q_1)$ lead to

$$C = -2\frac{G_F^2}{E_k^2}\int \frac{dE_1 dE_2 dE_3}{(2\pi)^3}\delta(E_k - E_3 - E_1 + E_2)$$
$$\times\Big((1-f_{(N),1})f_{(N),2}\left\{1-f_3, f\right\} - f_{(N),1}(1-f_{(N),2})\left\{f_3, 1-f\right\}\Big)$$
$$\times\Big((C_V + C_A)^2\Big(E_1E_2E_3E_k D_1(q_1,q_2,q_3,k) + E_2E_k D_2(q_2,k;q_1,q_3)$$
$$+ E_1E_3 D_2(q_1,q_3;q_2,k) + D_3(q_1,q_2,q_3,k)\Big)$$
$$+ (C_V - C_A)^2\Big(E_1E_2E_3E_k D_1(q_1,q_2,q_3,k) - E_1E_k D_2(q_1,k;q_2,q_3)$$
$$- E_2E_3 D_2(q_2,q_3;q_1,k) + D_3(q_1,q_2,q_3,k)\Big)$$
$$- M_N^2(C_V^2 - C_A^2)\Big(E_3E_k D_1(q_1,q_2,q_3,k) - D_2(q_1,q_2;q_3,k)\Big)\Big), \tag{93}$$

where $q_{1,2} = \sqrt{E_{1,2}^2 - M_N^2}$, $q_3, k = \sqrt{E_{3,k}^2}$, and $D_{1,2,3}$ are expressions previously discussed by Dolgov, Hansen and Semikoz in [22] (see also [57]), and we list them explicitly in Appendix D. In the above expressions one recognizes the usual loss and gain terms. In the one-flavor limit the anti-commutators become trivial and we recover the standard Boltzmann collision term for neutrino-nucleon scattering. The antineutrino collision term $\bar{C}^T$ can be obtained from C in (93) with the replacements $f_i \leftrightarrow \bar{f}_i^T$ and $C_A \to -C_A$.

Neutrino-electron processes

The neutrino collision term induced by ν-$e^\pm$ processes is given by

$$C = -\frac{G_F^2}{E_k^2}\int \frac{dE_1 dE_2 dE_3}{\pi^3}\delta(E_k - E_3 - E_1 + E_2)\ (1-f_{(e),1})f_{(e),2}\ \times$$
$$\times\Big(\Big(E_1E_2E_3E_k\, D_1(q_1,q_2,q_3,k) + E_2E_k D_2(q_2,k;q_1,q_3) + E_1E_3 D_2(q_1,q_3;q_2,k)$$
$$+ D_3(q_1,q_2,q_3,k)\Big)\left\{Y_L(1-f_3)Y_L, f\right\}$$
$$+ \Big(E_1E_2E_3E_k D_1(q_1,q_2,q_3,k) - E_1E_k D_2(q_1,k;q_2,q_3) - E_2E_3 D_2(q_2,q_3;q_1,k)$$
$$+ D_3(q_1,q_2,q_3,k)\Big)\left\{Y_R(1-f_3)Y_R, f\right\}$$
$$- \frac{m_e^2}{2}\Big(E_3E_k D_1(q_1,q_2,q_3,k) - D_2(q_1,q_2;q_3,k)\Big)\sum_{I=L,R}\left\{Y_I(1-f_3)Y_{J\neq I}, f\right\}\Big)$$
$$+ \left\{\ E_{2,3} \to -E_{2,3}, \qquad f_{2,3} \to (1-\bar{f}_{2,3})\ \right\} + \left\{\ E_{1,2} \to -E_{1,2}, \qquad f_{1,2} \to (1-\bar{f}_{1,2})\ \right\} + \text{gain}, \tag{94}$$

where $q_{1,2} = \sqrt{E_{1,2}^2 - m_e^2}$, $q_3, k = \sqrt{E_{3,k}^2}$, "gain" denotes the corresponding gain terms for which the overall sign is flipped and all $f \leftrightarrow (1-f)$ (including barred occurrences of f_i), and the polynomial functions $D_{1,2,3}$ are given in Appendix D. The explicit loss terms in (94) correspond to $\nu(k)e^-(q_2) \to \nu(q_3)e^-(q_1)$ scattering. The additional loss term expressions indicated implicitly in next-to-last and second-to-last lines in (94) represent $\nu(k)e^+(q_1) \to \nu(q_3)e^+(q_2)$ scattering and $\nu(k)\bar{\nu}(q_3) \to e^+(q_2)e^-(q_1)$ pair processes, respectively. Note that in these terms, the sign flips of energies affect the energy-conserving delta functions as well as the overall sign of some of the terms proportional to D_2. The antineutrino collision term $\bar{C}^T$ is obtained from (94) by the replacements $f_i \leftrightarrow \bar{f}_i^T$ and $Y_L \leftrightarrow Y_R$. If we neglect the off-diagonal densities ($f_{a\neq b} = 0$, $\bar{f}_{a\neq b} = 0$), the anti-commutators become trivial and we reproduce the results of Refs. [21, 22] for the diagonal entries C_{aa} of the collision term.

Charged-current processes

The processes $\nu(k)n(q_2) \leftrightarrow e^-(q_3)p(q_1)$ lead to the neutrino collision term

$$
\begin{aligned}
C = -2\frac{G_F^2}{E_k^2}\int \frac{dE_1 dE_2 dE_3}{(2\pi)^3}\delta(E_k - E_3 - E_1 + E_2)& \\
\times \Big((1-f_{(p),1})f_{(n),2}(1-f_{(e),3})\ \{I_e, f\} - f_{(p),1}(1-f_{(n),2})f_{(e),3}\ \{I_e, 1-f\}\Big)& \\
\times \Big((1+g_A)^2\Big(E_1E_2E_3E_kD_1(q_1,q_2,q_3,k) + E_2E_kD_2(q_2,k;q_1,q_3)& \\
+ E_1E_3D_2(q_1,q_3;q_2,k) + D_3(q_1,q_2,q_3,k)\Big)& \\
+ (1-g_A)^2\Big(E_1E_2E_3E_kD_1(q_1,q_2,q_3,k) - E_1E_kD_2(q_1,k;q_2,q_3)& \\
- E_2E_3D_2(q_2,q_3;q_1,k) + D_3(q_1,q_2,q_3,k)\Big)& \\
+ M_pM_n(g_A^2-1)\Big(E_3E_kD_1(q_1,q_2,q_3,k) - D_2(q_1,q_2;q_3,k)\Big)\Big),& \qquad (95)
\end{aligned}
$$

where $q_1 = \sqrt{E_1^2 - M_p^2}$, $q_2 = \sqrt{E_2^2 - M_n^2}$, $q_3 = \sqrt{E_3^2 - M_e^2}$, $k = \sqrt{E_k^2}$, $D_{1,2,3}$ are given in Appendix D, and the flavor projector I_e is defined in Eq. (48).

The antineutrino collision term $\bar{C}^T$ induced by the processes $\bar{\nu}(k)p(q_2) \leftrightarrow e^+(q_3)n(q_1)$ can be obtained from C in (95) with the replacements $f \leftrightarrow \bar{f}^T$, $f_e \to \bar{f}_e$, $f_n \leftrightarrow f_p$, and $g_A \to -g_A$. Moreover, $\bar{C}^T$ receives a contribution induced by neutron decay, which can be obtained from C in (95) with the replacements $f \leftrightarrow \bar{f}^T$, $f_e \to 1 - f_e$, $f_n \leftrightarrow f_p$, and $g_A \to -g_A$, and $E_3 \to -E_3$.

