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Rényi entropy of free $(2,0)$ tensor multiplet and its supersymmetric counterpart

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Abstract: We compute the Rényi entropy and the supersymmetric Rényi entropy for the six-dimensional free $(2,0)$ tensor multiplet. We make various checks on our results, and they are consistent with the previous results about the $(2,0)$ tensor multiplet. As a by-product, we have established a canonical way to compute the Rényi entropy for p-form fields in d-dimensions.

1. Introduction

Entanglement entropy (EE) and Rényi entropy have been intensively studied in recent years. They not only play important roles in quantum information theory and condensed matter physics, but also bring new insights into high energy physics. For instance, in the context of conformal field theories the entanglement entropy in even dimensions is related to the conformal anomaly [1], while both entanglement entropy and Rényi entropy can be computed holographically [2,3]. Combining the ideas of supersymmetric localization [4–6] and Rényi entropy, one can also define the supersymmetric refinement of the ordinary Rényi entropy on the branched sphere [7]. Interestingly, the supersymmetric Rényi entropy enjoys universal relations with central charges in even dimensions, which provides a new way to derive the Hofman-Maldacena bounds [8].

Let us briefly review these concepts. Suppose the space on which the theory is defined can be divided into a piece A and its complement $\bar{A} = B$, and correspondingly the Hilbert space factorizes into a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. The density matrix over the whole Hilbert space is ρ; then the reduced density matrix is

$$\rho_A := \mathrm{tr}_B \rho\,. \tag{1.1}$$

The entanglement entropy is the von Neumann entropy of ρ_A,

$$S_E := -\mathrm{tr}\rho_A \log \rho_A\,, \tag{1.2}$$

while the Rényi entropies are defined to be

$$S_n := \frac{1}{1-n} \log \mathrm{tr}(\rho_A)^n\,. \tag{1.3}$$

Assuming that a satisfactory analytic continuation of S_n can be obtained, we can alternatively express the entanglement entropy as the $n \to 1$ limit of the Rényi entropies:

$$\lim_{n\to 1} S_n = S_E\,. \tag{1.4}$$

In ref. [1], Casini, Huerta and Myers proposed a way to compute the Rényi entropy of a general CFT for a spherical entangling surface Σ in flat space $\mathbb{R}^{1,d-1}$. One first considers the causal development $\mathcal{D}$ of the ball enclosed by Σ and then observes that $\mathcal{D}$ can be conformally mapped to a "hyperbolic cylinder", $\mathbb{R} \times \mathbb{H}^{d-1}$.

The curvature scale of $\mathbb{H}^{d-1}$ matches the radius of Σ, R, and the vacuum of the CFT in the original spacetime is mapped to a thermal bath in the hyperbolic cylinder with temperature

$$T_0 = \frac{1}{2\pi R} \ . \tag{1.5}$$

Further a unitary transformation connects the original reduced density matrix ρ and the thermal density matrix ρ_{therm}:

$$\rho = U^{-1} \rho_{\text{therm}} U \ . \tag{1.6}$$

Since the definition (1.3) is invariant under unitary transformation, the Rényi entropy across Σ in flat space is then equal to the Rényi entropy of ρ_{therm}. Taking the n-th power of ρ_{therm} is equivalent to setting the temperature to $T = T_0/n$. A similar Weyl transformation can map the hyperbolic cylinder to de Sitter space. Writing the trace in terms of Euclidean path integral, we have

$$S_n = \frac{1}{1-n} \left(\log Z_n[\mathbb{S}^d_n] - n \log Z_1[\mathbb{S}^d] \right) \tag{1.7}$$

$$= \frac{1}{1-n} \left(\log Z_n[\mathbb{S}^1_n \times \mathbb{H}^{d-1}] - n \log Z_1[\mathbb{S}^1 \times \mathbb{H}^{d-1}] \right) \ . \tag{1.8}$$

These identities are generally true for both non-supersymmetric and supersymmetric Rényi entropies of all CFTs. However, these quantities defined in (1.7) and (1.8) in general are UV divergent. One therefore has to extract "universal part" free of ambiguities. More precisely, by "universal part" we mean the finite term in odd dimensions but the log term in even dimensions. For further discussions on the universal terms in both odd and even dimensions we refer to [1].

