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Heterotic Non-Abelian String of a Finite Length

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Abstract

We consider non-Abelian strings in $\mathcal{N}=2$ supersymmetric QCD with the U(N) gauge group and $N_f=N$ quark flavors deformed by a mass term for the adjoint matter. This deformation breaks $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$. Dynamics of orientational zero modes on the string world sheet are described then by CP($N-1$) model with $\mathcal{N}=(0,2)$ supersymmetry. We study the string of a finite length L assuming compactification on a cylinder (periodic boundary conditions). The world-sheet theory is solved in the large-N approximation. At $N=\infty$ we find a rich phase structure in the (L, u) plane where u is a deformation parameter. At large L and intermediate u we find a phase with broken Z_{2N} symmetry, N vacua and a mass gap. At large values of L and u still larger we have the Z_{2N}-symmetric phase with a single vacuum and massless fermions. In both phases $\mathcal{N}=(0,2)$ supersymmetry is spontaneously broken. We also observe a phase with would-be broken SU(N) symmetry at small L (it is broken only for $N=\infty$). In the latter phase the mass gap vanishes and the vacuum energy is zero in the leading $1/N$ approximation. We expect that at large but finite N corrections $O(1/N)$ will break $\mathcal{N}=(0,2)$ supersymmetry. Simultaneously, the phase transitions will become rapid crossovers. Finally we discuss how the observed rich phase structure matches the $\mathcal{N}=(2,2)$ limit in which the world-sheet theory has a single phase with the mass gap independent of L.

1 Introduction

Recently there was a considerable progress in the study of long confining strings of a fixed length both on lattices [1, 2] and by constructing the effective theory on the string world sheet, see [3, 4]. In our recent paper [5] we initiated a study of a closed non-Abelian string of a finite length L assuming compactification on a cylinder with circumference L (periodic boundary conditions).

Non-Abelian strings were first found in $\mathcal{N} = 2$ supersymmetric gauge theories [6, 7, 8, 9]. Later this construction was generalized to a wide class of non-Abelian gauge theories, both supersymmetric and non-supersymmetric, see [10, 11, 12, 13]. Both Abelian and non-Abelian strings have translational modes associated with broken translation symmetries. The main feature of the non-Abelian strings is the occurrence of extra moduli: orienational zero modes associated with the color flux rotation in the internal space. Dynamics of these orientational moduli is described by two-dimensional CP($N-1$) model on the string world-sheet. The translational modes are completely decoupled.

In [5] we studied both non-supersymmetric case as well as 1/2-BPS string in $\mathcal{N} = 2$ supersymmetric QCD. For non-supersymmetric case we found a phase transition in the world-sheet theory in the $N = \infty$ limit. At large L this theory develops a mass gap and is in the Coulomb/confinement phase. Finite-length effects are exponentially suppressed. At small lengths it is in the deconfinement phase.

$\mathcal{N} = 2$ supersymmetric QCD has eight supercharges and, since our strings are 1/2-BPS, the world-sheet CP($N-1$) model has $\mathcal{N} = (2, 2)$ supersymmetry. In this case we found a single phase with a mass gap and unbroken supersymmetry [5]. The mass gap turns out to be independent of the string length.

If we introduce a mass term for the adjoint matter in the bulk we break bulk $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. The string remains BPS saturated [14]. It was conjectured by Edalati and Tong [15] and confirmed in [16] that the target space in the deformed model is $CP(N-1) \times C$. The right-handed supertranslational modes become coupled to superorientational ones, and the world sheet theory becomes heterotic model with $\mathcal{N} = (0, 2)$ supersymmetry. It is important that this is a nonminimal model (cf. [17]) well defined for all N.

In this paper we solve the above heterotic $\mathcal{N} = (0, 2)$ CP($N-1$) model on a cylinder with circumference L in the large-N approximation, assuming periodic boundary conditions. Our solution is drastically different from the one obtained in the $\mathcal{N} = (2,2)$ case. First of all we observe three distinct phases instead of one. Two phases (III and IV in Fig. 1) preserve the SU(N) global symmetry. The finite-L effects are exponentially suppressed at large L and intermediate values of the deformation parameter u, in much the same way as in non-supersymmetric theory [5]. The parameter of deformation u is related to the mass of the adjoint field in the bulk SQCD. The theory in this phase has mass gap and N vacua; the discrete chiral Z_{2N} symmetry is spontaneously broken down to Z_2.

As we increase u still keeping L large the theory undergoes a third order phase transition into a phase with a single vacuum and unbroken Z_{2N}. This is a phase with massless fermions. A sketch of the full phase diagram of the world sheet theory in the (L, u) plane is shown in Fig. 1.

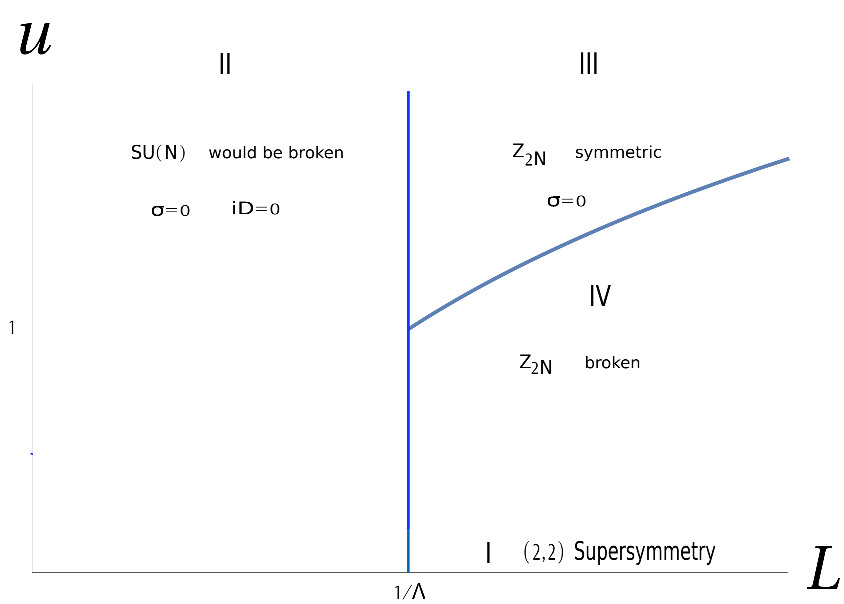

Figure 1: (I) $u < 1/N^2$ region corresponds to the $\mathcal{N} = (2,2)$ solution regardless of L; (II) $u \gg 1/N^2$ and $L < 1/\Lambda$ region corresponds to the would be broken SU(N) phase (n^l fields develop VEV); (III) $L > 1/\Lambda$ and large u region represents the Z_{2N}-symmetric phase with massless fermions; (IV) $L > 1/\Lambda$ and moderate u region represents Z_{2N}-broken phase with massive bosons and fermions.

As was the case for non-supersymmetric theory, we find a phase with would-be broken SU(N) symmetry at small L. In the latter phase a mass gap is zero in the leading approximation. Moreover, we find that the vacuum energy also vanishes at $N = \infty$. We expect corrections of higher order in $1/N$ (or, perhaps, exponential corrections e^{-N}) to break $\mathcal{N} = (0, 2)$ supersymmetry and lift the vacuum energy. We stress that SU(N) is broken only when $N = \infty$. At large but finite N this and other phase transitions turn into rapid crossovers.

We discuss how this rich phase structure evolves to the $\mathcal{N} = (2, 2)$ picture with a single phase in the limit of zero deformation, $u = 0$.

