Four wave mixing as a probe of the vacuum

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Phys. Rev. D **93**, 125032 — Published 30 June 2016

DOI: 10.1103/PhysRevD.93.125032

Four Wave Mixing as a Probe of the Vacuum

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(Dated: June 17, 2016)

Much attention has been paid to the quantum structure of the vacuum. Higher order processes in Quantum Electrodynamics are strongly believed to cause polarization and even breakdown of the vacuum in the presence of strong fields soon to be accessible in high intensity laser experiments. Less explored consequences of strong field electrodynamics include effects from Born-Infeld type of electromagnetic theories, a nonlinear electrodynamics that follows from classical considerations as opposed to coupling to virtual fluctuations. In this article, I will demonstrate how Vacuum Four Wave Mixing has the possibility to differentiate between the these two types of vacuum responses: quantum effects on one hand and nonlinear classical extensions on the other.

PACS numbers: Valid PACS appear here

I. INTRODUCTION

The technology of high intensity lasers has been progressing rapidly. The next generation of high power lasers about to come on-line such as ELI, capable of providing up to 10^{25} Wcm^{-2} field strengths, will open up previously unexplored energy regimes. The high intensity, low frequency domain that will soon be made accessible due to these next generation laser facilities will provide a new parameter space in which to explore aspects of non-perturbative Quantum Electrodynamics. From a fundamental physics perspective, this will not only shed much needed light on the behavior of non-perturbative regimes of quantum field theories, but also provide a space in which to search for new physics in these high field intensity environments [1].

A general framework in terms of effective Lagrangians can be developed to describe non-perturbative QED, as well as other nonlinear theories of electrodynamics such as Born-Infeld (BI) type theories. Electrodynamics can be thought of as the study of gauge potentials whose kinematics satisfy the Bianchi Identities, and whose dynamics follow from the extremes of an action constructed from invariant quantities with respect to the symmetries of the theory[2]. In the case of electrodynamics, these symmetries include gauge, Lorentz, and parity invariance, as well as causal behavior. This leads to a very general action constructed from the only two independent quantities, $\mathcal{F} = \frac{1}{4}F_{ab}F^{ab} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2)$ & $\mathcal{G} = \frac{1}{4}F_{ab}\tilde{F}^{ab} = -\mathbf{E}\cdot\mathbf{B}$, that are invariant under these transformations.

$$\mathcal{L} = -\mathcal{F} + c_1\mathcal{F}^2 + c_2\mathcal{G}^2 + ... \tag{1}$$

Note that parity invariance disallows odd powers of $\mathcal{G}$ as it is a pseudoscalar. Maxwell's electrodynamics are encapsulated in the quadratic term, $\mathcal{L} = -\mathcal{F}$, resulting in equations of motion that are linear in the fields. Linear equations of motion result in theories where the law of superposition holds, a reflection of the fact that it is possible to disentangle charges from fields. This property breaks down in a number of scenarios.

One well known scenario occurs when Maxwell's theory is coupled to relativistic electrically charged sources, such as electrons described by the Dirac equation. In this case, the electromagnetic fields contribute to the electron's mass, magnetic moment, and charge. Conversely, we can consider how the ground, or zero particle, state of the electron quantum field can affect the classical electromagnetic field equations. In this paradigm, the electron vacuum field can be viewed as a high band gap insulator, $E_g \propto 2\,m_e$, which can be polarized like any other dielectric medium. This dielectric response was first calculated by Euler & Heisenberg (EH)[3] and reproduced in the full context of QED by Schwinger[4].

$$c_1 = \frac{8\alpha^2}{45m^4} \qquad c_2 = \frac{14\alpha^2}{45m^4}. \tag{2}$$

Another expected extension to Maxwell's theory comes from Born-Infeld electrodynamics[5], a nonlinear classical field theory that accepts additional, less emphasized, symmetries found in the source-free Maxwell's equations as canon. By constraining nonlinear electromagnetic theory to respect either electromagnetic duality[6], or a vacuum response which lacks birefringence[7][8], separately leads to the same electrodynamics.

$$\mathcal{L} = -\beta^2\sqrt{1 + \frac{2\mathcal{F}}{\beta^2} - \frac{\mathcal{G}^2}{\beta^4}} + \beta^2 \tag{3}$$

An additional consequence of these restrictions is the appearance of a critical maximal field strength, β, analogous to the Schwinger Field of QED. Solving the equations of motion for a point charge leads to a finite field strength everywhere, in stark contrast to Maxwell's equations.