The concept supersymmetric Rényi entropy was first studied in three dimensions [7, 10, 11], and later generalized to four dimensions [8, 12, 13] and five dimensions [14, 15]. By turning on certain background gauge fields (chemical potentials) and using the supersymmetric localization technique, one can calculate the partition function Z_n on the n-branched sphere, and define the supersymmetric Rényi entropy:

$$S_n^{SUSY} = \frac{\log Z_n(\mu(n)) - n \log Z_1(0)}{1-n} \, . \tag{1.9}$$

It is a supersymmetric refinement of the ordinary Rényi entropy, which is in general non-supersymmetric because of the conic singularity. The quantities defined by (1.9) are in general UV divergent, but one can extract universal parts free of ambiguities. This becomes particularly clear in even dimensions. For instance, for $\mathcal{N} = 4$ SYM in four dimensions, the log coefficient of supersymmetric Rényi entropy as a function of n and three chemical potentials μ_1, μ_2, μ_3 (corresponding to three $U(1)$ Cartans of $SO(6)$ R-symmetry respectively) has been shown to be protected from the interactions [12]. It also received a precise check from the holographic computation of the BPS 3-charge topological AdS black hole in five-dimensional gauged supergravity [12]. The above facts indicate that supersymmetric Rényi entropy may be used as a new robust observable to understand superconformal field theories (SCFTs).

Our main concern in this work is six-dimensional $(2,0)$ superconformal field theories. While it is easy to identify a free Abelian tensor model that realizes $(2,0)$ superconformal symmetry, the existence of interacting $(2,0)$ theories was only inferred from their embedding into particular constructions in string theory [16–18]. It is not yet understood how to formulate interacting $(2,0)$ SCFTs. Furthermore, it is believed that the relevant or marginal deformations preserving $(2,0)$ supersymmetry do not exist [19, 20]. Therefore, as the initial start we will mainly focus on free Abelian tensor multiplets in the present work.

There are some previous works on the Rényi entropy of $6d$ CFTs. In [21] and [22], the shape dependence of entanglement entropy for general $6d$ CFTs was initially studied [1] and some concrete results of entanglement entropy for $(2,0)$ theories were also presented in [22]. Based on these works, in [24] the authors investigated the shape dependence of Rényi entropy, where they pointed out that the n-dependence of Rényi entropy across a non-spherical entangling surface (with vanishing extrinsic curvature) is actually determined by that across a spherical one in flat space.

In this note we provide the first direct field-theoretic calculation of the Rényi entropy and the supersymmetric Rényi entropy for the most interesting six-dimensional SCFTs, the $(2,0)$ theories. We compute the Rényi entropy S_n explicitly by carefully analyzing the contribution of the two-form field with self-dual strength. The result receives quite a few consistency checks. Namely, our Rényi entropy result at $n = 1$ is consistent with [22, 25], while the first derivative $\partial_n S_n$ at $n = 1$ and the two-point correlator of stress tensor [26] satisfy the relation proposed in [27]. Furthermore, the second derivative $\partial_n^2 S_n$ at $n = 1$ and the three-point correlator of stress tensor [26] satisfy the relation proposed in [28]. We also obtain the supersymmetric counterparts of the Rényi entropy for the tensor multiplet, which behave reasonably simple as a function of n because of supersymmetry.

The note is organized as follows. In Section 2, we calculate the Rényi entropy for the six-dimensional $(2,0)$ tensor multiplet, and perform various checks on the result. In Section 3, we turn on the chemical potential and compute the supersymmetric Rényi entropy for several cases. We conclude in Section 4 and leave the explicit analysis of Killing spinor equations in Appendix A.

2. Rényi entropy of $6d$ free CFTs

The Rényi entropy of $6d$ CFTs in $\mathbb{R}^6$ associated with a $4d$ spherical entangling surface $\mathbb{S}^4$ can be computed using the thermal partition function on a hyperbolic space

[1] See [23] for further investigations.

$\mathbb{S}^1 \times \mathbb{H}^5$. The partition function $Z(\beta)$ on $\mathbb{S}^1_\beta \times \mathbb{H}^5$ can be computed from the heat kernel of the gapless Laplacian Δ,

$$\log Z(\beta) = \frac{1}{2}\int_0^\infty \frac{dt}{t} K_{\mathbb{S}^1_\beta \times \mathbb{H}^5}(t) \ , \tag{2.1}$$

where $K(t)$ is defined as the trace of the kernel of the operator Δ

$$K(t) := \mathrm{Tr}(e^{-t\Delta}) = \int \mathrm{d}^6 x \sqrt{g}\, K(x,x,t) \ , \quad K(x,y,t) := \langle x|e^{-t\Delta}|y\rangle \ . \tag{2.2}$$

Since $\mathbb{S}^1_\beta \times \mathbb{H}^5$ is a direct product, the kernel is factorized,

$$K_{\mathbb{S}^1_\beta \times \mathbb{H}^5}(t) = K_{\mathbb{S}^1_\beta}(t)\, K_{\mathbb{H}^5}(t) \ . \tag{2.3}$$

The heat kernel on $\mathbb{S}^1$ is given by

$$K_{\mathbb{S}^1_\beta}(t) = \frac{\beta}{\sqrt{4\pi t}} \sum_{m\neq 0, \in \mathbb{Z}} e^{\frac{-\beta^2 m^2}{4t}} \ . \tag{2.4}$$

The hyperbolic space $\mathbb{H}^5$ is homogeneous and therefore the volume V_5 factorizes

$$K_{\mathbb{H}^5}(t) = \int d^5 x \sqrt{g}\ K_{\mathbb{H}^5}(x,x,t) := V_5\, K_{\mathbb{H}^5}(0,t) \ , \tag{2.5}$$

where $V_5 = \pi^2 \log(R/\epsilon)$ is the regularized volume of $\mathbb{H}^5$. ϵ is the UV cutoff of the theory and R is the curvature radius of the hyperbolic space.

