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Composite gauge-bosons made of fermions

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Abstract

We construct a class of Abelian and non-Abelian local gauge theories that consist only of matter fields of fermions. The Lagrangian is local and does not contain an auxiliary vector field nor a subsidiary condition on the matter fields. It does not involve an extra dimension nor supersymmetry. This Lagrangian can be extended to non-Abelian gauge symmetry only in the case of $SU(2)$ doublet matter fields. We carry out explicit diagrammatic computation in the leading $1/N$ order to show that massless spin-one bound states appear with the correct gauge coupling. Our diagram calculation exposes the dynamical features that cannot be seen in the formal auxiliary vector-field method. For instance, it shows that the s -wave fermion-antifermion interaction in the 3S_1 channel $(\psi\gamma_\mu\psi)$ alone cannot form the bound gauge bosons; the fermion-antifermion pairs must couple to the d -wave state too. One feature common to our class of Lagrangian is that the Noether current does not exist. Therefore it evades possible conflict with the no-go theorem of Weinberg and Witten on formation of the non-Abelian gauge bosons.

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I. INTRODUCTION

The U(1) gauge theory normally consists of a gauge field and matter fields. The Lagrangian is invariant under the simultaneous gauge transformation of the gauge field and the matter fields. After this was generalized to non-Abelian group[1], we learned that the non-Abelian extension underlies dynamics of the fundamental particles.

Let us take a side step and ask out of curiosity the following question: Is it possible to construct a gauge-invariant Lagrangian with matter fields alone? For instance, can we construct a *local* field theory with the electron-positron field alone such that it is invariant under the space-time dependent rotation $\psi(x) \rightarrow e^{i\alpha(x)}\psi(x)$ even in the absence of an auxiliary gauge field ? If the particles are bosons, the CP^N/CP^{N-1} model[2] would probably be the best known example of this type. Its supersymmetric extension was also discussed.[4] In the case that the matter fields are fermions alone, the history actually goes much further back to the work by Bjorken[5], but the work along this line has not been fruitful.¹

The method of the auxiliary vector-fields was often used in the past to proceed in this kind of argument. It introduces nonpropagating gauge fields at start and their kinetic energy terms are added later by the loop contribution, ending up with the Lagrangian of matter and propagating gauge fields. Many argued that the nonpropagating gauge field implanted as an auxiliary field in Lagrangian should be interpreted as turning into a bound state once it has acquired its kinetic energy from the loop contributions. But it is an inevitable consequence of gauge invariance of Lagrangian that such an auxiliary field, elementary or otherwise, ought to acquire a gauge invariant kinetic energy term $-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}$ after loops are included. Wouldn't it be more illuminating if composition of the massless vector-state can be seen explicitly in terms of the constituent matter fields ? Such a diagrammatic computation was indeed made by Haber, Hinchliffe and Ravinovici[6] for the CP^{N-1} model many years ago. Unfortunately, this demonstration cannot be repeated when the constituents are fermions, since a simple *local* gauge-invariant Lagrangian corresponding to that of the CP^N model has not been known in the case of fermion constituents.

More recently, attempt has been made to introduce composite gauge bosons through the fifth dimension of the Randall-Sundrum model [13]. The gauge bosons live in the branes

¹ A review is found for some of early history at the beginning of the Reference [4] including references.

and can be interpreted as composite wholly or partially. This is a new class or concept of composite gauge bosons. Models were built and phenomenology was discussed for possible extensions of the standard model along this line.[14, 15].

In this paper we would like to focus on the dynamics of formation of composite gauge bosons at an elementary level of particle physics. Many of us have the underlying conviction or speculation that when Lagrangian is locally gauge invariant, gauge bosons must emerge as composite states even if they are not placed as elementary particles. We would like to see it with our model Lagrangians in an explicit diagrammatic way. In order to separate the issue from the argument based on the auxiliary vector field trick, we study the Lagrangians consisting of fermion fields alone. Furthermore, since our Lagrangian consists only of fermions, supersymmetry is not relevant to our argument, barring the nonlinear realization[7]. We stay in the space-time of dimension four all the time. We have no need of an extra dimension explicitly or implicitly. Given our Lagrangian, we can carry through diagram calculation in the leading $1/N$ order with no further approximation or assumption. In this way we can observe how the composite gauge bosons are made of their constituents dynamically. Our reasoning for construction of the Lagrangian is simple and resorts to no sophisticated mathematical argument or technique.

The primary purpose of this paper is to give model Lagrangians that advocate inevitability of gauge bosons in gauge symmetric theories. Although application of our class of model Lagrangians to the real world is not our primary concern at this moment, short comments are made at the end on issues in electroweak phenomena. At the end, looking back the history of “compositeness” including findings in some supersymmetric theories, we wonder if it is really a meaningful concept at a fundamental level.

At present, we do not have in mind an immediate application of our model Lagrangian to particle phenomenology. The gauge bosons have been generally accepted as the “elementary” particles and, experimentally, there is no compelling reason of compositeness for them. Therefore we shall not pursue experimental relevance of our models seriously in this paper. Our emphasis at present is primarily on their theoretical implications in composite gauge bosons in general. When Yang and Mills introduced the non-Abelian gauge field theory[1], it had no immediate application. Even the ρ -meson was not known at that time although the concept of the weak intermediate bosons was entertained by theorists. The Yang-Mills theory became a subject of intense phenomenological interest only after the Higgs mechanism[9],

Weinberg’s “A Model of Leptons”[10] and quantum chromodynamics came up at the stage unexpectedly one after another. If we recall this history, we may have chance to see some feature of our models develop into a subject of experimental interest as the Large Hadron Collider upgrades luminosity and energy further.

We organize the paper as follows: In Section II, following the footstep of the CP^N model, we introduce the U(1) gauge model of charged Dirac fields alone. We emphasize that, in contrast to the CP^N model, one cannot write a local Lagrangian of fermion fields alone with the so-called auxiliary field trick. In Section III we show that the Noether current is inevitably absent in the gauge theories that consist of matter fields alone. In Section IV, we show dynamics of the U(1) gauge-boson formation first in the bosonic matter model and then in the fermionic matter model. We introduce, as usual, the N families of matter fields and take the large N limit in order to solve the models explicitly in a compact form. We find that a massless bound state appears in the 3S_1 channel of elastic fermion-antifermion scattering, but that the fermion-antifermion pair must interact in the 3D_1 channel as well in order to form the massless bound state of spin-one. In Section V we extend our models to the non-Abelian gauge symmetry. Choosing the matter fields in SU(2) doublet, we can build a non-Abelian model with Dirac fields. Computing the elastic scattering amplitude, we find the non-Abelian gauge bosons in the SU(2)-triplet channel as bound states with the correct self-couplings as required by the non-Abelian gauge invariance. In our class of models, the SU(2)-doublet matter plays a special role; it is impossible to extend the model to matter fields of general SU(2) multiplets nor to general Lie groups. The special role of the SU(2) doublet is discussed in the text and also with two examples in one of the Appendices. In the final Section VI, we discuss on relevance of the missing Noether currents to the no-go theorem of Weinberg and Witten[11]. We conclude with comments on possible relevance to the electroweak phenomenology and on historical mutation of the concept of compositeness.

II. U(1) MODELS

We proceed by following an elementary line of argument. The first step is to construct a local Lagrangian $L(\psi, \bar{\psi})$ such that

$$L(e^{i\alpha(x)}\psi(x), e^{-i\alpha(x)}\bar{\psi}(x)) = L(\psi(x), \bar{\psi}(x)), \quad (1)$$

where $L(\psi(x), \bar{\psi}(x))$ depends on space-time coordinates x_μ only through the unconstrained fields $\psi(x)/\bar{\psi}(x)$. We cannot construct such a Lagrangian backward from the QED Lagrangian by integrating out the gauge field $A_\mu(x)$: We would need a gauge fixing to integrate over $A_\mu(x)$, but fixing a gauge breaks manifest gauge invariance. We make our search here with the CP^N model as a guide.

Quantum electrodynamics cannot be modified or extended in our way if both renormalizability and locality are required in the space-time of (3+1) dimensions. We do not consider here genuinely or intrinsically nonlocal field theories in which the fundamental fields and/or interaction contains nonlocality.² In contrast to nonlocality, unrenormalizability can be controlled formally by dimensional regularization or by a covariant cutoff in phenomenology. Therefore, we abandon here renormalizability in (3+1) dimensions for the moment and move to a world off (3+1) dimensions or consider a covariant cutoff theory in (3+1) dimensions.

A. Boson matter

In order to construct a local Lagrangian with fermion matter fields alone, we first reexamine the gauge invariance of the bosonic matter model, namely the CP^N model, from a slightly different viewpoint.

In the CP^N model the gauge noninvariance of the free Lagrangian L_0 due to $\partial^\mu \phi$ under $\phi \rightarrow e^{i\alpha(x)}\phi$ must be counterbalanced with that of the interaction L_{int} . Therefore L_{int} must have at least the same number of derivatives as L_0 . Since L_0 and L_{int} have the same space-time dimension, we must introduce an inverse of $(\phi^* \phi)$ in L_{int} to make up for the dimension due to ∂^μ in the numerator of L_{int} . Keeping the number of ∂^μ in L_{int} the smallest, we reach almost uniquely the simplest form of the gauge-invariant Lagrangian made of the matter fields alone as

$$L_{tot} = L_0 + L_{int}, \quad (2)$$

where L_0 is the standard free Lagrangian,

$$L_0 = \sum_{i=1}^N \partial^\mu \phi_i^* \partial_\mu \phi_i - \sum_{i=1}^N m^2 \phi_i^* \phi_i, \quad (3)$$

² For example, the field theories once considered by Yukawa[8] and his followers.

and the interaction Lagrangian L_{int} is given by

$$L_{int} = \lambda \frac{\sum_{i=1}^N (\phi_i^* \overset{\leftrightarrow}{\partial} \phi_i) \sum_{j=1}^N (\phi_j^* \overset{\leftrightarrow}{\partial}_\mu \phi_j)}{4 \sum_{k=1}^N (\phi_k^* \phi_k)}, \quad (\lambda \rightarrow 1). \quad (4)$$

The indices (i, j, k) run from 1 to N so that the model be solvable in the leading order of $1/N$. They are referred to as the *copy* indices hereafter. From time to time, however, the summation over the copy indices will be suppressed unless we need to remind of it.

Under the local U(1) gauge transformation, the fields transform with a space-time dependent phase $\alpha(x)$ common to all the copy index i as

$$\phi_i \rightarrow e^{i\alpha(x)} \phi_i, \quad \text{and} \quad \phi_i^* \rightarrow e^{-i\alpha(x)} \phi_i^*. \quad (5)$$

For the total Lagrangian, each of L_0 and L_{int} varies nontrivially under the gauge transformation Eq.(5), but the variations δL_0 and δL_{int} are so made as to be proportional to each other:

$$\begin{aligned} \delta L_0 &= -i \left(\sum_i \phi_i^* \overset{\leftrightarrow}{\partial}_\mu \phi_i \right) \partial^\mu \alpha + \left(\sum_i \phi_i^* \phi_i \right) \partial^\mu \alpha \partial_\mu \alpha, \\ \delta L_{int} &= -\lambda \delta L_0. \end{aligned} \quad (6)$$

These gauge variations cancel each other between L_0 and L_{int} for

$$\lambda = 1 \quad (\text{gauge limit}). \quad (7)$$

If we remove the mass term and impose the constraint $\sum_i \phi_i^* \phi_i = N/2f$ in Eq.(4), we recognize this Lagrangian (with $\lambda = 1$) as that of the CP^{N-1} model [2]. However, we have introduced N copies solely for the purpose of the computational ease of the leading $1/N$ expansion. Our interest is not in the SU(N) symmetry among the different copies while numerous works were done on the CP^{N-1} model with rich physical consequences in the 1970's and 80's.

As far as U(1) gauge invariance is concerned, we may add to Eq.(2) the terms that are gauge invariant by themselves. For instance, nonderivative ϕ^4 -couplings such as

$$L'_{int} = - \sum_{i,j=1}^N \lambda_{ij} (\phi_i^* \phi_i) (\phi_j^* \phi_j), \quad (8)$$

where λ_{ij} are arbitrary real constants. However, in the leading $1/N$ order the interactions such as L'_{int} do not affect on the bound-state formation.³ Therefore we leave out such

³ Because we compute the bound state of spin-one, not of spin-zero.

interactions hereafter. It is reassuring to see later that the vector bound state comes out massless with the correct gauge coupling irrespective of the additional gauge-invariant interactions such as L'_{int} .

B. Fermionic model

Following the reasoning outlined above, we can obtain with a little stretch of imagination a fermionic extension of the bosonic model Lagrangian Eq.(2). Since the free Lagrangian L_0 contains only one first-derivative of ψ , the interaction L_{int} can counterbalance the gauge variation of L_0 with only one first-derivative of field. Just as in the bosonic case, we need to introduce the inverse of the scalar density $\bar{\psi}\psi$ in L_{int} in order to match the dimension. Following this reasoning as in the bosonic model, we reach the Lagrangian $L_0 + L_{int}$,

$$\begin{aligned} L_0 &= \sum_i \bar{\psi}_i (i \not{\partial} - m) \psi_i, \\ L_{int} &= -i\lambda \frac{\sum_i (\bar{\psi}_i \gamma_\mu \psi_i) \sum_j (\bar{\psi}_j \overset{\leftrightarrow}{\partial}^\mu \psi_j)}{2 \sum_k \bar{\psi}_k \psi_k}, \quad (\lambda \rightarrow 1), \end{aligned} \quad (9)$$

where the gauge invariance is realized at $\lambda = 1$. Under the gauge transformation,

$$\begin{aligned} \psi &\rightarrow e^{i\alpha(x)} \psi \\ \bar{\psi} &\rightarrow \bar{\psi} e^{-i\alpha(x)}, \end{aligned} \quad (10)$$

the Lagrangian of Eq.(9) is invariant by cancellation between the gauge variations of L_0 and L_{int} at $\lambda = 1$:

$$\begin{aligned} \delta L_0 &= -\bar{\psi} (\not{\partial} \alpha) \psi, \\ \delta L_{int} &= \lambda \bar{\psi} (\not{\partial} \alpha) \psi. \end{aligned} \quad (11)$$

We may add to L_{int} the self-gauge-invariant terms such as

$$L'_{int} = -\frac{fm}{4} (\bar{\psi} \gamma_\mu \psi) \frac{1}{(\bar{\psi} \psi)} (\bar{\psi} \gamma^\mu \psi), \quad (12)$$

where insertion of the fermion mass m is just to make the constant f dimensionless. The constant f is unconstrained by gauge invariance. After we compute for the massless bound state with L_{int} of Eq.(9) alone, we shall examine how the interactions like L'_{int} affect its mass and coupling. Since they will turn out to be irrelevant to determination of the mass

and coupling of the massless bound state, we shall not include them in our diagram calculation. Before diagram calculation, some may suspect that the fermion-antifermion interaction through $\propto (\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\gamma^\mu\psi)$ might be responsible for or relevant to binding a gauge boson. It is wrong. Such an interaction does not exist in our L_{int} . Even if one includes it in L_{int} , it does not participate in formation of the massless gauge boson nor in determination of the gauge coupling, as we shall see later.