It is worth remarking that the nonlinearities present in BI Electrodynamics arise through an entirely different paradigm as those found in EH: they are not caused by

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coupling with virtual fluctuations of the electron's quantum field vacuum state. If one does seek a microscopic explanation, BI electrodynamics can be viewed as arising in the context of certain string theories. In particular, if the universe is regarded as a D3-brane, open strings attached to the brane may couple to a $U(1)$ field at the end of the string, so that the lowest-order contribution to the partition function, after integrating out the string degrees of freedom allowed by the Dirichlet boundary conditions in the path integral, is given by an action containing the BI Lagrangian[9]. If we expand this Lagrangian around weak field strengths, we recover the Maxwell Lagrangian plus corrections of the same form as above.

$$\mathcal{L} \approx -\mathcal{F} + \frac{\mathcal{F}^2}{2\beta^2} + \frac{\mathcal{G}^2}{2\beta^2} \tag{4}$$

There have been numerous previous and ongoing attempts to measure possible nonlinearities that may exist in electrodynamics. One experiment in particular, PVLAS[10], captures some crucial ingredients that will be useful for future discussion. The PVLAS experiment has set the existing experimental limit on QED type vacuum nonlinearities. Their null results have verified the absence of nonlinearities up to field strengths of 4.3×10^{13} Vcm^{-1}, compared to the Schwinger Field of 1.3×10^{16} Vcm^{-1}.

The PVLAS experiment is comprised of a low intensity laser repeatedly passing through, and perpendicular to, a strong magnetic field.

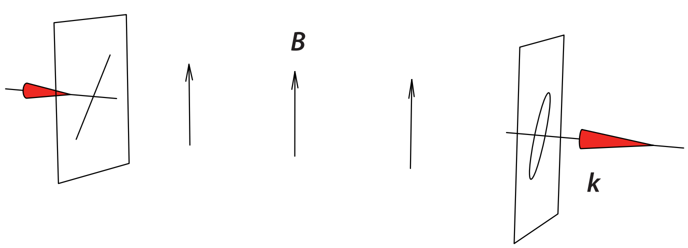


FIG. 1: Basic schematic of the PVLAS experiment. Due to the birefringent nature of the vacuum in the presence of the background magnetic field, the incoming laser suffers a rotation of its polarizaion, as well as an induced ellipsoidity.

Because of the strong magnetic field, QED predicts that the vacuum will become birefringent: the component of the laser's electric field propagating parallel and perpendicular to the magnetic field will travel at different velocities, causing the two components to become out of phase, and the polarization of the beam to rotate. This difference in propagation velocities can be traced back mathematically to the difference between c_1 and c_2 found in the EH Lagrangian. BI Electrodynamics, on the other hand, predicts that while the beam will experience a non-trivial index of refraction, because of the equality of coefficients of the nonlinear terms, its vacuum response will be isotropic. There will be no rotation of polarization. Thus, this experiment can separate the effects of these two types of nonlinearities. It simply measures the birefringence of the QED vacuum and is completely insensitive to BI effects. It would be advantageous to have an experimental setup that could effectively filter out either QED or BI effects, but in general be sensitive to both[11].

A number of experiments have been proposed to probe electromagnetic nonlinearities using high intensity lasers. Most of the proposed experiments, however, suffer from the same insensitivities as PVLAS. Both Heinzl *et al*[12] and Piazza *et al* [13], for example, exploit the birefringence of the quantum vacuum to obtain their experimental signature. To be sensitive to BI type nonlinearities, other types of optical processes are necessary. One possible process that will be discussed in this paper is Four Wave Mixing (FWM).