In the $\mathcal{N} = (2, 2)$ problem supersymmetry is unbroken and we deal with a single phase with an L independent mass gap (the latter property is dictated by holomorphy [5]). The limit $u \to 0$ turns out to be rather subtle. It turns out that the relevant parameter which ensures the saddle point approximation used in the large-N method is uN^2 rather than N. If the deformation parameter $u \sim 1$ the large-N limit ensures the validity of the quasiclassical approximation in the effective one-loop action. However, at extremely small u this approximation breaks down. To get a smooth $u \to 0$ limit we quantize the holonomy of the two-dimensional gauge potential around the compact spatial dimension of the string. The Polyakov line

$$\exp\left(i \int dx_k A_k\right) \tag{1.1}$$

depends only on time. Hence we consider a quantum-mechanical problem averaging this operator over the appropriate wave functions. This gives us the desired smooth $u \to 0$ limit and we recover the $\mathcal{N} = (2, 2)$ result in the narrow strip $u \sim 1/N^2$, see Fig. 1.

The paper is organized as follows. In Sec. 2 we review the large-N solution of the heterotic CP($N-1$) model on the infinite two-dimensional plane. In Sec. 3 we present the large-N solution for the heterotic string compactified on a cylinder. In Sec. 4 we discuss the SU(N)-symmetric phase with broken Z_{2N} symmetry while in Sec. 5 we consider the Z_{2N} unbroken phase. In Sec. 6 we deal with the would be broken SU(N) phase. In Sec. 7 we discuss the $u \to 0$ limit at small L. Section 8 summarizes our conclusions.

2 Heterotic $\mathcal{N} = (0,2)$ CP$(N-1)$ model at $L = \infty$

The heterotic $\mathcal{N} = (0,2)$ CP$(N-1)$ model at $L = \infty$ was solved in [18] in the large-N limit. In this section we will briefly review this solution. The bosonic part of the action in the gauged formulation is

$$S_b = \int d^2x \Big[|\nabla_k n^l|^2 + 2|\sigma|^2 |n^l|^2 + iD(|n^l|^2 - r_0) + 4|\omega|^2|\sigma|^2 \Big] , \qquad (2.1)$$

where n^l ($l = 1, ...N$) is a complex N-vector parametrizing the orientational modes. Moreover,

$$\nabla_k = \partial_k - iA_k \, .$$

Here A_k is the gauge potential, σ is a complex scalar field. The fields A_k, σ and D belong to the gauge (vector) multiplet. These fields come without kinetic terms and are auxiliary. Moreover, r_0 is a coupling constant, while ω is the $\mathcal{N} = (2,2)$ deformation parameter. Eliminating D leads to the constraint

$$|n^l|^2 = r_0 \, . \qquad (2.2)$$

The fermionic part of the action is

$$\begin{aligned} S_f \;&=\; \int d^2x \Big[\bar{\xi}_{lR} i(\nabla_0 - i\nabla_3)\xi^l_R + \bar{\xi}_{lL} i(\nabla_0 + i\nabla_3)\xi^l_L \\ &+\; i\sqrt{2}\sigma\bar{\xi}_{lR}\xi^l_L + i\sqrt{2}\bar{n}^l(\lambda_R\xi^l_L - \lambda_L\xi^l_R) \\ &+\; i\sqrt{2}\sigma^\star\bar{\xi}_{lL}\xi^l_R + i\sqrt{2}(\bar{\lambda}_L\bar{\xi}^l_R - \bar{\lambda}_R\bar{\xi}^l_L)n^l \\ &+\; \frac{1}{2}\bar{\zeta}_R i\partial_L\zeta_R + (i\sqrt{2}\omega\bar{\lambda}_L\zeta_R + \text{H.c.}) \Big] , \end{aligned} \qquad (2.3)$$

where $\xi^l_{R,L}$ are fermionic superpartners of n^l (superorientational modes of the string), $\lambda_{R,L}$ are auxiliary fermions from the vector superfield, while ζ_R is the right-handed supertranslational mode. In the $\mathcal{N} = (2,2)$ model it was decoupled. We do not include the bosonic translational modes describing shifts of the string center. Nor do we include the left-handed supertranslational mode ζ_L, because both decouple not only in the $\mathcal{N} = (2,2)$ but in the $\mathcal{N} = (0,2)$ model as well [15, 16].

The terms containing ζ_R or ω break $\mathcal{N}=(2,2)$ down to $\mathcal{N}=(0,2)$. The deformation parameter ω is complex and scales with N as [18]

$$\omega \sim \sqrt{N} \,. \tag{2.4}$$

It is determined by the mass parameter of the adjoint matter in the bulk theory [16].

Integrating over $\lambda_{L,R}$ leads to the constraints

$$\begin{aligned} \bar{n}^l \xi_L^l &= 0 \,, \\ \bar{\xi}_R n^l &= \omega \zeta_R \,. \end{aligned} \tag{2.5}$$

Integrating over σ implies

$$\sigma = -\frac{i}{\sqrt{2}(r_0 + 2|\omega|^2)} \bar{\xi}_{lL} \xi_R^l \,. \tag{2.6}$$

Note that this model has an axial $U(1)$ symmetry broken by the chiral anomaly down to Z_{2N} much in the same way as in the $\mathcal{N}=(2,2)$ model [19]. We find that σ develops a vacuum expectation value (VEV) which results in a spontaneous breaking of the discrete Z_{2N} down to Z_2. Moreover as can be seen from (2.6), a non-zero VEV of the σ field corresponds to a non-zero fermion bilinear condensate $\langle \bar{\xi}_{lL} \xi_R^l \rangle$.

Since both fields n^l and ξ^l appear in the action quadratically we can integrate them out. This produces the product of two determinants,

$$\det{}^{-N}\left(-\partial_i^2 + iD + 2|\sigma|^2\right) \det{}^{N}\left(-\partial_i^2 + 2|\sigma|^2\right) . \tag{2.7}$$

The first determinant comes from the boson n^l fields, while the second comes from the fermion ξ^l fields. Note that if $D=0$ the two contributions obviously cancel each other, and supersymmetry is unbroken. Also, the non-zero values of $iD + 2|\sigma|^2$ and $2|\sigma|^2$ can be interpreted as non-zero values of the masses of the n^l and ξ^l fields, respectively. We put $A_k = 0$.

The final expression for the effective potential is (see [18])

$$\begin{aligned} V_{\text{eff}} &= \int d^2x \frac{N}{4\pi} \left[-\left(iD + 2|\sigma|^2\right) \ln \frac{iD + 2|\sigma|^2}{\Lambda^2} + iD \right. \\ &+ \left. 2|\sigma|^2 \ln \frac{2|\sigma|^2}{\Lambda^2} + 2|\sigma|^2 u \right], \end{aligned} \tag{2.8}$$

where the logarithmic ultraviolet divergence of the coupling constant is traded for the finite scale Λ of the asymptotically free CP$(N-1)$ model. We also introduced a dimensionless deformation parameter

$$u = \frac{8\pi}{N}|\omega|^2\,, \tag{2.9}$$

which does not scale with N.

To find the saddle point we minimize the potential with respect to D and σ, which yields the following set of equations:

$$\begin{aligned} \ln\frac{iD+2|\sigma|^2}{\Lambda_{CP}^2} &= 0\,, \\ \ln\frac{iD+2|\sigma|^2}{2|\sigma|^2} &= u\,. \end{aligned} \tag{2.10}$$

The solution to these equations is

$$iD = \Lambda^2(1-e^{-u})\,, \quad \text{and} \quad 2|\sigma|^2 = \Lambda^2 e^{-u}\,. \tag{2.11}$$

The value of D in this solution does not vanish, implying that supersymmetry is spontaneously broken. We see that σ develops a VEV giving masses to the n^l fields and their fermion superpartners ξ^l. More exactly, the solution for σ can also be written as

$$\sqrt{2}\sigma = \Lambda\exp\left(-\frac{u}{2}+\frac{2\pi i k}{N}\right)\,, \quad k=0,...,N-1\,, \tag{2.12}$$

where the phase factor is not seen in Eq. (2.10). It comes as a result of a chiral anomaly which breaks the chiral U(1) symmetry, U(1) $\to Z_{2N}$. The field σ has the chiral charge 2. Thus a non-zero VEV of $|\sigma|$ ensures that Z_{2N} symmetry is broken down to Z_2 and there are N vacua presented in (2.12).