Our fermionic Lagrangian Eq.(9) is obviously unrenormalizable in the space-time dimension of four just like that of the CP^N model. As we know, the only renormalizable U(1) gauge field theory with a charged fermion is quantum electrodynamics: It needs the propagating gauge field A_μ explicitly in Lagrangian.

C. Auxiliary vector-field trick

Our bosonic Lagrangian Eq.(2) with $\lambda = 1$ takes the same form as what we could obtain by starting with the gauge-invariant Lagrangian of a nonpropagating auxiliary gauge field A_μ ,

$$L_{aux} = \sum_i (\partial_\mu - ieA_\mu)\phi_i^* (\partial^\mu + ieA^\mu)\phi_i - m^2\phi_i^*\phi_i. \quad (13)$$

Either by integrating Eq.(13) over A^μ or by substituting the equation of motion for A^μ ,

$$eA_\mu = \frac{i}{2} \left(\sum_i \phi_i^* \overleftrightarrow{\partial}_\mu \phi_i \right) / \left(\sum_j \phi_j^* \phi_j \right), \quad (14)$$

we obtain for $m^2 \rightarrow 0$ the CP^N Lagrangian (before imposing the constraint and turning it into CP^{N-1})[12].

When we compute by the loop correction the dimension-four operator of A^μ for the effective action, we obtain the “kinetic energy term” $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. One cannot obtain anything other than the gauge invariant FF term (“the Maxwell term”) since the Lagrangian Eq.(13) is gauge invariant by construction. Whether this appearance of the FF term is to be interpreted as “generation of a bound state” or not should be subject to debate. If we accepted such interpretation, a massless spin-one state would emerge irrespectively of strength of the interaction e^2 which is implanted in Eq.(13) at the beginning. After rescaling of the A_μ field, the physical coupling of A_μ to ϕ/ϕ^* is fixed to some number, which is independent of e in the one-loop and logarithmically divergent in the world of dimension four. Turning of the

field A_μ into a massless boson is guaranteed once the field is introduced as an auxiliary field. In contrast, in our model the strength of interaction L_{int} must be tuned to the optimum value ($\lambda = 1$) in order to make the bound state massless. In this way we see that masslessness of the vector bound-state is a dynamical consequence of gauge invariance rather than a kinematical outcome.

Substitution of the equation of motion Eq.(14) also needs scrutiny: If one computes $\partial_\mu F^{\mu\nu}$ with this A^μ , one would obtain $\partial_\mu F^{\mu\nu} = 0$ instead of $\partial_\mu F^{\mu\nu} = J^\nu$. Therefore, the field A^μ of Eq.(14) is not acceptable as the composite gauge field. One would need contributions from loops to write a dynamical gauge field that obeys the correct equation of motion. We do not know how to write such an object in a local composite field.

What would happen if one attempts to introduce the auxiliary field A_μ in the fermionic model ? For the fermionic matter, the Lagrangian with a nonpropagating auxiliary field is simply equal to

$$L_{aux} = \sum_i \bar{\psi}_i (i \not{D} + e \not{A} - m) \psi_i, \quad (15)$$

The equation of motion with respect to A_μ is trivially equal to $\sum_i \bar{\psi}_i \gamma_\mu \psi_i = 0$ and provides us nothing. As for the functional integration over the auxiliary field A_μ , one cannot carry it out at the tree level since the auxiliary Lagrangian Eq.(15) is not quadratic in A_μ unlike that of the bosonic model. When the two-point loop-diagrams of $A_\mu A_\nu$ is computed, the local limit of the two-point functions ought to be proportional to $F_{\mu\nu} F^{\mu\nu}$ by the underlying gauge invariance. But we cannot obtain a compact local Lagrangian of the matter fields alone such as ours out of the auxiliary Lagrangian of Eq.(15).

The auxiliary vector-field trick bypasses the important part of dynamics of the matter fields. In contrast, our explicit Lagrangian models provide dynamical details of binding which are missing in the auxiliary field trick or else very different from it.

III. NOETHER CURRENT

When we attempt to write a conserved current in our models, we encounter one peculiar problem: We are unable to construct a conserved current with the prescription of the Noether theorem. In fact, such a current simply does not exist.

According to the general prescription, the Noether current J_μ^N is obtained when Lagrangian is invariant under a set of space-time independent phase transformations of fields.

In the bosonic model, it would be generated by the transformation,

$$\phi_i \rightarrow (1 + i\alpha)\phi_i \text{ and } \phi_i^* \rightarrow (1 - i\alpha)\phi_i^*, \quad (16)$$

where α is infinitesimal and *independent* of space-time. The variation δL_{tot} of $O(\alpha)$ under this transformation leads to divergence of the Noether current through the identification,

$$\partial^\mu J_\mu^N = -\delta L_{tot}/\delta\alpha. \quad (17)$$

Using the equation of motion in the right-hand side, one ought to obtain the Noether current J_μ^N as

$$J_\mu^N = -i \sum_i \left(\frac{\partial L_{tot}}{\partial(\partial^\mu \phi_i)} \phi_i - \phi_i^* \frac{\partial L_{tot}}{\partial(\partial^\mu \phi_i^*)} \right). \quad (18)$$

When we follow this standard procedure in our models, we find that the right-hand side of Eq.(18) is identically zero in the gauge symmetry limit by cancellation between the contributions from L_0 and L_{int} :

$$J_\mu^N = i(1 - \lambda) \sum_i (\phi_i^* \overset{\leftrightarrow}{\partial}_\mu \phi_i), \quad (19)$$

where the term proportional to λ comes from L_{int} and the gauge symmetry holds at $\lambda = 1$. One may be puzzled when one thinks of perturbative calculation: Since ϕ and ϕ^* always appear pairwise in product in the Lagrangian, one may assign the conserved U(1) charge ± 1 to ϕ and ϕ^* . Then this charge ought to be conserved in all diagrams of physical processes such as scattering and decay even in the gauge symmetry limit where the Noether current disappears.

The same happens in the fermionic model too. Just as in the bosonic model, the conserved Noether current disappears in the gauge symmetry limit:

$$J_\mu^N = (1 - \lambda) \sum_i \bar{\psi}_i \gamma_\mu \psi_i. \quad (20)$$

The current $\sum \bar{\psi}_i \gamma_\mu \psi_i$ is not the Noether current. It is a general property of the gauge theories having no gauge field that the Noether current is identically zero; $J_\mu^N \equiv 0$. It is easy to trace the root cause of absence of the Noether current to local gauge invariance itself. An almost trivial proof is given in the Appendix A. The proof can be easily extended to the non-Abelian models. It has an important implication in the non-Abelian case: If the Noether current existed, generation of the massless gauge bosons would face a potential conflict with the no-go theorem of Weinberg and Witten[11].

Unlike the Noether current, the conserved energy-momentum tensor exists in the Abelian and non-Abelian gauge theories of matter fields alone. For the fermionic U(1) model with the Lagrangian of Eq.(9), the conserved energy-momentum tensor is given by

$$T^{\mu\nu} = i \sum_i \bar{\psi}_i \gamma_\mu \partial_\nu \psi_i - \frac{i\lambda(\sum_i \bar{\psi}_i \gamma^\mu \psi_i)(\sum_j \bar{\psi}_j \overset{\leftrightarrow}{\partial}^\nu \psi_j)}{2 \sum_k (\bar{\psi}_k \psi_k)} - g^{\mu\nu} L_{tot}. \quad (21)$$

It is manifestly gauge invariant with the matter fields alone.

IV. COMPOSITE U(1) GAUGE BOSON

It is natural to wonder if our U(1) models contain a gauge boson as a composite state even though we have not placed it by hand. In order to answer to this question, we carry out diagram calculation in this section in order to exhibit the dynamical mechanism of forming the composite gauge boson. We compute our models perturbatively in the $1/N$ expansion: We sum an infinite series of the leading $1/N$ order terms and show explicitly that a massless vector boson indeed appears as a pole in scattering amplitudes with the properties required by gauge symmetry both in the bosonic and the fermionic model. In the case of the CP^{N-1} model in which $\phi^* \phi$ is subject to a constraint, this diagram computation was done by Haber *et al* [6]. Our primary interest is in the fermionic model, which is technically complex since channel coupling occurs between the 3S_1 and 3D_1 channels. Unlike the formal argument based on the auxiliary vector-field trick[16], the diagrammatic computation allows us to see explicitly how a massless bound state is formed dynamically with the matter particles. For instance, when we examine elastic fermion-antifermion scattering of $J^{PC} = 1^{--}$, we find that the massless bound state appears in the 3S_1 channel, not in the 3D_1 channel. That is, the bound state couples with the fermions through the vertex $\bar{\psi} \gamma_\mu \psi$, not through $\bar{\psi} \overset{\leftrightarrow}{\partial}_\mu \psi$. Nonetheless, the interactions of both types are needed to form a massless bound state.

A. Gauge boson in bosonic model

We start with our U(1) bosonic model for study of a composite gauge boson before our study of the fermionic model since the computation is simpler for the bosonic model, yet it demonstrates essential aspects of the diagram calculation.

We consider the two-body $\phi^+\phi^-$ scattering in p -wave ($J^{PC} = 1^{--}$), treating all N copies of the fields ($i = 1, \dots, N$) as independent. We show that a pole of a massless bound-state appears in this channel. Then we proceed to make sure that the pattern and magnitude of the coupling of this bound state indeed obey what we expect for the $U(1)$ gauge boson.

We study the p -wave amplitude for the two-body scattering,

$$\phi_i^+(p_1) + \phi_i^-(p_2) \rightarrow \phi_j^+(p_3) + \phi_j^-(p_4). \quad (22)$$

We compute the amplitude in the leading $1/N$ order since a compact explicit solution can be obtained only in this order. In the scattering Eq.(22), the copy indices are chosen to be the same for the initial particles and also for the final particles. In the diagram calculation, L_{int} is separated from L_{tot} in Eq.(4) and treated as *the interaction*. While this statement sounds trivial, we point out one subtlety. That is, when we carry out perturbative calculation by splitting the Lagrangian into L_0 and L_{int} , we have fixed once for all the gauge ambiguity of our Lagrangian Eq.(2). That is, when we write the propagator of ϕ/ϕ^* in the momentum space as $1/(p^2 - m^2)$, we need no more gauge fixing since there is no A^μ field in the Lagrangian. With this separation, the fields obey the equation of motion of L_0 that violates gauge symmetry. Consequently the Noether current of L_0 is the conserved current in diagrams. For the purpose of visualizing how the gauge-invariance limit is reached, we float λ in L_{int} as a free parameter until we set it to unity at the end of calculation.

In the diagram calculation of the leading $1/N$ order, we normal-order the operator $\phi^*\phi$ in the denominator of L_{int} and expand it around its vacuum value as

$$\begin{aligned} 1/\sum \phi^*\phi &= 1/\left(\sum \langle 0|\phi^*\phi|0\rangle + \sum : \phi^*\phi : \right) \\ &= \frac{1}{\sum \langle 0|\phi^*\phi|0\rangle} \sum_{n=0}^{\infty} (-1)^n \left(\frac{\sum : \phi^*\phi :}{\sum \langle 0|\phi^*\phi|0\rangle} \right)^n, \end{aligned} \quad (23)$$

where the summation \sum with no index attached means the summation over the copy index $i(= 1, \dots, N)$. This separation of the vacuum value is important to handle the denominator of L_{int} in a systematic $1/N$ expansion.[6] The vacuum expectation value $\langle 0|\phi^*\phi|0\rangle$ is infinite in the (3+1) space-time, so it is regularized dimensionally as

$$\begin{aligned} \sum \langle 0|\phi^*\phi|0\rangle &= \lim_{x \rightarrow 0} \sum \langle 0|T(\phi^*(x)\phi(0))|0\rangle, \\ &= \frac{N\Gamma(1 - D/2)}{(4\pi)^{D/2}(m^2)^{1-D/2}}, \end{aligned} \quad (24)$$

where N copies of bosons contribute to the vacuum value of the scalar density. The space-time dimension D is set to four eventually. We denote this vacuum-expectation-value by I_0^b hereafter,

$$I_0^b \equiv \sum \langle 0 | \phi^* \phi | 0 \rangle. \quad (25)$$

Now we are ready to compute for the two-body scattering of Eq.(22). The great simplification of the leading $1/N$ order is that for elastic scattering we have only to sum the chain of the bubble diagrams, as shown in Fig. 1, in which the copy index i runs within a loop of each bubble.

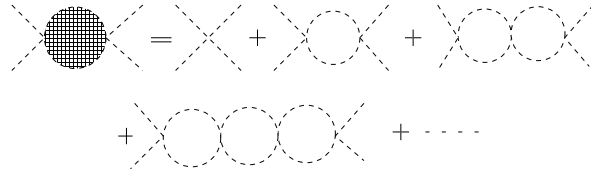


FIG. 1: The chain of the bubble diagrams for the elastic boson scattering.