The remainder of this article will be dedicated to describing how FWM can not only detect, but also distinguish between Born-Infeld type theories and QED related nonlinearities. The absence of birefringence found in BI, indicative of a larger class of polarization dependence, will be instrumental in distinguishing different nonlinear theories from one another. The ratio of the coefficients of the nonlinear terms in the Lagrangian will be crucial in determining the polarization of electromagnetic radiation produced in nonlinear optical processes.

First, a textbook example of Four Wave Mixing[14], a third order nonlinear optical effect, will be described as proof of principle. These ideas will be applied to more experimentally realizable set-ups that utilize FWM to produce detectable amounts of electromagnetic radiation.

II. FOUR WAVE MIXING

Four wave mixing is a third order nonlinear optical process in which the polarized nonlinear medium oscillates at the three incoming waves' frequencies and, in turn, produces electromagnetic radiation at sums and differences of the incoming frequencies. This is simple to imagine in terms of a real nonlinear optical material, where actual molecules are participating in this harmonic motion. In the present cases, the picture of what is actually oscillating is less clear. In the case of QED, virtual fluctuations of the electron's quantum field are driven at the incoming beams frequencies, and in turn re-radiate as a fourth beam. In the BI paradigm, the fields themselves act as the oscillating medium, and as a direct source of radiation. It is a testament to the method of effective Lagrangians that these two disparate physical processes can be expressed in the same formalism.

With FWM in mind as a destination, a suitable wave equation can be developed[15]. The general electromagnetic Lagrangian can be rewritten in terms of two parameters, ξ, that sets the scale of the nonlinearity, and Υ, that represents the ratio of the coefficients of the two nonlinear terms. For QED, $\Upsilon = 7/4$, and for BI, $\Upsilon = 1$.

$$\mathcal{L} = -\mathcal{F} + \xi(\mathcal{F}^2 + \Upsilon\mathcal{G}^2) \tag{5}$$

With these definitions, the Euler-Lagrange equations produce the two dynamical Maxwell-like equations of motion,

$$\partial_b \frac{\partial \mathcal{L}}{\partial F_{ab}} = 0 \Rightarrow \partial_b F^{ab} = \xi \partial_b (\mathcal{F} F^{ab} + \Upsilon \mathcal{G} \tilde{F}^{ab}) \equiv \partial_b P^{ab}, \tag{6}$$

along with the two equations that follow from the Bianchi identities,

$$\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0. \tag{7}$$

The polarization tensor, analogous to the field tensor, encodes the vacuum's induced electric polarization and magnetization, $P^{0i} = P^i$ & $P^{ij} = \frac{1}{2}\epsilon^{ijk} M_k$, and is written out explicitly as

$$\mathbf{P} = \xi[2(E^2 - B^2)\mathbf{E} + 4\Upsilon(\mathbf{E}\cdot\mathbf{B})\mathbf{B}]$$

$$\mathbf{M} = \xi[-2(E^2 - B^2)\mathbf{B} + 4\Upsilon(\mathbf{E}\cdot\mathbf{B})\mathbf{E}]. \tag{8}$$

The wave equation for the electric and magnetic fields is derived by the usual procedure of applying the derivative operator, ∂_c to the Bianchi identity and inserting the dynamical equations.

$$\partial^2 F^{ab} = \partial^b \partial_c P^{ca} - \partial^a \partial_c P^{cb} \tag{9}$$

The electric field component of the wave equation is

$$\Box \mathbf{E} = [\nabla(\nabla \cdot \mathbf{P}) - \frac{\partial}{\partial t}(\frac{\partial}{\partial t}\mathbf{P} + \nabla \times \mathbf{M})]. \tag{10}$$

For the case of FWM, there will be three 'strong' beams producing a fourth, weaker in field strength, by order ξ. The resulting effect on the original three beams is of order ξ^2 and will be neglected. The direction and frequency of the produced beam, k_4, can be determined through energy and momentum conservation.

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 \qquad \omega_1 + \omega_2 = \omega_3 + \omega_4 \tag{11}$$

It is plain to see that the direction of the produced radiation, set by momentum conservation considerations and independent of the medium, will be identical for both QED & BI nonlinearities.

