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Refinement of the Bousso-Engelhardt Area Law

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Past holographic screens are codimension-one surfaces of indefinite signature that are foliated by marginally anti-trapped surfaces called leaves. Future holographic screens are defined similarly except with marginally trapped leaves. Bousso and Engelhardt recently showed that the leaves of past and future holographic screens have monotonic area. We prove a stronger area law that shows that subregions of leaves also have monotonic area. For every past and future holographic screen, there exists a family of leaf-orthogonal curves called the fibration of the screen. Any region in a leaf can be translated along the fibration to a leaf of larger area. Our result states that the area of the subregion grows as it is translated.

Introduction

Black hole thermodynamics [1–7] is a critical principle that has guided the development of quantum gravity over the past few decades. In particular, Hawking's area theorem displayed parallels between the area of the event horizon of a black hole and entropy. This identification of entropy with area is the heart of the holographic behavior [8, 9] exhibited by gravity.

Recently, Bousso and Engelhardt [10, 11] proved an area law for surfaces called past and future holographic screens that arise in a more general setting than the spacetimes of black holes. These objects are not defined by the global notion of an event horizon and thus provide an example of "quasi-locally" defined surfaces with thermodynamic behavior.

Holographic screens are well-motivated from considerations in quantum gravity. The covariant entropy bound suggests [14, 15] that holographic screens play a role in general spacetimes that is analogous to the AdS boundary[1] in the context of the AdS/CFT correspondence [12, 13]. This hypothesis is supported by the recent demonstration that holographic entanglement entropy [17, 18] can be defined for regions on past and future holographic screens in a way that is consistent with many known properties of entanglement entropy [19].

Below we show that the Bousso-Engelhardt area law can be refined into a more local form. The original area law of [10, 11] states that preferred codimension-2 surfaces called leaves have monotonic area. We show that arbitrary subregions of leaves also have monotonic area. From the point of view of the holographic principle, this provides evidence that degrees of freedom of a holographic description for arbitrary spacetimes are locally distributed and satisfy a local version of the second law of thermodynamics.

[1] Ref. [16] studies a related construction.

Holographic Screens and Area Laws

Fix a globally hyperbolic spacetime of dimension D satisfying the genericity conditions stated in [11]. A *past holographic screen* is a codimension-1 submanifold H of the spacetime that is foliated by marginally anti-trapped surfaces called *leaves*. The foliation into leaves is unique: other splittings of H cannot satisfy the marginally anti-trapped condition. A *future holographic screen* is instead foliated by marginally trapped surfaces.

The area law of [10, 11] is a statement about the evolution of leaves comprising a past or future holographic screen H. We denote the leaves of H by σ_r where r is a smooth parameter. In our notation, we can express the Bousso-Engelhardt area law as the statement that $\|\sigma_r\|$ is monotonic where $\|\cdot\|$ denotes the area functional. By convention, we will always choose the parameter r so that $\|\sigma_r\|$ is increasing.

On a particular leaf σ, let k and l denote the two future-directed null vector fields orthogonal σ. The condition that σ be marginally anti-trapped or marginally trapped can be written in terms of the null expansions θ^k and θ^l in the two directions:

Marginally Anti-Trapped	Marginally Trapped	
$\theta^k = 0$	$\theta^k = 0$	(1)
$\theta^l > 0$	$\theta^l < 0$	

In particular, the marginal condition that $\theta^k = 0$ means that σ is the area-maximizing surface on the geodesic congruence generated by k and $-k$.

We define a vector field h on H by requiring that h is orthogonal to every leaf and by the normalization condition $dr(h) = 1$. The integral curves of h are called the *fibration*[2] of H. If we extend the definition of k and

[2] Note that h need not have definite signature. This is the key distinguishing feature between past (and future) holographic

l to all of H, then $h = \alpha l + \beta k$ where α and β are smooth functions on H. The Bousso-Engelhardt area law was proven by showing that α never changes sign from which equation 1 implies that leaves have increasing area.

Our area law extends this result as follows. Suppose that A_0 is a region in σ_0. We can translate A_0 to a region A_r in σ_r by following the fibration from points in A_0 to σ_r. We will prove that the area of A_r is increasing. This conclusion relies on the fact that the area increase associated with zig-zagging along l and k is a first order effect in r, while the failure of such a zig-zag procedure to follow the fibration is at most a second-order effect.

Relation to the Screen Entanglement Conjecture

Holographic entanglement entropy proposals [17, 18] have recently been conjecturally generalized beyond the context of AdS/CFT by employing past or future holographic screens in arbitrary spacetimes [19]. The proposed construction is to anchor extremal surfaces to the boundaries of subregions of leaves. The properties of past and future holographic screens are sufficient to ensure that the areas of these extremal surfaces satisfy expected properties of entanglement entropy like strong subadditivity. The statement that one fourth of the area of such extremal surfaces is in fact the entanglement entropy of a subsystem in a quantum theory holographically defining the spacetime in which the screen lies is called the "screen entanglement conjecture."