Let us define with the S-matrix the two-body scattering amplitude $T(p_3, p_4; p_1, p_2)$ as

$$\langle p_3, p_4 | S - 1 | p_1, p_2 \rangle = i(2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) T(p_3, p_4; p_1, p_2). \quad (26)$$

The amplitude T has the Lorentz structure of the form

$$T(p_4, p_3; p_1, p_2) = (p_3 - p_4)^\mu (p_1 - p_2)^\nu T(q)_{\mu\nu}, \quad (27)$$

where $q = p_1 + p_2 = p_3 + p_4$ and the one-particle states are normalized as $\langle p_i | p_j \rangle = 2E_i \delta(\mathbf{p}_i - \mathbf{p}_j)$ so that the amplitude $T(p_3, p_4; p_2, p_1)$ is a Lorentz scalar. For the elastic scattering in the leading $1/N$ order, it is sufficient to keep only the first term of the expansion Eq.(23) in the denominator of L_{int} . The normal-ordered product $(\sum : \phi^* \phi :)$ starts contributing to the next-to-leading order of $1/N$ in the elastic scattering.

The amplitude $T(q)_{\mu\nu}$ starts with a contact interaction term with no bubble, the first term in the right-hand side of Fig. 1, which is equal to

$$T_{\mu\nu}^0 = \frac{\lambda}{2I_0^b} g_{\mu\nu}, \quad (28)$$

where the superscript “zero” of $T_{\mu\nu}^0$ indicates the zero-loop contribution of $O(\lambda)$. The bubble summation can be carried out by solving the algebraic equation (Fig. 2),

$$T(q)_{\mu\nu} = T_{\mu\nu}^0 + K(q)_{\mu\kappa} T_\nu^\kappa(q). \quad (29)$$

where the kernel $K(q)_{\mu\kappa}$ is given by the single bubble diagram in which the copy index flows around the loop. Eq.(29) will become powerful later when we sum the corresponding series in the fermionic model in which two eigenchannels contribute and entangle in formation of a bound state.

Straightforward computation gives us the kernel as

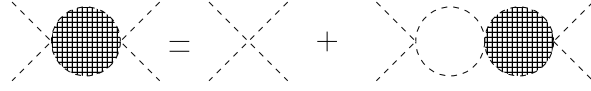


FIG. 2: The iteration equation of bubbles into a chain.

$$K_{\mu\kappa}(q) = \frac{\lambda N \Gamma(1 - D/2)}{(4\pi)^{D/2} (m^2)^{1-D/2} I_0^b} \left(g_{\mu\kappa} + \frac{1 - D/2}{6m^2} (g_{\mu\kappa} q^2 - q_\mu q_\kappa) \right) + O(q^4). \quad (30)$$

Since we want to extract the pole and residue of a massless bound state at $q^2 = 0$, we need $K_{\mu\kappa}(q)$ only to the orders no higher than $O(q^2)$. The factor outside the large bracket in Eq.(30) is simply equal to λ when Eq.(24) is substituted for I_0^b so that

$$K_{\mu\kappa}(q) = \lambda \left(g_{\mu\kappa} + \frac{1 - D/2}{6m^2} (g_{\mu\kappa} q^2 - q_\mu q_\kappa) \right) + O(q^4). \quad (31)$$

Note here that $K_{\mu\kappa}(q)$ does not satisfy the transversality, $q^\mu K_{\mu\kappa} \neq 0$. This is not violation of gauge invariance. In the standard Lagrangian where the elementary A_μ field is present, one would need the $A_\mu A^\mu \phi^* \phi$ term to realize transversality of the photon self-energy, $q^\mu \Pi(q)_{\mu\kappa} = 0$, namely, gauge invariance. The term needed for transversality does exist in our model, but it is tucked away elsewhere at this stage. As we shall see in a moment, it is this nontransversality of $K_{\mu\kappa}(q)$ that makes the composite boson massless.⁴

Let us substitute Eq.(31) in the iteration equation Eq.(29) and move the term $\lambda g_{\mu\kappa}$ of the kernel $K_{\mu\kappa}(q)$ to the left-hand side. We may drop the term proportional to $q_\mu q_\mu$ by use of $q \cdot (p_1 - p_2) = 0 = q \cdot (p_3 - p_4)$ on the external boson lines. Then Eq.(29) turns into

$$(1 - \lambda) T(q)_{\mu\nu} = T_{\mu\nu}^0 + \frac{\lambda(1 - D/2)q^2}{6m^2} T(q)_{\mu\nu} + O(q^4). \quad (32)$$

Now we go to the gauge limit $\lambda \rightarrow 1$. Since $T_{\mu\nu}^0$ is independent of q , Eq.(32) tells us that in this limit there is a pole at $q^2 = 0$ in the amplitude $T(q)_{\mu\nu}$ as

$$T(q)_{\mu\nu} = -\frac{6m^2}{(1 - D/2)q^2} T_{\mu\nu}^0 + O(q^2), \quad (\lambda = 1). \quad (33)$$

⁴ This is the case in the CP^{N-1} model analyzed in [6] too.

When the parameter λ is off the gauge limit ($\lambda \neq 1$), the pole is located away from zero at $q^2 = [6(1 - \lambda)/\lambda(1 - D/2)]m^2$ so that the bound state would be either a massive vector boson or a tachyon. We extract the residue of the pole at $q^2 = 0$ for $\lambda = 1$ and compare this residue with what we would obtain from the Feynman diagram of the standard U(1) gauge Lagrangian of the charged scalar fields,

$$L_{tot} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial^\mu\phi^* - ieA^\mu\phi^*)(\partial_\mu\phi + ieA_\mu\phi) - m^2\phi^*\phi. \quad (34)$$

By equating our residue with that of Feynman diagram, we obtain the coupling e^2 of our model as

$$e^2 = \frac{3(4\pi)^{D/2}(m^2)^{2-D/2}}{N\Gamma(2 - D/2)}. \quad (35)$$

When we approach the space-time dimension of $D = 4$, this coupling can be expressed in terms of the logarithmic cutoff of divergence as

$$e^2 = \frac{48\pi^2}{N \ln(\bar{\Lambda}^2/m^2)}, \quad (36)$$

where $\ln \bar{\Lambda}^2 = (2 - D/2)^{-1} + \ln 4\pi - \gamma_E$ (γ_E = Euler constant). The sign of e^2 comes out to be correctly positive. It is amusing to observe that the factor $(1 - D/2)$ in the denominator of Eq.(33) is combined with $\Gamma(1 - D/2)$ in $1/I_b^0$ of $T_{\mu\nu}^0$ to turn into $\Gamma(2 - D/2)$, which is the logarithmic divergence in the space-time dimension of $D = 4$. That is, a quadratic divergence $\Gamma(1 - D/2)$ metamorphoses into a logarithmic divergence as it can happen in the dimensional regularization.

If we started with the auxiliary A_μ field and generate the $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ to the leading $1/N$ order, we would obtain the coupling constant identical with Eq.(36) after rescaling A_μ by wave-function renormalization.[4] This equality is not unexpected since the one-loop self-energy diagram of the auxiliary A_μ field leading to Eq.(36) is identical with the bubble diagram of the p -wave $\phi^\dagger\phi$ scattering in the leading $1/N$ order. There is no guarantee that this equality holds beyond the leading $1/N$ order since noncontact interactions enter the scattering amplitude while the self-energy diagram remains the two-point function.

In order to claim that the massless bound state discovered above is indeed the U(1) gauge boson, we must show that other couplings of this state obey the pattern required for the gauge boson. One may bypass this part by resorting to the gauge invariance that has been embedded in the Lagrangian of our model. But we show here explicitly how the U(1)-gauge invariance arises diagrammatically for the coupling of the massless bound state.

Absence of the coupling $eA_\mu\partial^\mu(\phi^*\phi)$ is obvious since the form of our L_{int} requires the bound state to couple with ϕ^*/ϕ through $(\phi^* \overset{\leftrightarrow}{\partial}^\mu \phi)$ not through $\partial^\mu(\phi^*\phi)$. This is also required by C -invariance of our Lagrangian. However, there must exist the coupling $e^2\phi^*\phi A_\mu A^\mu$, where A_μ is the effective gauge field and e^2 is to be given by Eq.(35). Aside from this coupling, there should be no coupling of dimension four such as a nonderivative quartic coupling of A_μ .

The coupling of $\phi^*\phi A_\mu A^\mu$ requires a little computation. Here the first nontrivial term of the expansion of $1/(\phi^*\phi)$ enters the computation,

$$-\frac{\lambda}{4I_0^2}(\phi^* \overset{\leftrightarrow}{\partial}^\mu \phi)(\phi^* \overset{\leftrightarrow}{\partial}_\mu \phi)(:\phi^*\phi:). \quad (37)$$

In the leading $1/N$ order, we attach a chain of the bubble diagrams to $(\phi^* \overset{\leftrightarrow}{\partial}^\mu \phi)$ and another chain to $(\phi^* \overset{\leftrightarrow}{\partial}_\mu \phi)$ to form the composite A^μ and A_μ bosons, respectively. (See Fig. 3.) Then we equate this diagram at the poles of the A^μ and A_μ bosons to the diagram of Fig. 4 which is obtained with the interaction $e^2\phi^*\phi A_\mu A^\mu$ of the standard U(1) gauge Lagrangian, Eq.(34).

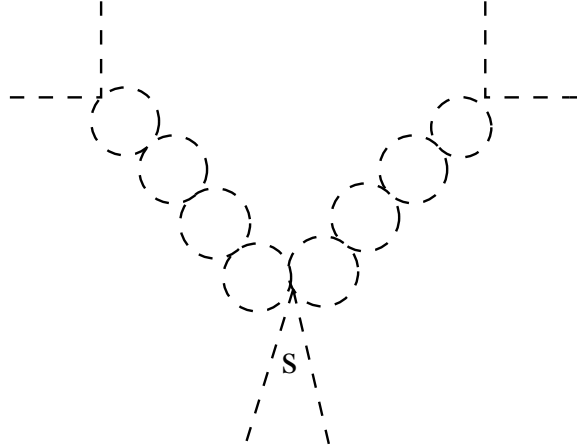


FIG. 3: The diagram for formation of $\phi^*\phi A_\mu A^\mu$ coupling. The $\phi^*\phi$ pair arises from the six-body interaction of Eq.(37) at the center. The letter S denotes that the external $\phi^*\phi$ pair at the center is in the scalar state $\phi^*\phi$, not in the vector state $\phi^* \overset{\leftrightarrow}{\partial}_\mu \phi$.

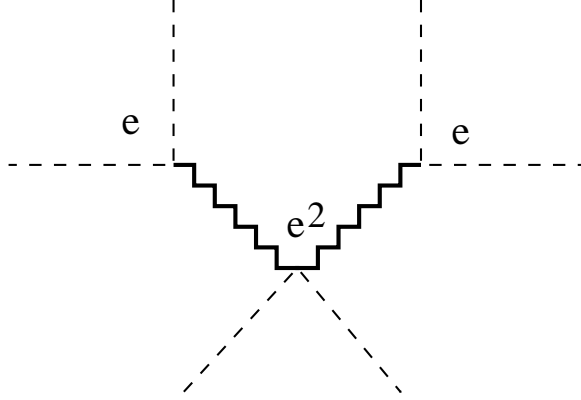


FIG. 4: The corresponding Feynman diagram for $e^2 \phi^* \phi A_\mu A^\mu$.

This calculation gives us the relation

$$e^4 = \left(\frac{3(4\pi)^{D/2}(m^2)^{2-D/2}}{N\Gamma(2-D/2)} \right)^2. \quad (38)$$

Two powers e^2 out of e^4 in Eq.(38) are to be attributed to the couplings of the $\phi^* \phi$ pairs with A_μ and with A_ν at the outer ends of two bubble chains in Fig. 3. The remaining e^2 is to be assigned to the four-body $A_\mu A^\mu \phi^* \phi$ coupling at the center. Therefore, the coupling e^4 of Eq.(38) is precisely what we want to see.

Absence of the triple self-coupling of A_μ is a consequence of C -invariance. Diagrammatically, this is assured in the U(1) model by cancellation between a pair of diagrams where the two chains are interchanged. Since they do not cancel in the non-Abelian models and there is some subtlety, we add a few comments here in anticipation of the non-Abelian cases. The relevant diagram is depicted in Fig. 5.

If we indeed compute this coupling with individual diagrams, we must be careful about the surface-term ambiguity. The triangular loop at the center is linearly divergent in the space-time dimension of four and therefore its constant term is ambiguous by the surface term of loop-integral. The value depends on how the loop-momentum is routed just as in the chiral anomaly or the finite part of the electron self-energy in QED. To fix this finite ambiguity, one must impose invariance and/or symmetry that must be preserved in theory. In this case C invariance of L_{tot} and/or the Bose statistics of the composite A_μ fixes the ambiguity. With the right choice of the routing momentum, a pair of triangular loop diagrams cancel each other and turn the net triple self-coupling to zero in the U(1) model.

In comparison, we need explicit computation of diagrams to show vanishing of the net

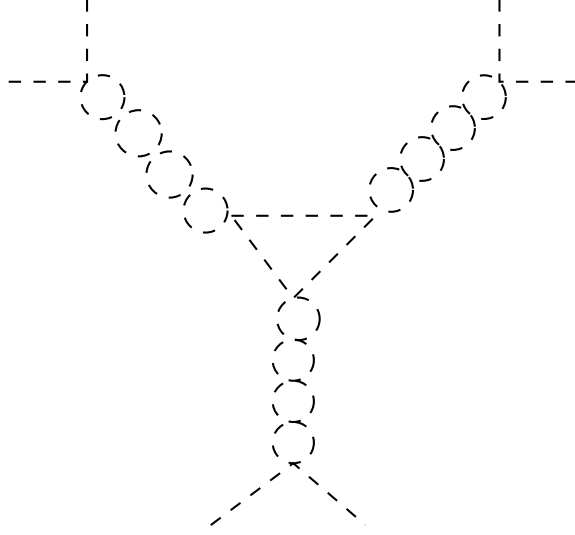


FIG. 5: The triple self-coupling of the composite A_μ , which can appear potentially from the center of the diagrams containing three chains of ϕ^*/ϕ bubbles.

quartic self-coupling although there is no subtlety of the surface-term ambiguity. In the presence of the six-body coupling of Eq.(37), three classes of loop diagrams can potentially contribute to the quartic self-coupling of the composite gauge boson in the leading $1/N$ order (Fig. 6).