Neutrino-neutrino processes

Neutrino scattering off neutrinos and antineutrinos induces the collision term

$$
\begin{aligned}
C = -2\frac{G_F^2}{E_k^2}\int \frac{dE_1 dE_2 dE_3}{(2\pi)^3}\,\delta(E_k - E_3 - E_1 + E_2)& \\
\times\Bigg(\Big(E_1E_2E_3E_kD_1(q_1,q_2,q_3,k) + E_2E_kD_2(q_2,k;q_1,q_3) + E_1E_3D_2(q_1,q_3;q_2,k) + D_3(q_1,q_2,q_3,k)\Big)& \\
\times\Big\{\Big(\mathrm{tr}((1-f_1)f_2) + (1-f_1)f_2\Big)(1-f_3)\,,\ f\Big\}& \\
+\Big(E_1E_2E_3E_kD_1(q_1,q_2,q_3,k) - E_1E_kD_2(q_1,k;q_2,q_3) - E_2E_3D_2(q_2,q_3;q_1,k) + D_3(q_1,q_2,q_3,k)\Big)& \\
\times\Big\{\Big(\mathrm{tr}\big(\bar{f}_2(1-\bar{f}_1)\big) + \bar{f}_2(1-\bar{f}_1)\Big)(1-f_3) + \Big(\mathrm{tr}\big((1-f_3)(1-\bar{f}_1)\big) + (1-f_3)(1-\bar{f}_1)\Big)\bar{f}_2\,,\ f\Big\}\Bigg) + \mathrm{gain}\,,& \qquad (96)
\end{aligned}
$$

where all $q_i = \sqrt{E_i^2}$ and "gain" denotes the corresponding gain terms for which the overall sign is flipped and all $f \leftrightarrow (1-f)$ (including barred occurrences of

f_i). The second and third lines in (96) correspond to $\nu(k)\nu(q_2) \to \nu(q_3)\nu(q_1)$ scattering. The fourth and fifth lines in (96) represent $\nu(k)\bar{\nu}(q_2) \to \nu(q_3)\bar{\nu}(q_1)$. The antineutrino collision term $\bar{C}^T$ induced by the processes $\bar{\nu}\bar{\nu} \to \bar{\nu}\bar{\nu}$ and $\bar{\nu}\nu \to \bar{\nu}\nu$ can be obtained from C in (96) with the replacements $f \leftrightarrow \bar{f}^T$ (including all occurrences of f_i). If we neglect the off-diagonal densities ($f_{a\neq b} = 0, \bar{f}_{a\neq b} = 0$), the anti-commutators become trivial and we reproduce the results of Refs. [21, 22] for the diagonal entries C_{aa} of the collision term. [58]

VII. DISCUSSION AND CONCLUSIONS

In this work we have derived the collision terms entering the quantum kinetic equations that describe the evolution of Dirac or Majorana neutrinos in a thermal bath. We include electroweak processes involving neutrino scattering off other neutrinos, electrons, and nucleons, as well as $\nu\bar{\nu} \leftrightarrow e^+e^-$ pair processes.

Throughout our analysis we have kept track of both flavor and spin neutrino degrees of freedom. We have first provided general, rather formal, expressions for the collision terms, valid in principle for any geometry, including anisotropic environments such as supernovae and accretion disks in neutron-star mergers. Our results generalize earlier work by Sigl and Raffelt [1], in which spin coherence effects were neglected. When including spin degrees of freedom, the gain and loss potentials $\Pi^{\pm}$ become $2n_f \times 2n_f$ matrices, whose diagonal $n_f \times n_f$ blocks describe the collisions of each spin state, and whose off-diagonal $n_f \times n_f$ blocks describe interference effects in the scattering involving coherent superpositions of the two spin states. Since for Dirac neutrinos the "wrong helicity" states (R-handed neutrinos and L-handed antineutrinos) do not interact in the massless limit, in this case the gain and loss potentials greatly simplify, as only the upper $n_f \times n_f$ block survives (see Section IV). Our results are in qualitative agreement with the ones in Ref. [35], where collision terms involving flavor and helicity coherence have been studied in the context of kinetic equations for leptogenesis.

The main results of this paper are:

- Within the field-theoretic framework, we have derived general expressions for the neutrino collision terms, valid in anisotropic environments. The lengthy results for the gain/loss potentials are given in Sect. III, while the collision terms are presented in Appendix C. Compared to previous literature, new terms involving spin coherence appear. After using the constraints from the energy-momentum conservation, all the terms (including the "standard" ones that do not involve spin) can be expressed as five-dimensional integrals. These are intractable at the moment in codes describing astrophysical objects, but will be required for a detailed study of neutrino transport in the future, especially to assess the impact of the so-called "halo" on collective neutrino oscillations in supernovae [48].

- After presenting general results, we have focused on two limiting cases of great physical interest. First, in Sect. V we have taken the one-flavor limit and illustrated the structure of the collision term for the two spin degrees of freedom (neutrino and antineutrino in the Majorana case). Here we have provided simple expressions for the diagonal and off-diagonal entries of the gain and loss potential. As expected, the diagonal terms are proportional to the square moduli of the amplitudes describing neutrino and antineutrino scattering off the target particles in the medium ($|A_{\mp}|^2$). On the other hand, the off-diagonal terms are proportional to the product $A_-^* A_+$ of scattering amplitudes for neutrino and antineutrino. In this section we have also estimated the damping rate for spin coherence due to neutrinos (antineutrinos) scattering off nucleons, under the assumption that collisions involve small energy transfer compared to typical energies of the system. Finally, we have shown that coherence "transfer" among momentum modes is enforced by the collision term even in the case of "flavor-blind" interactions, as long as the collisions involve energy transfer.

- Next, in Sect. VI we have considered the isotropic limit relevant for the description of neutrinos in the early universe. In this case, following Ref. [22], we were able to analytically evaluate most of the collision terms, leaving just two-dimensional integrals for computational implementation. These latter expressions generalize earlier results found in Refs. [21, 22]. In fact, our results encode the same scattering kernels as in Ref. [22], but multiplied by the appropriate products of density matrices that realize the "non-Abelian" Pauli-blocking first described in Ref. [2]. Our flavor-diagonal collision terms reproduce the results of Ref. [22]. The resulting collision terms in the isotropic limit are amenable for computational implementation in studies of neutrino transport in the early universe, and its impact on primordial lepton number asymmetries and Big Bang nucleosynthesis.

In summary, this work completes the derivation of neutrino QKEs from field theory in general anisotropic environments, started in Ref. [9], by including the collision terms. While the computational implementation of these collision terms will be challenging, here we have provided the needed theoretical background. We have also gained insight on the structure of the collision term by discussing in some detail the one-flavor limit, relevant for neutrino-antineutrino conversion in compact objects, and the isotropic limit, relevant for the physics of the early universe.

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Appendix A: Kinematics of ultra-relativistic neutrinos

1. Basis vectors

For ultra-relativistic neutrinos of momentum $\vec{k}$, it is useful to express all Lorentz tensors in terms of a basis formed by two light-like four-vectors $\hat{\kappa}^\mu(k) = (\mathrm{sgn}(k^0), \hat{k})$ and $\hat{\kappa}'^\mu(k) = (\mathrm{sgn}(k^0), -\hat{k})$ ($\hat{\kappa}\cdot\hat{\kappa} = \hat{\kappa}'\cdot\hat{\kappa}' = 0$, $\hat{\kappa}\cdot\hat{\kappa}' = 2$) and two transverse four vectors $\hat{x}_{1,2}(k)$ such that $\hat{\kappa}\cdot\hat{x}_i = \hat{\kappa}'\cdot\hat{x}_i = 0$ and $\hat{x}_i\cdot\hat{x}_j = -\delta_{ij}$. It also useful to define $\hat{x}^\pm \equiv \hat{x}_1 \pm i\hat{x}_2$ so that $\hat{x}^+\cdot\hat{x}^- = -2$. Note, that $k^\mu \to -k^\mu$ means (with our choice of basis) that $\hat{\kappa} \to -\hat{\kappa}$ and $\hat{x}^\pm \to \hat{x}^\mp$, i.e. $(\hat{\kappa}\wedge\hat{x}^\pm) \to -(\hat{\kappa}\wedge\hat{x}^\mp)$, where $(\hat{\kappa}\wedge\hat{x}^\pm)_{\mu\nu} \equiv (\hat{\kappa}_\mu\hat{x}^\pm_\nu - \hat{\kappa}_\nu\hat{x}^\pm_\mu)$. This can be seen directly in terms of spherical coordinates:

$$
\vec{\hat{k}} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} , \quad \vec{\hat{x}}_1 = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} ,
$$
$$
\vec{\hat{x}}_2 = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} , \quad \vec{\hat{x}}^\pm = \begin{pmatrix} \cos\theta\cos\varphi \mp i\sin\varphi \\ \cos\theta\sin\varphi \pm i\cos\varphi \\ -\sin\theta \end{pmatrix} . \tag{A1}
$$

Under parity, the angles change as $\theta \to \pi - \theta$ and $\varphi \to \pi + \varphi$. Therefore, $\cos\theta \to -\cos\theta$, $\sin\theta \to +\sin\theta$ and $\cos\varphi \to -\cos\varphi$, $\sin\varphi \to -\sin\varphi$, leading to $\vec{\hat{k}} \to -\vec{\hat{k}}$, $\vec{\hat{x}}_1 \to +\vec{\hat{x}}_1$, $\vec{\hat{x}}_2 \to -\vec{\hat{x}}_2$ and $\vec{\hat{x}}^\pm \to \vec{\hat{x}}^\mp$.