Let the beams be plane waves whose amplitudes change relatively slowly compared to their harmonic component. All non-resonant terms, those not satisfying energy momentum conservation as required by Equation (11), will rapidly oscillate and average to zero, leaving only the single term on the right hand side of Equation (10).

$$\Box \mathbf{E}_4 = \xi \omega_4^2 \mathbf{G}\, E_1\, E_2\, E_3^*\, e^{i(\mathbf{k}_4\cdot\mathbf{r} - \omega_4 t)} \tag{12}$$

Its solution in the far field approximation is

$$\mathbf{E}_4 = \frac{\xi \omega_4^2}{r} \mathbf{G} e^{i(\mathbf{k}_4\cdot\mathbf{r}-\omega_4 t)} \int_V (E_1\, E_2\, E_3^*)\,|_{t_R}\, e^{ik_4(\hat{\mathbf{k}}_4 - \hat{\mathbf{r}})\cdot\mathbf{r}'} dV', \tag{13}$$

where the integration is performed over the interaction region at the retarded time. In the case of incoming plane waves, the direction of propagation is defined by a delta function in the $\mathbf{k}_4$ direction. More realistic beam profiles will produce more of a directional spread, but for an order of magnitude prediction this assumption is sufficient.

1. *Textbook Example*

For the first case, consider a one dimensional degenerate FWM that produces radiation at the same frequency as the incoming waves. In this case, beams 1 and 3 will

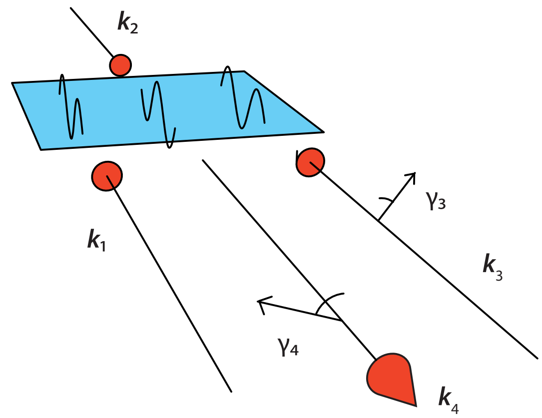
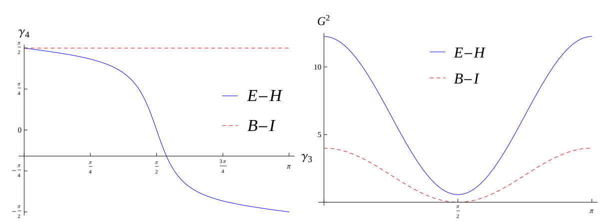


FIG. 2: One-dimensional FWM setup. The polarizations of the individual beams, γ_i, are defined with respect to the $\hat{\mathbf{z}}$ axis such that they are positive in the clockwise direction when looking along the $\mathbf{k}_i$ direction.

propagate in the positive $\hat{\mathbf{x}}$ direction, beam 2 will be in the negative $\hat{\mathbf{x}}$ direction, and beam 4 will be produced in the negative $\hat{\mathbf{x}}$ direction.



(a) Orientation (b) Magnitude

FIG. 3: One-dimensional Polarization Dependence. The polarization, γ_4, and magnitude, G^2, of the produced beam are plotted as a function of the polarization of the third beam, γ_3, where the polarizations of the first two beams are held fixed at $\gamma_1 = 0$ and $\gamma_2 = \pi/2$.

Calculation of the polarization vector direction is involved, and a good description can be found in Lundström[16] as well as the appendix. As an example, consider the case where $\gamma_1 = 0$ & $\gamma_2 = \pi/2$ while γ_3 is the independant variable:

$$\mathbf{G}_{1\mathrm{d}} = 2\Upsilon\cos\gamma_3\hat{\mathbf{y}} + (\Upsilon - 1)\sin\gamma_3\hat{\mathbf{z}}. \tag{14}$$

There are some interesting results to take note of in this example. From studying Fig. 3, it is plain to see that the two types of nonlinearities predict very different polarization behavior for the produced radiation. Because of the coefficient of the second term in Equation (14), the polarization of the BI produced radiation is always constrained to be in the $\hat{\mathbf{y}}$ direction. This interesting confinement is particular to these experimental conditions and would not necessarily happen for different incoming directions and polarizations. In an actual experiment, ideally the two effects to produce radiation polarized at right angles to one another. Unfortunately, at the angle at which this occurs, $\gamma_3 \simeq \pi/2$, the magnitude of $\mathbf{G}_{1\mathrm{d}}$ for both the BI and QED is lowest.