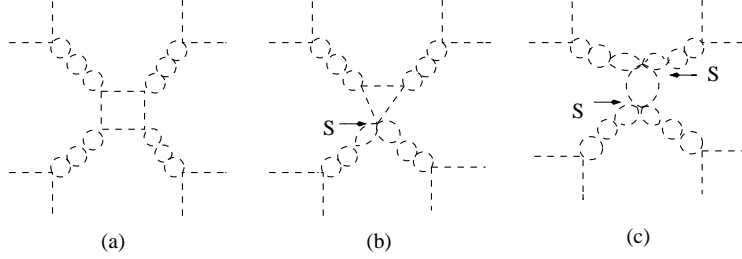


FIG. 6: Three classes of diagrams can contribute to the quartic self-coupling of composite A_μ . The letter S for the six-body ϕ^*/ϕ interaction point in the loop at center denotes that the $\phi^*\phi$ pair is in the scalar state.

The square box diagrams (6a) alone do not cancel among themselves. When we add all three classes of the diagrams, however, they sum up to zero at the zero external momentum limit where the on-shell quartic coupling constant is defined. Up to an overall constant, the cancellation occurs among the three types of diagrams in Fig. 6 as

$$\propto \left(\frac{1}{4} - \frac{1}{2} + \frac{1}{4} \right) \frac{1}{\Gamma(2 - D/2)}, \quad (39)$$

where the first, second and third terms in the bracket are from the three types of diagrams, Figs. 6a, 6b and 6c, respectively. Of course, this cancellation is not an accident. Its origin is traced back to the U(1) gauge invariance incorporated in the Lagrangian.⁵

Our fundamental Lagrangian L_{tot} is invariant under the gauge transformation $\phi(x) \rightarrow e^{i\alpha(x)}\phi(x)$ and the conjugate. Once a massless vector bound-state emerges with the effective coupling $ie(\phi^* \overleftrightarrow{\partial}_\mu \phi)A^\mu$, the only way to be compatible with the gauge invariance is that the additional interaction $e^2\phi^*\phi A_\mu A^\mu$ exists for this effective A_μ field and that A_μ transforms as $eA_\mu \rightarrow eA_\mu + i\partial_\mu\alpha$ under $\phi(x) \rightarrow e^{i\alpha(x)}\phi(x)$. As far as the interactions of dimension four are concerned, there is no other way known to us that satisfies the U(1) gauge invariance incorporated in L_{tot} . As for the self-couplings of A_μ , we would have to satisfy U(1) gauge invariance with the A_μ fields alone without derivatives. That is, there is no room to accommodate nonderivative self-interaction of A_μ in dimension four. When we argue in this way, gauge invariance of the composite A_μ coupling is an inevitable and trivial consequence of the gauge symmetry of L_{tot} , once a massless spin-one bound-state emerges with the coupling $ie(\phi^* \overleftrightarrow{\partial}_\mu \phi)A^\mu$. When we take this viewpoint, the crucial step is whether or not a massless bound state of spin-one is indeed formed out of the interactions among the matter fields themselves. The rest may be interpreted as logical inevitability.

Before closing this subsection, we comment on the interactions of dimension higher than four (in the world of space-time dimension four or 3+1). The interaction $(\phi^*\phi)^2 A_\mu A^\mu$ has dimension six. It can arise from the third term ($n = 2$) of the expansion of the denominator $1/(\phi^*\phi)$ in Eq.(4), that is,

$$L_{int} = \frac{1}{4(I_0^b)^3} (: \phi^* \phi :)^2 (\phi^* \overleftrightarrow{\partial}_\mu \phi) (\phi^* \overleftrightarrow{\partial}^\mu \phi). \quad (40)$$

By attaching the chains of the ϕ bubbles to $(\phi^* \overleftrightarrow{\partial}_\mu \phi)$ and $(\phi^* \overleftrightarrow{\partial}^\mu \phi)$, then going to the gauge-boson mass shells on the chains, we can extract the effective interaction of dimension six for the composite gauge boson,

$$L_{int} = \frac{e^2}{I_0^b} (\phi^* \phi)^2 A_\mu A^\mu, \quad (41)$$

⁵ We freely switch between $\phi^*\phi$ and $:\phi^*\phi:$ in this calculation since our computation of the couplings involves only those diagrams in which a ϕ/ϕ^* particle emitted from one L_{int} annihilates at a another L_{int} in the center of diagram. See Figs. 6b and 6c. The normal ordering makes no difference in Figs. 6b nor 6c for this reason.

where the coupling e^2 is given by Eq.(35). This coupling is not gauge invariant by itself. However, there is another effective coupling of dimension six, which contains only a single A_μ . We can compute it with the interaction of Eq.(40) and put it in the form of effective interaction,

$$L_{int} = \frac{ie}{I_0^b}(\phi^*\phi)(\phi^* \overleftrightarrow{\partial}_\mu \phi)A^\mu. \quad (42)$$

When the two interactions Eqs.(41) and (42) of dimension six are combined and added to the first term of the expansion of $1/(\phi^*\phi)$,

$$\frac{1}{4I_0^b}(\phi^* \overleftrightarrow{\partial}_\mu \phi)(\phi^* \overleftrightarrow{\partial}^\mu \phi), \quad (43)$$

the sum total is gauge invariant. That is, when all the couplings of $O(1/I_0^b)$, Eqs.(41), (42) and (43) are combined, the interaction of dimension six for the effective field A_μ is gauge invariant. The combined effective interaction can be cast into the form

$$L_{int}^{eff} = \frac{1}{4I_0^b}(\phi^* \overleftrightarrow{D}^\mu \phi)(\phi^* \overleftrightarrow{D}_\mu \phi), \quad (44)$$

where $D_\mu = \partial_\mu + ieA_\mu$ and $(\phi^* \overleftrightarrow{D}_\mu \phi) \equiv \phi^* D_\mu \phi - (D_\mu \phi)^* \phi$. The interaction of Eq.(44) illustrates what happens for the effective interactions of higher dimension in general. It is obvious from the dimensional reason that L_{int}^{eff} must be inversely proportional to powers of I_0^b . Although I_0^b is formally proportional to m^2 in the dimensional regularization, it is quadratically divergent in the cutoff ($\sim N\overline{\Lambda}^2$) in the world of $D = 4$. If we give a physical meaning to the cutoff, therefore, the interactions of dimension six are suppressed by $O(p^2/N\overline{\Lambda}^2)$ in the region of energy scale $O(p^2)$ relative to those of dimension four. Meanwhile the divergences of $O(N \ln \overline{\Lambda}^2)$ are absorbed into the gauge coupling e^2 as we have seen in Eq.(35). Therefore, if our model should turn out to be phenomenologically relevant in one way or another, its cutoff $\overline{\Lambda}$ would place these higher-dimensional interaction under control. Whether these interactions can generate anything phenomenologically interesting or not is a separate question.

We can cast the amplitudes of higher-dimension processes in the standard U(1)-gauge theory with the elementary gauge boson into the form of effective interactions. However, such effective interactions are generally not identical with the higher dimensional interactions that have been obtained above from our Lagrangian Eq.(2). The loop-diagram amplitudes produced by the standard U(1) gauge theory do exist equally in our model since the gauge

boson exists as a composite. Our model contains the additional terms that are generated by matter fields and suppressed by the large cutoff scale of I_b . Physics is generally different in these orders from the standard gauge theory of elementary gauge boson. If our model were identical with the standard U(1) gauge theory, it would be perfectly renormalizable in our world of dimension four. But that is not the case: Our model contains the higher-dimensional local interactions that are additional to the standard gauge theory and suppressed by powers of $1/I_b = O(\bar{\Lambda}^2)$.

B. Gauge boson in fermionic model

Computation of the massless bound state is technically a little complex in the fermionic model since there exist two channels of $J^{PC} = 1^{--}$. We compute the elastic scattering of fermion-antifermion,

$$f^+(p_1, s_1) + f^-(p_2, s_2) \rightarrow f^+(p_3, s_3) + f^-(p_4, s_4) \quad (45)$$

in the leading $1/N$ order with the Lagrangian Eq.(9). The copy indices are chosen to be the same for the initial f^+f^- and for the final f^+f^- . We shall suppress spin indices $s_i (i = 1, \dots, 4)$ in the following since they are obvious in most places. We leave out the *self-gauge invariant* interactions such as Eq.(12). Although those interactions certainly contribute to the fermion-fermion scattering in general, we show later that omission of such interactions does not affect the properties of the massless bound state.

We follow our path taken for the bosonic model: We separate $\bar{\psi}\psi$ in the denominator of L_{int} into sum of the vacuum expectation values and the normal-ordered products: $\bar{\psi}\psi$ and then expand it in the power series of $\sum : \bar{\psi}\psi : / \sum \langle 0 | \bar{\psi}\psi | 0 \rangle$. The vacuum expectation value $\langle 0 | \bar{\psi}\psi | 0 \rangle$ is divergent and dimensionally regularized as

$$\begin{aligned} \sum \langle 0 | \bar{\psi}\psi | 0 \rangle &= - \lim_{x \rightarrow 0} \text{tr} \langle 0 | T(\psi(x) \bar{\psi}(0)) | 0 \rangle, \\ &= - \frac{4Nm\Gamma(1 - D/2)}{(4\pi)^{D/2}(m^2)^{1-D/2}}, \end{aligned} \quad (46)$$

where the trace (tr) in the first line of the right-hand side refers to the spinor indices of ψ and $\bar{\psi}$. We shall denote the right-hand side of Eq.(46) by I_0^f hereafter as

$$I_0^f \equiv \langle 0 | \bar{\psi}\psi | 0 \rangle = -4mI_0^b. \quad (47)$$

I_0^f is opposite in sign to I_0^b of the boson Eq.(25) and its dimension is three instead of two.

Now we proceed to compute the two-body scattering amplitude of $J^{PC} = 1^{--}$. There exist two eigenchannels in the fermion scattering. The fermion-antifermion pair is in the configuration of $\bar{v}_{-\mathbf{p}}\gamma u_{\mathbf{p}}$ in one channel and in $2\mathbf{p}\bar{v}_{-\mathbf{p}}u_{\mathbf{p}}$ in the other in the center-of-momentum frame. The spins of $\bar{v}_{-\mathbf{p}}$ and $u_{\mathbf{p}}$ are combined into a triplet in both cases so that they make the 3S_1 and 3D_1 states of f^+f^- , respectively. With our choice of L_{int} in Eq.(9), the fermion-antifermion pair turns from $\bar{\psi}\gamma_\mu\psi$ on one side to $(\bar{\psi}\overset{\leftrightarrow}{\partial}^\mu\psi)$ on the other, or conversely from $(\bar{\psi}\overset{\leftrightarrow}{\partial}_\mu\psi)$ to $\bar{\psi}\gamma^\mu\psi$ at every interaction point in the chain of bubbles.

Let us define the Lorentz scalar amplitude $T(p_1, p_2; p_3, p_4)$ with the S-matrix as

$$\langle p_3, p_4 | S - 1 | p_1, p_2 \rangle = i(2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) T(p_3, p_4; p_1, p_2), \quad (48)$$

where the one-fermion states are so normalized that the amplitude $T(p_3, p_4; p_1, p_2)$ is a Lorentz scalar and its Lorentz structure is given in the (2×2) matrix form by

$$T = \begin{pmatrix} \bar{u}_{p_3}\gamma_\mu v_{p_4}, \bar{u}_{p_3}(p_3 - p_4)_\mu v_{p_4}/m \end{pmatrix} \begin{pmatrix} T_{11}^{\mu\nu}(q) & T_{12}^{\mu\nu}(q) \\ T_{21}^{\mu\nu}(q) & T_{22}^{\mu\nu}(q) \end{pmatrix} \begin{pmatrix} \bar{v}_{p_2}\gamma_\nu u_{p_1} \\ \bar{v}_{p_2}(p_1 - p_2)_\nu u_{p_1}/m \end{pmatrix}, \quad (49)$$

where $q = (p_1 + p_2) = (p_3 + p_4)$. The perturbation series for $T(q)^{\mu\nu}$ starts with the tree diagram, which gives $-(\lambda/2I_0^f)g_{\mu\nu}$ to the off-diagonal elements of $T_{\mu\nu}^0$:

$$T_{\mu\nu}^0 = -\frac{1}{2I_0^f} \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} g_{\mu\nu}. \quad (50)$$

Summation of the bubble chains can be carried out by solving the matrix equation,

$$T(q)_{\mu\nu} = T_{\mu\nu}^0 + K(q)_{\mu\kappa} T_\nu^\kappa(q), \quad (51)$$

where the kernel $K(q)_{\mu\kappa}$ is the 2×2 matrix of the four single-bubble diagrams that connect between γ_μ -type vertex (3S_1) and the $\overset{\leftrightarrow}{\partial}_\mu$ -type vertex (3D_1). (See Fig. 7.)

$$K^{\mu\kappa}(q) = \begin{pmatrix} K(q)_{11} & K(q)_{12} \\ K(q)_{21} & K(q)_{22} \end{pmatrix}^{\mu\kappa}. \quad (52)$$

In order to extract the mass and coupling of the composite boson from $T(q)_{\mu\nu}$, we need $(I - K(q))_{\mu\kappa}$ near $q^2 = 0$ in Eq.(51). To be more specific, the terms of $g_{\mu\kappa}$ and $(q^2 g_{\mu\kappa} - q_\mu q_\kappa)$ for K_{ij} . In fact, for the off-diagonal elements K_{12} and K_{21} , all we need is the leading terms

$$\begin{pmatrix} \text{diagram 1} & \text{diagram 2} \\ \text{diagram 3} & \text{diagram 4} \end{pmatrix} = \begin{pmatrix} 0 & \text{diagram 5} \\ \text{diagram 6} & 0 \end{pmatrix} + \begin{pmatrix} \text{diagram 7} & \text{diagram 8} \\ \text{diagram 9} & \text{diagram 10} \end{pmatrix} \begin{pmatrix} \text{diagram 11} & \text{diagram 12} \\ \text{diagram 13} & \text{diagram 14} \end{pmatrix}$$

FIG. 7: Iteration of bubble diagrams for fermion scattering. The letters γ and d denote that the fermion pair at the interaction point is $\bar{\psi}\gamma_\mu\psi$ and $\bar{\psi}\overset{\leftrightarrow}{\partial}_\mu\psi$, respectively.

that give $K_{12}K_{21} = O(q^2)$. By straightforward diagram computation, we find the relevant terms of $K^{\mu\kappa}(q)$ near $q^2 = 0$ as

$$\begin{aligned}
K^{\mu\kappa}(q)_{11} &= \lambda \left(g^{\mu\kappa} + \frac{\Gamma(2 - D/2)}{6m^2\Gamma(1 - D/2)} (g^{\mu\kappa}q^2 - q^\mu q^\kappa) \right), \\
&= K^{\mu\kappa}(q)_{22} \\
K^{\mu\kappa}(q)_{12} &= -\lambda \left(\frac{\Gamma(-D/2)}{\Gamma(1 - D/2)} - 2 \right) g^{\mu\kappa}, \\
K^{\mu\kappa}(q)_{21} &= -\lambda \left(\frac{\Gamma(2 - D/2)}{6m^2\Gamma(1 - D/2)} \right) (g^{\mu\kappa}q^2 - q^\mu q^\kappa). \tag{53}
\end{aligned}$$

We have kept Γ -functions above since they are partially canceled with $\Gamma(1 - D/2)$ coming from $1/I_0^f$ of T^0 when $(I - K)^{-1}$ is operated on T^0 later. The terms in Eq.(53) that turn out to determine the pole and residue of the massless bound state are the first term $\lambda g^{\mu\kappa}$ of the diagonal element $K(q)_{11}^{\mu\kappa} (= K(q)_{22}^{\mu\kappa})$ and the off-diagonal element $K(q)_{12}^{\mu\kappa} \neq 0$ at $q^2 = 0$.

