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Phys. Rev. D **93**, 106004 — Published 20 May 2016

DOI: 10.1103/PhysRevD.93.106004

Numerical Study of the Simplest String Bit Model

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Abstract

String bit models provide a possible method to formulate string as a discrete chain of point-like string bits. When the bit number M is large, a chain behaves as a continuous string. We study the simplest case that has only one bosonic bit and one fermionic bit. The creation and annihilation operators are adjoint representations of $U(N)$ color group. We show that the supersymmetry reduces the parameter number of a Hamiltonian from seven to three and, at $N=\infty$, ensures continuous energy spectrum, which implies the emergence of one spatial dimension. The Hamiltonian H_0 is constructed so that in large N limit it produces a worldsheet spectrum with one grassmann worldsheet field. We concentrate on numerical study of the model in finite N. For the Hamiltonian H_0, we find that the would-be ground energy states disappear at $N=(M-1)/2$ for odd $M \leq 11$. Such a simple pattern is spoiled if H has an additional term $\xi\Delta H$ which does not affect the result of $N=\infty$. The disappearance point moves to higher (lower) N when ξ increases (decreases). Particularly, the $\pm(H_0-\Delta H)$ cases suggest a possibility that the ground state could survive at large M and $M \gg N$. Our study reveals that the model has stringy behavior: when N is fixed and large enough, the ground energy decreases linearly with respect to M and the excitation energy is roughly of order M^{-1}. We also verify that a stable system of Hamiltonian $\pm H_0+\xi\Delta H$ requires $\xi \geq \mp 1$.

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1 Introduction

The idea of string bits, proposed over two decades ago [1], is one approach to formulate string theory. In this formulation, strings in D-dimensional spacetime are chain-like objects comprised of point-like entities, string bits, moving in space of $d = D - 2$ dimensions. The dynamics of the string bits is chosen to retain the Galilei symmetry described by the group Galilei $(d, 1)$. While one spatial coordinate is missing and the Lorentz invariance is not built in a priori, both of them are regained in the critical dimension when the number of string bits is large enough. Thereby, string theory emerges. Since the physics in $(d+1)$-dimensional space is described by physics in d-dimensional space, the string bit models provide an implementation of 't Hooft's holography hypothesis [2–4].

Such an idea is motivated by discretization of continuous string. Consider a string in lightcone coordinates [5, 6],

$$x^{\pm} = \frac{x^0 \pm x^1}{\sqrt{2}}, \quad \mathbf{x} = \left(x^2, \cdots, x^{d+1}\right),$$

where $\mathbf{x}$ is the transverse coordinates, the Hamiltonian of the string reads [7, 8]

$$P^- = \frac{1}{2}\int_0^{P^+} d\sigma \, [\mathbf{p}^2 + T_0^2 \mathbf{x}'^2], \tag{1}$$

where $P^{\pm}$ are the momenta conjugate to $x^{\mp}$ coordinates. In analogy to (1), a harmonic chain of M string bits each of which has mass m is described by the Hamiltonian

$$H = \frac{1}{2m}\sum_{k=1}^{M}\left[\mathbf{p}_n^2 + T_0^2\left(\mathbf{x}_{n+1} - \mathbf{x}_n\right)^2\right]. \tag{2}$$

Under the Galilei transformation $x^k \to x^k + V^k x^+$, the time-like coordinate x^+ and the mass of each string bit are invariant. Consequently, $P^+ = Mm$ can be considered as the Newtonian mass of the bit-chain. For $M \to \infty$, P^+ behaves like a continuous variable whose conjugate can be interpreted as the missing coordinate x^-. If the bound states for a many-bit system are closed linear chains and the excitation energies scale as $1/M$ for large M, Lorentz invariance is regained and leads to a Poincaré invariant dispersion relation $P^- = \left(\mathbf{P}^2 + \mu^2\right)/(2P^+)$. It is noteworthy that such bound states can be achieved in the context of 't Hooft large N limit [9, 10].