Any Lorentz vector V^μ has light-like and space-like components defined as $V^\kappa \equiv \hat{\kappa}\cdot V$ and $V^i \equiv \hat{x}^i\cdot V$, respectively. For the components of the derivative operator we adopt the notation $\partial^\kappa \equiv \hat{\kappa}\cdot\partial$, $\partial^i \equiv \hat{x}^i\cdot\partial$.

2. Vector and tensor components of two-point functions and self-energies

The decomposition of the neutrino Green function $G^{(\nu)}(k)$ into independent spinor and Lorentz structures is discussed in Ref. [9]. It takes the form

$$
\begin{aligned}
G^{(\nu)} &= \begin{pmatrix} \frac{1}{2} i G_T^{L,\mu\nu} S^L_{\mu\nu} & G_V^L\cdot\sigma \\ G_V^R\cdot\bar{\sigma} & \frac{1}{2} i G_T^{R,\mu\nu} S^R_{\mu\nu} \end{pmatrix} \\
&= \left[\left(G_V^R\right)^\mu \gamma_\mu - \frac{i}{4}\left(G_T^L\right)^{\mu\nu}\sigma_{\mu\nu}\right] P_L \\
&\quad + \left[\left(G_V^L\right)^\mu \gamma_\mu + \frac{i}{4}\left(G_T^R\right)^{\mu\nu}\sigma_{\mu\nu}\right] P_R ,
\end{aligned} \tag{A2}
$$

where $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and σ^i are the usual Pauli matrices. Additionally,

$$
\begin{aligned}
S^L_{\mu\nu} &= -\frac{i}{4}\left(\sigma_\mu\bar{\sigma}_\nu - \sigma_\nu\bar{\sigma}_\mu\right) , \\
S^R_{\mu\nu} &= \frac{i}{4}\left(\bar{\sigma}_\mu\sigma_\nu - \bar{\sigma}_\nu\sigma_\mu\right) , \\
-\frac{1}{2}\sigma_{\mu\nu}P_L &= \begin{pmatrix} S^L_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} , \\
\frac{1}{2}\sigma_{\mu\nu}P_R &= \begin{pmatrix} 0 & 0 \\ 0 & S^R_{\mu\nu} \end{pmatrix} .
\end{aligned} \tag{A3}
$$

The first line in Eq. (A2) corresponds to the notation of Ref. [9], while the second line makes explicit use of the four-dimensional Dirac matrices. For additional relations between the four and two component representations, see e.g. [59].

Similarly, any self-energy diagram carries spinor indices. It can be written as follows,

$$
\begin{aligned}
\hat{\Pi} &= \begin{pmatrix} \Pi_S + \frac{1}{2} i \Pi_T^{L,\mu\nu} S^L_{\mu\nu} & \Pi_L\cdot\sigma \\ \Pi_R\cdot\bar{\sigma} & \Pi_S^\dagger + \frac{1}{2} i \Pi_T^{R,\mu\nu} S^R_{\mu\nu} \end{pmatrix} \\
&= \left[\Pi_S + \Pi_R^\mu\gamma_\mu - \frac{i}{4}\left(\Pi_T^L\right)^{\mu\nu}\sigma_{\mu\nu}\right] P_L \\
&\quad + \left[\Pi_S^\dagger + \Pi_L^\mu\gamma_\mu + \frac{i}{4}\left(\Pi_T^R\right)^{\mu\nu}\sigma_{\mu\nu}\right] P_R ,
\end{aligned} \tag{A4}
$$

where the first line corresponds to the notation of Ref. [9], while the second line makes explicit use of the four-dimensional Dirac matrices. The vector and tensor spinor components of Π can be isolated with the following projections:

$$
\begin{aligned}
\Pi^\alpha_{L,R} &= \frac{1}{2}\,\mathrm{Tr}\left[\hat{\Pi}\,\gamma^\alpha P_{L,R}\right] , \\
\left(\Pi_T^L\right)_{\mu\nu} &= \frac{i}{2}\,\mathrm{Tr}\left[\hat{\Pi}\,\sigma_{\mu\nu}P_L\right] , \\
\left(\Pi_T^R\right)_{\mu\nu} &= -\frac{i}{2}\,\mathrm{Tr}\left[\hat{\Pi}\,\sigma_{\mu\nu}P_R\right] .
\end{aligned} \tag{A5}
$$

Furthermore, the Lorentz vector and tensor objects $\Pi^\alpha_{L,R}$, $(\Pi_T^{L,R})_{\mu\nu}$ can be decomposed in terms of the basis vectors $\hat{\kappa}, \hat{\kappa}', \hat{x}_{1,2}$. The quantities $\Pi^\kappa_{L,R}$ and P_T that appear in the collision term correspond to specific components $\Pi^\alpha_{L,R}$ and $(\Pi_T^{L,R})_{\mu\nu}$,

$$\begin{aligned}\left(\Pi_T^R\right)_{\mu\nu} &= e^{-i\varphi}(\hat\kappa'\wedge\hat x^-)_{\mu\nu}\; P_T \;+\; \dots\,,\\ \left(\Pi_T^L\right)_{\mu\nu} &= e^{i\varphi}(\hat\kappa'\wedge\hat x^+)_{\mu\nu}\; P_T^\dagger \;+\; \dots\,,\\ \Pi^\mu_{L,R} &= \frac{1}{2}\Pi^\kappa_{L,R}\,\hat\kappa'^{\mu} \;+\; \dots\end{aligned} \tag{A6}$$

The components relevant for the collision term are obtained by the contractions

$$\begin{aligned}\Pi^\kappa_{L,R} &= \hat\kappa_\mu\;\Pi^\mu_{L,R}\,, \qquad\qquad \hat\kappa_\mu \simeq \frac{k_\mu}{|\vec k|}\,,\\ P_T &= -\frac{e^{i\varphi}}{8}\,(\hat\kappa\wedge\hat x^+)^{\mu\nu}\;\left(\Pi_T^R\right)_{\mu\nu}\,,\\ P_T^\dagger &= -\frac{e^{-i\varphi}}{8}\,(\hat\kappa\wedge\hat x^-)^{\mu\nu}\;\left(\Pi_T^L\right)_{\mu\nu}\,.\end{aligned} \tag{A7}$$

So in summary, to obtain the quantities relevant for the collision term we need the following projections (traces are only on spinor indices; $\Pi^\kappa_{L,R}$ and P_T are matrices in flavor space):

$$\Pi^\kappa_{L,R} = \frac{1}{2}\hat\kappa_\mu\,\mathrm{Tr}\left[\hat\Pi\,\gamma^\mu P_{L,R}\right]\,, \tag{A8a}$$

$$P_T = \frac{ie^{i\varphi}}{16}\,(\hat\kappa\wedge\hat x^+)^{\mu\nu}\;\mathrm{Tr}\left[\hat\Pi\,\sigma_{\mu\nu}P_R\right]\,, \tag{A8b}$$

$$P_T^\dagger = -\frac{ie^{-i\varphi}}{16}\,(\hat\kappa\wedge\hat x^-)^{\mu\nu}\;\mathrm{Tr}\left[\hat\Pi\,\sigma_{\mu\nu}P_L\right]\,. \tag{A8c}$$

3. Tensor components and duality properties

The projections and traces are greatly simplified by noting the following identities. Denote by $T^{\alpha\beta}_\pm$ any self-dual $(+)$ or anti self-dual $(-)$ antisymmetric tensor, with dual tensor $T^\star$ defined by

$$T^\star_{\mu\nu} = \frac{i}{2}\epsilon_{\mu\nu\alpha\beta}T^{\alpha\beta}\,. \tag{A9}$$

Then one has:

$$\begin{aligned}\gamma^\mu\sigma_{\alpha\beta}T_\pm^{\alpha\beta} &= 4i\,T_\pm^{\mu\mu'}\gamma_{\mu'}\,P_{R/L}\,,\\ \sigma_{\alpha\beta}T_\pm^{\alpha\beta}\gamma^\mu &= -4i\,T_\pm^{\mu\mu'}\gamma_{\mu'}\,P_{L/R}\,.\end{aligned} \tag{A10}$$

Using the above identities one can perform all the needed projections in a straightforward way (at most four gamma matrices and a γ_5 appear in the traces).