The area law proven in this paper applies to subregions of leaves, the same objects to which an entanglement entropy-like quantity was assigned in [19]. Suppose that A_0 is a region in the leaf σ_0 and A_r is the result of translating A_0 along the fibration to σ_r. Let $S(A_r)$ denote the screen entanglement entropy of A_r as defined above via the extremal surface anchored to ∂A_r. With the exception of cases that are topologically nontrivial, $S(A_r)$ satisfies a "Page bound": $S(A_r) \leq \min(\|A_r\|, \|\sigma_r \setminus A_r\|)$. Our area law applies to the evolution of the subregions A_r and A_r^C and thus causes the Page bound to become less restrictive whenever r is increased. This does not prove that $S(A_r)$ increases monotonically.

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screens and related objects including future outer trapping horizons and dynamical horizons [20–23]. Past and future holographic screens can be regarded as a synthesis such ideas with those of [14].

Proof of the Area Law for Subregions

From here on we will assume that H is a past holographic screen. Our argument can be modified to the case of a future holographic screen in an obvious way. Because H is a past screen,

$$\begin{aligned} \theta^k &= 0 \\ \theta^l &> 0. \end{aligned} \tag{2}$$

Moreover, we now have $\alpha > 0$ on all of H.

To carefully study the evolution of areas of regions in leaves, it is convenient to consider the null surfaces passing through a leaf σ_r. First, extend k and l to a tubular neighborhood of H by following along the geodesics generated by k and l. Now let N_r denote the null surface obtained by starting from points on σ_r and following the integral curves of k in both the $+k$ and $-k$ directions. Let L_r^+ denote the null surface obtained by starting at σ_r and following the integral curves of l only in the $+l$ direction.

We now fix an (arbitrarily chosen) reference leaf σ_0. There exists an $r_0 > 0$ such that if $0 < r < r_0$, it is possible to define a "zig-zag" map $f_r : \sigma_0 \to \sigma_r$ as follows. If $p \in \sigma_0$, follow L_0^+ from p along a generator of L_0^+ (i.e. along the integral curve of l that p lies on) until L_0^+ intersects a generator of N_r. Then, follow the N_r generator to σ_r. Bousso and Engelhardt established that f_r is well-defined for sufficiently small r (this is why we restrict to $r < r_0$). f_r is, in fact, a diffeomorphism between σ_0 and σ_r.

Considering equation 2 and the fact that $\alpha > 0$, the zig-zag construction of f_r implies that if A_0 is a $D-2$ dimensional submanifold of σ_0,

$$\frac{d}{dr}\Big|_{r=0} \|f_r(A_0)\| = \int_{A_0} \sqrt{g^{\sigma_0}}\, \alpha\, \theta^l \; > 0. \tag{3}$$

The area law of Bousso and Engelhardt is obtained in the case where $A_0 = \sigma_0$ because f_r is surjective.

Aside from the case where $A_0 = \sigma_0$, the fact that $\|f_r(A_0)\|$ is an increasing function of r is an unattractive area law. One issue is that the definition of the function f_r involves the choice of the reference leaf (i.e. the choice of $r = 0$). Moreover, the family of regions $\{f_r(A_0) \mid r \in [0, r_0)\}$ cannot necessarily be extended to all r.

Fortunately, as described above, there is a simpler way to carry subregions from one leaf to the next. Let A_{r_1} be a $D-2$ dimensional submanifold of the leaf σ_{r_1}. Define $A_{r_2} \subset \sigma_{r_2}$ by starting from points in A_{r_1} and following along the fibration of H (i.e. the integral curves of h) by parameter $r_2 - r_1$. Note that this procedure gives a well-defined region $A_r \subset \sigma_r$ for the entire range of r. We now prove that $\|A_r\|$ is an increasing function.

First, the following Lemma shows that f_r behaves similarly to h-translation for small r:

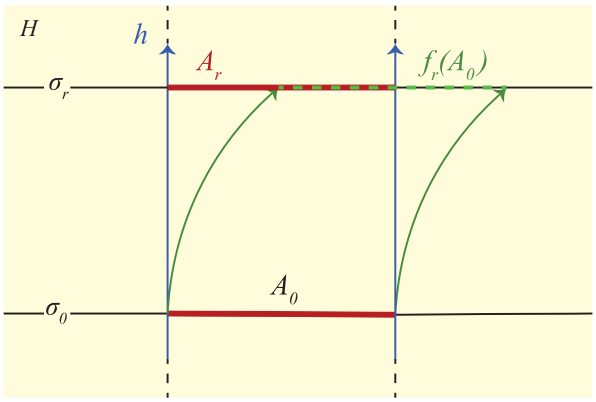

FIG. 1. We show that A_r has monotonic area by comparing A_r with the region $f_r(A_0)$. As depicted here, A_r and $f_r(A_0)$ are identical at linear order in r.

Lemma 1. If $p_0 \in \sigma_0$, let $\gamma : [0, r_0) \to H$ be the curve on H defined by $\gamma(r) = f_r(p_0)$. Then, the tangent vector of γ at $r = 0$ is $h(p_0)$.