However, the Hamiltonian (2) for a bosonic closed string bit chain leads to inevitable instability. The ground state energy of such a system in the limit $M \to \infty$ is given by

$$E_G = \frac{2dT_0M}{m\pi} - \frac{\pi d T_0}{6Mm} + \mathcal{O}\left(M^{-3}\right).$$

The first term can be dropped as the bit number is conserved in string interaction[11]. Because of the negative $\mathcal{O}\left(M^{-1}\right)$ term, a long closed bit chain tends to split into multiple smaller chains for a lower energy state. This instability issue can be fixed by introducing supersymmetry[12–17]. In supersymmetry, string bits are multiplets with both bosonic and fermionic degrees of freedom [18, 19]. It turns out that, for models with d bosonic and s fermionic worldsheet degrees of freedom, the ground energy becomes

$$E_G = \frac{(s-d)\pi T_0}{6Mm}.$$

It implies that the system is stable for $s > d$ and unstable for $s < d$. The supersymmetric case $s = d$ gives rise to exact cancellation between bosonic and fermionic contributions for all M.

To set up the dynamics of the superstring bit model, we employ 't Hooft's large N limit and follow the standard second-quantized formalism. A general superstring bit annihilation operator is an $N \times N$ matrix denoted by

$$\left(\phi_{[a_1 \cdots a_n]}\right)_\alpha^\beta (\mathbf{x}), \quad n = 0, \cdots, s,$$

where each a_i is a spinor index running over s values, and $\alpha, \beta = 1, \cdots, N$ are color indices for the adjoint representation of the color group $SU(N)$. ϕ is bosonic for even n and fermionic for odd n. The square bracket in the subscript denotes complete antisymmetric relation among a_i indices. For superstring theory, the Poincaré symmetry demands $s = d = 8$.

In [20] Thorn and one of us studied the simplest case of the model with $d = 0,\ s = 1$, where there are N^2 bosonic annihilation operators $\left(a_\alpha^\beta\right)$ and N^2 fermionic annihilation operators $\left(b_\alpha^\beta\right)$, with corresponding creation operators defined as $\bar{a}_\alpha^\beta \equiv \left(a_\beta^\alpha\right)^\dagger$ and $\bar{b}_\alpha^\beta \equiv \left(b_\beta^\alpha\right)^\dagger$. These operators satisfy the (anti-)commutation relations:

$$\left[a_\alpha^\beta, \bar{a}_\gamma^\delta\right] = \delta_\alpha^\delta \delta_\gamma^\beta, \quad \left\{b_\alpha^\beta, \bar{b}_\gamma^\delta\right\} = \delta_\alpha^\delta \delta_\gamma^\beta, \tag{3}$$

and all others vanishing. With these creation operators, we can build trace states as follows. Introduce the vacuum state $|0\rangle$ annihilated by all the a_α^β and b_α^β. We can act on $|0\rangle$ with a sequence of $\bar{a}$ and $\bar{b}$ to obtain a non-vacuum state with color indices. Finally, we take trace of the creation operators to obtain a color-singlet state. Each creation operator in the trace state is interpreted as a string bit. Trace states with even number of $\bar{b}$ are bosonic states while those with odd number of $\bar{b}$ are fermionic states. To give a few examples, $\operatorname{Tr}\bar{a}^3|0\rangle$, $\operatorname{Tr}\bar{a}^2 \operatorname{Tr}\bar{a}|0\rangle$, and $\operatorname{Tr}\bar{a}\bar{b}^2|0\rangle$ are 3-bit bosonic trace states; $\operatorname{Tr}\bar{a}\bar{b}|0\rangle$ and $\operatorname{Tr}\bar{a}\operatorname{Tr}\bar{b}|0\rangle$ are 2-bit fermionic trace states. Note that, because of the property of trace and the anticommutation relation in (3), some of such expressions are not valid trace state, for example, $\operatorname{Tr}\bar{b}\bar{b}|0\rangle = -\operatorname{Tr}\bar{b}\bar{b}|0\rangle = 0$. Clearly, the number of trace states increases exponentially as M increases. In Appendix B, we provide a formula to count the single trace states and an algorithm to calculate the number of trace states, including both single and multiple trace states. In Appendix A, we list all the different bosonic trace states from 1 bit to 7 bits.