Furthermore, notice also that $S^{\mu\nu}_{L/R}$ (introduced in (A3)) are (anti)self-dual, i.e.

$$\begin{aligned}S_L^{\mu\nu} &= -(S_L^{\mu\nu})^\star = -\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}S_{L,\rho\sigma}\,,\\ S_R^{\mu\nu} &= (S_R^{\mu\nu})^\star = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}S_{R,\rho\sigma}\,,\end{aligned} \tag{A11}$$

which may be checked by explicit computation using the Lie algebra of the Pauli matrices. Similarly, the wedge products satisfy the following duality relations

$$\begin{aligned}(\hat\kappa\wedge\hat x^\pm)^{\mu\nu} &= \pm\left((\hat\kappa\wedge\hat x^\pm)^{\mu\nu}\right)^\star\\ &= \pm\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}(\hat\kappa\wedge\hat x^\pm)_{\rho\sigma}\,,\\ (\hat\kappa'\wedge\hat x^\pm)^{\mu\nu} &= \mp\left((\hat\kappa'\wedge\hat x^\pm)^{\mu\nu}\right)^\star\\ &= \mp\frac{i}{2}\epsilon^{\mu\nu\rho\sigma}(\hat\kappa'\wedge\hat x^\pm)_{\rho\sigma}\,,\end{aligned} \tag{A12}$$

and from these relations it directly follows that

$$\begin{aligned}\epsilon_{\alpha\beta\gamma\mu}(\hat\kappa\wedge\hat x^\pm)^{\mu\nu} = \pm i\Big(&\delta^\nu_\alpha\delta^\rho_\beta\delta^\sigma_\gamma + \delta^\nu_\gamma\delta^\rho_\alpha\delta^\sigma_\beta\\ &+\delta^\nu_\beta\delta^\rho_\gamma\delta^\sigma_\alpha\Big)(\hat\kappa\wedge\hat x^\pm)_{\rho\sigma}\,,\end{aligned} \tag{A13a}$$

$$(\hat\kappa_i\wedge\hat x_i^-)^{\mu\nu}(\hat\kappa_j\wedge\hat x_j^+)_{\mu\nu} = 0\,, \tag{A13b}$$

where we used the notation $\hat\kappa_i \equiv \hat\kappa(q_i^\mu)$, $\hat x_j^\pm \equiv \hat x^\pm(q_j^\mu)$, etc.

Appendix B: Green's functions to $O(\epsilon^0)$ in power counting

In this appendix we summarize the form of the two-point functions for neutrino and matter fields (e,n,p) to leading order in our power counting, i.e. to $O(\epsilon^0)$. We use these expression to evaluate the collision potentials $\Pi^\pm$ to $O(\epsilon^2)$.

The collision term involves the $^\pm$ components of the Green's functions, defined in terms of the statistical (F) and spectral (ρ) functions as:

$$G^\pm = -\frac{i}{2}\,\rho\;\pm F\,. \tag{B1}$$

Using the fact that to leading order the neutrino spectral function has only vector components,

$$\rho^{(\nu)}_{ab}(k,x) = 2i\pi\delta(k^2)\,\mathrm{sgn}(k^0)\,\not{k}\,\delta_{ab}\,, \tag{B2}$$

we can write (see (A2))

$$\begin{aligned}\left(G_V^{L,R}\right)^\pm_\mu(k) &= k_\mu\,\left(\bar G_V^{L,R}\right)^\pm(k)\,,\\ \left(G_T^{L,R}\right)^\pm_{\mu\nu} &= \pm\left(F_T^{L,R}\right)_{\mu\nu}\,,\\ \left(F_T^L\right)_{\mu\nu} &= e^{-i\varphi}(\hat\kappa\wedge\hat x^-)_{\mu\nu}\,\Phi\,,\\ \left(F_T^R\right)_{\mu\nu} &= e^{i\varphi}(\hat\kappa\wedge\hat x^+)_{\mu\nu}\,\Phi^\dagger\,.\end{aligned} \tag{B3}$$

The explicit form of $\left(\bar G_V^{L,R}\right)^\pm(k)$ and $\Phi(k)$ for Majorana neutrinos is [60]

$$\left(\bar{G}_V^L\right)^+(k) = 2\pi\delta(k^2)\Big[\theta(k^0)(1-f(\vec{k})) - \theta(-k^0)\bar{f}(-\vec{k})\Big]\,, \tag{B4a}$$

$$\left(\bar{G}_V^L\right)^-(k) = 2\pi\delta(k^2)\Big[\theta(k^0)f(\vec{k}) - \theta(-k^0)(1-\bar{f}(-\vec{k}))\Big]\,, \tag{B4b}$$

$$\left(\bar{G}_V^R\right)^+(k) = 2\pi\delta(k^2)\Big[\theta(k^0)(1-\bar{f}^T(\vec{k})) - \theta(-k^0)f^T(-\vec{k})\Big]\,, \tag{B4c}$$

$$\left(\bar{G}_V^R\right)^-(k) = 2\pi\delta(k^2)\Big[\theta(k^0)\bar{f}^T(\vec{k}) - \theta(-k^0)(1-f^T(-\vec{k}))\Big]\,, \tag{B4d}$$

$$\Phi(k) = -2\pi|\vec{k}|\delta(k^2)\left[\theta(k^0)\phi(\vec{k}) + \theta(-k^0)\phi^T(-\vec{k})\right]\,. \tag{B4e}$$

In the above equations the transposition operation acts on flavor indices.

In summary, the neutrino Green's functions to $O(\epsilon^0)$ can be written as:

$$\left(G^{(\nu)}\right)^{\pm} = \Big[\left(\bar{G}_V^R\right)^{\pm}k^\mu\gamma_\mu \mp \frac{i}{4}\Phi e^{-i\varphi}(\hat{\kappa}\wedge\hat{x}^-)^{\mu\nu}\sigma_{\mu\nu}\Big]P_L + \Big[\left(\bar{G}_V^L\right)^{\pm}k^\mu\gamma_\mu \pm \frac{i}{4}\Phi^\dagger e^{i\varphi}(\hat{\kappa}\wedge\hat{x}^+)^{\mu\nu}\sigma_{\mu\nu}\Big]P_R\,. \tag{B5}$$

Finally, for unpolarized target particles of mass m, and spin 1/2 (denoted generically by ψ, where $\psi = e, n, p$) the Wigner transformed two-point functions read [9, 16]:

$$\begin{aligned} G^{(\psi)+}(p) &= 2\pi\delta(p^2 - m_\psi^2)(\not{p} + m_\psi) \\ &\quad\times\Big[\theta(p^0)(1-f_\psi(\vec{p})) - \theta(-p^0)\bar{f}_\psi(-\vec{p})\Big]\,, \\ G^{(\psi)-}(p) &= 2\pi\delta(p^2 - m_\psi^2)(\not{p} + m_\psi) \\ &\quad\times\Big[\theta(p^0)f_\psi(\vec{p}) - \theta(-p^0)(1-\bar{f}_\psi(-\vec{p}))\Big]\,, \end{aligned} \tag{B6}$$

where $f_\psi(\vec{p})$ and $\bar{f}_\psi(\vec{p})$ are the "target" particle and antiparticle distributions. In order to isolate the spinor structure, we use the notation $G^{(\psi)\pm}(p) = (\not{p} + m_\psi)\,\bar{G}^{(\psi)\pm}(p)$.

Appendix C: Collision term for ν-N, ν-e, and charged current processes

In this appendix we present the results for the "assembled" collision terms C and C_ϕ induced by neutrino-nucleon, neutrino-electron, and charged-current processes, assembling the gain and loss potentials of Section III C according to Eq. (42). We refrain from displaying the expressions for the collision terms induced by neutrino-neutrino processes: these are quite lengthy but can be obtained straightforwardly in the same way as for the other processes.