2.1 Complex scalar

The heat kernel of a complex scalar on $\mathbb{H}^5$ is given by

$$K^s_{\mathbb{H}^5}(t) = \frac{V_5}{(4\pi t)^{5/2}}\left(2 + \frac{4t}{3}\right) \ . \tag{2.6}$$

The free energy can be computed by (2.1)

$$F^s(\beta) := -\log Z(\beta) = \frac{\pi V_5}{1890\beta^5}(-8\pi^2 - 7\beta^2) \ , \tag{2.7}$$

and the Rényi entropy is given by

$$S^s_n := \frac{nF(\beta = 2\pi) - F(\beta = 2n\pi)}{1-n} = \frac{(n+1)(3n^2+1)(3n^2+2)V_5}{15120\pi^2 n^5} \ . \tag{2.8}$$

This reproduces the Rényi entropy result of conformal scalars first presented in [29].

2.2 Weyl fermion

The heat kernel of a Weyl fermion on $\mathbb{H}^5$ is given by (we have taken into account $2^{[\frac{6}{2}]}/2 = 4$ components)

$$K^f_{\mathbb{H}^5}(t) = \frac{V_5}{(4\pi t)^{5/2}} \left(4 + \frac{20t}{3} + 3t^2\right) \ . \tag{2.9}$$

The free energy can be computed by (2.1) with anti-periodic boundary conditions along $\mathbb{S}^1$

$$F^f(\beta) = \frac{-V_5}{60480\pi\beta^5}(496\pi^4 + 980\pi^2\beta^2 + 945\beta^4) \ , \tag{2.10}$$

and the Rényi entropy is given by

$$S^f_n = \frac{(n+1)\,(1221n^4 + 276n^2 + 31)\,V_5}{120960\pi^2 n^5} \ . \tag{2.11}$$

This reproduces the Rényi entropy of massless fermions first presented in [28].

2.3 Two-form

One has to be careful about the heat kernel computation of Rényi entropy for p-form fields. To do this, we employ the general results of eigenvalue distributions for Hodge-de Rham operator of p-form fields in N-dimensional hyperbolic space [30],

$$\mathrm{Tr}\, K_{\mu_1\cdots\mu_p}{}^{\nu_1\cdots\nu_p}(x,x',t) = V_N \int_0^\infty d\lambda\ \mu(\lambda)\, e^{-[\lambda^2+(\rho-p)^2]t} \ , \tag{2.12}$$

where $\rho := \frac{N-1}{2}$ and the trace has been taken for both indices $\mu_1\cdots\mu_p$ and coordinates x. The eigenvalue distribution $\mu(\lambda)$ is given by

$$\mu(\lambda) = \frac{c_N}{\Omega_{N-1}} \frac{\pi\, g(p)}{\left[2^{N-2}\Gamma(\frac{N}{2})\right]^2 [\lambda^2 + (\rho-p)^2]} \prod_{j=0}^{\frac{N-1}{2}} (\lambda^2 + j^2) \qquad (N \text{ odd}) \ , \tag{2.13}$$

where c_N, $g(p)$ and Ω_N are defined as

$$c_N := \frac{2^N}{4\pi} \ , \quad g(p) := \frac{(N-1)!}{p!(N-p-1)!} \ , \quad \Omega_N := \frac{2\pi^{\frac{N+1}{2}}}{\Gamma(\frac{N+1}{2})} \ . \tag{2.14}$$

One may want to first reproduce the known Rényi entropy result of gauge field in $4d$ by using (2.12). In that case, $p = 1$ and $N = 3$. Therefore,

$$c_3 = \frac{2}{\pi} \ , \quad g(p=1) = 2 \ , \quad \Omega_2 = 4\pi \ . \tag{2.15}$$