Let us examine the pole and residue of the matrix amplitude $T_{\mu\nu}$ at $q^2 = 0$ by solving Eq.(51) as

$$T_{\mu\nu} = \left(\frac{1}{I - K} \right)_\mu^\kappa T_{\kappa\nu}^0. \tag{54}$$

Since the external fermion lines are on mass shell, the terms proportional to $q_\mu q_\kappa$ in $K_{\mu\kappa}$ has been removed by use of the Dirac equation and the mass shell condition on the external lines. We then approach the gauge symmetry limit $\lambda = 1$ of $T = (I - K)^{-1}T^0$. The result is

$$T(q)_{\mu\nu} = \frac{(4\pi)^{D/2}(m^2)^{2-D/2}}{\Gamma(2 - D/2)} \begin{pmatrix} \frac{3}{4q^2} & \frac{C}{m^2} \\ \frac{C}{m^2} & \frac{C}{m^2} \end{pmatrix} g_{\mu\nu}, \tag{55}$$

where

$$C = \frac{D(D-2)}{32(D+1)}. \quad (56)$$

A pole appears only in the (11)-matrix element at the upper left corner in Eq.(55) and the other entries are regular at $q^2 = 0$.

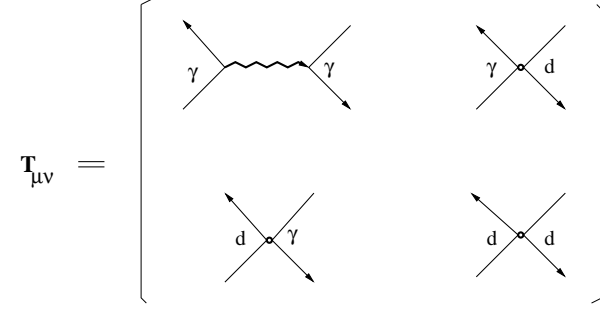


FIG. 8: The massless bound state appears only in the upper left corner, which is the 3S_1 channel.

It means that bound state appears in the channel of $\bar{\psi}\gamma_\mu\psi \rightarrow \bar{\psi}\gamma_\mu\psi$, that is, in the 3S_1 channel, not in the 3D_1 channel.⁶ If either end of the chain is $\bar{\psi} \overset{\leftrightarrow}{\partial}_\mu \psi$, no massless pole appears in such a chain.

By comparing the matrix element $T_{11}^{\mu\nu}$ with the one-photon pole diagram of the standard U(1) gauge interaction $-e\bar{\psi}\gamma_\mu\psi A^\mu$, we can identify the gauge coupling e^2 with the residue at the pole to obtain

$$e^2 = \frac{3(4\pi)^{D/2}(m^2)^{(2-D/2)}}{4N\Gamma(2-D/2)}. \quad (57)$$

or in terms of the covariant ultraviolet cutoff in the space-time of $D = 4$,

$$e^2 = \frac{12\pi^2}{N \ln(\bar{\Lambda}^2/m^2)}. \quad (58)$$

This is the parallel of Eq.(35) in the bosonic model. While the quartic divergence ($\propto \Gamma(-D/2) \sim \Lambda^4$) and quadratic divergence ($\sim \Lambda^2$) are present in $T(q)_{\mu\nu}$, they do not enter the residue of the pole at $q^2 = 0$. Therefore, the coupling e^2 involves only the logarithmic divergence ($\sim 1/N \ln \Lambda^2$) as it does for the bosonic model.

As we have pointed out, we may add to our fermionic model the interaction L'_{int} of Eq.(12) which is gauge invariant by itself. Let us denote the shifts of the matrices $K(q)$ and T^0 due

⁶ This has nothing to do with the d -wave threshold behavior $\sim |\mathbf{p}|^l (l=2)$. The threshold behaviors reside in the spinorial factors in Eq.(49) and have been separated out in defining $T(q)_{ij}^{\mu\nu}$.

to L'_{int} as $K(q) \rightarrow K(q) + \Delta K(q)$ and $T^0 \rightarrow T^0 + \Delta T^0$. Near $q^2 = 0$, these shifts are given by

$$\Delta T_{\mu\nu}^0 = \frac{1}{2I_0^f} g_{\mu\nu} \begin{pmatrix} fm & 0 \\ 0 & 0 \end{pmatrix}. \quad (59)$$

and

$$\Delta K^{\mu\kappa} = \frac{1 - D/2}{6m^2} \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} (g^{\mu\kappa} q^2 - q^\mu q^\kappa). \quad (60)$$

It is not difficult to see that these modifications, Eqs.(59) and (60), do not alter either the location of the pole at $q^2 = 0$ nor its residue. In terms of diagrams, we can visualize the effect of Eqs.(59) and (60) as follows: We should first notice the fact that the newly added bubble consisting of γ_μ on one end and γ_κ on the other end vanishes like $(g_{\mu\kappa} q^2 - q_\mu q_\kappa)$ at $q = 0$. Let us call this bubble as that of the type $\gamma_\mu \otimes \gamma_\kappa$. When the $\gamma_\mu \otimes \gamma_\kappa$ bubble enters the middle of the eigenchannel that produces the bound state, the chain would thus acquire a factor of $O(q^2)$ from this bubble. Therefore it cancels the pole and becomes irrelevant to formation of the massless bound state. The pole at $q^2 = 0$ is produced only by the $g_{\mu\kappa}$ term of $K(q)_{\mu\kappa}$ in the chain of bubbles of the type $\gamma_\mu \otimes \overset{\leftrightarrow}{\partial}_\nu$ and $\overset{\leftrightarrow}{\partial}_\mu \otimes \gamma_\nu$ alone. With the addition of L'_{int} , therefore, the massless pole is undisturbed and its residue is unaffected.

Let us move on to the self-coupling of the gauge field. Charge conjugation invariance forbids the triple self-coupling, but the quartic self-coupling is not forbidden by any discrete symmetry. Since the massless bound state couples only to the 3S_1 vertex, namely, to $\bar{\psi}\gamma_\mu\psi$, the relevant diagrams have a square box at the center with six permutations of the four γ -vertices, that is, the diagram of Fig 6a in which the boson lines are replaced by the fermion lines and the γ -matrices sit at the four corners of the box. However, sum of these box diagrams vanishes in the zero energy-momentum limit of the bound-state bosons, not just the leading divergent term ($\sim \ln \bar{\Lambda}^2$) but all finite terms as well in this limit. This fact is well-known as the gauge-invariance requirement $\sim e^4 F_\mu^\nu F_\nu^\kappa F_\kappa^\lambda F_\lambda^\mu$ on the photon-photon scattering amplitude in quantum electrodynamics.

For the diagrams corresponding to Fig 6b and 6c with the boson lines replaced by fermions, the two chains of bubbles are attached to the six-body fermion interaction. However, since the six-body fermion interaction is of the form $(\bar{\psi}\gamma_\mu\psi)(\bar{\psi}\overset{\leftrightarrow}{\partial}^\mu\psi)(\bar{\psi}\psi)$, one of the vector vertices starts with the γ -vertex but the other starts with the $\overset{\leftrightarrow}{\partial}$ -vertex. As we have already observed, the massless bound-state pole cannot appear in the latter chain. There-

fore, the massless vector bound state can be formed only in one of the two chains attached to the six-body interaction point, not in both. That is, only three massless bound states can be formed in Fig. 6b and two in Fig. 6c. Combining this observation with that for Fig. 6a above, we conclude that there exists no nonderivative quartic self-coupling of the massless U(1) bound-state in the fermion model either, just as gauge invariance requires.

The lowest possible couplings of higher dimension with fermion fields is the Pauli term $i\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}$. This coupling is gauge invariant by itself. With our interaction L_{int} , however, our composite boson does not have this coupling. To see this, recall the decomposition of the photon-fermion vertex for the fermion on mass shell, $i\bar{u}\sigma_{\mu\nu}q^\nu v' = \bar{u}(p+p')_\mu v' - 2m\bar{u}\gamma_\mu v'$. This relation tells that if the massless bound state had the Pauli-term interaction, we would have its pole in the channels of both $\bar{\psi} \overset{\leftrightarrow}{\partial}_\mu \psi$ and $\bar{\psi}\gamma_\mu\psi$. In our preceding study, however, we have found a massless pole only in $\bar{\psi}\gamma_\mu\psi$. That means no Pauli term.

The effective interaction $\bar{\psi}\psi A_\mu A^\mu$ is also of dimension five and not gauge invariant by itself. As in the bosonic model, If an interactions of A_μ appears with a dimension higher than four, it ought to appear in a gauge invariant combination since the underlying Lagrangian is gauge invariant. As for this specific interaction, the accompanying gauge-covariant partners are $\partial^\mu\bar{\psi}\partial_\mu\psi$ and $ie(\bar{\psi} \overset{\leftrightarrow}{\partial}_\mu \psi)A^\mu$. But we have already found that the coupling $(\bar{\psi} \overset{\leftrightarrow}{\partial}_\mu \psi)A^\mu$ does not exist in our model. Neither $\partial_\mu\psi\partial^\mu\psi$ in L_{tot} . Therefore the coupling $(\bar{\psi}\psi)A_\mu A^\mu$ can be generated as an effective interaction in our model.

One of the merits of our fermionic model is to reveal the dynamical details explicitly in regard to how the self-interaction of the constituent fermions conspires to generate the composite gauge boson. Specifically, the composite gauge boson is formed with fermions in the presence of the process of the transition between the 3S_1 and the 3D_1 channel. No massless bound state can be formed with the 3S_1 channel alone. There is no place to see this dynamics in the auxiliary field trick on fermions in which the auxiliary vector field has only the 3S_1 interaction.

V. NON-ABELIAN EXTENSIONS

It is possible to extend our U(1) models to non-Abelian models. The non-Abelian extension turns out to be quite easy if we choose matter fields in the SU(2) doublet. In this section we present the SU(2)-doublet model for both bosons and for fermions and compute

for the composite gauge bosons again in the leading $1/N$ order. Extension of our $U(1)$ models to a general Lie group or even to an $SU(2)$ representation other than the doublet encounters difficulty. This is not a simple technical difficulty, but it involves some problem at a fundamental level in our class of models. We explain this difficulty in the text, then go a little further with few examples of the bosonic models in Appendix B.

Those who approach the problem with the auxiliary field trick would trivially extend the $U(1)$ model to general groups and representations by simply replacing the two-by-two matrices $\frac{1}{2}\tau_a$ of $SU(2)$ with the $n \times n$ generator matrices T_a of a general Lie group. In our case, however, such simple substitution does not extend our models to those of general groups or representations.⁷ This is another indication of the fact that our models are physically different at some fundamental level from what the auxiliary field trick gives.

A. Non-Abelian bosonic model

Let us introduce N families of scalar boson fields in $SU(2)$ doublet,

$$\Phi^i = \begin{pmatrix} \phi_1^i \\ \phi_2^i \end{pmatrix}, \quad (i = 1, \dots, N), \quad (61)$$

and their conjugates $\Phi^{i\dagger}$, which we write in a row. The subscripts $(1, 2)$ are those of $SU(2)$. We shall suppress the copy index and/or the $SU(2)$ index wherever there is no confusion. Our bosonic Lagrangian is given simply by

$$\begin{aligned} L_0 &= \sum_i \partial^\mu \Phi^{i\dagger} \partial_\mu \Phi^i - \sum_i m^2 \Phi^{i\dagger} \Phi^i \\ L_{int} &= \lambda \frac{\sum_i (\Phi^{i\dagger} \boldsymbol{\tau} \overset{\leftrightarrow}{\partial}_\mu \Phi^i) \cdot \sum_j (\Phi^{j\dagger} \boldsymbol{\tau} \overset{\leftrightarrow}{\partial}^\mu \Phi^j)}{4 \sum_k (\Phi^{k\dagger} \Phi^k)}, \quad (\lambda \rightarrow 1) \end{aligned} \quad (62)$$

where i, j and k are copy indices and $\boldsymbol{\tau}$ denotes the Pauli matrices $\tau_a (a = 1, 2, 3)$.⁸ For the $SU(2)$ gauge invariance of $L_0 + L_{int}$, we give the proof here for the infinitesimal rotation,

$$\Phi \rightarrow (1 + \frac{i}{2} \boldsymbol{\tau} \cdot \boldsymbol{\alpha}) \Phi,$$

⁷ One well-known example of the special role of the $SU(2)$ may come to minds of some people. That is the instanton. The instanton is special to $SU(2)$, not extendable to $SU(N)$ ($N \geq 3$) or other general groups because of its topological property. In our case, however, topology is not an issue. Important is the *self-duality* of the group and the representation.