Substituting the solution (2.10) into (2.8) we obtain an expression for the vacuum energy density

$$V_{\rm vac} = \frac{N}{4\pi}\Lambda^2(1-e^{-u})\,, \tag{2.13}$$

which, as expected, vanishes in the limit $u\to 0$.

3 $\mathcal{N} = (0, 2)$ model on a cylinder

The $\mathcal{N} = (2, 2)$ model on a cylinder was solved in the large-N limit in [5]. In this section we apply the same approach to $\mathcal{N} = (0, 2)$ model assuming periodic boundary conditions both for bosons and fermions. Since the action (2.1) and (2.3) is quadratic in n^l and ξ^l we can integrate over these fields. We assume that the compact dimension in the bulk theory is x_1 and the string is wrapped around this dimension. We will assume a nontrivial holonomy (1.1) of A_k around this compact dimension. In the $A_0 = 0$ gauge we will look for a solution with $A_1 = \text{const}$.

First consider the case when neither of the fields n^l or ξ^l develop VEVs. The expression for the effective potential is easily found,

$$\begin{aligned} V &= \frac{N}{4\pi}\Big(iD - iD\ln\frac{m_b^2}{\Lambda^2} - m_f^2\ln\frac{m_b^2}{m_f^2} + m_f^2 u \\ &+ 8m_f^2\sum_{k=1}^{\infty}\frac{K_1(Lm_fk)}{Lm_fk}\cos LkA_1 \\ &- 8m_b^2\sum_{k=1}^{\infty}\frac{K_1(Lm_bk)}{Lm_bk}\cos LkA_1\Big)\,, \end{aligned} \tag{3.1}$$

where we use an effective mass notation for the bosonic n^l and fermionic ξ^l fields,

$$m_b^2 = iD + 2|\sigma|^2, \qquad m_f^2 = 2|\sigma|^2,\ . \tag{3.2}$$

Here $K_1(z)$ is the modified Bessel function of the second kind and the deformation parameter u is related to the parameter ω as in (2.9). The first line in (3.1) is the same as the one found in the case of the $L = \infty$ string (2.8), while the second and third lines represent contributions arising due to the finite length of the string. The potential (3.1) is periodic in the phase LA_1, with the period 2π, so we can assume that $0 \le LA_1 < 2\pi$.

3.1 Saddle point approximation

To find VEVs of A_1, of σ and iD we take derivatives of (3.1) with respect to these fields. Then we obtain three equations,

$$\begin{aligned}
V_{N,A_1} &= m_b \sum_{k=1}^{\infty} K_1(Lm_b k) \sin LkA_1 - m_f \sum_{k=1}^{\infty} K_1(Lm_f k) \sin LkA_1 \,, \\
V_{N,\sigma^\star} &= 2\sigma \left[-\ln \frac{m_b^2}{m_f^2} + 4\sum_{k=1}^{\infty} K_0(Lm_b k) \cos LkA_1 \right. \\
&- \left. 4\sum_{k=1}^{\infty} K_0(Lm_f k) \cos LkA_1 + u \right] , \\
V_{N,iD} &= -\ln \frac{m_b^2}{\Lambda^2} + 4\sum_{k=1}^{\infty} K_0(Lm_b k) \cos LkA_1 \,.
\end{aligned} \tag{3.3}$$

One can see that the first equation is satisfied when either $A_1 = 0$ or $A_1 = \pi/L$. However, unlike the bosonic theory [5], $A_1 = 0$ corresponds to the maximum of potential. The energy is lower if $LA_1 = \pi$. This can be easily understood. Consider the second and third lines in (3.1),

$$V_A \sim [m_f K_1(Lm_f) - m_b K_1(Lm_b)] \cos(LA_1) \,. \tag{3.4}$$

On the one hand we know from the definition that $m_b \geq m_f$. On the other hand $K_1(x)$ decreases exponentially at large values of the argument. Thus, at least for large L the potential $E_A = c \times \cos(LA_1)$, where $c > 0$. Hence we conclude that the minimum of the potential is at $LA_1 = \pi$. This conclusion is also supported by a numerical calculation, see Figs. 2,3. Below we assume that

$$LA_1 = \pi \,. \tag{3.5}$$

As can be seen from the graphs in Figs. 2, 3 no solution with lower energy exists for sufficiently small L and/or high enough value of the deformation parameter. To explore this issue we need to find approximate analytical solutions.

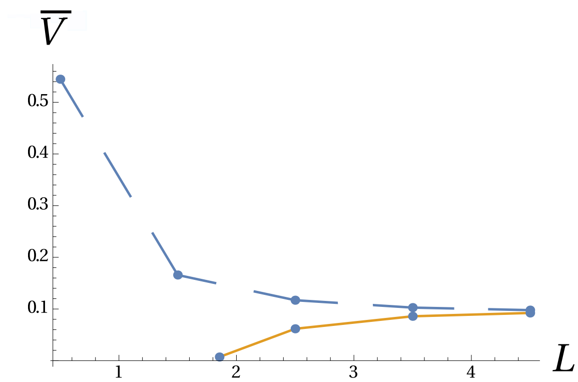

Figure 2: $\bar{V} \equiv 4\pi V$ vs string length L at the value of deformation parameter $u = 0.1$. Solid line corresponds to $A_1 = \pi/L$, while dashed line correcponds to $A_1 = 0$.

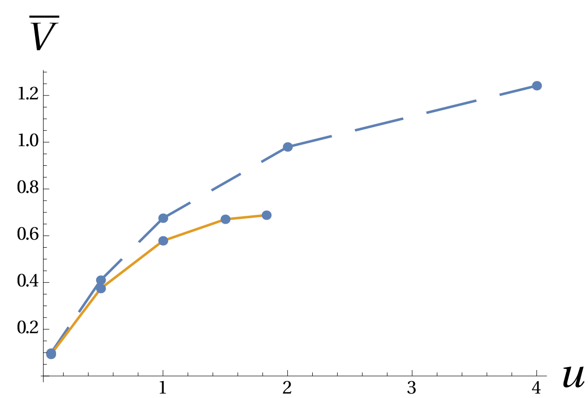

Figure 3: $\bar{V} \equiv 4\pi V$ vs deformation parameter u at the sting length $L = 4.5$.

4 Z_{2N} broken phase

Consider first the large-L domain or, more precisely, $L \gg 1/\Lambda$. In addition we assume that u is not very large. Then we use the second and third equations in (3.3) to find the expressions for masses. Next, we use (3.1) to find the vacuum energy.

We will show below that in the limit of large $L\Lambda \gg 1$ and intermediate u we have $Lm_{b,f} \gg 1$. If so, to find the boson and fermion masses we can apply the asymptotic behavior of the modified Bessel functions,

$$K_0(z) \approx K_1(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z} \,. \tag{4.1}$$

Assuming that $LA_1 = \pi$ we arrive at the following expressions for masses:

$$\begin{aligned} m_b^2 &\approx \Lambda^2 \left(1 - \sqrt{\frac{8\pi}{\Lambda L}} e^{-\Lambda L} \right) \,, \\ m_f^2 &\approx \Lambda^2 e^{-u} \left\{ 1 - \sqrt{\frac{8\pi}{\Lambda L}} e^{\frac{u}{4}} e^{-\Lambda L e^{-u/2}} \right\} \,. \end{aligned} \tag{4.2}$$

If L is large, $L\Lambda \gg 1$, and the value of u is neither too large nor too small, exponential corrections are small and $m_{b,f}$ are of order of Λ. This justifies our approximation. As was already mentioned, m_b and m_f have a meaning of masses for bosons n^l and fermions ξ^l. Thus we have a non-vanishing mass gap in this phase.