The eigenvalue distribution $\mu(\lambda)$ is

$$\mu(\lambda) = \frac{\lambda^2 + 1}{\pi^2} \ . \tag{2.16}$$

The heat kernel of a 1-form field on $\mathbb{H}^3$ is given by

$$K^v_{\mathbb{H}^3}(t) = V_3 \frac{2t+1}{4\pi^{3/2}t^{3/2}} \ . \tag{2.17}$$

With this kernel the Rényi entropy can be computed by evaluating the partition function (2.1)

$$S^v_{d=4} = \frac{(n+1)\,(31n^2+1)\,V_3}{360\pi n^3} \ . \tag{2.18}$$

After adding up with a constant discrepancy between entanglement entropy $S_{n\to 1}$ and $4a_4$, where a_4 is the standard gauge field a-anomaly coefficient, we are able to reproduce the known Rényi entropy of a $4d$ gauge field

$$S^v_{d=4} = \frac{(91n^3+31n^2+n+1)\,V_3}{360\pi n^3} \ . \tag{2.19}$$

Notice that the physical degrees of freedom of the $4d$ Abelian gauge field (photon) is two, hence we learn from the $4d$ exercise that the formula (2.12) actually takes into account all the physical degrees of freedom for the p-form field.

Now we turn to two-form field in $6d$, which corresponds to $p=2$ and $N=5$. In this case,

$$c_5 = \frac{8}{\pi} \ , \quad g(p=2) = 6 \ , \quad \Omega_4 = \frac{8\pi^2}{3} \ . \tag{2.20}$$

The eigenvalue distribution is

$$\mu(\lambda) = \frac{(\lambda^2+1)(\lambda^2+4)}{2\pi^3} \ . \tag{2.21}$$

The kernel of a 2-form field on $\mathbb{H}^5$ is given by

$$K^v_{\mathbb{H}^5}(t) = V_5 \frac{2t(8t+5)+3}{16\pi^{5/2}t^{5/2}} \ . \tag{2.22}$$

With this kernel the Rényi entropy is obtained as

$$S^v_{d=6} = \frac{(n+1)\,(877n^4+37n^2+2)\,V_5}{5040\pi^2 n^5} \ . \tag{2.23}$$

Notice that this should not be the final result of the Rényi entropy for a 2-form field in 6d, as we have not yet considered the boundary contribution on the entangling surface [31]. Since the boundary modes are staying on the fixed surface under the n-orbifolding, their contribution is independent of n and then becomes a constant shift. Therefore we can compute it at $n=1$ and use the result for general n.

Let us first figure out the precise value of the complete entanglement entropy of a 2-form field. Recall that the universal relation between the a-anomaly and the entanglement entropy for a general CFT in even dimensions has been shown in Ref. [1]

$$S_{EE} = (-1)^{\frac{d}{2}-1}\, 4\, a_d \log\left(\frac{R}{\epsilon}\right) , \tag{2.24}$$

where a_d is the A-type trace anomaly in d dimensions, i.e.,[2]

$$\langle T^{\mu}{}_{\mu}\rangle = \sum b_n I_n - 2(-1)^{\frac{d}{2}}\, a_d\, E_d\,. \tag{2.25}$$

E_d and I_n are the Euler density and the Weyl invariants of weight $-d$ in d dimensions respectively. Due to the linear relation (2.24), one can make use of the known results of the a-anomalies, the ratios between the a-anomalies of one complex scalar field, one Weyl fermion and one two-form field [25],[3]

$$a_6^s : a_6^f : a_6^v = \frac{5\times 2}{72} : \frac{191}{72\times 2} : \frac{221}{4}\ . \tag{2.26}$$

On the other hand, the ratios between the entanglement entropies of one complex scalar (2.8), one Weyl fermion (2.11) and one two-form field are supposed to be

$$S^s_{n\to 1} : S^f_{n\to 1} : S^v_{n\to 1} = \frac{2}{756}\frac{V_5}{\pi^2} : \frac{191}{7560}\frac{V_5}{\pi^2} : S^v_{n\to 1}\ . \tag{2.27}$$

Demanding that (2.26) and (2.27) are equal (because of (2.24)), one can solve

$$S^v_{n\to 1} = \frac{221}{210}\frac{V_5}{\pi^2}\ . \tag{2.28}$$

Now comparing this "correct" value with the $n\to 1$ value of direct heat kernel result (2.23), we obtain the discrepancy

$$\Delta S^v = \left(\frac{221}{210} - \frac{229}{630}\right)\frac{V_5}{\pi^2} = \frac{434}{630}\frac{V_5}{\pi^2}\ . \tag{2.29}$$

Therefore, the correct Rényi entropy of a 2-form field in 6d should be

$$S^v_n = S^v_{d=6} + \Delta S^v = \frac{(n+1)\,(37n^2+2) + 877n^4 + 4349n^5}{5040n^5}\frac{V_5}{\pi^2}\ . \tag{2.30}$$

This is one of our new results.