⁸ This bosonic Lagrangian as well as its Abelian version appears in the earlier paper[4].

$$\Phi^\dagger \rightarrow \Phi^\dagger (1 - \frac{i}{2} \boldsymbol{\tau} \cdot \boldsymbol{\alpha}), \quad (63)$$

where $\boldsymbol{\alpha}$ is a space-time dependent vector function. Let us compute the variations $L_0 \rightarrow L_0 + \delta L_0$ and $L_{int} \rightarrow L_{int} + \delta L_{int}$ separately and confirm cancellation to $O(\alpha)$ between the two variations. For L_0 , it is easy to obtain

$$\delta L_0 = -\frac{i}{2} (\Phi^\dagger \boldsymbol{\tau} \overset{\leftrightarrow}{\partial}_\mu \Phi) \cdot \partial^\mu \boldsymbol{\alpha} + O(\alpha^2). \quad (64)$$

We need a little care in computation of δL_{int} . To the order $O(\alpha)$, it is not difficult to obtain the transformation,

$$(\Phi^\dagger \boldsymbol{\tau} \overset{\leftrightarrow}{\partial}^\mu \Phi) \rightarrow \Phi^\dagger U^\dagger \boldsymbol{\tau} U \partial^\mu \Phi - (\partial^\mu \Phi^\dagger) U^\dagger \boldsymbol{\tau} U \Phi + 2i(\Phi^\dagger \Phi) \partial^\mu \boldsymbol{\alpha} + O(\alpha^2), \quad (65)$$

where $U = 1 + i\boldsymbol{\tau} \cdot \boldsymbol{\alpha}/2$. The third term proportional to $\partial^\mu \boldsymbol{\alpha}$ in the right-hand side has been obtained by use of the relation,

$$\boldsymbol{\tau}(\boldsymbol{\tau} \cdot \partial^\mu \boldsymbol{\alpha}) + (\boldsymbol{\tau} \cdot \partial^\mu \boldsymbol{\alpha}) \boldsymbol{\tau} = 2\partial^\mu \boldsymbol{\alpha}. \quad (66)$$

Since an isoscalar product remains unchanged under global SU(2) rotations, it holds for arbitrary SU(2)-doublet functions, A, B, C and D , that

$$\left((UA)^\dagger \boldsymbol{\tau} UB \right) \cdot \left((UC)^\dagger \boldsymbol{\tau} UD \right) = (A^\dagger \boldsymbol{\tau} B) \cdot (C^\dagger \boldsymbol{\tau} D). \quad (67)$$

Thanks to this relation, when we take product of Eq.(65) with itself in L_{int} , four products made of the first two terms are invariant by themselves as

$$(\Phi^\dagger U^\dagger \boldsymbol{\tau} U \partial_\mu \Phi) \cdot (\Phi^\dagger U^\dagger \boldsymbol{\tau} U \partial^\mu \Phi) = (\Phi^\dagger \boldsymbol{\tau} \partial_\mu \Phi) \cdot (\Phi^\dagger \boldsymbol{\tau} \partial^\mu \Phi), \quad (68)$$

and so forth. The product of the third term with itself is $O(\alpha^2)$. In the cross products of the first two terms with the third term $2i(\Phi^\dagger \Phi) \partial^\mu \boldsymbol{\alpha}$, we may set $U = 1$ since we are computing to $O(\alpha)$. Dividing these terms of $O(\alpha)$ in the numerator of δL_{int} by $4(\Phi^\dagger \Phi)$, we obtain that the variation of L_{int} is equal to

$$+\frac{i}{2} \lambda (\Phi^\dagger \boldsymbol{\tau} \overset{\leftrightarrow}{\partial}_\mu \Phi) \cdot \partial^\mu \boldsymbol{\alpha} + O(\alpha^2), \quad (69)$$

which cancels δL_0 for $\lambda = 1$.

The proof to all orders of $\boldsymbol{\alpha}$ is not difficult though a bit tedious. We can carry it out with brute force using the local rotation matrix U for the SU(2) doublet matter fields,

$$U = \cos \alpha + i(\hat{\boldsymbol{\alpha}} \cdot \boldsymbol{\tau}) \sin \alpha, \quad (70)$$

where $\hat{\alpha} = \alpha/\alpha$. Alternatively, in the case of bosons, we could introduce the auxiliary fields and integrate over them to reach the Lagrangian Eq.(62). Operationally, this turns out to be a much simpler avenue. While its physical meaning is subject to debate or some people feel it questionable, we can use the auxiliary field method as a mathematical tool of manipulation without a problem. If one wants to proceed along that line, one starts with

$$L_{tot} = (\partial^\mu + i\mathbf{A}^\mu\Phi)^\dagger \cdot (\partial_\mu + i\mathbf{A}_\mu)\Phi - m^2\Phi^\dagger\Phi + \frac{1}{2}\mu^2\mathbf{A}^\mu\mathbf{A}_\mu, \quad (71)$$

where $\mathbf{A}^\mu = \frac{1}{2}\tau_a A_a^\mu$. Although we do not really need it here, we have added the mass term μ^2 to \mathbf{A}^μ for gauge fixing, which is to be removed after functional interaction is completed.

Having seen the Lagrangian of Eq.(62), it is tempting to speculate that if the isospin $\frac{1}{2}\tau_a$ is replaced by the $n \times n$ matrices of the generator T_a of some other group G , we could obtain the non-Abelian extension to the case where the matter fields form the n -dimensional multiplets of group G . Namely,

$$L_{tot} = \sum_i \partial^\mu \Phi^{i\dagger} \partial_\mu \Phi^i - m^2 \Phi^{i\dagger} \Phi^i + \lambda \frac{\sum_i (\Phi^{i\dagger} T_a \overset{\leftrightarrow}{\partial}_\mu \Phi^i) \cdot \sum_j (\Phi^{j\dagger} T_a \overset{\leftrightarrow}{\partial}^\mu \Phi^j)}{\sum_k (\Phi^{k\dagger} \Phi^k)}, \quad (\lambda \rightarrow 1), \quad (72)$$

where $T_a \neq \frac{1}{2}\tau_a$. Unfortunately, this does not work. *The Lagrangian of Eq.(72) is not gauge invariant.* We can pinpoint the step where the proof fails in this attempt: The relation of Eq.(66) is crucial in achieving non-Abelian gauge invariance in the Lagrangian Eq.(62). This relation holds only for the SU(2) doublet.

Some may yet wonder why one cannot resort to the auxiliary field trick starting with

$$L_{tot} = (\partial^\mu + i\mathbf{A}^\mu\Phi)^\dagger \cdot (\partial_\mu + i\mathbf{A}_\mu)\Phi - m^2\Phi^\dagger\Phi, \quad (73)$$

where $\mathbf{A}_\mu = T_a A_\mu^a$. The equation of motion for the auxiliary field A_μ^a is to be obtained by solving

$$-i(\Phi^\dagger T_a \overset{\leftrightarrow}{\partial}_\mu \Phi) + \Phi^\dagger \{T_a, T_b\} \Phi A_\mu^b = 0. \quad (74)$$

The $n \times n$ matrix $\{T_a, T_b\}$ is not proportional to a unit matrix except in the case of $T_a = \frac{1}{2}\tau_a$. In fact, its determinant is zero in most cases. Consequently, the set of the algebraic equations Eq.(74) is generally unsolvable. This same problem derails an attempt to integrate over the field A_μ^a to get an effective action in terms of Φ and Φ^\dagger alone. We have illustrated this difficulty by two examples in Appendix B.

When one attempts diagram calculation with the wrong Lagrangian of Eq.(72), one could tune the location of a pole in the chain of the bubble diagrams to zero by setting λ off

unity. However, when one proceeds to calculate the coupling of $\Phi^\dagger \Phi A_\mu A^\mu$ (see Fig. 3), the Lagrangian of Eq.(72) would generate the form

$$\Phi^\dagger \Phi \mathbf{A}_\mu \cdot \mathbf{A}^\mu, \quad (75)$$

where the structure $\mathbf{A}_\mu \cdot \mathbf{A}^\mu$ arises from the denominator of L_{int} and enters the center of the triangular loop in Fig. 3. However, the correct non-Abelian structure for these couplings ought to be

$$\Phi^\dagger \{T_a, T_b\} \Phi A_\mu^a A^{\mu b}. \quad (76)$$

This conflict is another manifestation of the fact that the Lagrangian of Eq.(72) is not gauge invariant.

These arguments are more than what we really need, but they hopefully clarify the special role of the SU(2) doublet matter fields when we attempt to write a *local* non-Abelian gauge invariant Lagrangian with matter fields alone. We have not succeeded in finding such a Lagrangian in a reasonably simple form except for the SU(2) doublet matters.

B. Non-Abelian fermionic model

The non-Abelian extension is possible for the fermionic model if one follows the bosonic model given above. For the SU(2) gauge group where the Dirac fields form SU(2) doublets with N copies,

$$\begin{aligned} \Psi^i &= \begin{pmatrix} \psi_1^i \\ \psi_2^i \end{pmatrix} \\ \bar{\Psi}^i &= (\bar{\psi}_1^i, \bar{\psi}_2^i), \quad (i = 1, 2 \dots N), \end{aligned} \quad (77)$$

the gauge invariant Lagrangian is given by

$$\begin{aligned} L_0 &= \sum_{i=1} \bar{\Psi}^i (i \not{\partial} - m) \Psi^i \\ L_{int} &= -i\lambda \frac{\sum_i (\bar{\Psi}^i \boldsymbol{\tau} \gamma_\mu \Psi^i) \cdot \sum_j (\bar{\Psi}^j \boldsymbol{\tau} \overleftrightarrow{\partial}^\mu \Psi^j)}{2 \sum_k (\bar{\Psi}^k \Psi^k)}, \quad (\lambda \rightarrow 1). \end{aligned} \quad (78)$$

Gauge invariance can be proved in parallel to the bosonic model although the auxiliary field method never leads us to this Lagrangian. To the first order in $\boldsymbol{\alpha}(x)$ under the space-time

dependent rotation $\Psi \rightarrow \exp(i\boldsymbol{\tau} \cdot \boldsymbol{\alpha}(x)/2)\Psi$ and its conjugate, the gauge variations are given by

$$\begin{aligned}\delta L_0 &= -\frac{1}{2}(\bar{\Psi}\gamma_\mu\boldsymbol{\tau}\Psi) \cdot \partial^\mu \boldsymbol{\alpha} + O(\alpha^2), \\ \delta L_{int} &= -\lambda\delta L_0. \quad (\lambda \rightarrow 1)\end{aligned}\tag{79}$$

We can prove the gauge invariance to all orders of $\boldsymbol{\alpha}(x)$ using Eq.(70). In fact, a brute-force proof to all orders of α is mathematically less cumbersome for the fermionic model than for the bosonic model.

Just as in the case of bosonic matters, this simple form of the non-Abelian model is possible only for the doublet matter fields in SU(2) gauge symmetry. It should be emphasized that our non-Abelian fermionic model cannot be obtained from the Lagrangian of nonpropagating auxiliary vector fields.

C. Noether current

As it happens in the Abelian models, the Noether current does not exist in our bosonic nor fermionic non-Abelian models. The reason is the same as in the Abelian case: For the Lagrangians with the matter fields alone, the contributions to the Noether current from L_0 and L_{int} cancel each other as a very consequence of gauge invariance. The proof in Appendix A can be trivially extended to the non-Abelian models. Even without such a general proof, the Noether currents off the gauge symmetry limit, which are given below, clearly show their absence in the gauge symmetry limit.

The Noether current exists off the gauge symmetry limit. Following the standard prescription, we obtain the Noether currents from our Lagrangians of Eqs.(62) and (78) in the form,

$$\begin{aligned}\mathbf{J}_\mu^N &= i(1-\lambda)\Phi^\dagger \frac{\boldsymbol{\tau}}{2} \overleftrightarrow{\partial}_\mu \Phi, \quad (\text{bosonic}) \\ \mathbf{J}_\mu^N &= (1-\lambda)\bar{\Psi} \frac{\boldsymbol{\tau}}{2} \gamma_\mu \Psi. \quad (\text{fermionic})\end{aligned}\tag{80}$$

As for the energy-momentum tensor, the conserved tensor operator exists for any value of λ just as in the U(1) models.

D. Composite gauge bosons

In the case of the SU(2)-doublet matter fields, the non-Abelian diagram calculation is almost identical with the Abelian one. The only difference is in the insertion of the $\boldsymbol{\tau}$ matrix at every point of vectorial interactions in Fig. 1 and Fig. 7. The massless composite bosons emerge in the $J^{PC} = 1^{--}$ channels of the adjoint representation of SU(2). In the case of fermion matter the composite massless bosons appear in the 3S_1 eigenchannel, that is, they couple only through $\bar{\Psi}\boldsymbol{\tau}\gamma_\mu\Psi$. The correct properties of the massless bound states are confirmed just as in the Abelian cases.

We summarize the difference of the SU(2)-doublet models from the Abelian models:

(A) For the non-Abelian models of SU(2)-doublet matter fields, the vacuum expectation value $I_0^b = \langle 0|\Phi^\dagger\Phi|0\rangle$ and $I_0^f = \langle 0|\bar{\Psi}\Psi|0\rangle$ are twice as large as their Abelian values, respectively, since both the upper and lower components of the doublet matter contribute.

(B) The bubble diagrams entering the kernel K of the iteration equation are scaled upward by the same factor of two since a trace is taken within the bubble loop; $\text{tr}(\tau_a \cdot \tau_b) = 2\delta_{ab}$.

(C) Since the multiplication of the factor two in (A) and (B) occurs in both the numerator and the denominator of the kernel K in Eq.(31) and Eq.(53), it keeps the kernel K unchanged from the Abelian value. Meanwhile, the lowest-order T-matrix, T^0 , is scaled down by factor two since it is inversely proportional to I_0^b (I_0^f). So is the amplitude $T = (I - K)^{-1}T_0$.

Since the kernel $K^{\mu\nu}$ remains unchanged, $(I - K)$ is still transverse and starts with a term proportional to $g^{\mu\nu}q^2 - q^\mu q^\nu$ with the same nonvanishing coefficient. Consequently the solution for the iterated amplitude T takes the same form as in the corresponding Abelian models, but the residue at $q^2 = 0$ is half as large, reflecting the fact that the lowest-order term T^0 is scaled down by factor two.

Summing up this argument, the location of the pole at $q^2 = 0$ remains the same and its residue is scaled down by factor two, relative to the Abelian models, for both the bosonic and the fermionic model. We describe below some more details specific to each of the non-Abelian models.