From (4.2) we find VEVs of D and σ,

$$\begin{aligned} iD &\approx \Lambda^2 \left\{ 1 - e^{-u} - \sqrt{\frac{8\pi}{\Lambda L}} \left(e^{-\Lambda L} - e^{-3u/4} e^{-\Lambda L e^{-u/2}} \right) \right\} \,, \\ \sqrt{2}\sigma &\approx \Lambda \, e^{-\frac{u}{2}} \left\{ 1 - \sqrt{\frac{2\pi}{\Lambda L}} e^{\frac{u}{4}} e^{-\Lambda L e^{-u/2}} \right\} e^{\frac{2\pi i k}{N}} \,, \end{aligned} \tag{4.3}$$

where $k = 0, ..., (N-1)$.

The presence of non-zero D signals that $\mathcal{N} = (0,2)$ supersymmetry is spontaneously broken. The vacuum energy is

$$E \approx \frac{NL\Lambda^2}{4\pi} \left\{ 1 - e^{-u} + \frac{2}{\Lambda L} \sqrt{\frac{8\pi}{\Lambda L}} \left(e^{-\Lambda L} - e^{-u/4} e^{-\Lambda L e^{-u/2}} \right) \right\} \,. \tag{4.4}$$

The phase of σ in (4.3) is determined by the same phase factor as in (2.12). We see that we have N degenerative vacua, in much the same way as in the infinite volume case. The degeneracy is not due to supersymmetry but due to the fact that the discrete chiral Z_{2N} symmetry is broken down to Z_2.

Our approximation assumes that both boson and fermion masses are large as compared to $1/L$. However, from (4.2) we see that m_f exponentially decreases at large u. Our approximation breaks down when we increase u above the curve

$$L\Lambda \sim e^{\frac{u}{2}}. \tag{4.5}$$

We will see in Sec. 5 that in fact on this curve σ becomes zero and the theory goes into Z_{2N}-symmetric phase.

4.1 Quantum mechanics: the $u \to 0$ limit

It was shown in [5] that the VEV of the σ field in the CP$(N-1)$ model with $\mathcal{N} = (2,2)$ supersymmetry does not depend on the string length. Since in [5] L is not a holomorphic parameter, $\mathcal{N} = (2,2)$ supersymmetry forbids the effective twisted superpotential (which determines the σ VEV) to depend on L.

The fact that L is not a holomorphic parameter in $\mathcal{N} = (2,2)$ CP$(N-1)$ model is not a universal statement. Examples are known when L in combination with another variable form a holomorphic quantity. For instance, in the case of $\mathcal{N} = 1$ supergravity on $R^3 \times S^1$ considered in [20] the radius of S^1 is combined with the dual photon field into one holomorphic parameter which does enter the expression for the superpotential.

Our problem, however, does not fall in the above class. In $\mathcal{N} = (2,2)$ CP$(N-1)$ there is no additional field to partner with the parameter L to make it holomorphic. The conserved R charge in this model plays a custodial role, see [5] and precludes L dependence of the superpotential.

More explicitly, one can expect that the effective twisted superpotential can depend on dimensionless parameter σL, however U$(1)_R$ symmetry forbids this dependence. This is because σ has U$(1)_R$ charge equal to 2 while L is neutral [1]. The L independence of the σ condensate ensues.

However, in the heterotic CP$(N-1)$ model supersymmetry is spontaneously broken. Thus one can expect the σ VEV to depend on the string length. This is what we observe in Eq. (4.3). However, one can note that the expressions for the boson and fermion masses (4.2) in the limit of vanishing u do not reduce to those obtained in the CP$(N-1)$ model with

[1] U$(1)_R$ symmetry is broken by chiral anomaly, however one can compensate for this breaking if one assigns R charge equal to 2 to CP$(N-1)$ scale Λ.

$\mathcal{N} = (2,2)$ supersymmetry. It depends on the string length even if $u = 0$. What is happening?

To resolve this puzzle in this section we note that the $u \to 0$ limit turns out to be in conflict with the quasiclassical approximation in the one-loop effective action which we use in the large-N analysis. We will see below that the relevant parameter is uN^2. Thus, the change of regime we expect to detect occurs at $u \sim 1/N^2$ and is not seen in the standard treatment. We must remember that the value of LA_1 is in its turn determined by a quantum-mechanical problem. In other words, we must take into consideration fluctuations of this quantal variable.

To detect this change of regimes we must consider a quantum-mechanical problem for the Polyakov line (1.1) and average operators $\cos(LkA_1)$ that appear in the equations defining masses (3.3) over the ground state wave function. The equations for the masses in the small-u limit become

$$\begin{aligned} \ln\frac{m_b^2}{\Lambda^2} &= 4\sum_{k=1}^{\infty} K_0(Lm_b k)\chi_k\,, \\ \ln\frac{m_f^2}{\Lambda^2} &= 4\sum_{k=1}^{\infty} K_0(Lm_f k)\chi_k - u\,. \end{aligned} \tag{4.6}$$

where the χ_k is the average value of the operator $\cos(LkA_1)$ defined as

$$\chi_k = \int_{-\pi}^{\pi} L\, dA_1 |\psi|^2 \cos(LkA_1)\,. \tag{4.7}$$

Here ψ is the ground state wave function in quantum mechanics for LA_1.

In this way we obtain the masses

$$\begin{aligned} m_{b\pi}^2 &\approx \Lambda^2\left(1+\sqrt{\frac{8\pi}{\Lambda L}}e^{-\Lambda L}\chi_1\right)\,, \\ m_{f\pi}^2 &\approx \Lambda^2\left(1+\sqrt{\frac{8\pi}{\Lambda L}}e^{-\Lambda L}\left(1+\frac{u\Lambda L}{2}-\frac{3u}{4}\right)\chi_1 - u\right)\,, \end{aligned} \tag{4.8}$$

where we expand the expressions for masses m_b and m_f at large L and small u. This expressions imply a smooth $\mathcal{N} = (2,2)$ limit if χ_1 vanishes with u.

From equation (3.1) one can read off the action for the A_1 quantal variable,

$$S = \int dt \Bigg[\frac{L\dot{A}_1^2}{4e^2} \quad + \quad \frac{LN}{4\pi} \Bigg(8m_f^2 \sum_{k=1}^{\infty} \frac{K_1(Lm_f k)}{Lm_f k} \cos(LkA_1)$$

$$-8m_b^2 \sum_{k=1}^{\infty} \frac{K_1(Lm_b k)}{Lm_b k} \cos(LkA_1) \Bigg) \Bigg] . \tag{4.9}$$

In the large-L limit the equation for the wave function is given by

$$\frac{d^2\psi}{d\phi^2} + (\lambda - 2q\cos(2\phi))\psi = 0 \, , \tag{4.10}$$

where $\phi = LA_1/2$, and the parameter q is defined as follows:

$$q = \frac{uN^2 e^{-\Lambda L}}{(2\pi\Lambda L)^{3/2}} \Lambda L \, , \tag{4.11}$$

(please, observe its explicit dependence on uN^2). This is the Mathieu equation. The solution for the wave function can be found numerically. The averaged value of $\cos(LA_1)$ is

$$\begin{aligned}
\chi_1 &= -0.99 \quad \text{at} \quad \Lambda L = 5 \quad \text{and} \quad uN^2 = 10^9 \\
\chi_1 &= -0.85 \quad \text{at} \quad \Lambda L = 5 \quad \text{and} \quad uN^2 = 10^5 \\
\chi_1 &= -10^{-3} \quad \text{at} \quad \Lambda L = 5 \quad \text{and} \quad uN^2 = 10^1 \, .
\end{aligned} \tag{4.12}$$

Thus we see that for large values of the deformation parameter the averaging plays almost no role, and the saddle point approximation works well. However, as the deformation parameter gets smaller the averaged value of cosine vanishes and the expression for fermion mass reduces to that obtained in the $\mathcal{N} = (2,2)$ model.