Neutrino-nucleon scattering processes

Neutrino-nucleon scattering $\nu(k)N(q_2) \to \nu(q_3)N(q_1)$ induces the following contributions to C and C_ϕ in (42):

$$\begin{aligned} C = -\frac{2G_F^2}{|\vec{k}|}\int \widetilde{dq}_1\widetilde{dq}_2\widetilde{dq}_3\,(2\pi)^4 \\ \times\Bigg(\mathcal{M}_R(q_1,q_2,q_3,k)\Big((1-f_{(N),1})f_{(N),2}\,\{1-f_3, f\} - f_{(N),1}(1-f_{(N),2})\,\{f_3, 1-f\}\Big) \\ - 4(C_V^2 + C_A^2)\left(f_{(N),2} - f_{(N),1}\right)\,\Big(\mathcal{M}_T(q_1,q_2,q_3,k)\,\phi_3\phi^\dagger + \mathcal{M}_T^*(q_1,q_2,q_3,k)\,\phi\phi_3^\dagger\Big)\Bigg), \end{aligned} \tag{C1}$$

$$\begin{aligned} C_\phi = -\frac{2G_F^2}{|\vec{k}|}\int \widetilde{dq}_1\widetilde{dq}_2\widetilde{dq}_3\,(2\pi)^4 \\ \times\Bigg(\mathcal{M}_R(q_1,q_2,q_3,k\,)\Big((1-f_{(N),1})f_{(N),2} + (f_{(N),1} - f_{(N),2})f_3\Big)\,\phi \\ + \mathcal{M}_L(q_1,q_2,q_3,k)\,\phi\,\Big((1-f_{(N),1})f_{(N),2} + (f_{(N),1} - f_{(N),2})\bar{f}_3^T\Big) \\ - 4(C_V^2 + C_A^2)\mathcal{M}_T(q_1,q_2,q_3,k)\,\Big((f_{(N),2} - f_{(N),1})(f\,\phi_3 + \phi_3\,\bar{f}^T) + 2f_{(N),1}(1-f_{(N),2})\phi_3\Big)\Bigg), \end{aligned} \tag{C2}$$

where $\mathcal{M}_{R,L,T}(q_1,q_2,q_3,k)$ given in (61) and $\mathcal{M}_T^*$ denotes the complex conjugate of $\mathcal{M}_T$, whose only differences are $\hat{x}^\pm \leftrightarrow \hat{x}^\mp$ and the sign of the phase.

The contribution to $\bar{C}^T$ in (42) can be obtained from C in Eq. (C1) with the following substitutions: $f_i \leftrightarrow \bar{f}_i^T$, $\phi_j \leftrightarrow \phi_j^\dagger$, $\mathcal{M}_R \leftrightarrow \mathcal{M}_L$, and $\mathcal{M}_T \leftrightarrow \mathcal{M}_T^*$.

Neutrino-electron processes

In terms of the matrix elements $\mathcal{M}_I^L(q_1, q_2, q_3, k)$, $\mathcal{M}_I^R(q_1, q_2, q_3, k)$, $\mathcal{M}_m(q_1, q_2, q_3, k)$ defined in (67) and $\mathcal{M}_T(q_1, q_2, q_3, k)$ defined in (61), the collision terms C and C_ϕ read

$$
\begin{aligned}
C = &-\frac{16 G_F^2}{|\vec{k}|} \int \widetilde{dq_1} \widetilde{dq_2} \widetilde{dq_3} (2\pi)^4 \\
&\times \sum_{I=L,R} \Bigg((1-f_{(e),1}) f_{(e),2} \Big\{ Y_I (1-f_3) \big(2 Y_I \mathcal{M}_I^R(q_1, q_2, q_3, k) - Y_{J \neq I} \mathcal{M}_m(q_1, q_2, q_3, k) \big), f \Big\} \\
&\quad - f_{(e),1} (1-f_{(e),2}) \Big\{ Y_I f_3 \big(2 Y_I \mathcal{M}_I^R(q_1, q_2, q_3, k) - Y_{J \neq I} \mathcal{M}_m(q_1, q_2, q_3, k) \big), 1-f \Big\} \\
&\quad - \Big((f_{(e),2} - f_{(e),1}) Y_I \phi_3 Y_I \mathcal{M}_T(q_1, q_2, q_3, k) \Big) \phi^\dagger - \phi \Big((f_{(e),2} - f_{(e),1}) Y_I \phi_3 Y_I \mathcal{M}_T(q_1, q_2, q_3, k) \Big)^\dagger \Bigg) \\
&+ \{ \ q_{2,3} \to -q_{2,3}, \qquad f_{2,3} \to (1 - \bar{f}_{2,3}), \qquad \phi_3 \to -\phi_3^T \ \} \\
&+ \{ \ q_{1,2} \to -q_{1,2}, \qquad f_{1,2} \to (1 - \bar{f}_{1,2}) \ \}
\end{aligned}
\tag{C3}
$$

and

$$
\begin{aligned}
C_\phi = &-\frac{16 G_F^2}{|\vec{k}|} \int \widetilde{dq_1} \widetilde{dq_2} \widetilde{dq_3} (2\pi)^4 \sum_{I=L,R} \Bigg(\big((1-f_{(e),1}) f_{(e),2} Y_I + (f_{(e),1} - f_{(e),2}) Y_I f_3 \big) \\
&\quad \times \big(2 Y_I \mathcal{M}_I^R(q_1, q_2, q_3, k) - Y_{J \neq I} \mathcal{M}_m(q_1, q_2, q_3, k) \big) \phi \\
&+ \phi \big((1-f_{(e),1}) f_{(e),2} Y_I + (f_{(e),1} - f_{(e),2}) Y_I \bar{f}_3^T \big) \big(2 Y_I \mathcal{M}_I^L(q_1, q_2, q_3, k) - Y_{J \neq I} \mathcal{M}_m(q_1, q_2, q_3, k) \big) \\
&- \Big(\big(f (f_{(e),2} - f_{(e),1}) + 2 f_{(e),1} (1-f_{(e),2}) \big) Y_I \phi_3 Y_I + (f_{(e),2} - f_{(e),1}) Y_I \phi_3 Y_I \bar{f}^T \Big) \mathcal{M}_T(q_1, q_2, q_3, k) \Bigg) \\
&+ \{ \ q_{2,3} \to -q_{2,3}, \qquad f_{2,3} \to (1 - \bar{f}_{2,3}), \qquad \phi_3 \to -\phi_3^T, \qquad \bar{f}_3^T \to (1 - f_3^T) \ \} \\
&+ \{ \ q_{1,2} \to -q_{1,2}, \qquad f_{1,2} \to (1 - \bar{f}_{1,2}) \ \} .
\end{aligned}
\tag{C4}
$$

Note that these expressions display explicitly the effect of the process $\nu(k) e^-(q_2) \to \nu(q_3) e^-(q_1)$, while the impact of neutrino scattering off positrons and pair processes is obtained by simple substitutions, as indicated above.

The antineutrino collision term $\bar{C}^T$ in (42) can be obtained from C in Eq. (C3) with the following substitutions: $f_i \leftrightarrow \bar{f}_i^T$, $\phi_j \leftrightarrow \phi_j^\dagger$, $Y_R \leftrightarrow Y_L$, and $\mathcal{M}_T \leftrightarrow \mathcal{M}_T^*$.

Charged-current processes

The contributions to the collision term from charged-current neutrino absorption and emission $\nu(k) n(q_2) \leftrightarrow e^-(q_3) p(q_1)$ are

$$
\begin{aligned}
C = &-\frac{2 G_F^2}{|\vec{k}|} \int \widetilde{dq_1} \widetilde{dq_2} \widetilde{dq_3} (2\pi)^4 \mathcal{M}_R^{CC}(q_1, q_2, q_3, k) \\
&\times \Big((1-f_{(p),1}) f_{(n),2} (1-f_{(e),3}) \{ I_e, f \} \ - \ f_{(p),1} (1-f_{(n),2}) f_{(e),3} \{ I_e, 1-f \} \Big)
\end{aligned}
\tag{C5}
$$

and

$$C_\phi = -\frac{2G_F^2}{|\vec{k}|}\int \widetilde{dq_1}\widetilde{dq_2}\widetilde{dq_3}(2\pi)^4\Big(\mathcal{M}_R^{CC}(q_1,q_2,q_3,k)\left((1-f_{(p),1})f_{(n),2}+(f_{(p),1}-f_{(n),2})f_{(e),3}\right)[I_e\,\phi]$$
$$+\mathcal{M}_L^{CC}(q_1,q_2,q_3,k)\,[\phi\, I_e]\,\left((1-f_{(n),1})f_{(p),2}+(f_{(n),1}-f_{(p),2})\bar{f}_{(e),3}\right)\Big) \tag{C6}$$

where the flavor projector I_e is defined in Eq. (48) and the matrix elements $\mathcal{M}_{R,L}^{CC}(q_1,q_2,q_3,k)$ are given in (64).