A. FWM in Three Dimensions

The previous example, while illustrating the effect, does not lend itself well to an actual experiment. An immediate issue is that the produced radiation must be generated in a direction different from the incoming beams in order to set up a suitable detector. This can only be accomplished by fully utilizing the three dimensions. In addition, the experiment should also produce at least a modest value for the magnitude of $\mathbf{G}_{3d}$.

One such set up, as illustrated in Fig. 4, has been previously described[17],[18] with

$$\mathbf{k}_1 = k\hat{\mathbf{x}}$$
$$\mathbf{k}_2 = k\hat{\mathbf{y}}$$
$$\mathbf{k}_3 = \frac{k}{2}\hat{\mathbf{z}}$$
$$\mathbf{k}_4 = k(\hat{\mathbf{x}} + \hat{\mathbf{y}} - \frac{1}{2}\hat{\mathbf{z}}). \tag{15}$$

The polarization vector is computed in a similar fashion to the one dimensional case discussed above. Rotating the coordinate system so that the $\hat{\mathbf{x}}'$ is parallel to $\mathbf{k}_4$ causes $\mathbf{G}_{3d}$ to lie entirely in the $\hat{\mathbf{y}}' - \hat{\mathbf{z}}'$ plane, yielding

$$\mathbf{G}_{3d}{}' = \frac{1}{48\sqrt{5}}[(-20+\Upsilon)\cos\gamma_2 + 8(2-\Upsilon)\sin\gamma_2]\hat{\mathbf{y}}' + \frac{1}{24\sqrt{5}}[(20-\Upsilon)\cos\gamma_2 + 2(2-\Upsilon)\sin\gamma_2]\hat{\mathbf{z}}'. \tag{16}$$

This calculation was performed with both γ_1 and γ_3 equal to zero.

Feasibility calculations utilizing the Astra Gemini (10^{21} Wcm^{-2} at 3 shots per minute) system were performed by Lundin *et al*[17] taking into account noise considerations because of imperfect vacuum, such as ponderomotive forces, Compton scattering, and collective plasma effects. For the case of QED nonlinearites, they predict detectable levels of approximately 0.07 photons produced per shot, approximately three orders of magnitude above calculated noise levels.

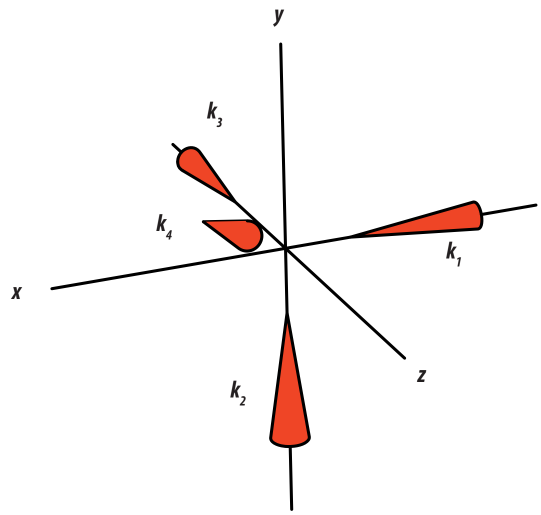
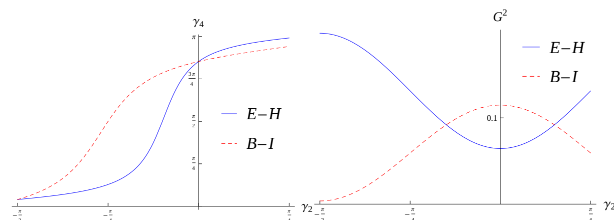


FIG. 4: Three-dimensional FWM setup. The polarization vectors, γ_1 & γ_2, are defined the same as before while γ_3 is defined with respect to the $\hat{\mathbf{x}}$ axis such that it is positive in the clockwise direction when looking along the $\mathbf{k}_3$ direction.