A more transparent albeit qualitative analysis can be carried out if we use the harmonic oscillator approximation in our quantal problem. Then one can find the averaged value of $\cos LA_1$ analytically,

$$\chi_1 \approx -\sqrt{uN^2 e^{-\Lambda L}} \left(\frac{2\pi}{\Lambda L} \right)^{1/4} . \tag{4.13}$$

This result explicitly demonstrates vanishing of χ_1 as the deformation parameter uN^2 tends to zero. Thus we see that in the $u \to 0$ limit the solution of the $\mathcal{N} = (0,2)$ model tends to that of the $\mathcal{N} = (2,2)$ model in the interval $u \in [0, \text{const}/N^2]$.

5 The Z_{2N} unbroken phase

Now let us consider the region where u is large, i.e. $u \gg \log \Lambda L$, see Eq. (4.5). For the time being we assume that L is still large, $L \gg 1/\Lambda$. We can find approximate analytic solution for a curve in the $(L,\, u)$ plane at which the Z_{2N} broken phase with N distinct vacua ceases to exist (see the phase diagram in Fig. 1). This phase is terminated when the fermion mass (it is always smaller or equal to the boson mass) reaches zero as we increase u. Assuming that the fermion mass is close to zero so that $Lm_f \ll 1$ we can approximate the sums of the Bessel functions in (3.3). Noting that $\cos(\pi k) = (-1)^k$ we use (A.3) with $y = 0$ to obtain the following expression for the fermion mass

$$(Lm_f)^2 S_2 \approx S_1 + \gamma - \ln \frac{4\pi}{\Lambda L} - \frac{u}{2}\,, \tag{5.1}$$

where $S_{1,2}$ are defined in (A.3). Thus, the solution with non-zero m_f exists only below the curve

$$\Lambda L \approx 4\pi e^{u/2 - S_1 - \gamma}\,. \tag{5.2}$$

This formula gives a more accurate prediction for the curve (4.5) which was obtained in the previous section. Moreover, the minimal string length is $\Lambda L \approx 1.76$. Numerical calculation also shows that the fermionic mass goes to zero at finite values of both L and u, as can be seen from Fig. (4) and (5).

Moreover it is clear from Figs. 4 and 6 that as $L \gg 1/\Lambda$ the fermionic mass m_f tends to Λe^{-u} while iD tends to $\Lambda^2(1 - e^{-u})$, in agreement with (4.2) and (4.3), respectively. One can also note that $iD \to 0$ as $u \to 0$. This is expected since the $u = 0$ limit corresponds to the $\mathcal{N} = (2,2)$ model.

Above the curve (5.2), the only solution of the second equation in (3.3) is

$$\sigma = 0\,, \tag{5.3}$$

while the boson mass

$$m_b^2 \approx \Lambda^2 \left(1 - \sqrt{\frac{8\pi}{\Lambda L}} e^{-\Lambda L}\right) \tag{5.4}$$

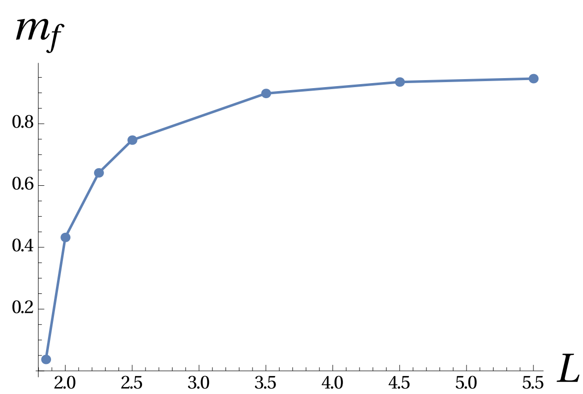

Figure 4: Fermion mass m_f vs string length L at the value of the deformation parameter $u = 0.1$.

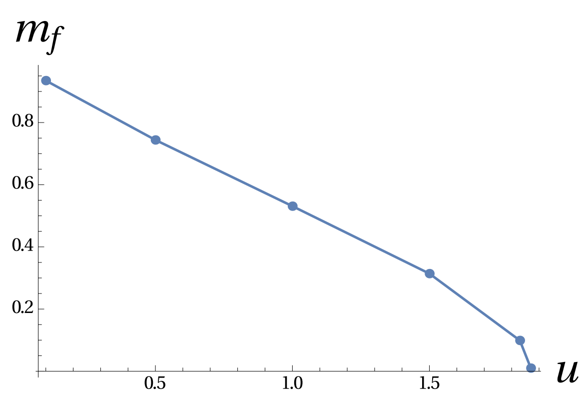

Figure 5: Fermion mass m_f vs deformation parameter u at $L = 4.5$.

is still given by the same expression as in the Z_{2N} broken phase, see (4.2).

Note that the Z_{2N} unbroken phase we have observed is quite remarkable. On the phase transition line N vacua fuse to one, a family of split Z_{2N}-symmetric vacua does not emerge. We will discuss this circumstance later.

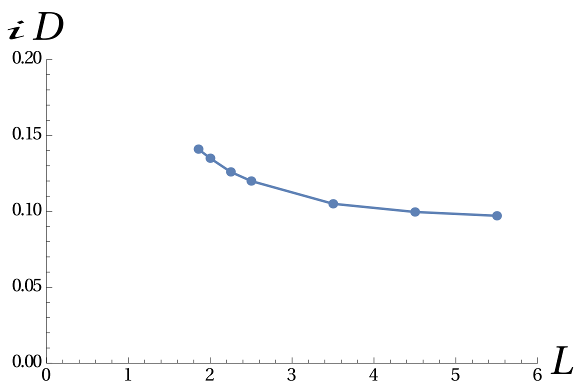

Figure 6: iD vs L at the value of the deformation parameter $u = 0.1$.

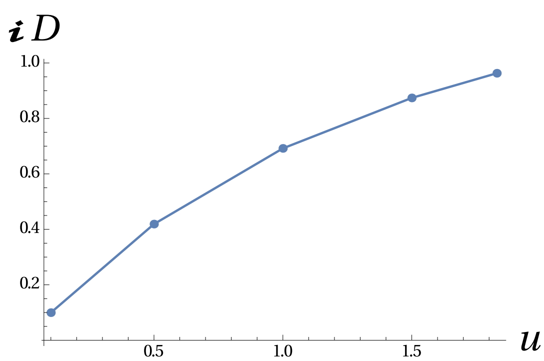

Figure 7: iD vs the deformation parameter u at $L = 4.5$.

5.1 The Lüscher term.

Using the expression (A.13) from Appendix we find that the vacuum energy in this phase is independent on u and given by

$$E \approx \frac{LN\Lambda^2}{4\pi}\left(1 + \frac{2}{\Lambda L}\sqrt{\frac{8\pi}{\Lambda L}}e^{-\Lambda L}\right) - \frac{\pi N}{6L}\,. \tag{5.5}$$

The second term here is the Lüscher term [22]. It arises due to massless fermions. Note, that it equals to half of what we found for non-supersymmetric theory [5] where it comes from bosons. The reason is that now the gauge holonomy is non-trivial, $A_1 = \pi/L$. Moreover, the same reason ensures that although the Lüscher term in (5.5) comes from fermions it still gives negative contribution to the energy as compared to the non-supersymmetric case.