The antineutrino collision term $\bar{C}^T$ induced by the processes $\bar{\nu}(k)p(q_2) \leftrightarrow e^+(q_3)n(q_1)$ can be obtained from C in (C5) with the replacements $f \to \bar{f}^T$, $f_e \to \bar{f}_e$, $f_n \leftrightarrow f_p$, and $\mathcal{M}_R^{CC} \to \mathcal{M}_L^{CC}$ $(g_A \to -g_A)$.

Appendix D: The DHS integrals

Using Mathematica we find (for all $q_i > 0$)

$$D_1(q_1,q_2,q_3,q_4) = \frac{4}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^2}\sin(\lambda q_1)\sin(\lambda q_2)\sin(\lambda q_3)\sin(\lambda q_4)$$
$$= \frac{1}{4}\Big(|q_1+q_2+q_3-q_4| + |q_1+q_2-q_3+q_4| + |q_1-q_2+q_3+q_4| + |q_2+q_3+q_4-q_1|$$
$$- |q_1+q_2-q_3-q_4| - |q_1-q_2+q_3-q_4| - |q_1-q_2-q_3+q_4| - (q_1+q_2+q_3+q_4)\Big), \tag{D1}$$

$$D_2(q_1,q_2;q_3,q_4) = \frac{4q_3q_4}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^2}\sin(\lambda q_1)\sin(\lambda q_2)\left[\cos(\lambda q_3)-\frac{\sin(\lambda q_3)}{\lambda q_3}\right]\left[\cos(\lambda q_4)-\frac{\sin(\lambda q_4)}{\lambda q_4}\right]$$
$$= \frac{1}{24}\Big(|q_1+q_2-q_3-q_4|^3 + |q_1-q_2+q_3-q_4|^3 - |q_1+q_2+q_3-q_4|^3 + |q_1-q_2-q_3+q_4|^3$$
$$- |q_1+q_2-q_3+q_4|^3 - |q_1-q_2+q_3+q_4|^3 - |-q_1+q_2+q_3+q_4|^3 + (q_1+q_2+q_3+q_4)^3\Big)$$
$$+ \frac{q_3q_4}{4}\Big(|q_1+q_2-q_3-q_4| - |q_1-q_2+q_3-q_4| + |q_1+q_2+q_3-q_4| - |q_1-q_2-q_3+q_4|$$
$$+ |q_1+q_2-q_3+q_4| - |q_1-q_2+q_3+q_4| - |-q_1+q_2+q_3+q_4| + q_1+q_2+q_3+q_4\Big)$$
$$+ \frac{q_3-q_4}{8}\Big(\mathrm{sgn}(q_1+q_2+q_3-q_4)\,(q_1+q_2+q_3-q_4)^2 + \mathrm{sgn}(q_1-q_2-q_3+q_4)\,(q_1-q_2-q_3+q_4)^2$$
$$-\mathrm{sgn}(q_1+q_2-q_3+q_4)\,(q_1+q_2-q_3+q_4)^2 - \mathrm{sgn}(q_1-q_2+q_3-q_4)\,(q_1-q_2+q_3-q_4)^2\Big)$$
$$+ \frac{q_3+q_4}{8}\Big(\mathrm{sgn}(q_1+q_2-q_3-q_4)\,(q_1+q_2-q_3-q_4)^2 + \mathrm{sgn}(q_1-q_2+q_3+q_4)\,(q_1-q_2+q_3+q_4)^2$$
$$-\mathrm{sgn}(q_1-q_2-q_3-q_4)\,(-q_1+q_2+q_3+q_4)^2 - (q_1+q_2+q_3+q_4)^2\Big), \tag{D2}$$

$$D_3(q_1,q_2,q_3,q_4) = \frac{4q_1q_2q_3q_4}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^2}\left[\cos(\lambda q_1)-\frac{\sin(\lambda q_1)}{\lambda q_1}\right]\left[\cos(\lambda q_2)-\frac{\sin(\lambda q_2)}{\lambda q_2}\right]\times$$
$$\times\left[\cos(\lambda q_3)-\frac{\sin(\lambda q_3)}{\lambda q_3}\right]\left[\cos(\lambda q_4)-\frac{\sin(\lambda q_4)}{\lambda q_4}\right]$$
$$= \frac{1}{120}\Big(q_1^5 + q_2^5 - 5\left(q_2^2+q_3^2+q_4^2\right)q_1^3 - 5\left(q_2^3+q_3^3+q_4^3\right)q_1^2 - 5q_2^3\left(q_3^2+q_4^2\right) - 5q_2^2\left(q_3^3+q_4^3\right)$$
$$+ (q_3+q_4)^3\left(q_3^2-3q_4q_3+q_4^2\right)\Big)$$

$$
\begin{aligned}
&+\frac{1}{480}\Big(\left|q_1-q_2-q_3-q_4\right|^5-\left|q_1+q_2-q_3-q_4\right|^5-\left|q_1-q_2+q_3-q_4\right|^5+\left|q_1+q_2+q_3-q_4\right|^5\\
&\quad-\left|q_1-q_2-q_3+q_4\right|^5+\left|q_1+q_2-q_3+q_4\right|^5+\left|q_1-q_2+q_3+q_4\right|^5\Big)\\
&+\frac{1}{24}\Big(\left(q_3q_4+q_2\left(q_3+q_4\right)-q_1\left(q_2+q_3+q_4\right)\right)\left|q_1-q_2-q_3-q_4\right|^3-6q_1q_2q_3q_4\left|q_1-q_2-q_3-q_4\right|\\
&\quad+\left(q_1\left(q_2+q_3-q_4\right)-q_2q_3+\left(q_2+q_3\right)q_4\right)\left|q_1-q_2-q_3+q_4\right|^3-6q_1q_2q_3q_4\left|q_1-q_2-q_3+q_4\right|\\
&\quad+\left(q_1q_2+q_3q_2+q_1q_3-\left(q_1+q_2+q_3\right)q_4\right)\left|q_1+q_2+q_3-q_4\right|^3-6q_1q_2q_3q_4\left|q_1+q_2+q_3-q_4\right|\\
&\quad+\left(q_2\left(q_3-q_4\right)+q_3q_4+q_1\left(q_2-q_3+q_4\right)\right)\left|q_1-q_2+q_3-q_4\right|^3-6q_1q_2q_3q_4\left|q_1-q_2+q_3-q_4\right|\\
&\quad+\left(q_2\left(q_4-q_3\right)-q_3q_4+q_1\left(q_2-q_3+q_4\right)\right)\left|q_1+q_2-q_3+q_4\right|^3-6q_1q_2q_3q_4\left|q_1+q_2-q_3+q_4\right|\\
&\quad+\left(q_3q_4-q_2\left(q_3+q_4\right)+q_1\left(-q_2+q_3+q_4\right)\right)\left|q_1-q_2+q_3+q_4\right|^3-6q_1q_2q_3q_4\left|q_1-q_2+q_3+q_4\right|\\
&\quad+\left(q_2\left(q_3+q_4\right)-q_3q_4+q_1\left(-q_2+q_3+q_4\right)\right)\left|q_1+q_2-q_3-q_4\right|^3-6q_1q_2q_3q_4\left|q_1+q_2-q_3-q_4\right|\Big)\\
&+\frac{1}{96}\Big(\mathrm{sgn}(q_1+q_2-q_3-q_4)\left(q_1+q_2-q_3-q_4\right)^2\Big(q_1^3+3q_1\left(q_2^2-6\left(q_3+q_4\right)q_2+q_3^2+q_4^2+6q_3q_4\right)\\
&\qquad+q_2^3+3\left(q_2-q_3-q_4\right)q_1^2-\left(q_3+q_4\right)^3-3q_2^2\left(q_3+q_4\right)+3q_2\left(q_3^2+6q_4q_3+q_4^2\right)\Big)\\
&\quad+\mathrm{sgn}(q_1+q_2-q_3+q_4)\left(q_1+q_2-q_3+q_4\right)^2\Big(q_3\left(q_1+q_2-q_3+q_4\right)^2-q_1\left(q_1+q_2-q_3+q_4\right)^2\\
&\qquad-q_2\left(q_1+q_2-q_3+q_4\right)^2-q_4\left(q_1+q_2-q_3+q_4\right)^2+12q_1q_2(q_3-q_4)+12(q_1+q_2)q_3q_4\Big)\\
&\quad+\mathrm{sgn}(q_1-q_2+q_3+q_4)\left(q_1-q_2+q_3+q_4\right)^2\Big(q_2\left(q_1-q_2+q_3+q_4\right)^2-q_1\left(q_1-q_2+q_3+q_4\right)^2\\
&\qquad-q_3\left(q_1-q_2+q_3+q_4\right)^2-q_4\left(q_1-q_2+q_3+q_4\right)^2+12q_1q_2(q_3+q_4)+12(q_2-q_1)q_3q_4\Big)\\
&\quad+\mathrm{sgn}(q_1+q_2+q_3-q_4)\left(q_1+q_2+q_3-q_4\right)^2\Big(3\left(q_2^2+6q_3q_2+q_3^2\right)q_4-q_1^3-3\left(q_2+q_3-q_4\right)q_1^2\\
&\qquad-3\left(q_2^2+6q_3q_2+q_3^2+q_4^2-6\left(q_2+q_3\right)q_4\right)q_1-\left(q_2+q_3\right)^3+q_4^3-3\left(q_2+q_3\right)q_4^2\Big)\\
&\quad+\mathrm{sgn}(q_1-q_2-q_3+q_4)\left(q_1-q_2-q_3+q_4\right)^2\Big(q_1^3+3\left(q_2^2+6q_3q_2+q_3^2\right)q_4-3\left(q_2+q_3-q_4\right)q_1^2\\
&\qquad+3\left(q_2^2+6q_3q_2+q_3^2+q_4^2-6\left(q_2+q_3\right)q_4\right)q_1-\left(q_2+q_3\right)^3+q_4^3-3\left(q_2+q_3\right)q_4^2\Big)\\
&\quad+\mathrm{sgn}(q_1-q_2+q_3-q_4)\left(q_1-q_2+q_3-q_4\right)^2\Big(q_1^3-3q_2\left(q_3^2-6q_4q_3+q_4^2\right)-3\left(q_2-q_3+q_4\right)q_1^2\\
&\qquad+3\left(q_2^2+6\left(q_4-q_3\right)q_2+q_3^2+q_4^2-6q_3q_4\right)q_1-q_2^3+\left(q_3-q_4\right)^3+3q_2^2\left(q_3-q_4\right)\Big)\\
&\quad+\mathrm{sgn}(q_1-q_2-q_3-q_4)\left(-q_1+q_2+q_3+q_4\right)^2\Big(3q_2\left(q_3^2+6q_4q_3+q_4^2\right)-q_1^3+3\left(q_2+q_3+q_4\right)q_1^2\\
&\qquad-3\left(q_2^2+6\left(q_3+q_4\right)q_2+q_3^2+q_4^2+6q_3q_4\right)q_1+q_2^3+\left(q_3+q_4\right)^3+3q_2^2\left(q_3+q_4\right)\Big)\Big).
\end{aligned}
\tag{D3}
$$