2.4 (2,0) tensor multiplet

A six-dimensional $(2,0)$ tensor multiplet includes five real scalars, two Weyl fermions and one 2-form field with self-dual strength. The 2-form field with self-dual strength can also be considered as a chiral 2-form field which has half of the degrees of freedom. Putting the contributions of all fields together, we get the Rényi entropy of the $(2,0)$ tensor multiplet

$$S_n^{(2,0)} = 5\times\frac{S^s_n}{2} + 2S^f_n + \frac{S^v_n}{2} = \frac{(n+1)(28n^2+3) + 313n^4 + 1305n^5}{2880n^5}\frac{V_5}{\pi^2}\ . \tag{2.31}$$

[2] We follow the convention in [1].

[3] The precise value of a-anomaly may be different under different normalizations. However, the ratios are independent of the normalization.

The entanglement entropy of the (2,0) tensor multiplet is given by

$$S_{EE}^{(2,0)} = S_{n\to 1}^{(2,0)} = \frac{7}{12}\frac{V_5}{\pi^2} \ . \tag{2.32}$$

The first and second derivatives at $n = 1$ are

$$\partial_n S_n^{(2,0)}\Big|_{n=1} = -\frac{1}{6}\frac{V_5}{\pi^2} \ , \quad \partial_n^2 S_n^{(2,0)}\Big|_{n=1} = \frac{4}{9}\frac{V_5}{\pi^2} \ . \tag{2.33}$$

A few consistency checks are in order.

- The entanglement entropy (2.32) is consistent with the result (2.12) in [22] for a spherical entangling surface $\mathbb{S}^4$, if one takes into account a factor difference $\frac{16}{7}\mathrm{N}^3$ [25] between the trace anomaly of (2,0) tensor multiplet and that of the large-N theory of coincident M5-branes. It is also consistent with (2.29) in [25] if one adopts the normalization condition $-\frac{1}{8\times 3!}\frac{1}{(4\pi)^3}\int_{\mathbb{S}^6} E_6 = 2$.

- The first derivative $-\frac{1}{6}\frac{V_5}{\pi^2}$ is consistent with the coefficient of the two-point correlator of the stress tensor in the (2,0) tensor multiplet, which is given by [26]

$$C_T = \frac{84}{\pi^6} \ . \tag{2.34}$$

  By consistency we mean the universal relation between the first derivative of Rényi entropy at $n = 1$ and C_T in any CFT as shown in [27]

$$S'_{n=1} = -\mathrm{Vol}(\mathbb{H}^{d-1})\frac{\pi^{d/2+1}\Gamma(d/2)(d-1)}{(d+1)!}C_T \ . \tag{2.35}$$

  Using (2.34) we indeed find that our result $\partial_n S_n^{(2,0)}\Big|_{n=1} = -\frac{1}{6}\frac{V_5}{\pi^2}$ satisfies the relation (2.35) for $d = 6$.

- The second derivative $\frac{4}{9}\frac{V_5}{\pi^2}$ is consistent with the coefficients of the three-point correlator of the stress tensor in the (2,0) tensor multiplet. The relation between $S''_{n=1}$ and the coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in the three-point correlator of the stress tensor was derived in [28]:

$$S''_{n=1} = \frac{4\pi^{d+1}}{3d^3(d+2)\,\Gamma(d-1)}\mathrm{Vol}(\mathbb{H}^{d-1})\left((4d^2-10d+8)\mathcal{A} - d\mathcal{B} - (10d-8)\mathcal{C}\right) \ . \tag{2.36}$$

  The coefficients $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ for the (2,0) tensor multiplet are given in [26]:

$$\mathcal{A} = -\frac{2^6\cdot 3^4}{5^2\pi^9} \ , \quad \mathcal{B} = -\frac{181\cdot 2^4\cdot 3^2}{5^2\pi^9} \ , \quad \mathcal{C} = -\frac{59\cdot 2^3\cdot 3^3}{5^2\pi^9} \ . \tag{2.37}$$

  As discussed in [28], for the complex scalar there is a mismatch between the known result for $S''_{n=1}$ and the one derived from the right hand side of (2.36). For

$d = 6$ this mismatch is compensated by a factor $\frac{113}{125}$. Since we have calculated the Rényi entropies for the complex scalar and for the $(2,0)$ tensor multiplet explicitly, the expected result for (2.36) is given by

$$\frac{113}{125} \cdot \frac{5}{2} \partial_n^2 S_n^s \Big|_{n\to 1} + \left[\partial_n^2 S_n^{(2,0)} \Big|_{n\to 1} - \frac{5}{2} \partial_n^2 S_n^s \Big|_{n\to 1} \right] = \frac{11}{25} \frac{V_5}{\pi^2} . \tag{2.38}$$

One can check that this indeed agrees with the right hand side of (2.36) by using (2.37).