The vacuum energy (5.5) can be compared to the vacuum energy in the Z_{2N} broken phase below the curve (5.2) in the limit of $Lm_f \ll 1$,

$$E \approx \frac{LN\Lambda^2}{4\pi}\left(1 + \frac{2}{\Lambda L}\sqrt{\frac{8\pi}{\Lambda L}}e^{-\Lambda L}\right) - \frac{\pi N}{6L} - \frac{NS_2}{4\pi L}(Lm_f)^4 \,. \tag{5.6}$$

The energy difference is approximately given by the last term above. Equation (5.1) tells us that the energy difference behaves as $\sim (L - L_c(u))^2$ near the phase transition curve, where $L_c(u)$ is given by (5.2).

In summary, we conclude that as we increase u and cross the curve (5.2) our system goes through a line of *third* order phase transitions into the phase with $\sigma = 0$. All N vacua coalesce in the σ plane and Z_{2N} symmetry is restored. In the infrared limit our theory in this phase flows to a conformal limit which is a free theory of massless fermions ξ^l.

5.2 What happens with the A_μ auxiliary field in the Z_{2N} unbroken phase

As we move into the Z_{2N} unbroken phase by increasing u we could, in principle, observe two distinct scenarios: the N former vacua of the Z_{2N} broken phase which fuse themselves into $\sigma = 0$ in phase III, in fact, split in energy, with $N-1$ of them becoming quasivacua, and only one of them remaining as the true vacuum. This phase would be quite similar to the Coulomb/confinement phase in the non-supersymmetric CP($N-1$) model [21, 5].

The second option is to have just a unique vacuum at $\sigma = 0$, with no accompanying family of quasivacua. One can decide between the two options by analyzing the auxiliary field A_μ.

We need to evaluate the two diagrams shown in Fig. (8). The first diagram comes from bosons n^l. In much the same way as in the non-supersymmetric CP($N-1$) model it produces a kinetic term for the photon in the Lagrangian,

$$\frac{1}{4e^2}F_{kl}^2\,, \tag{5.7}$$

where at large L the expression for the charge e^2 is given by

$$\frac{1}{e^2} \approx \frac{N}{12\pi\Lambda^2}\,. \tag{5.8}$$

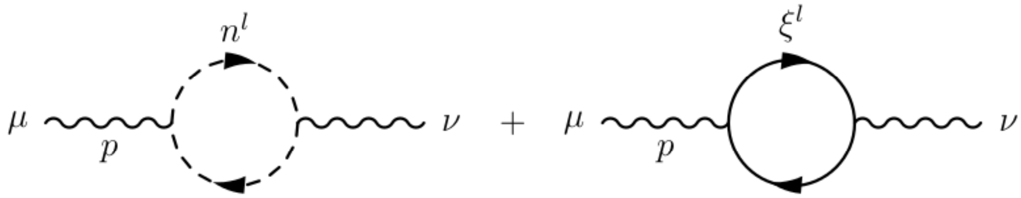

Figure 8: One loop diagrams that contribute to the photon kinetic term.

This makes U(1) gauge field dynamical [21]. In the non-supersymmetric model this leads to confinement of electric charges. The reason is that the static Coulomb potential in two dimensions is linear and ensures that the charged n^l states are linearly confined in the non-supersymmetric model [21]. Similar Coulomb/confining phase occur in the compactified non-supersymmetric CP($N-1$) model at large L [5]. Confinement of n^l states can be interpreted as a small split between quasivacua involved in the θ-angle evolution [25, 26]. In this picture the n^l states are interpreted as kinks interpolating between true vacuum an the first quasivacuum.

On the other hand, in our $\mathcal{N} = (0,2)$ theory we have also the second diagram coming from massless fermions. It produces a mass term for the photon

$$V(A_1) = \frac{N}{2\pi}\left(A_1 - \frac{\pi}{L}\right)^2 . \tag{5.9}$$

Evaluation of the coefficient $N/2\pi$ is presented in Appendix B. This term is a manifestation of the chiral anomaly and appears in much the same way as in the Schwinger model.

Therefore, the photon obtains a mass

$$m_\gamma \approx \sqrt{12}\Lambda . \tag{5.10}$$

The photon mass ensures the exponential fall-off of the electric potential between charged sources. Thus, there is no confinement in the $\sigma = 0$ phase of our (0,2) supersymmetric CP($N-1$) model.

This ensures the absence of fine vacuum structure with split quasivacua. In fact there is no θ dependence in the theory with massless fermions, and the argument of [25] does not apply. We have a single vacuum with the unbroken Z_{2N} symmetry and no family of quasivacua in the $\sigma = 0$ phase (i.e. phase III in Fig. 1). This is a new phase in the CP($N-1$) model which was not known before.

6 Would be broken SU(N) phase

Now let us consider the region of small L. At small L the theory enters a weak coupling regime so we expect the emergence of the classical picture in the limit $N \to \infty$. Classically CP($N-1$) model has $2(N-1)$ massless states which can be viewed as Goldstone states of the broken SU(N) symmetry. To study this possibility much in the same way as in [5, 27] we assume that one component of the field n^l, say $n^1 \equiv n$ can develop VEV and we integrate over all other components of n^l in the external fields n, σ D and A_1. However now in order not to break supersymmetry by the boundary conditions we have to leave out one component of ξ fields as well. Due to the constraint (2.5) we can choose these components to be $\xi^N_{L,R} \equiv \xi_{L,R}$. The expression for the energy is

$$
\begin{aligned}
E &= \frac{LN}{4\pi}\Bigg[iD - iD\ln\frac{m_b^2}{\Lambda^2} - m_f^2 \ln\frac{m_b^2}{m_f^2} + m_f^2 u \\
&+ 8m_f^2 \sum_{k=1}^{\infty} \frac{K_1(Lm_f k)}{Lm_f k}\cos(kLA_1) - 8m_b^2 \sum_{k=1}^{\infty} \frac{K_1(Lm_b k)}{Lm_b k}\cos(kLA_1)\Bigg] \\
&+ L\left[(m_b^2 + A_1^2)|n|^2 + i\sqrt{2}\sigma\bar{\xi}_R\xi_L + i\sqrt{2}\sigma^\star\bar{\xi}_L\xi_R\right] \\
&+ i\bar{\xi}_L\xi_L LA_1 - i\bar{\xi}_R\xi_R LA_1 \\
&+ N\left[\sqrt{m_f^2 + A_1^2} - \sqrt{m_b^2 + A_1^2}\right], \qquad (6.1)
\end{aligned}
$$

where the first two lines are the same as in (3.1), the third and fourth lines correspond to components which we left out of integration, and the last line gives the contribution due to omission of the zero modes.

6.1 Saddle point approximation

Proceeding as in the SU(N) symmetric case we obtain the following set equations that defines a stationary point

$$0 = (m_b^2 + A_1^2)n\,, \tag{6.2}$$

$$0 = \sqrt{2}\sigma\xi_L - \xi_R A_1 = \sigma^\star \xi_R + \xi_L A_1\,, \tag{6.3}$$

$$|n|^2 = \frac{N}{L}\Big[\frac{1}{2\sqrt{m_b^2+A_1^2}} + \frac{L}{4\pi}\ln\frac{m_b^2}{\Lambda^2} - \frac{L}{\pi}\sum_{k=1}^{\infty} K_0(Lm_b k)\cos(kLA_1)\Big], \tag{6.4}$$

$$\begin{aligned} 0 = N\Big[&\frac{2Lm_b}{\pi}\sum_{k=1}^{\infty} K_1(Lm_b k)\sin(kLA_1) - \frac{2Lm_f}{\pi}\sum_{k=1}^{\infty} K_1(Lm_f k)\sin(kLA_1), \\ - &\frac{A_1}{\sqrt{m_b^2+A_1^2}} + \frac{A_1}{\sqrt{m_f^2+A_1^2}}\Big] + 2LA_1|n|^2 + iL\bar{\xi}_L\xi_L - iL\bar{\xi}_R\xi_R \end{aligned} \tag{6.5}$$

$$\begin{aligned} 0 = Li\sqrt{2}\bar{\xi}_L\xi_R + 2\sigma\Big[&L|n|^2 + N\Big(- \frac{1}{2\sqrt{m_b^2+A_1^2}} + \frac{1}{2\sqrt{m_f^2+A_1^2}} \\ + &\frac{L}{\pi}\sum_{k=1}^{\infty} K_0(Lm_b k)\cos(kLA_1) - \frac{L}{\pi}\sum_{k=1}^{\infty} K_0(Lm_f k)\cos(kLA_1)\Big), \\ + &\frac{LN}{4\pi}\Big(u - \ln\frac{m_b^2}{m_f^2}\Big)\Big]\,. \end{aligned} \tag{6.6}$$