The Hamiltonian of the toy model in [20] is chosen to be a linear combination of single trace operators

$$\operatorname{Tr}\bar{a}^2a^2, \quad \operatorname{Tr}\bar{b}^2b^2, \quad \operatorname{Tr}\bar{b}^2a^2, \quad \operatorname{Tr}\bar{a}^2b^2, \quad \operatorname{Tr}\bar{a}\bar{b}ba, \quad \operatorname{Tr}\bar{a}\bar{b}ab, \quad \operatorname{Tr}\bar{b}\bar{a}ba, \quad \operatorname{Tr}\bar{b}\bar{a}ab, \tag{4}$$

with coefficients scaling as $1/N$. Such a choice ensures the action of the Hamiltonian to the trace states survive at large N limit. It then studied a special form of such Hamiltonian

$$H_0 = \frac{2}{N} \operatorname{Tr} \left[\left(\bar{a}^2 - i\bar{b}^2 \right) a^2 - \left(\bar{b}^2 - i\bar{a}^2 \right) b^2 + \left(\bar{a}\bar{b} + \bar{b}\bar{a} \right) ba + \left(\bar{a}\bar{b} - \bar{b}\bar{a} \right) ab \right], \tag{5}$$

which produces the Green-Schwarz Hamiltonian[18, 21] at $N = \infty$. By variational method, it shows that the ground states of the Hamiltonian only survive at $N > (M-1)/2$. Then a numerical study of the Hamiltonian at $M = 3$ is performed.

In this paper, we will investigate more general forms of supersymmetric Hamiltonian and their energy spectrum at large N limit. We will perform a numerical study of the Hamiltonian H_0 for $M \leq 11$. We will plot the energy levels as a function of N at fixed values of M and show numerically that the would-be ground state disappears at $N \leq (M-1)/2$ for odd $M \leq 11$. Such a pattern is spoiled when we add to H_0 an additional ΔH term, which does not affect the large N limit. For the Hamiltonians $\pm (H_0 - \Delta H)$, the disappearance of ground state occurs at $N < (M-1)/2$, which might suggest that the ground states can survive when M is large and N is much smaller than M. We will also plot the ground energy and excitation energy as a function of M at fixed N to check whether the system manifests stringy behavior. For stringy behavior, the ground energy should be a linear function of M with negative slope and the excitation energy proportional to M^{-1} with positive coefficient. It turns out that, for N large enough, the ground energies do drop almost linearly. For excitation energies, although there is not enough data for an unquestioned pattern, it still shows tendencies to go roughly as M^{-1} when N is large.

The rest of this paper is organized as follows. In Section 2, we discuss the general constraint on a supersymmetric Hamiltonian. In Section 3, we investigate the energy spectrum of the system in the large N limit. In Section 4, we compute the energy spectrum at finite N numerically and present the plots from the numerical study. Hamiltonian H_0 and its variations will be studied in the section. The main text is closed with a section of summary and conclusion. Finally, we include seven appendices covering techenical details.