(a) Orientation (b) Magnitude

FIG. 5: Three-dimensional Polarization Dependence. The polarization, γ_4, and magnitude, G^2, of the produced beam are plotted as a function of the polarization of the second beam, γ_2, where the polarizations of the other two beams are held fixed at zero angle.

As shown in Fig. 5, the effect of both types of nonlinearities are nearly identical for most of the input beam polarizations. However, for $\gamma_2 \simeq -\frac{\pi}{8}$, the two types of nonlinearities produce radiation polarized at right angles, offering a chance to distinguish between the two signals. A filter can be positioned to only allow one polarization through to the detector. In this sense, FWM provides a unique opportunity, not only test nonlinear electrodynamics, but also to distinguish between different types of nonlinearities.

III. CONCLUSION

An experiment has been presented that is sensitive to both types of nonlinearities, QED & BI. Because of the different resulting polarizations, it is possible to filter one from another. The open question at this point, is at what scale do BI effects take place? Unlike QED effects where the energy scale is set by the rest mass and charge of the electron, BI type nonlinearities do not have a definite, agreed upon energy scale at which they become important. A similar situation arises in special relativity, where belief in locality demands a finite maximal velocity, but offers no insight as to what that velocity might be.

Born and Infeld, by postulating that the electron's mass was entirely electromagnetic, arrived at a value of 1.2×10^{18} Vcm^{-1} [5] for their critical field strength. Further research on BI electrodynamics lay fallow for decades until its remarkable wave propagation properties were discovered [7],[8], [19], and research refocused on possible limits of field strengths attainable in nature. Rafelski *et al*[20], by considering the spectra of atoms with heavy nuclei, raised the lower bound to 1.7×10^{20} Vcm^{-1}. More recent work [21],[22] has refined their methods to explore spectra of light Hydrogen like atoms to obtain similar lower bounds.

With the noise limit of the Astra Gemini system as discussed in [17], a null result of the experiment proposed in this article would provide a lower bound for the BI critical field strength of approximately 5×10^{17} Vcm^{-1}. This result would be a welcome corroboration of the other lower bounds deduced through indirect methods. If BI theory is the effective electrodynamics of a stringy fundamental physics, its effects wouldn't be expected to become important until the Planck scale, 10^{60} Vcm^{-1}, although there is some discussion[23] that they could emerge at the slightly more modest scale of 10^{44} Vcm^{-1}.

Near future laser facilities can be expected to observe vacuum nonlinearities. ELI, in particular, is purposely being constructed in order to study high field physics. The author hopes this article will act as a reminder to researchers in this field that the phenomenology of vacuum nonlinearities such as FWM, vacuum birefringence, and diffraction[24] are all general features of nonlinear electrodynamics, and that quantum mechanics is only one of many such mechanisms that can produce these effects.

Acknowledgments

The author wishes to thank Dr. Gerardo Muñoz, Dr. John Markert, Dr. Michael Downer, and Dr. Orbach all for thoughtful discussion, help with proofreading, and their general support and encouragement. The author would also like to thank the referee for his/her useful comments and suggestions which greatly contributed to the final product. This work was supported in part by the U.S. Department of Energy, Office of Basic Energy Sciences, Division of Materials Science and Engineering under Award DE-SC0013599 as well as a Teaching Assistantship provided by the College of Natural Sciences at the University of Texas at Austin.

Appendix: Derivation of the Geometric Factor

The right hand side of Equation (9) can be organized as

$$\partial^2 F^{ab} = \xi G^{abcfrspqmn} \partial_c \partial_f F_{rs} F_{pq} F_{mn} \tag{A.1}$$

$$\begin{aligned} G^{abcfrspqmn} &= \frac{1}{2}(\delta^{rp}\delta^{sq}(\delta^{bf}\delta^{cm}\delta^{an} - \delta^{af}\delta^{bm}\delta^{cn}) \\ &+ \frac{\Upsilon}{2}\epsilon^{rspq}(\delta^{bf}\epsilon^{camn} - \delta^{af}\epsilon^{bcmn})). \end{aligned} \tag{A.2}$$