From (6.2) we conclude that $m_b = A_1 = 0$. Then (6.5) does not have a solution unless $\sigma = 0$. We also see that $\bar{\xi}_{L,R} = \xi_{L,R} = 0$ satisfies the above system of equations. We find that n^l field develops a vacuum expectation value

$$|n|^2 = \frac{N}{2\pi}\left(\ln\frac{4\pi}{\Lambda L} - \gamma\right)\,, \tag{6.7}$$

which implies in turn that this solution exists only for $\Lambda L \leq 7.05$. The energy is found to be zero as in the supersymmetric case, see phase I in Fig. (1).

This phase is similar to the dynamical regime we found previously in the non-supersymmetric CP($N-1$) model [5]. In particular, the VEV of n^l breaks global SU(N) symmetry implying the presence of $2(N-1)$ real massless degrees of freedom. As we already mentioned the dynamics of the

CP($N-1$) model in this phase is determined by quasiclassical approximation in the action (2.1). At small L the theory is at weak coupling because the inverse coupling constant r is determined by

$$r = \frac{N}{2\pi} \log \frac{1}{L\Lambda}. \tag{6.8}$$

The constant r grows large at small L.

However, we do not expect exactly massless modes to appear in 1 + 1 dimensions because of Mermin-Wagner-Coleman's theorem [23, 24]. We found the above solution in the leading order in $1/N$. It holds only in the limit $N = \infty$. Thus, we should expect higher order corrections to modify the result. In particular, the would-be Goldstone massless modes may acquire small masses suppressed in the large N limit. As a consequence the energy might be uplifted from zero.

The solution that we found is completely u-independent. Thus we expect that the vacuum energy in the would be broken phase is given by E_{br} which is independent on u and suppressed at large N.

7 Quantum mechanics at small L: $u \to 0$ limit

Now we have to study the limit $u \to 0$ at small L where the theory should match the $\mathcal{N} = (2, 2)$ CP($N-1$) model which has a single SU(N) symmetric (Z_{2N} broken) phase with the mass gap independent of L. Clearly the would be broken SU(N) phase cannot explain this limit because it is u-independent. Our analysis in this section has a qualitative nature. As we have already seen, for the case of large L the transition occurs at $uN^2 \sim 1$ where the large-N approximation strictly speaking is not applicable.

Below we argue that the SU(N) symmetric phase reappear again when we go to the limit of extremely small u keeping L small, $L \ll 1/\Lambda$. Assuming that both $Lm_{b,f} \ll 1$ in this phase we use (A.13) to find the expression for the potential valid for LA_1 close to π

$$V(\tilde{A}_1) \approx \frac{NL^2}{\pi} \tilde{A}_1^2 \left(m_b^2 - m_f^2 \right) S_2 \, , \tag{7.1}$$

where $\tilde{A}_1 \equiv A_1 - \pi/L$. By analogy with (5.1) one can find the expression for the bosonic mass

$$(Lm_b)^2 S_2 \approx S_1 + \gamma - \ln \frac{4\pi}{\Lambda L} \,. \tag{7.2}$$

Thus the expression for the potential is given by

$$V(\tilde{A}_1) \approx \frac{Nu}{2\pi} \tilde{A}_1^2 \,, \tag{7.3}$$

Hence, as u gets smaller the potential becomes weaker and flatter. When LA_1 gets close to 0 or 2π the above expression becomes invalid. The results of numerical calculations are given in Fig. (9). Two curves correspond to two values of deformation parameter $u = 0.05$ and $u = 0.1$ (dashed curve). One can see that the expression we derived is in a good agreement with numerical results. As u gets smaller the amplitude of the potential also decreases.

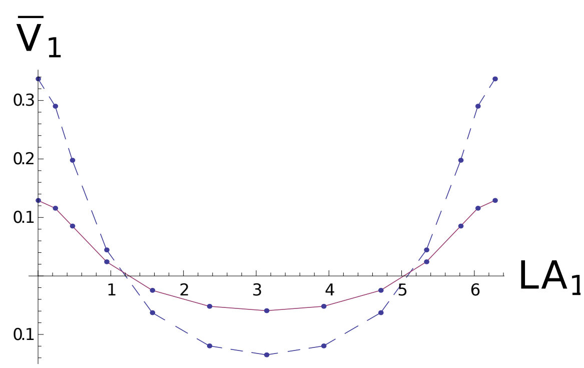

Figure 9: Dependence of potential $\bar{V}_1 \equiv \dfrac{\pi L^2}{2N} V$ on the deformation parameter u.

We see that in the limit $u \to 0$ the potential $V(\tilde{A}_1)$ becomes flat and we have to average over A_1 (instead of taking the saddle point value $A_1 = \pi/L$) in much the similar way as we did in Sec. 4.1 for the region of large L. The averaging procedure gives us $\mathcal{N} = (2,2)$ limit.

More exactly the vacuum energy in SU(N) symmetric phase at extremely small u is given by

$$E_{sym} \approx \frac{uN}{4\pi} \Lambda^2 L \,. \tag{7.4}$$

Comparing this with the vacuum energy E_{br} in the would be broken SU(N) phase which is independent of u we see that at very small critical $u_c \sim 1/N^2$, the energy in the SU(N) unbroken phase becomes lower then that in the SU(N) broken phase, and the system undergoes a phase transition into SU(N) symmetric phase, see Fig. 1. The SU(N) symmetric phase has a perfectly smooth $u \to 0$ limit.

8 Conclusions

To summarize, we find three different phases, see Fig 1. At large L and intermediate values of the deformation parameter u there is a phase with a mass gap, N vacua and broken discrete Z_{2N} symmetry. As we increase u a reach a critical value (which grows with L) we find a phase transition to the Z_{2N} symmetric phase, with a unique vacuum. The line separating these two SU(N) symmetric phases is a line of a third order phase transitions in the large N limit.

As the string under consideration gets shorter we find a phase transition to a phase with the would be broken SU(N) symmetry (phase II). In this phase we expect masses of the n fields to be much smaller than in two SU(N) symmetric phases. In fact, at $N = \infty$ they vanish. At small L and extremely small u we expect another phase transition from the would be broken SU(N) phase into the SU(N) unbroken phase which has a smooth $u \to 0$ limit.

Strictly speaking, our description of the underlying dynamics in terms of the phase transitions is valid only at $N = \infty$. At large but finite N one can expect that all phase transitions become rapid crossovers.