3. Supersymmetric Rényi entropy

Now we study the supersymmetric Rényi entropy of the free tensor multiplet. To preserve supersymmetry, one has to turn on an R-symmetry background gauge field to twist the boundary conditions for scalars and fermions along the replica direction [32]. In the manifold $\mathbb{S}^1_{\beta=2\pi n} \times \mathbb{H}^5$, this is equivalent to turning on an R-symmetry chemical potential along $\mathbb{S}^1_\beta$, therefore the heat kernel along the circle has a phase shift. The kernel (2.4) becomes

$$\widetilde{K}_{\mathbb{S}^1_\beta}(t) = \frac{\beta}{\sqrt{4\pi t}} \sum_{m\neq 0, \in \mathbb{Z}} e^{\frac{-\beta^2 m^2}{4t} + i2\pi m\mu - i\pi m f} , \tag{3.1}$$

where $f = 0$ for scalars and $f = 1$ for fermions. Using this twisted kernel one can perform the same computations of partition functions as what has been done in Section 2. These partition functions are functions of the chemical potential μ. In consideration of supersymmetry, μ has to be a function of n and it should be vanishing at $n = 1$ since we do not need additional background field for the manifold $\mathbb{S}^1_{\beta=2\pi} \times \mathbb{H}^5$, which is conformally equivalent to a round six-sphere. The explicit function $\mu(n)$ can be found by solving Killing spinor equation either on a branched $\mathbb{S}^6_n$ or on $\mathbb{S}^1_n \times \mathbb{H}^5$ since it is invariant under Weyl transformation.

The supersymmetric Rényi entropy is defined as

$$S_n^{SUSY} := \frac{nF_1(0) - F_n(\mu)}{1-n} . \tag{3.2}$$

Similar to the case in $4d$, it is also convenient to extract the extra contribution due to the nontrivial chemical potential,

$$\Delta S := S_n^{SUSY} - S_n . \tag{3.3}$$

For a complex scalar,

$$\Delta S^s(\mu) = \frac{V_5\, \mu^2(\mu+1)^2 \left(-2\mu^2 - 2\mu + 5n^2 + 1\right)}{360\pi^2 (n-1) n^5} . \tag{3.4}$$

For a Weyl fermion,

$$\Delta S^f(\mu) = \frac{V_5\,\mu^2\left(16\mu^4 - 20\mu^2 + 135n^4 - 50\left(2\mu^2 - 1\right)n^2 + 7\right)}{1440\pi^2(n-1)n^5} \ . \tag{3.5}$$

Notice that $\Delta S^f(\mu)$ is an even function of μ, but $\Delta S^s(\mu)$ is not. Also note that the effective chemical potential for a dynamical field depends on the product of the R-charge and the value of the background field.

3.1 R-symmetry chemical potential

Following the same way of the construction of $4d$ supersymmetric Rényi entropy, we may first look at the Killing spinors on a branched six-sphere $\mathbb{S}^6_n$. By solving the Killing spinor equations one can determine the R-symmetry chemical potential for the Killing spinor, $\mu(n)$. The explicit computation is performed in Appendix A, and we find that for $d = 6$:

$$\mu(n) = \frac{n-1}{2} \ . \tag{3.6}$$

This same result has been obtained by explicit computations in $d = 2, 3, 4, 5$ [7, 10, 12, 15, 33].

3.2 Supersymmetric Rényi entropy

The R-symmetry group of $6d$ $(2,0)$ theories is $SO(5)$, which has two $U(1)$ Cartans. To compute the supersymmetric Rényi entropy of the tensor multiplet we first look at the R-charges (k_1, k_2) of the component fields under the two Cartans, as listed in Table 1, where (ψ^1, ψ^2) and (ψ^3, ψ^4) are two Weyl fermions, and $\Phi^1 := \phi^1 + i\phi^2$ and $\Phi^2 := \phi^3 + i\phi^4$ are two complex scalars.

Table 1: charges under two $U(1)$ Cartans

	ψ^1	ψ^2	ψ^3	ψ^4	$B_{\mu\nu}$	Φ^1	Φ^2	ϕ^5
k_1	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	$+1$	0	0
k_2	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	$+1$	0

3.2.1 A single $U(1)$

If we only turn on a single $U(1)$ chemical potential, for instance $A^2 = 0$, by the constraint (3.6) of the Killing spinor equation, the background field should be

$$A^1 = n - 1 \ . \tag{3.7}$$

From Table 1, we see that there are two Weyl fermions charged $|k_1| = 1/2$ and one complex scalar charged $k_1 = 1$. The supersymmetric Rényi entropy is then computed by

$$S_1 = S^{(2,0)_n} + \Delta S^s(\mu = n-1) + 2\Delta S^f(\mu = (n-1)/2) = \frac{V_5}{\pi^2}\left(\frac{6n+1}{12n}\right) . \tag{3.8}$$