2 Supersymmetric Hamiltonian

In the toy model with $d = 0$, $s = 1$, while the spacetime supersymmetry is explicitly broken, there still exists a form of supersymmetry between bosonic and fermionic trace states. As the mathematical proof in Appendix B shows, the numbers of bosonic and fermionic trace states are equal at any value of M. This is not a coincidence. The physical interpretation is that the bit number operator $M = \operatorname{Tr} \left(\bar{a}a + \bar{b}b \right)$ commutes with the supersymmetry operator

$$Q = \exp \left(\frac{i\pi}{4} \right) \operatorname{Tr} \bar{a}b + \exp \left(-\frac{i\pi}{4} \right) \operatorname{Tr} \bar{b}a. \tag{6}$$

Also we notice that $M = Q^2$. A Hamiltonian H is supersymmetric if $[H, Q] = 0$. As we will show in the next section, a nice feature of supersymmetric Hamiltonian is that its excitation energy vanishes at large M.

Now, let's investigate possible forms of a supersymmetric Hamiltonian and generalizations of H_0. The general form of a Hermitian Hamiltonian built out of the trace operators in (4) reads

$$\begin{aligned} H &= \frac{1}{N}\Big[c_1 \operatorname{Tr} \bar{a}^2 a^2 + c_2 \operatorname{Tr} \bar{b}^2 b^2 + i z_1 \operatorname{Tr} \bar{a}^2 b^2 - i z_1^* \operatorname{Tr} \bar{b}^2 a^2 \\ &\quad + c_3 \operatorname{Tr} \bar{a}\bar{b}ba + c_4 \operatorname{Tr} \bar{b}\bar{a}ab + z_2 \operatorname{Tr} \bar{a}\bar{b}ab + z_2^* \operatorname{Tr} \bar{b}\bar{a}ba\Big], \end{aligned} \tag{7}$$

where c_i are real and z_i are complex. Imposing the constraint $[H, Q] = 0$ yields[1]

$$\begin{cases} \Im z_1 &= \Im z_2 \\ c_1 - c_2 &= 2\Re z_2 \\ c_3 - c_4 &= 2\Re z_1 \\ c_1 + c_2 &= c_3 + c_4 \end{cases}, \tag{8}$$

which implies that a supersymmetric Hamiltonian can be written as

$$\begin{aligned} H &= H_0 + \frac{2\xi}{N} \operatorname{Tr}\left(\bar{a}\bar{b}ba + \bar{b}\bar{a}ab + \bar{a}^2 a^2 + \bar{b}^2 b^2\right) \\ &\quad + \frac{2\eta}{N} \operatorname{Tr}\left(\bar{b}^2 a^2 + \bar{a}^2 b^2 + i\bar{a}\bar{b}ab - i\bar{b}\bar{a}ba\right) \\ &\quad + \frac{2\zeta}{N} \operatorname{Tr}\left(i\bar{b}^2 a^2 - i\bar{a}^2 b^2 - \bar{a}\bar{b}ba + \bar{b}\bar{a}ab\right), \end{aligned} \tag{9}$$

where ξ, η, ζ are real parameters. Note that each term in (9) is Hermitian and supersymmetric.

The Hamiltonian H_0 is the special case of (9) when $\xi = \eta = \zeta = 0$. But we can also obtain a generalization of H_0 by keeping a twisted ξ term. As noted in [20], we are free to add the terms

$$\Delta H' = \frac{1}{N} \operatorname{Tr}\left[2\xi_1 \bar{a}\bar{b}ba + 2\xi_2 \bar{b}\bar{a}ab + (\xi_1 + \xi_2)\left(\bar{a}^2 a^2 + \bar{b}^2 b^2 - \tilde{M}\right)\right], \tag{10}$$

to a Hamiltonian without affecting the large N limit. Here, $\tilde{M}$ is a supersymmetric term given by[2]

$$\tilde{M} = \operatorname{Tr}\left(\bar{a}a + \bar{b}b\right) - \frac{1}{N}\left(\operatorname{Tr}\bar{a}\operatorname{Tr}a + \operatorname{Tr}\bar{b}\operatorname{Tr}b\right).$$

Setting $\xi_1 - 1 = \xi_2 + 1 = \xi$, we obtain a supersymmetric $\Delta H'$ term which equals the ξ term in (9) minus a $\tilde{M}$ term. Therefore, H_0 can be generalized to

$$H = H_0 + \xi \Delta H, \tag{11}$$

[1] Appendix D details the calculation of $[H, Q]$.