As discussed in [22], there are four different cases of physical interest for which these expressions simplify considerably. They are listed in (A.15)-(A.25) of that paper and can be checked explicitly using e.g. Mathematica. For completeness, let us repeat those expressions here:

Assuming for all cases (without loss of generality) that $q_1 > q_2$ and $q_3 > q_4$, we have [22]:

Case 1: : $q_1 + q_2 > q_3 + q_4$, $q_1 + q_4 > q_2 + q_3$ and $q_1 \leq q_2 + q_3 + q_4$

$$D_1 = \frac{1}{2}\left(q_2+q_3+q_4-q_1\right), \tag{D4a}$$

$$D_2 = \frac{1}{12}\left((q_1-q_2)^3+2(q_3^3+q_4^3) -3(q_1-q_2)(q_3^2+q_4^2)\right), \tag{D4b}$$

$$
\begin{aligned}
D_3 = \frac{1}{60}\Big(&q_1^5-5q_1^3q_2^2+5q_1^2q_2^3-q_2^5-5q_1^3q_3^2+5q_2^3q_3^2\\
&+5q_1^2q_3^3+5q_2^2q_3^3-q_3^5-5q_1^3q_4^2+5q_2^3q_4^2\\
&+5q_3^3q_4^2+5q_1^2q_4^3+5q_2^2q_4^3+5q_3^2q_4^3-q_4^5\Big).
\end{aligned}
\tag{D4c}
$$

The unphysical case $q_1 > q_2 + q_3 + q_4$ yields $D_1 = D_2 = D_3 = 0$ here.

Case 2: : $q_1 + q_2 > q_3 + q_4$ and $q_1 + q_4 < q_2 + q_3$

$$D_1 = q_4\,, \tag{D5a}$$

$$D_2 = \frac{q_4^3}{3}\,, \tag{D5b}$$

$$D_3 = \frac{q_4^3}{30}\left(5q_1^2 + 5q_2^2 + 5q_3^2 - q_4^2\right)\,. \tag{D5c}$$

Case 3: : $q_1 + q_2 < q_3 + q_4\,,\quad q_1 + q_4 < q_2 + q_3$ and $q_3 \le q_1 + q_2 + q_4$

$$D_1 = \frac{1}{2}\left(q_1 + q_2 + q_4 - q_3\right)\,, \tag{D6a}$$

$$D_2 = \frac{1}{12}\left(-(q_1 + q_2)^3 - 2q_3^3 + 2q_4^3 + 3(q_1 + q_2)(q_3^2 + q_4^2)\right)\,, \tag{D6b}$$

and D_3 equals Eqn. (D4c) with variables $q_1 \leftrightarrow q_3$, $q_2 \leftrightarrow q_4$ exchanged. The unphysical case $q_3 > q_1 + q_2 + q_4$ yields $D_1 = D_2 = D_3 = 0$ here.

Case 4: : $q_1 + q_2 < q_3 + q_4$ and $q_1 + q_4 > q_2 + q_3$

$$D_1 = q_2\,, \tag{D7a}$$

$$D_2 = \frac{q_2}{6}\left(3q_3^2 + 3q_4^2 - 3q_1^2 - q_2^2\right)\,, \tag{D7b}$$

$$D_3 = \frac{q_2^3}{30}\left(5q_1^2 + 5q_3^2 + 5q_4^2 - q_2^2\right)\,. \tag{D7c}$$