3.2.2 Two same $U(1)$'s

Now we turn on two $U(1)$ chemical potentials with the same value, $A^1 = A^2$. Since the Killing spinors are charged under both two Cartans (we only consider the Killing spinor with R-charges of the same sign $|k_1 + k_2| = 1$), under the constraint (3.6) the background field should be

$$A^1 = A^2 = \frac{n-1}{2} . \tag{3.9}$$

From Tabel 1, we see that there are one Weyl fermion charged $|k_1 + k_2| = 1$ and two complex scalars charged +1, so the supersymmetric Rényi entropy is

$$S_2 = S_n^{(2,0)} + 2\Delta S^s(\mu = (n-1)/2) + \Delta S^f(\mu = (n-1)/2) = \frac{(91n^3 + 19n^2 + n + 1)\, V_5}{192\pi^2 n^3} . \tag{3.10}$$

3.2.3 Two generic $U(1)$'s

We can also consider two $U(1)$ chemical potentials given by

$$A^1 = (n-1)a\,, \quad A^2 = (n-1)(1-a)\,, \tag{3.11}$$

where a is a real deformation parameter. In this case, there are one complex scalar with chemical potential A^1, one complex scalar with chemical potential A^2 and two Weyl fermions with chemical potential $(A^1 - A^2)/2$ and $(A^1 + A^2)/2$ respectively. The supersymmetric Rényi entropy is

$$\begin{aligned} S_2 &= S_n^{(2,0)} + \Delta S^s(\mu = A^1) + \Delta S^s(\mu = A^2) + \Delta S^f(\mu = \frac{A^1 - A^2}{2}) + \Delta S^f(\mu = \frac{A^1 + A^2}{2}) \\ &= \frac{(C_3 n^3 + C_2 n^2 + C_1 n + C_0)\, V_5}{12\pi^2 n^3}\,, \end{aligned} \tag{3.12}$$

where the coefficients are

$$\begin{aligned} C_0 &= a^2 - 2a^3 + a^4\,, \\ C_1 &= a - 4a^2 + 6a^3 - 3a^4\,, \\ C_2 &= 1 + 3a^2 - 6a^3 + 3a^4\,, \\ C_3 &= 6 - a + 2a^3 - a^4\,. \end{aligned} \tag{3.13}$$

For $a = \frac{1}{2}$, the result is the same as Eq. (3.10) for the case with two same $U(1)$'s.

4. Discussion

In this note we have discussed the Rényi entropy and the supersymmetric Rényi entropy for the six-dimensional $(2,0)$ tensor multiplet with a spherical entangling surface. The results are consistent with the existing results in the literature.

It would be interesting to go further to compute the supersymmetric Rényi entropy for the interacting $(2,0)$ theory and finally establish the $\text{TBH}_7/\text{qSCFT}_6$ correspondence following the same spirit of $\text{TBH}_4/\text{qSCFT}_3$ [10] and $\text{TBH}_5/\text{qSCFT}_4$ [12].

As a by-product, we have calculated the Rényi entropy for a 2-form field in six dimensions. This approach could be generalized to other cases with gauge symmetry, for instance higher forms or graviton. This opens up the possibility of computing the (supersymmetric) Rényi entropy for more field theories.

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A. Killing spinors on $\mathbb{S}^6_n$

As discussed in the introduction, in order to compute the supersymmetric Rényi entropy, we need to solve the Killing spinor equation on the branched sphere. Let us consider the branched six-sphere $\mathbb{S}^6_n$ given by the metric

$$\frac{ds_6^2}{\ell^2} = d\psi^2 + \sin^2\psi\left[d\chi^2 + \sin^2\chi\left(d\rho^2 + \sin^2\rho\left(d\theta^2 + n^2\sin^2\theta\, d\tau^2 + \cos^2\theta\, d\phi^2\right)\right)\right], \quad \text{(A.1)}$$

where n is the branching parameter. The vielbeins are chosen to be

$$\begin{aligned} e^1 &= \ell\, d\psi\,, & e^4 &= \ell \sin\psi \sin\chi \sin\rho\, d\theta\,, \\ e^2 &= \ell \sin\psi\, d\chi\,, & e^5 &= n\ell \sin\psi \sin\chi \sin\rho \sin\theta\, d\tau\,, \\ e^3 &= \ell \sin\psi \sin\chi\, d\rho\,, & e^6 &= \ell \sin\psi \sin\chi \sin\rho \cos\theta\, d\phi\,. \end{aligned} \quad \text{(A.2)}$$