Proof. We will begin by introducing a set of convenient coordinates. Fix a coordinate chart on σ_0 for a neighborhood of p_0. We denote these coordinates by x^i, $i \in \{1, \dots D-2\}$ and require that p_0 corresponds to the origin of $\mathbf{R}^{D-2}$. Extend to coordinates $\{(x^i, r)\}$ on H by following the integral curves of h from x^i by parameter r to reach the point labeled by (x^i, r). Note that this point will lie in σ_r. Finally, extend to coordinates $\{(x^i, r, z)\}$ by starting from the point (x^i, r) and following the integral curves of k by affine parameter z. Note that H is the $z = 0$ hypersurface.

Because $\alpha \neq 0$, we can put $l\big|_H = \frac{1}{\alpha}h - \frac{\beta}{\alpha}k$. Thus, in the coordinates (x^i, r, z) constructed above, we have

$$\begin{aligned} h &= (\mathbf{0}, 1, 0) \\ l\big|_{z=0} &= (\mathbf{0}, \frac{1}{\alpha}, -\frac{\beta}{\alpha}) \\ k\big|_{z=0} &= (\mathbf{0}, 0, 1) \end{aligned} \tag{4}$$

where $\mathbf{0}$ denotes $D-2$ zeros. The curve $\gamma(r)$ also takes a simple form in our coordinates: because f_r maps points in σ_0 to points in σ_r, we have

$$\gamma(r) = (x^i(r), r, 0) \tag{5}$$

where $x^i(r)$ is a curve in $\mathbf{R}^{D-2}$. Our Lemma will be proven by showing that $\dot{x}^i(0) = 0$.

Let $\xi_0(\lambda)$ and $\zeta_r(\lambda)$ denote, respectively, the geodesics generated by l and k from the points $\gamma(0) = (\mathbf{0}, 0, 0)$ and $\gamma(r) = (x^i(r), r, 0)$. The zig-zag definition of f_r implies that ξ_0 and ζ_r have an intersection: there exist functions $\lambda_1(r)$ and $\lambda_2(r)$ such that

$$\xi_0(\lambda_1(r)) = \zeta_r(\lambda_2(r)). \tag{6}$$

Meanwhile, equation 4 implies that

$$\begin{aligned} \xi_0(\lambda_1(r)) &= \Big(\mathbf{0}, \frac{1}{\alpha_0}\lambda_1(r), -\frac{\beta_0}{\alpha_0}\lambda_1(r)\Big) + O\big(\lambda_1(r)^2\big) \\ \zeta_r(\lambda_2(r)) &= \Big(x^i(r), r, \lambda_2(r)\Big) + O\big(\lambda_2(r)^2\big) \end{aligned} \tag{7}$$

where $\alpha_0 = \alpha(r = 0)$ and $\beta_0 = \beta(r = 0)$. Comparing the r and z components of equation 7 now gives

$$\begin{aligned} \lambda_1(r) &= \alpha_0\, r + O\big(\lambda_1(r)^2, \lambda_2(r)^2\big) \\ \lambda_2(r) &= -\beta_0\, r + O\big(\lambda_1(r)^2, \lambda_2(r)^2\big) \end{aligned} \tag{8}$$

which then implies that

$$x^i(r) = O\big(\lambda_1(r)^2, \lambda_2(r)^2\big) = O\big(r^2\big). \tag{9}$$

We conclude that $\dot{x}^i(r = 0) = 0$. □

Theorem 1. Let $A_0 \subset \sigma_0$ be a $D-2$ dimensional submanifold of σ_0 and define A_r as the result of translating A_0 along the integral curves of h by parameter r. Then, A_r has strictly increasing area.

Proof. Take $r \in [0, r_0)$. We have

$$\Big|\|f_r(A_0)\| - \|A_r\|\Big| \leq \|f_r(A_0)\,\Delta\,A_r\| \tag{10}$$

where Δ denotes the symmetric difference of sets: $A \Delta B = (A \setminus B) \cup (B \setminus A)$. Now Lemma 1 and the compactness of σ_r implies that

$$\frac{d}{dr}\Big|_{r=0}\Big(\|f_r(A_0)\,\Delta\,A_r\|\Big) = 0.$$

Noting that both sides of equation 10 are nonnegative for all r and are zero at $r = 0$, we conclude that

$$\frac{d}{dr}\Big|_{r=0}\left(\Big|\|f_r(A_0)\| - \|A_r\|\Big|\right) = 0. \tag{11}$$

But equation 3 implies that $\|f_r(A_0)\|$ is increasing at $r = 0$ so we must have that $\|A_r\|$ is also increasing at $r = 0$.

While we have only proven that A_r has increasing area at $r = 0$, we can define a zig-zag function analogous to f from any reference leaf and repeat all arguments above for any r. Thus, we conclude that A_r has strictly increasing area. In fact, equations 3 and 11 show that

$$\frac{d}{dr}\|A_r\| = \int_{A_r} \sqrt{g^{\sigma_r}}\,\alpha\,\theta^l \;> 0.$$

□

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