[2] [20] uses the bit operator $M = \operatorname{Tr}\left(\bar{a}a + \bar{b}b\right)$ instead of $\tilde{M}$ in $\Delta H'$. Our calculation shows in order that $\Delta H'$ vanishes in large N limit, M must be replaced by $\tilde{M}$.

where

$$\Delta H = \frac{2}{N} \operatorname{Tr}\left[\bar{a}\bar{b}ba + \bar{b}\bar{a}ab + \bar{a}^2 a^2 + \bar{b}^2 b^2 - \tilde{M} \right].$$

In (11) H_0 makes $\mathcal{O}(1)$ contribution while ΔH makes only $\mathcal{O}\left(\frac{1}{N}\right)$ contribution. The values of ξ are constrained by the requirement that a well-defined Hamiltonian should be stable for large M. The $\operatorname{Tr} \bar{a}^2 a^2$ term can produce about M^2 terms by attacking to the trace state $\operatorname{Tr} \bar{a}^M |0\rangle$. This would cause a dangerous instability if the coefficient of $\operatorname{Tr} \bar{a}^2 a^2$ is negative. To maintain a positive $\operatorname{Tr} \bar{a}^2 a^2$ term, we must choose $\xi \geq -1$. Therefore, we obtain a form of well-defined Hamiltonian

$$H = H_0 + \xi \Delta H, \quad \xi \geq -1. \tag{12}$$

In addition to (12), there exist another form of supersymmetric Hamiltonian. As suggested in [20], we can replace H_0 with $-H_0$ and obtain

$$H = -H_0 + \xi \Delta H, \quad \xi \geq 1, \tag{13}$$

where the constraint $\xi \geq 1$ comes from the stability condition.

One might wonder if there exist other supersymmetric operators that are capable of stabilizing $-H_0$ and make only $\mathcal{O}\left(\frac{1}{N}\right)$ contributions. As suggested by [1], one possibility is to use $\operatorname{Tr} \bar{a}a\bar{a}a$ operator which also produces about M^2 terms when acting on $\operatorname{Tr} \bar{a}^M |0\rangle$. A combination as

$$H' = \frac{2}{N} \operatorname{Tr}\left(\bar{a}a\bar{a}a + \bar{b}b\bar{a}a - \bar{a}b\bar{b}a \right)$$

meets such a requirement. However, as Appendix E shows, H' equals ΔH for all trace states, i.e,

$$\left(H' - \Delta H \right) |\text{Any trace state}\rangle = 0.$$

While we are not sure if there exist other variations of H_0, for the time being, we leave the question for further research and only study Hamiltonians as (12) and (13) in this paper.

3 Energy Spectrum in Large N Limit

In this section, we will study the energy spectrum of our toy string bit model in the large N limit by both analytic and numerical methods. We first show that the supersymmetry guarantees the excitation energy to be vanishing at large M and then present the energy spectrum graphically.

3.1 General H

For convenience, we introduce a super creation operator using a Grassmann anti-commuting number θ,

$$\psi(\theta) = \bar{a} + \bar{b}\theta, \quad \bar{b} = -\frac{d}{d\theta}\psi, \quad \bar{a} = \left(1 - \theta \frac{d}{d\theta} \right) \psi.$$