The nonvanishing components of the spin connections are

$$
\begin{aligned}
&\omega_\chi^{21} = -\omega_\chi^{12} = \cos\psi\,, && \omega_\theta^{41} = -\omega_\theta^{14} = \cos\psi\sin\chi\sin\rho\,,\\
&\omega_\rho^{31} = -\omega_\rho^{13} = \cos\psi\sin\chi\,, && \omega_\theta^{42} = -\omega_\theta^{24} = \cos\chi\sin\rho\,,\\
&\omega_\rho^{32} = -\omega_\rho^{23} = \cos\chi\,, && \omega_\theta^{43} = -\omega_\theta^{34} = \cos\rho\,,\\
&\omega_\tau^{51} = -\omega_\tau^{15} = n\cos\psi\sin\chi\sin\rho\sin\theta\,, && \omega_\phi^{61} = -\omega_\phi^{16} = \cos\psi\sin\chi\sin\rho\cos\theta\,,\\
&\omega_\tau^{52} = -\omega_\tau^{25} = n\cos\chi\sin\rho\sin\theta\,, && \omega_\phi^{62} = -\omega_\phi^{26} = \cos\chi\sin\rho\cos\theta\,,\\
&\omega_\tau^{53} = -\omega_\tau^{35} = n\cos\rho\sin\theta\,, && \omega_\phi^{63} = -\omega_\phi^{36} = \cos\rho\cos\theta\,,\\
&\omega_\tau^{54} = -\omega_\tau^{45} = n\cos\theta\,, && \omega_\phi^{64} = -\omega_\phi^{46} = -\sin\theta\,.
\end{aligned}
\tag{A.3}
$$

The Killing spinor equation for the round $\mathbb{S}^6$

$$\nabla_\mu\zeta := \partial_\mu\zeta + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\zeta = -\frac{i}{2\ell}\gamma_\mu\zeta \tag{A.4}$$

has the solution

$$\zeta = e^{-\frac{i}{2}\gamma_1\psi}\, e^{\frac{1}{2}\gamma_{12}\chi}\, e^{\frac{1}{2}\gamma_{23}\rho}\, e^{\frac{1}{2}\gamma_{34}\theta}\, e^{\frac{1}{2}\gamma_{45}\tau}\, e^{\frac{1}{2}\gamma_{36}\phi}\,\zeta_0\,, \tag{A.5}$$

where ζ_0 is a constant spinor, and $\gamma_{ij} := \frac{1}{2}(\gamma_i\gamma_j - \gamma_j\gamma_i)$. For the branched sphere $\mathbb{S}_n^6$, the Killing spinor equation becomes

$$\partial_\mu\zeta + \frac{1}{4n}\omega_\mu^{ab}\gamma_{ab}\zeta = -\frac{i}{2n\ell}\gamma_\mu\zeta\,. \tag{A.6}$$

In a special choice of the basis

$$
\begin{aligned}
\gamma_1 &= \sigma_1\otimes\mathbb{I}\otimes\mathbb{I}\,,\\
\gamma_2 &= \sigma_2\otimes\mathbb{I}\otimes\mathbb{I}\,,\\
\gamma_3 &= \sigma_3\otimes\sigma_1\otimes\mathbb{I}\,,\\
\gamma_4 &= \sigma_3\otimes\sigma_2\otimes\mathbb{I}\,,\\
\gamma_5 &= \sigma_3\otimes\sigma_3\otimes\sigma_1\,,\\
\gamma_6 &= \sigma_3\otimes\sigma_3\otimes\sigma_2\,,
\end{aligned}
\tag{A.7}
$$

where σ_i $(i = 1, 2, 3)$ are the Pauli matrices, and the constant spinor

$$\zeta_0 = \begin{pmatrix} 0\\ c_1\\ c_1\\ 0\\ 0\\ c_2\\ c_2\\ 0 \end{pmatrix} \qquad (c_1, c_2 : \text{constants})\,, \tag{A.8}$$

Eq. (A.6) is equivalent to the following Killing spinor equation, if we turn on a background gauge field A_μ and require that the solution ζ for the round $\mathbb{S}^6$ remains as a solution on the branched sphere $\mathbb{S}^6_n$:

$$D_\mu\zeta := \partial_\mu\zeta + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\zeta + iA_\mu\zeta = -\frac{i}{2\ell}\gamma_\mu\zeta\,, \tag{A.9}$$

where the background gauge field is

$$A = \frac{n-1}{2}d\tau\,. \tag{A.10}$$

If we choose a different Killing spinor with the same γ-matrices and the same constant spinor ζ_0:

$$\widetilde{\zeta} = e^{\frac{i}{2}\gamma_1\psi}\, e^{\frac{1}{2}\gamma_{12}\chi}\, e^{\frac{1}{2}\gamma_{23}\rho}\, e^{\frac{1}{2}\gamma_{34}\theta}\, e^{\frac{1}{2}\gamma_{45}\tau}\, e^{\frac{1}{2}\gamma_{36}\phi}\,\zeta_0\,, \tag{A.11}$$

it satisfies another Killing spinor equation

$$D_\mu\widetilde{\zeta} := \partial_\mu\widetilde{\zeta} + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\widetilde{\zeta} + iA_\mu\widetilde{\zeta} = \frac{i}{2\ell}\gamma_\mu\widetilde{\zeta} \tag{A.12}$$

with the same background gauge field A_μ as before.

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