Expressing the field tensors on the r.h.s. of (A.1) as plane wave fields described in the one and three dimensional example respectively yields the following geometrical factors. To transform to the primed coordinate system in the three dimensional case, the appropriate rotational matrix needs to be applied to $\mathbf{G}_{3d}$.

$$\begin{aligned} \mathbf{G}_{1d} = &\{\Upsilon\,[\cos\gamma_3 \sin(\gamma_1 + \gamma_2) + \cos\gamma_1 \sin(\gamma_2 + \gamma_3)] \\ &- \sin\gamma_1 \cos(\gamma_2 + \gamma_3) - \sin\gamma_3 \cos(\gamma_1 + \gamma_2)\}\mathbf{e}_{\mathrm{y}} \\ &+ \{\Upsilon\,[\sin\gamma_3 \sin(\gamma_1 + \gamma_2) + \sin\gamma_1 \sin(\gamma_2 + \gamma_3)] \\ &+ \cos\gamma_1 \cos(\gamma_2 + \gamma_3) + \cos\gamma_3 \cos(\gamma_1 + \gamma_2)\}\mathbf{e}_{\mathrm{z}}. \end{aligned} \tag{A.3}$$

$$\mathbf{G}_{3d} = -\frac{1}{2}\{[(\frac{1}{2}\sin\gamma_3 - \cos\gamma_3)\cos(\gamma_1+\gamma_2) + (\frac{1}{8}\sin\gamma_2 - \frac{1}{4}\cos\gamma_2)\cos(\gamma_1+\gamma_3) + (\frac{1}{4}\sin\gamma_1 + \frac{1}{8}\cos\gamma_1)\sin(\gamma_2+\gamma_3)]$$
$$+ \Upsilon[(-\sin\gamma_3 - \frac{1}{2}\cos\gamma_3)\sin(\gamma_1+\gamma_2) + (-\frac{1}{8}\cos\gamma_2 - \frac{1}{4}\sin\gamma_2)\sin(\gamma_1+\gamma_3) + (\frac{1}{4}\cos\gamma_1 - \frac{1}{8}\sin\gamma_1)\cos(\gamma_2+\gamma_3)]\}\mathbf{e}_x$$
$$- \frac{2}{9}\{[(\frac{1}{2}\cos\gamma_3 - \sin\gamma_3)\cos(\gamma_1+\gamma_2) + (\frac{1}{8}\cos\gamma_2 - \frac{1}{4}\sin_2)\cos(\gamma_1+\gamma_3) + (-\frac{1}{4}\cos\gamma_1 - \frac{1}{8}\sin\gamma_1)\sin(\gamma_2+\gamma_3)]$$
$$+ \Upsilon[(\cos\gamma_3 + \frac{1}{2}\sin\gamma_3)\sin(\gamma_1+\gamma_2) + (\frac{1}{8}\sin\gamma_2 + \frac{1}{4}\cos\gamma_2)\sin(\gamma_1+\gamma_3) + (\frac{1}{4}\sin\gamma_1 - \frac{1}{8}\cos\gamma_1)\cos(\gamma_2+\gamma_3)]\}\mathbf{e}_y$$
$$+ \frac{4}{9}\{[\frac{1}{2}(\cos\gamma_3 + \sin\gamma_3)\cos(\gamma_1+\gamma_2) + \frac{1}{8}(\sin\gamma_2 + \cos\gamma_2)\cos(\gamma_1+\gamma_3) + \frac{1}{8}(\cos\gamma_1 - \sin\gamma_1)\sin(\gamma_2+\gamma_3)]$$
$$+ \Upsilon[\frac{1}{2}(\sin\gamma_3 - \cos\gamma_3)\sin(\gamma_1+\gamma_2) + \frac{1}{8}(\sin\gamma_2 - \cos\gamma_2)\sin(\gamma_1+\gamma_3) + \frac{1}{8}(-\cos\gamma_1 - \sin\gamma_1)\cos(\gamma_2+\gamma_3)]\}\mathbf{e}_z \quad \text{(A.4)}$$

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