[1] G. Sigl and G. Raffelt, Nucl. Phys. **B406**, 423 (1993).
[2] G. Raffelt, G. Sigl, and L. Stodolsky, Phys. Rev. Lett. **70**, 2363 (1993), Erratum-ibid. 98 (2007) 069902, arXiv:hep-ph/9209276.
[3] B. H. J. McKellar and M. J. Thomson, Phys. Rev. **D49**, 2710 (1994).
[4] R. Barbieri and A. Dolgov, Nucl. Phys. **B349**, 743 (1991).
[5] K. Enqvist, K. Kainulainen, and J. Maalampi, Nucl. Phys. **B349**, 754 (1991).
[6] M. A. Rudzsky, Astrophys. Space Sci. **165**, 65 (1990).
[7] P. Strack and A. Burrows, Phys. Rev. **D71**, 093004 (2005), arXiv:hep-ph/0504035.
[8] C. Volpe, D. Väänänen, and C. Espinoza, Phys. Rev. **D87**, 113010 (2013), arXiv:1302.2374 [hep-ph].
[9] A. Vlasenko, G. M. Fuller, and V. Cirigliano, Phys. Rev. **D89**, 105004 (2014), arXiv:1309.2628 [hep-ph].
[10] Y. Zhang and A. Burrows, Phys. Rev. **D88**, 105009 (2013), arXiv:1310.2164 [hep-ph].
[11] V. Cirigliano, G. M. Fuller, and A. Vlasenko, Phys. Lett. **B747**, 27 (2015), arXiv:1406.5558 [hep-ph].
[12] J. Serreau and C. Volpe, Phys. Rev. **D90**, 125040 (2014), arXiv:1409.3591 [hep-ph].
[13] A. Kartavtsev, G. Raffelt, and H. Vogel, Phys. Rev. **D91**, 125020 (2015), arXiv:1504.03230 [hep-ph].
[14] A. Dobrynina, A. Kartavtsev, and G. Raffelt, (2016), arXiv:1605.04512 [hep-ph].
[15] J. S. Schwinger, J. Math. Phys. **2**, 407 (1961).
[16] L. V. Keldysh, Sov. Phys. JETP **20**, 1018 (1965), [*Zh. Eksp. Teor. Fiz.* **47** (1965) 1515–1527].
[17] R. A. Craig, J. Math. Phys. **9**, 605 (1968).
[18] E. Calzetta and B. L. Hu, Phys. Rev. **D37**, 2878 (1988).
[19] J. Berges, AIP Conf. Proc. **739**, 3 (2005), arXiv:hep-ph/0409233.
[20] In the early universe, the small lepton number implies $\Sigma_{\rm forward} \sim G_F n_e \ll m_\nu \sim \Delta m_\nu$. This is not the case in supernovae.
[21] S. Hannestad and J. Madsen, Phys. Rev. **D52**, 1764 (1995), arXiv:astro-ph/9506015.
[22] A. D. Dolgov, S. H. Hansen, and D. V. Semikoz, Nucl. Phys. **B503**, 426 (1997), arXiv:hep-ph/9703315.
[23] G. Mangano, G. Miele, S. Pastor, T. Pinto, O. Pisanti, and P. D. Serpico, Nucl. Phys. **B729**, 221 (2005), arXiv:hep-ph/0506164.
[24] E. Grohs, G. M. Fuller, C. T. Kishimoto, M. W. Paris, and A. Vlasenko, Phys. Rev. **D93**, 083522 (2016), arXiv:1512.02205 [astro-ph.CO].
[25] E. Grohs, G. M. Fuller, C. T. Kishimoto, and M. W. Paris, JCAP **1505**, 017 (2015), arXiv:1502.02718 [astro-ph.CO].
[26] S. Reddy, M. Prakash, and J. M. Lattimer, Phys. Rev. **D58**, 013009 (1998), arXiv:astro-ph/9710115.
[27] S. W. Bruenn, Astrophys. J. Suppl. **58**, 771 (1985).
[28] D. L. Tubbs and D. N. Schramm, Astrophys. J. **201**, 467 (1975), Erratum-ibid. 205 (1976) 308.
[29] For Dirac neutrinos the kinetic Lagrangian reads $\mathcal{L}_{\rm Kin} = i\,\bar\nu \partial\!\!\!/ \nu \;-\; \bar\nu\, m\, \nu$, where m is a generic complex matrix.
[30] We neglect here correlations that pair particles and antiparticles of opposite momenta [8, 12]. The coupling of these new densities to the standard density matrices has been worked out explicitly in Ref. [12]. We neglect these terms as their effect primarily generates coherence of opposite-momentum neutrinos only for very long-wavelength modes, with $\lambda_{\rm deBroglie} \sim \lambda_{\rm scale-hight}$, where $\lambda_{\rm scale-hight}$ is the length scale characterizing a given astrophysical environment. Significant feedback effects from the long-wavelength modes could alter the analysis presented here. However, a detailed study of this point goes beyond the scope of this work.
[31] The interchange $i \leftrightarrow j$ in the definition of antiparticle distribution matrices is chosen so that under unitary transformations $\nu' = U\nu$, f and $\bar f$ transform in the same way, i.e. $f' = U f U^\dagger$.
[32] C. Giunti, C. Kim, and U. Lee, Phys. Rev. **D45**, 2414 (1992).
[33] Compared to Ref. [9], in Eq. (21b) the appearance of the additional phase $\varphi(k)$ (azimuthal angle of $\vec{k}$) is due to our

choice of coordinates (A1), as was explained in Ref. [11].

[34] Notice that the arrangement of the L, R components here differs from the one in the matrix $\hat{\Pi}$ in Eqn. (19) above. The reason for this is that the Lorentz scalar components of the equations of motion (15) (ultimately determining the kinetic equations for F_L and F_R) involve derivatives acting on $F_{L,R}$ as well as products of the type $F_{L,R} \cdot \Sigma_{R,L}$ and $F_{L,R} \cdot \Pi^{\kappa}_{R,L}$. This can be verified either by direct matrix multiplication [9] or by taking the appropriate traces. Therefore, the "R" chiral components of the self-energy affect the dynamics of the "L" density F_L, and vice versa. With the choice made in Eqs. (19) and (22) one then obtains kinetic and shell conditions in the compact $2n_f \times 2n_f$ matrix form given below in (24a) and (24b).

[35] P. S. Bhupal Dev, P. Millington, A. Pilaftsis, and D. Teresi, Nucl. Phys. **B886**, 569 (2014), arXiv:1404.1003 [hep-ph].

[36] P. Millington and A. Pilaftsis, Phys. Rev. **D88**, 085009 (2013), arXiv:1211.3152 [hep-ph].

[37] L. Wolfenstein, Phys. Rev. **D17**, 2369 (1978).

[38] S. P. Mikheyev and A. Y. Smirnov, Sov. J. Nucl. Phys. **42**, 913 (1985), [*Yad. Fiz.* **42** (1985) 1441–1448].

[39] A. Balantekin and Y. Pehlivan, J. Phys. G: Nucl. Phys. **34**, 47 (2007), arXiv:astro-ph/0607527.

[40] M. T. Keil, G. G. Raffelt, and H.-T. Janka, Astrophys. J. **590**, 971 (2003), arXiv:astro-ph/0208035.

[41] A. Mezzacappa, Ann. Rev. Nucl. Part. Sci. **55**, 467 (2005).

[42] K. Kotake, K. Sato, and K. Takahashi, Rept. Prog. Phys. **69**, 971 (2006), arXiv:astro-ph/0509456.

[43] T. D. Brandt, A. Burrows, C. D. Ott, and E. Livne, Astrophys. J. **728**, 8 (2011), arXiv:1009.4654 [astro-ph.HE].

[44] C. I. Ellinger, G. Rockefeller, C. L. Fryer, P. A. Young, and S. Park, (2013), arXiv:1305.4137 [astro-ph.SR].

[45] J. F. Cherry, M.-R. Wu, J. Carlson, H. Duan, G. M. Fuller, and Y.-Z. Qian, Phys. Rev. **D85**, 125010 (2012), arXiv:1109.5195 [astro-ph.HE].

[46] S. Sarikas, I. Tamborra, G. Raffelt, L. Hudepohl, and H.-T. Janka, Phys. Rev. **D85**, 113007 (2012), arXiv:1204.0971 [hep-ph].

[47] A. Mirizzi and P. D. Serpico, Phys. Rev. **D86**, 085010 (2012), arXiv:1208.0157 [hep-ph].

[48] J. F. Cherry, J. Carlson, A. Friedland, G. M. Fuller, and A. Vlasenko, Phys. Rev. **D87**, 085037 (2013), arXiv:1302.1159 [astro-ph.HE].

[49] A. Vlasenko, G. M. Fuller, and V. Cirigliano, (2014), arXiv:1406.6724 [astro-ph.HE].

[50] From now on we suppress the explicit x-dependence in all Wigner transforms.

[51] H. Duan, G. M. Fuller, and Y.-Z. Qian, Ann. Rev. Nucl. Part. Sci. **60**, 569 (2010), arXiv:1001.2799 [hep-ph].

[52] R. A. Harris and L. Stodolsky, Phys. Lett. **B116**, 464 (1982).

[53] L. Stodolsky, Phys. Rev. **D36**, 2273 (1987).

[54] P. B. Arnold, D. T. Son, and L. G. Yaffe, Phys. Rev. **D59**, 105020 (1999), arXiv:hep-ph/9810216.

[55] W. Botermans and R. Malfliet, Phys. Lett. **B215**, 617 (1988).

[56] G. G. Raffelt, *Stars as Laboratories for Fundamental Physics* (Univ. of Chicago Press, 1996).

[57] A. D. Dolgov, S. H. Hansen, and D. V. Semikoz, Nucl. Phys. **B543**, 269 (1999), arXiv:hep-ph/9805467.

[58] For the $\nu_a \nu_a \to \nu_a \nu_a$ processes we agree with the overall factor of Ref. [22] which is twice as large as in Ref. [21].

[59] H. K. Dreiner, H. E. Haber, and S. P. Martin, Phys. Rept. **494**, 1 (2010), arXiv:0812.1594 [hep-ph].

[60] The Dirac case can be recovered by the replacements $f \to f_{LL}$, $\bar{f} \to f_{RR}$ in $\bar{G}_V^L$, $\bar{f}^T \to f_{RR}$, $f^T \to \bar{f}_{LL}$ in $\bar{G}_V^R$, $\phi \to f_{LR}$, $\phi^T \to \bar{f}^{LR}$ in Φ, and $\phi^\dagger \to f_{RL}$, $\phi^* \to \bar{f}^{RL}$ in $\Phi^\dagger$.