The bosonic model

We compute the chain of bubble diagrams as shown in Fig. 1 where the $\boldsymbol{\tau}$ -matrices are inserted at every point of interaction. The residue at the massless pole is compared with

that of the corresponding Feynman diagram computed with the standard Lagrangian of the SU(2) gauge symmetry,

$$L_{int} = ig_2 \left(\Phi^\dagger \mathbf{A}^\mu \partial_\mu \Phi - \partial_\mu \Phi^\dagger \mathbf{A}^\mu \Phi \right) + g_2^2 \Phi^\dagger (\mathbf{A}^\mu \cdot \mathbf{A}_\mu) \Phi, \quad (81)$$

where $\mathbf{A}^\mu = \frac{1}{2} \tau_a A_a^\mu$. We obtain the gauge coupling of the composite SU(2) gauge bosons \mathbf{A}^μ to the matter fields,

$$\frac{g_2^2}{4\pi} = \frac{96\pi^2}{N \ln(\bar{\Lambda}^2/m^2)}, \quad (82)$$

when it is expressed with the cutoff $\bar{\Lambda}$ in the space-time dimension of four.⁹ Recall that the standard definition of g_2 accompanies the generators $\frac{1}{2}\boldsymbol{\tau}$ instead of just $\boldsymbol{\tau}$. (See the definition of \mathbf{A}^μ following Eq.(81). In the leading $1/N$ order, the magnitude of coupling Eq.(82) coincides with what one would obtain in the auxiliary field trick since it comes from the same single bubble diagram with $\boldsymbol{\tau}$ on the both ends.

The four-body interaction $\Phi^\dagger \Phi A_\mu A^\mu$ can be computed with the second term of the expansion for $1/(\Phi^\dagger \Phi)$ around its vacuum value in L_{int} , namely,

$$-\frac{1}{4(I_0^b)^2} (\Phi^\dagger \boldsymbol{\tau} \overset{\leftrightarrow}{\partial}_\mu \Phi) \cdot (\Phi^\dagger \boldsymbol{\tau} \overset{\leftrightarrow}{\partial}^\mu \Phi) (: \Phi^\dagger \Phi :). \quad (83)$$

Attaching chains of bubbles to $(\Phi^\dagger \boldsymbol{\tau} \overset{\leftrightarrow}{\partial}^\mu \Phi)$ and $(\Phi^\dagger \boldsymbol{\tau} \overset{\leftrightarrow}{\partial}_\mu \Phi)$ of this interaction and approaching the zero momentum limit, we obtain g_2^4 , of which g_2^2 is assigned to the gauge couplings of two composite gauge bosons with the external $\Phi^\dagger \boldsymbol{\tau} \Phi$ at the outer ends of the chains and the remaining g_2^2 is assigned to the $\Phi^\dagger \Phi A_\mu A^\mu$ coupling. This step is a repeat of what we have done for the Abelian model depicted in Fig. 3 and Fig. 4. Going through this computation, we find that the resulting g_2^2 for $\Phi^\dagger (\mathbf{A}^\mu \cdot \mathbf{A}_\mu) \Phi$ is equal to the value given in Eq.(82), as we expect.

For the non-Abelian gauge bosons, there must be the triple self-coupling and the quartic self-coupling. They are computed with the diagrams of Fig.5 and Fig.6 after inserting the τ -matrices appropriately. The triple self-coupling diagrams, of course, do not cancel among themselves in the non-Abelian case. Charge conjugation invariance allows the triple self-coupling since the non-Abelian charge flowing in the opposite directions in a pair of triangular diagrams survives with $\tau_a \tau_b - \tau_b \tau_a = 2i\epsilon_{abc} \tau_c \neq 0$. Paying attention to the subtlety of the

⁹ For $\bar{\Lambda}$, see Eq.(36) and the line following it.

linear divergence that has been cautioned earlier, we find that the value obtained for the triple self-coupling agrees with what the SU(2) gauge symmetry requires by $-\frac{1}{4}\mathbf{G}_{\mu\nu} \cdot \mathbf{G}^{\mu\nu}$. The quartic self-coupling arises from the diagrams with four-corner, three-corner and two-corner loops at the center (i.e., Fig. 6a, 6b, and 6c) and survives in the limit of zero external momenta. They have the correct magnitude and group structure as required by the SU(2) gauge symmetry.

All this should not be surprising after we have found a triplet of spin-one massless bound states out of the manifestly gauge invariant Lagrangian. Once we have found that the effective fields of these bound states couple with the matter fields in the form

$$L_{int} = ig_2(\Phi^\dagger \mathbf{A}^\mu \partial_\mu \Phi - \partial_\mu \Phi^\dagger \mathbf{A}^\mu \Phi), \quad (84)$$

with $\mathbf{A}^\mu = \frac{1}{2}\tau_a A_a^\mu$, all other couplings of \mathbf{A}_μ necessary to satisfy the SU(2) gauge invariance ought to be generated by loop and chain diagrams in the same $1/N$ order. Otherwise the models would violate the SU(2) gauge invariance that has been embedded in Lagrangian at the beginning. We know no other way to be compatible with the SU(2) gauge symmetry once the interaction of Eq.(84) emerges.

The fermionic model

Let us turn to the fermionic model. While presence of two $J^{PC} = 1^{--}$ channels requires 2×2 matrix calculation, the diagram computation of the bound-state generation is identical with that of the Abelian case except for insertion of the $\boldsymbol{\tau}$ matrices into the 2×2 matrix equation of Fig. 7 after replacing the boson lines with the fermion lines. Massless bound states appear in the 3S_1 channel here again and the squared SU(2) gauge coupling expressed in g_2^2 is larger than that of the U(1) fermionic model by factor two just as in the bosonic case:

$$\frac{g_2^2}{4\pi} = \frac{24\pi^2}{N \ln(\bar{\Lambda}^2/m^2)}, \quad (85)$$

where the coupling g_2 is defined by

$$L_{int} = -g_2 \bar{\Psi} \gamma_\mu \mathbf{A}^\mu \Psi. \quad (86)$$

When we work on the other couplings of dimension four, we do not encounter any complication new to the non-Abelian symmetry. The reason is that the massless bound states couple to the matter fields only through the vertex of $(\bar{\Psi} \gamma_\mu \boldsymbol{\tau} \Psi)$, not through $(\bar{\Psi} \boldsymbol{\tau} \overleftrightarrow{\partial}_\mu \Psi)$. Therefore

the computation of the triple and quartic self-couplings can be carried out in the same way as in the U(1) model. The relevant diagrams are those of Fig. 5 and Fig. 6 where the boson lines are replaced with the fermion lines. Since the composite bound states generated in the chains of bubbles couple with the fermions only through $(\bar{\Psi}\boldsymbol{\tau}\gamma_\mu\Psi)$, not through $(\bar{\Psi}\boldsymbol{\tau}\overset{\leftrightarrow}{\partial}^\mu\Psi)$, the vertices of the triangle (Fig. 5) and the box (Fig. 6a) at the center of diagram are only those of γ_μ , not of $\overset{\leftrightarrow}{\partial}_\mu$. The diagrams of Fig. 6b and Fig. 6c do not contribute since the six-body interaction $(\bar{\Psi}\boldsymbol{\tau}\gamma_\mu\Psi)(\bar{\Psi}\boldsymbol{\tau}\overset{\leftrightarrow}{\partial}^\mu\Psi)(\bar{\Psi}\Psi)$ is incapable of producing two composite bosons. (Recall the argument in the Abelian fermionic model.) As for the fermionic triangular and box diagrams corresponding to Figs. 5 and 6a, the same large- N computation was actually carried out twenty years ago in a similar model[20] that contains an explicit gauge-symmetry breaking but only through the gauge boson mass. We do not repeat the calculation of the self-couplings for the non-Abelian fermionic model here. The bottom line is that the same coupling g_2 as the matter-gauge-boson coupling of Eq.(85) appears in the self-interaction of the gauge bosons as we expect.

All these beautiful outcomes conforming to non-Abelian gauge symmetry are manifestation of gauge invariance that is embedded in the Lagrangian at the beginning. Hoping that we are not overly repetitious, we emphasize that once the massless bound states of spin-one appear and their effective fields \mathbf{A}^μ couple with the matter fields like $g_2\bar{\Psi}\gamma_\mu\mathbf{A}^\mu\Psi$, the bound states must be gauge bosons and the associated gauge self-couplings of \mathbf{A}^μ in $-\frac{1}{4}\mathbf{G}_{\mu\nu}\mathbf{G}^{\mu\nu}$ must be generated in order to satisfy SU(2) gauge invariance of L_{tot} . We know no other way to realize the non-Abelian gauge invariance.

VI. DISCUSSION

We start the final section with an obvious observation common to all of our models. In our models we cannot introduce an elementary gauge field by the method of the substitution rule $\partial_\mu \rightarrow \partial_\mu + ieA_\mu$ in our Lagrangian. The reason is obvious by the structure of the models: This substitution operation is nothing other than one special gauge transformation. Take for example the fermion fields ψ in our U(1) Lagrangian. The substitution $\partial_\mu\psi \rightarrow (\partial_\mu + ieA_\mu)\psi$ is realized by the rotation

$$\psi(x) \rightarrow \exp(ie \int^x A_\mu(y)dy^\mu)\psi(x). \quad (87)$$

Since Eq.(87) is one of the gauge transformations with

$$\alpha(x) = e \int^x A_\mu(y) dy^\mu, \quad (88)$$

the function $\alpha(x)$ is canceled out between L_0 and L_{int} by gauge invariance and disappears from Lagrangian entirely. Therefore the elementary A_μ field cannot be introduced into our Lagrangians in this way. Inability to introduce the elementary A_μ field in our Lagrangians by the so-called substitution rule is closely in parallel with vanishing of the Noether current.

The next observation concerns the no-go theorem of Weinberg and Witten. The theorem was stated in the following way [11]:

Theorem *A theory that allows the construction of a Lorentz-covariant conserved four-vector current J^μ cannot contain massless particles of spin $j > 1/2$ with nonvanishing values of the conserved charge $\int J^0 d^3x$.*

The proof is simple. Fix first the Lorentz scalar value of the matrix element $\langle p' | J_\mu | p \rangle$ for the massless spin-one particle in the forward limit $p' \rightarrow p$. Then make a Lorentz transformation and examine its rotational property around the momentum \mathbf{p} in the brick-wall frame ($\mathbf{p}' = -\mathbf{p}$). We need the conserved current J_μ that provides the Lorentz scalar charge $\int J_0 d^3x$.

The theorem holds whether the massless boson is elementary or composite. As was emphasized by the authors[11], however, the theorem does not apply to the standard non-Abelian gauge bosons (without spontaneous symmetry breaking). The catch is in the word “Lorentz-covariant”. The state of zero helicity does not exist for massless gauge bosons. In order to make the theory manifestly Lorentz covariant and gauge invariant at the same time, one has to fix a gauge by introducing an unphysical ghost state in the Lagrangian. Otherwise, one cannot carry out diagram calculation. Fixing a gauge by a subsidiary condition either violates manifest gauge invariance or introduces a state that does not exist as a physical particle state. Therefore, Lorentz scalar charges that meet the conditions of the Theorem do not exist in the standard non-Abelian gauge theory.¹⁰

What should we do with this theorem for our non-Abelian models ? If we could write the

¹⁰ If one takes the purist viewpoint that the initial and final states of the matrix element $\langle p' | J_\mu | p \rangle$ must be asymptotic states, the theorem does not apply to the non-Abelian gauge theory like QCD, which is singular in the infrared limit so that one-gluon states are not asymptotic states. Our non-Abelian models contain $N(\rightarrow \infty)$ doublets of matter particles so that the infrared limit is nonsingular, i.e., not confining.

non-Abelian Noether currents with the matter fields alone, we would potentially interfere with this theorem. However, the Lorentz-covariant conserved currents do not exist in our models. They exist only off the gauge symmetry limit ($\lambda \neq 1$) and disappear as we go to the gauge symmetry limit of $\lambda = 1$, only at which point the vector bound states become massless. We thus circumvent the theorem. Is this really the answer to the potential conflict of the composite non-Abelian gauge bosons with the Weinberg-Witten theorem ? To be frank, the present author is not totally comfortable with this answer. But it appears in our models that generation of the massless non-Abelian composite bosons evades the conflict with the Weinberg-Witten theorem.¹¹

It is explicitly visible in our models that gauge invariance requires that the force in the 1^{--} channel be attractive ($\lambda > 0$) and that the bound state in this channel be massless ($\lambda \rightarrow 1$). Repulsive forces ($\lambda < 0$) cannot be gauge invariant. We are tempted to speculate that *even if gauge fields are not introduced explicitly, gauge bosons must appear as composite states if a theory is gauge invariant*. While it sounds like a trivial proposition, it is desirable to elevate it to a rigorous theorem of field theory.

One obvious question is whether our models have anything to do with the real world. At an early stage of the electroweak theory, people discussed the possibility of composite W and Z.[18, 19] A quarter century ago the present author also joined to propose an unrenormalizable phenomenological model of composite W and Z bosons which an explicit symmetry breaking enters only through the W/Z masses [20, 21]. It was the time right after the experimental confirmation of the W and Z bosons by accelerator[22, 23]. At that time very little was known experimentally about the properties of W and Z . One sensitive theoretical test was to study how much deviation from the gauge symmetry could be accommodated for the self-couplings of dimension four through their loop contributions[24]. More general test irrespective of sources was proposed [25] and is still being used for experimental test of the minimal standard model. Now the Higgs boson has been discovered with its properties roughly in agreement with the theoretical expectation, the next step is to raise precision in the interaction of W and Z by direct measurement. The early indication of the two-photon anomaly at 750GeV is one example that may open up a new window. However, since the

¹¹ The W and Z bosons in the extra dimension model[14] are the lowest lying Kaluza-Klein modes with mass so that they do not conflict with the theorem.

invariant mass of 750GeV is near the upper end of the two-photon phase space in the current data and “the anomaly” is still no more than a three-standard-deviation effect even with the ATLAS and CMS data combined, we need to wait some time before a consensus is reached among experimentalists on this anomaly. Both experimentalists and phenomenologists are working toward to this goal [26, 27].

When our model is expressed as a composite gauge theory with the effective fields \mathbf{A}_μ , difference from the minimal standard model would appear in the interactions of dimension higher than four which are suppressed by powers of $p^2/\bar{\Lambda}^2$ at $|p^2| < \bar{\Lambda}^2$. When experiment explores the region of energies comparable or higher than $\bar{\Lambda}$, shall we be able to discriminate directly our model Lagrangian from the standard model of W and Z . But we have no theoretical basis to speculate on magnitude of $\bar{\Lambda}$ at present.