We then choose

$$|\theta_1\theta_2\cdots\theta_M\rangle = \mathrm{Tr}\left[\psi\left(\theta_1\right)\psi\left(\theta_2\right)\cdots\psi\left(\theta_M\right)\right]|0\rangle \tag{14}$$

to be a basis of M-bit single trace states. A general single trace energy eigenstate at large N reads

$$|E\rangle = \int d^M\theta\,\Psi\left(\theta_1\cdots\theta_M\right)|\theta_1\theta_2\cdots\theta_M\rangle\,, \tag{15}$$

where $\Psi\left(\theta_1\cdots\theta_M\right)$ is the wave function in terms of θ_i. Under the cyclic transformation, $\theta_i \to \theta_{i+1}$, $|\theta_1\cdots\theta_M\rangle$ is invariant and the Jacobi $d^M\theta$ obtain a factor of $(-1)^{M-1}$. It follows that we can constrain the wave function by a cyclic symmetry,

$$\Psi\left(\theta_1\theta_2\cdots\theta_M\right) = (-1)^{M-1}\,\Psi\left(\theta_M\theta_1\cdots\theta_{M-1}\right). \tag{16}$$

In the basis (14), the leading term of trace operators in (4) can be expressed in terms of θ_i and $\frac{d}{d\theta_i}$, as shown in Eq. (9) to (16) of [20], by which we rewrite (7) in large N limit as

$$H|\theta_1\cdots\theta_M\rangle = \hat{h}|\theta_1\cdots\theta_M\rangle + \mathcal{O}\left(\frac{1}{N}\right),$$

$$\begin{aligned}\hat{h} \;=\; & \sum_{k=1}^{M}\Big[iz_1\theta_{k+1}\theta_k - iz_1^\dagger\frac{d}{d\theta_k}\frac{d}{d\theta_{k+1}} + z_2\theta_k\frac{d}{d\theta_{k+1}} \\ & +z_2^\dagger\theta_{k+1}\frac{d}{d\theta_k} + \left(-2c_1+c_3+c_4\right)\theta_k\frac{d}{d\theta_k} \\ & +\left(c_1+c_2-c_3-c_4\right)\theta_k\frac{d}{d\theta_k}\theta_{k+1}\frac{d}{d\theta_{k+1}}\Big] + c_1M.\end{aligned} \tag{17}$$

Performing Integration by parts as

$$\int d^M\theta\,\Psi\left(\theta_1\cdots\theta_M\right)\hat{h}\,|\theta_1\theta_2\cdots\theta_M\rangle = \int d^M\theta\,h\Psi\left(\theta_1\cdots\theta_M\right)|\theta_1\theta_2\cdots\theta_M\rangle\,,$$

we obtain

$$\begin{aligned}h \;=\; & \sum_{k=1}^{M}\Big[iz_1\theta_{k+1}\theta_k - iz_1^\dagger\frac{d}{d\theta_k}\frac{d}{d\theta_{k+1}} - z_2\theta_k\frac{d}{d\theta_{k+1}} \\ & -z_2^\dagger\theta_{k+1}\frac{d}{d\theta_k} + \left(2c_1-c_3-c_4\right)\theta_k\frac{d}{d\theta_k}\Big] \\ & +\left(c_3+c_4-c_1\right)M,\end{aligned}$$

where for simplicity we drop the quartic term, which vanishes automatically under the supersymmetry constraint (8). We then introduce the Fourier transforms

$$
\begin{aligned}
\alpha_n &= \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \theta_k e^{2\pi i k n/M}, \quad \beta_n = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \frac{d}{d\theta_k} e^{2\pi i k n/M}, \quad n = 0, \dots M-1, \\
\theta_k &= \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} \alpha_n e^{-2\pi i k n/M}, \quad \frac{d}{d\theta_k} = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} \beta_n e^{-2\pi i k n/M}, \quad k = 1, \dots M,
\end{aligned}
$$

satisfying

$$
\{\alpha_n, \beta_m\} = \delta_{m+n,M} + \delta_{m,0}\delta_{n,0}.
$$

A little algebra yields

$$
\begin{aligned}
h &= \sum_{n=1}^{M-1} \Big[\left(z_1 \alpha_n \alpha_{M-n} + z_1^\dagger \beta_n \beta_{M-n} \right) \sin \frac{2n\pi}{M} \\
&\quad + 2 \left(c - \Re \left(z_2 e^{2\pi i n/M} \right) \right) \alpha_n \beta_{M-n} \Big] \\
&\quad + 2 \left(c - \Re z_2 \right) \alpha_0 \beta_0 + \left(c_1 - 2c \right) M,
\end{aligned}
$$

where we have defined $c = c_1 - \frac{1}{2}(c_3 + c_4)$. Note that we have $c = \Re z_2$ under the supersymmetry constraint (8).