Acknowledgments

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Appendix A:
Relations for modified Bessel functions

In this Appendix we derive all the relations for the sums of modified Bessel functions of the second kind used in the text. We will use the following asymptotic behavior

$$K_1(z) \to \frac{1}{z} \quad \text{as} \quad z \to 0\,, \tag{A.1}$$

as well as the properties of derivatives

$$K_0(z)' = -K_1(z) \quad \text{and} \quad K_1'(z) = -K_0(z) - \frac{K_1(z)}{z}\,, \tag{A.2}$$

and the following approximations, valid to order $O(y^2, z^2)$ (see formula 8.526 in [28])

$$\begin{aligned}
\sum_{k=1}^{\infty} K_0(zk)\cos(yk) &= \frac{\gamma}{2} + \frac{1}{2}\ln\frac{z}{4\pi} + \frac{\pi}{2\sqrt{z^2+y^2}} + S_0(2y^2 - z^2) + \delta_0\,, \\
\sum_{k=1}^{\infty} K_0(zk)(-1)^k\cos(yk) &= \frac{\gamma}{2} + \frac{1}{2}\ln\frac{z}{4\pi} + \frac{S_1}{2} + \frac{S_2}{2}(2y^2 - z^2) + \delta_1\,,
\end{aligned} \tag{A.3}$$

where $\delta_{0,1} \sim y^2 z^2$ and we defined the sums

$$\begin{aligned}
S_0 &= \sum_{l=1}^{\infty} \frac{\pi}{(2\pi l)^3} \approx 0.015\,, \quad S_1 = \sum_{l=1}^{\infty} \frac{1}{l(2l-1)} \approx 1.386\,, \\
S_2 &= \sum_{l=1}^{\infty} \frac{1}{\pi^2(2l-1)^3} \approx 0.107\,.
\end{aligned} \tag{A.4}$$

To find the sum involving cosine we notice that on one hand

$$\frac{d}{dz}\left(z\sum_{k=1}^{\infty}\frac{K_1(zk)}{k}\cos(yk)\right) = -z\sum_{k=1}^{\infty}K_0(zk)\cos(yk)\,, \tag{A.5}$$

and on the other hand

$$\frac{d}{dy}\left(\sum_{k=1}^{\infty}\frac{K_1(zk)}{k}\cos(yk)\right) = -\sum_{k=1}^{\infty}K_1(zk)\sin(yk)\,, \tag{A.6}$$

moreover the following relation also holds

$$\frac{d}{dz}\left(\sum_{k=1}^{\infty} K_0(zk)\cos(yk)\right) = -\frac{d}{dy}\left(\sum_{k=1}^{\infty} K_1(zk)\sin(yk)\right), \tag{A.7}$$

where we used (A.2) several times.

First using (A.5) and the expansion from (A.3) we find to order $O(y^2, z^2)$

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{K_1(zk)}{k}\cos(yk) &\approx -\frac{\pi\sqrt{z^2+y^2}}{2z} - \frac{z(2\gamma-1)}{8} - \frac{z}{4}\ln\frac{z}{4\pi} \\ &- S_0 z y^2 + \frac{f_1(y)}{z}\end{aligned} \tag{A.8}$$

where $f_1(y)$ depends on y.

Now using (A.7) and approximation (A.3) we find that

$$\sum_{k=1}^{\infty} K_1(zk)\sin(yk) \approx \frac{\pi y}{2z\sqrt{z^2+y^2}} - \frac{y}{2z} + 2S_0 z y + f_2(z), \tag{A.9}$$

where $f_2(z)$ is a function which depends on z. Since LHS vanishes when $y = 0$ and $z \neq 0$ we conclude that $f_2(z) = 0$. Now from (A.6) we find that

$$\sum_{k=1}^{\infty}\frac{K_1(zk)}{k}\cos(yk) \approx -\frac{\pi\sqrt{z^2+y^2}}{2z} + \frac{y^2}{4z} - S_0 z y^2 + f_3(z), \tag{A.10}$$

where $f_3(z)$ depends on z.

To fix $f_1(y)$ and $f_3(z)$ we use the property (A.1) and find that

$$\sum_{k=1}^{\infty}\frac{K_1(zk)}{k}\cos(yk) \to \sum_{k=1}^{\infty}\frac{\cos(yk)}{zk^2} = \frac{1}{z}\left(\frac{y^2}{4} - \frac{\pi y}{2} + \frac{\pi^2}{6}\right). \tag{A.11}$$

Thus we conclude that

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{K_1(zk)}{k}\cos(yk) &\approx -\frac{\pi\sqrt{z^2+y^2}}{2z} + \frac{y^2}{4z} + \frac{\pi^2}{6z} - S_0 z y^2 \\ &- \frac{z(2\gamma-1)}{8} - \frac{z}{4}\ln\frac{z}{4\pi}.\end{aligned} \tag{A.12}$$

In a similar way we find that

$$\sum_{k=1}^{\infty} \frac{K_1(zk)}{k}(-1)^k \cos(yk) \approx -\frac{z(2S_1+2\gamma-1)}{8} - \frac{z}{4}\ln\frac{z}{4\pi} - \frac{\pi^2}{12z} + \frac{y^2}{4z} - \frac{S_2}{2}zy^2 \,. \tag{A.13}$$

Appendix B: Photon mass

In this Appendix we derive an expression for the photon mass. Due to gauge invariance both the diagrams in Fig. (8) have to be of the form

$$\Pi_{ij} = \Pi(p^2)\left(p^2\delta_{ij} - p_i p_j\right) \,. \tag{B.1}$$

Below we show that for the second diagram $\Pi(p^2)$ has a pole which means that photons acquire mass. We put $p_1 = 0$ and evaluate Π_{11}:

$$\Pi_{11} = -\frac{1}{L}\Big[\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dq_0}{2\pi}\frac{2q_1^2 - 2q_0(p_0+q_0) - 2m_1^2}{(q_0^2+q_1^2+m_1^2)(p_0^2+2p_0q_0+q_0^2+q_1^2+m_1^2)} - [m_1 \leftrightarrow m_2]\Big] \,, \tag{B.2}$$

where m_1 is the fermion mass, which we put to zero at the end, m_2 is the mass of Pauli-Villars regulator, and q_1 is a discrete momentum

$$q_1 = \frac{2\pi k}{L} + A_1 = \frac{\pi}{L}(2k+1) \,. \tag{B.3}$$

We introduce Feynman parameter x and substitute integration variable $q_0 = l - p_0 x$

$$\Pi_{11} = -\frac{1}{L}\Big[\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^1 \frac{dl dx}{2\pi}\frac{2q_1^2 - 2m_1^2 + 2p_0^2 x(1-x) - 2l^2}{[l^2+m_1^2+q_1^2+xp_0^2-x^2p_0^2]^2} - [m_1 \leftrightarrow m_2]\Big] \,, \tag{B.4}$$

where terms linear in l drop out. Integrating over l one finds

$$\Pi_{11} = \frac{1}{L}\Big[\sum_{k=-\infty}^{\infty}\int_0^1 dx \frac{m_1^2}{[m_1^2+q_1^2+xp_0^2-x^2p_0^2]^{3/2}} - [m_1 \leftrightarrow m_2]\Big] \,, \tag{B.5}$$

and since $m_1 = 0$ the first term vanishes and only the contribution from the regulator remains. To integrate over x we use third Euler's substitution

$$\sqrt{-p_0^2 x^2 + p_0^2 x + m^2 + q_1^2} = \sqrt{-p_0^2 (x - x_1)(x - x_2)} = t(x - x_1)\,. \qquad \text{(B.6)}$$

One can easily check that neither of the roots belong to the interval $x \in [0, 1]$ and thus this substitution is justified. After integration we obtain the following sum

$$\Pi_{11} = -\frac{1}{L} \sum_{k=-\infty}^{\infty} \frac{m_2^2}{(q_1^2 + m_2^2 + \frac{p_0^2}{4})\sqrt{q_1^2 + m_2^2}} \approx -\frac{1}{L} \sum_{k=-\infty}^{\infty} \frac{m_2^2}{(q_1^2 + m_2^2)^{3/2}}\,, \quad \text{(B.7)}$$

where we ignore p_0 compared to m_2. Evaluating this sum (see Appendix in [5]) we finally obtain (setting $m_2 \to \infty$)

$$\Pi_{11} = -\frac{1}{\pi}\,, \qquad \text{(B.8)}$$

which tells us that $\Pi(p^2)$ indeed contains a pole

$$\Pi(p^2) = -\frac{1}{\pi p^2} \qquad \text{(B.9)}$$

and the photon becomes massive.

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