We conclude with one disturbing question to which we give no good answer. Is it really possible to tell experimentally or even theoretically whether a given particle is elementary or composite ? This is a nagging question that confronted theorists[28] at the height of nuclear democracy in the early 1960’s. Theorists proposed various criteria of compositeness, but no consensus emerged. Although we have started with the matter fields alone and constructed the massless gauge bosons explicitly as their bound states, can’t we describe exactly the same physics with some other Lagrangian in which all particles are elementary ? Can we really answer the question of elementarity vs compositeness once for all ?

The following theorem was given by Kamefuchi, O’Rafeartaigh and Salam[29] in 1961: If a composite local operator carries all quantum numbers of a given particle in regard to space-time (J^{PC}) and other properties (charge, isospin etc), it gives the same S matrix amplitudes on the particle mass shell up to overall normalizations. Difference shows up only off the mass shell. But the “off-shell amplitudes” are not really scattering amplitudes of the particle, but include continuum contributions. According to this theorem, therefore, the definition of particle fields is infinitely ambiguous with respect to their continua. When a different particle field is used, its interaction Lagrangian takes a different form. To avoid this ambiguity and the issue of renormalizability, we were tempted to replace the field theory with the S-matrix theory in the 1960’s so as to deal only with the on-shell amplitudes and the observables. As we know, it led us to the dual resonance model and then back to Lagrangian theory of strings with the Nambu-Goto action.

Meanwhile, the present author has been brought attention to one interesting observation

in supersymmetric theory. Along the line of the Olive-Montonen conjecture, Seiberg and Witten[30] showed in the $N = 2$ supersymmetric theory that the strong and weak coupling limits are dual to each other. To be more specific, the roles of a particle and a soliton of the same spin-parity are interchanged between the strong and weak limits of coupling. Since solitons are composite in everyone's picture, in such theories elementarity vs compositeness loses its absolute meaning. It depends on the strength of coupling. The similar duality was shown earlier for a model of $N=4$ too.[31] Proof of this duality relies on the simple holomorphy special to supersymmetry. If something similar holds in nonsupersymmetric theory as well, the meaning of elementarity and compositeness of particles would finally disappear and the naming would become just for a matter of convenience; if Lagrangian takes the simplest form with a certain choice for a set of particle fields, one would call such particles as *elementary* for convenience.

Appendix A: Nonexistence of Noether current

The Noether current does not exist in the theories that satisfy local gauge invariance with matter fields alone. The proof is almost trivial. We give it here only for the $U(1)$ bosonic model since extension to fermions and non-Abelian theories is straightforward.

Under the $U(1)$ gauge transformation, the Lagrangian satisfies the local invariance,

$$L(e^{-i\alpha(x)}\phi^*, e^{i\alpha(x)}\phi) = L(\phi^*, \phi), \quad (\text{A1})$$

where $\alpha(x)$ is an arbitrary function of space-time that satisfies mild conditions such as differentiability. The copy index i ($= 1, \dots, N$) has been suppressed in Eq.(A1). For the infinitesimal $\alpha(x)$, gauge invariance requires

$$\begin{aligned} -i\left(\phi^* \frac{\partial L}{\partial \phi^*} + \partial_\mu \phi^* \frac{\partial L}{\partial (\partial_\mu \phi^*)}\right)\alpha + i\left(\frac{\partial L}{\partial \phi} \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\mu \phi\right)\alpha \\ + i\left(-\phi^* \frac{\partial L}{\partial (\partial_\mu \phi^*)} + \frac{\partial L}{\partial (\partial_\mu \phi)} \phi\right)\partial_\mu \alpha = 0. \end{aligned} \quad (\text{A2})$$

Since $\alpha(x)$ and $\partial_\mu \alpha(x)$ are two independent functions when $\alpha(x)$ is an arbitrary function of x_μ , the condition of Eq.(A2) requires that the terms proportional to $\alpha(x)$ and to $\partial_\mu \alpha(x)$ must be separately equal to zero. After use of the equations of motion, the coefficient of $\alpha(x)$ equal zero gives

$$-\partial_\mu \left(\phi^* \frac{\partial L}{\partial \phi_\mu^*}\right) + \partial_\mu \left(\phi \frac{\partial L}{\partial \phi_\mu}\right) = 0. \quad (\text{A3})$$

Normally this would be the statement of conservation of the Noether current, $\partial^\mu J_\mu^N = 0$. However, the third term proportional to $\partial_\mu \alpha(x)$ in Eq.(A2) gives

$$-\frac{\partial L}{\partial(\partial_\mu \phi^*)}\phi^* + \frac{\partial L}{\partial(\partial_\mu \phi)}\phi = 0. \quad (\text{A4})$$

This is nothing other than the statement of

$$J_\mu^N \equiv 0 \quad (\text{A5})$$

at all space-time locations. In the case that the elementary gauge field A_μ exists in Lagrangian, the gauge transformation $A_\mu \rightarrow A_\mu + i\partial_\mu \alpha$ generates an additional term proportional to $\partial_\mu \alpha(x)$ and adds to the third term in Eq.(A2) to cancel exactly the variation due to ϕ/ϕ^* . This cancellation is nothing other than gauge invariance itself. Consequently, Eq.(A5) does not follow in the conventional gauge theory. Extension of this proof to the fermion models and the non-Abelian models is just as simple and easy.

Despite this general proof of $J_\mu^N \equiv 0$, some may wonder if it is possible to define a conserved current in the gauge symmetry limit by factoring out $(1 - \lambda)$ from the current J_μ defined by Eq.(19) off the gauge limit ($\lambda \neq 1$) and then going to the limit of $\lambda = 1$. If physics is somehow “continuous” in this respect in the neighborhood of $\lambda = 1$, this might allow us to circumvent the difficulty. That is, choose as a conserved current simply the current

$$J'_\mu = i \sum_i (\phi_i^* \overleftrightarrow{\partial}_\mu \phi_i), \quad (\text{A6})$$

so that the charge is $Q \equiv \int J'_0 d^3x$. This charge is not gauge invariant, but let us leave it aside for a moment. If one computes by brute force the divergence of this current J'_μ with the equation of motion, one would not be led to $\partial^\mu J'_\mu = 0$. Instead one would end up with the trivial circular identity as follows: Since $\partial^\mu J'_\mu = i \sum_i (\phi_i^* \square \phi_i - \square \phi_i^* \phi_i)$, one multiplies the equation of motion for ϕ_i with the field ϕ_i^* and subtracts the corresponding bilinear object with $\phi_i \leftrightarrow \phi_i^*$. Then the result is a trivial identity: $i \sum_i (\phi_i^* \square \phi_i - \phi_i^* \square \phi_i) = i \sum_i (\phi_i^* \square \phi_i - \square \phi_i^* \phi_i)$. Therefore the conclusion from this exercise is as follows: Only when one violates gauge invariance by staying away from the symmetry limit ($\lambda \neq 1$), can the Noether theorem define a conserved current in the familiar form with strength reduced by $(1 - \lambda)$.

The same happens for our fermion model. Just as in the bosonic model, the current

$\sum_i \bar{\psi}_i \gamma_\mu \psi_i$ is not the conserved Noether current in the gauge symmetry limit.¹² The equation of motion of L_{tot} does not allow us to compute $\partial^\mu (\bar{\psi} \gamma_\mu \psi)$ in the gauge symmetry limit: Such computation drives us around a circular loop just as in the case of bosons.

In the perturbative diagram calculation which is performed in the interaction picture, however, the fields obey the equation of *free* motion. Therefore $\phi^* \overset{\leftrightarrow}{\partial}_\mu \phi$ and $\bar{\psi} \gamma_\mu \psi$ are both divergence free, that is, conserved currents.

Appendix B: Difficulty in general non-Abelian models

The local Lagrangian of matter fields alone has been easily obtained by the auxiliary gauge fields method for the SU(2) model with the doublet matter. But we cannot extend it to other groups and representations. We show it here with two explicit examples.

Let us start with the Lagrangian of the nonpropagating auxiliary gauge fields,

$$L = \Phi^\dagger (\overset{\leftarrow}{\partial}^\mu - i\mathbf{A}^\mu) (\partial_\mu + i\mathbf{A}_\mu) \Phi - m^2 \Phi^\dagger \Phi + \frac{1}{2} \mu^2 A_{a,\mu} A_a^\mu, \quad (\mu^2 \rightarrow 0) \quad (\text{B1})$$

where Φ and Φ^\dagger are the column and row fields belonging to the n -dimensional representation of group G . We have absorbed the coupling e into \mathbf{A}_μ . Let the group G be induced by the generators T_a ($a = 1, \dots, k$), which are $n \times n$ matrices. We represent the nonpropagating gauge fields A_a^μ ($a = 1 \dots k$) in the $n \times n$ matrices,

$$\mathbf{A}^\mu = \sum_{a=1}^k T_a A_a^\mu. \quad (\text{B2})$$

The Lagrangian Eq.(B1) is invariant under the local gauge transformation,

$$\begin{aligned} \Phi &\rightarrow U\Phi, \\ A^\mu &\rightarrow UA^\mu U^\dagger - i(\partial^\mu U)U^\dagger. \end{aligned} \quad (\text{B3})$$

where $U = \exp(iT_a \alpha_a)$. In order to integrate the exponentiated action of L over A_a^μ , we combine the terms bilinear and linear in A_a^μ into a quadrature and “shift the origin”. In the case of the SU(2)-doublet matter fields, we see with $\{\tau_a, \tau_b\} = 2\delta_{ab}$ that the coefficients of the bilinear terms of A_a^μ are simply $\delta_{ab} \Phi^\dagger \Phi$ so that no diagonalization is needed for symmetrized product of the generators $\{T_a, T_b\} = \frac{1}{4} \{\tau_a, \tau_b\} = \frac{1}{2} \delta_{ab}$. Upon integration over

¹² Unlike the corresponding object in the bosonic case, this current is gauge invariant.

A_a^μ , the denominator of $L_{int}(\Phi^\dagger, \Phi)$ comes out to be the singlet $\Phi^\dagger\Phi$, as given in Eq.(62). Upon integration, an additional term

$$-2\text{tr}\ln(\Phi^\dagger\Phi) \quad (\text{B4})$$

appears in the effective action. But we may remove this term since it is gauge-invariant by itself. We retain the remainder as the gauge-invariant Lagrangian in terms of Φ^\dagger/Φ .

However, this procedure does not work in the cases other than the SU(2) doublet. When $\{T_a, T_b\} \not\propto \delta_{ab}I$, it happens that the integral over \mathbf{A}_μ is generally impossible. Even if it were possible, the *trace-log* term would not be invariant by itself under rotations of group G, not even under global rotations. While the whole action is gauge invariant, it is not separately so for the effective Lagrangian and the *trace-log* term. Unfortunately, this is what happens in the cases other than the SU(2) doublet. We show two simple examples below.

Let us first examine the case of the real triplets of SU(2). In this case the coefficient of the bilinear terms of A_a^μ ($a = 1, 2, 3$) is written in terms of the 3×3 matrices $(T_a)_{bc} = -i\varepsilon_{abc}$ and the matter fields $\Phi = (\phi_1, \phi_2, \phi_3)^t$ and $\Phi^\dagger = \Phi^t$. The bilinear terms of A_a^μ is given by

$$(\Phi^t T_a T_b \Phi) A_a^\mu A_{b,\mu}. \quad (\text{B5})$$

It can be diagonalized by the orthogonal transformation $A'_\mu = \mathbf{O} A_\mu$ into

$$(A_1'^\mu, A_2'^\mu, A_3'^\mu) \begin{pmatrix} \Phi^t\Phi & 0 & 0 \\ 0 & \Phi^t\Phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A_{1\mu}' \\ A_{2\mu}' \\ A_{3\mu}' \end{pmatrix}. \quad (\text{B6})$$

When this is placed in the action and exponentiated, we cannot integrate it over the third component of \mathbf{A}'_μ since the action is flat along that direction (at $\mu \rightarrow 0$). The action blows up as $\mu \rightarrow 0$ and there is no way to keep it well-defined.

How about the SU(3)-triplet matter fields as the next-to-simplest example? For the triplet matter fields, the bilinear terms in A_a^μ ($a = 1, \dots, 8$) can be written as

$$A_\mu^a M_{ab} A^{b,\mu}, \quad (\text{B7})$$

where $M_{ab} = \frac{1}{8}\Phi^\dagger\{\lambda_a, \lambda_b\}_+\Phi$ is a symmetric matrix under $a \leftrightarrow b$. The matrix M_{ab} can be diagonalized into \mathbf{D} by some orthogonal rotation \mathbf{O} as

$$(A'_\mu)^t \mathbf{O}^t M \mathbf{O} A'^\mu = A'_{a,\mu} D_{aa} A_a'^\mu. \quad (\text{B8})$$

Can the diagonal matrix \mathbf{D} be proportional to the unit matrix? If so, the functional integral over A_μ^a would produce a denominator common to all a in L_{int} just as in the case of $SU(2)$. But that is obviously not the case: If $\mathbf{D} \propto \mathbf{I}$, then $M_{ab} = (\mathbf{ODO}^t)_{ab}$ would also have to be proportional to δ_{ab} even before the rotation. We can easily see by simple inspection using the representation $T_a = \frac{1}{2}\lambda_a$ familiar to physicists, that M_{ab} is not proportional to an 8×8 unit matrix. Consequently the resulting Lagrangian in terms of matter fields alone would not take a form as compact as in the $SU(2)$ doublet case, if one could write it at all.¹³

These two examples show that the auxiliary field method can lead to a simple *local* field theory only for the $U(1)$ and the $SU(2)$ -doublet models of bosonic matter fields.

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¹³ This does not conflict with what Rabinovici and Smolkin [16] did for general Lie groups: They integrate over the matter fields in one loop for a general group and representation to show that the $-\frac{1}{4}G_{\mu\nu}G^{\mu\nu}$ is indeed generated. Their purpose is to see whether or not this *Maxwell term* can be generated by loops upon integrating over matter fields in the auxiliary vector-field Lagrangian. They did not address to finding of a *local* non-Abelian gauge invariant Lagrangian written in matter fields alone.

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