We now find the ladder operators of h, which we denote as L_k. We use the ansatz $L_k = a\alpha_k + b\beta_k$ and impose the constraint

$$
[h, L_k] = \epsilon_k L_k. \tag{18}
$$

By direct calculation, we have

$$
[h, a\alpha_k + ib\beta_k] = 2 \left(a d_k + b z_1 \sin \frac{2k\pi}{M} \right) \alpha_k + 2 \left(a z_1^\dagger \sin \frac{2k\pi}{M} - b d_{M-k} \right) \beta_k,
$$

where $d_k \equiv c - \Re \left(z_2 e^{2\pi i k/M} \right)$. Constraint (18) yields

$$
\begin{cases}
2 \left(a d_k + b z_1 \sin \frac{2k\pi}{M} \right) & = a\epsilon_k \\
2 \left(a z_1^\dagger \sin \frac{2k\pi}{M} - b d_{M-k} \right) & = b\epsilon_k
\end{cases} \tag{19}
$$

Let's first consider $k = 0$ case. If $d_0 \equiv c - \Re z_2 \neq 0$, there are two solutions:

$$
\begin{aligned}
\text{when } a \neq 0,\ b &= 0, \quad \epsilon_0 = 2 \left(\Re z_2 - c \right); \\
\text{when } a = 0,\ b &\neq 0, \quad \epsilon_0 = -2 \left(\Re z_2 - c \right).
\end{aligned}
$$

The corresponding ladder operators are α_0 and β_0 respectively. If $c - \Re z_2 = 0$, i.e., the supersymmetry case, then a, b can be any value and $\epsilon_0 = 0$, which implies there is no ladder operator for $k = 0$. In the supersymmetry case, the linear combination $\exp\left(\frac{i\pi}{4}\right)\alpha_0 + \exp\left(-\frac{i\pi}{4}\right)\beta_0$ is just the supersymmetry operator (6).

For $k \neq 0$, we solve for ϵ_k,

$$\epsilon_k^{\pm} = 2\Im z_2 \sin\frac{2k\pi}{M} \pm 2\sqrt{\left(c - \Re z_2 \cos\frac{2k\pi}{M}\right)^2 + |z_1|^2 \sin^2\frac{2k\pi}{M}}.$$

In general, ϵ_k is finite at large M and the energy levels are discrete. But under the supersymmetry constraint (8),

$$\epsilon_k^{\pm} = 4\left(-\Im z_1 \cos\frac{\pi k}{M} \pm \sqrt{(\Re z_2)^2 \sin^2\frac{k\pi}{M} + |z_1|^2\cos^2\frac{k\pi}{M}}\right)\sin\frac{k\pi}{M}, \tag{20}$$

which vanishes for finite k at large M. Therefore, supersymmetry ensures continuous energy spectrum and stringy behavior.

3.2 $H = H_0$

In the case of $H = H_0$, we have $c_1 = -c_2 = c_3 = -c_4 = c = 2$, $z_1 = z_2 = 2$, and

$$\epsilon_k^{\pm} = \pm 8 \sin\frac{k\pi}{M}, \quad r_k^{\pm} \equiv \frac{a}{b} = \tan\frac{k\pi}{M} \pm \sec\frac{k\pi}{M}, \quad k = 1, \cdots, M-1.$$

As $r_{M/2}^{+} = \infty$ and $r_{M/2}^{-} = 0$, we choose the raising and lowering operators to be

$$L_k^{+} = \alpha_k + \frac{1}{r_k^{+}}\beta_k, \quad L_k^{-} = r_k^{-}\alpha_k + \beta_k, \quad k = 1, \cdots, M-1.$$

Now we can construct the ground function, which is annihilated by all lowering operators. Observing that

$$L_k^{-}\left(1 + r_k^{-}\alpha_k\alpha_{M-k}\right) \;=\; L_{M-k}^{-}\left(1 + r_k^{-}\alpha_k\alpha_{M-k}\right) = 0,$$

and that α_0 commutes with all L_k^{-}, we obtain ground wave functions,

$$\Phi_M^b = \prod_{k=1}^{\lfloor M/2\rfloor}\left(1 + r_k^{-}\alpha_k\alpha_{M-k}\right), \quad \Phi_M^f = \alpha_0 \prod_{k=1}^{\lfloor M/2\rfloor}\left(1 + r_k^{-}\alpha_k\alpha_{M-k}\right)$$

with $\lfloor M/2 \rfloor$ the integral part of $M/2$. Clearly Φ_M^b is bosonic and Φ_M^f is fermionic. A direct calculation shows they have the same eigenvalue

$$E_G = -4\sum_{k=1}^{M-1}\sin\frac{k\pi}{M} = -4\cot\frac{\pi}{2M}. \tag{21}$$

For each $k < M/2$, we have four different choices to attack the ground functions, *i.e.*, using 1, L_k^+, L_{M-k}^+, and $L_k^+ L_{M-k}^+$, which correspond to energy level increasing by 0, ϵ_k^+, ϵ_k^+, and $2\epsilon_k^+$. For $k = M/2$, there are two choices to attack Φ_M, by 1 and $L_{M/2}^+$, with energy increment of 0 and $\epsilon_{M/2}^+$. Therefore, for each choice of ground function, the energy levels can be written as

$$\begin{aligned} E\left(\{\eta_k\}\right) &= E_G + 8\sum_{k=1}^{\lfloor M/2\rfloor} \sin\frac{k\pi}{M} + 8\sum_{k=1}^{\lfloor M/2\rfloor} \eta_k \sin\frac{k\pi}{M} \\ &= 8\sum_{k=1}^{\lfloor M/2\rfloor} \eta_k \sin\frac{k\pi}{M} + \begin{cases} 0 & \text{for odd } M \\ 4 & \text{for even } M \end{cases} \end{aligned} \tag{22}$$

$$\eta_k = -1,\ 0,\ 0,\ 1,\ \text{for } k < M/2;\ \eta_{M/2} = -1, 0. \tag{23}$$

Here we reproduced equation (94) and (95) of [20] with a different approach.

Now consider the cyclic constraint (16). The eigenfunctions should be changed by a factor of $(-1)^{M-1}$ under the transformation $\alpha_k \to \exp\left(2ik\pi/M\right)\alpha_k$ and $\beta_k \to \exp\left(2ik\pi/M\right)\beta_k$. Clearly the ground eigenfunction Φ_M is invariant under the transformation and L_k^+ changes as $L_k^+ \to \exp\left(2ik\pi/M\right) L_k^+$, from which it follows η_k must satisfy

$$\sum_{\eta_k=0}^{M/2} k = \begin{cases} nM, & \text{for odd } M \\ \left(n+\frac{1}{2}\right)M, & \text{for even } M \end{cases}, \quad n = 0, 1, 2, \cdots . \tag{24}$$

This constraint has several interesting consequences.

- For odd M, the lowest energy state of M-bit system is comprised of M-bit single trace states, which are generated by setting all η_k to -1, i.e.,

  $$E_{\min} = E_{\min}^{(1)} = -4\cot\frac{\pi}{2M} = -\frac{8M}{\pi} + \frac{2\pi}{3M} + \mathcal{O}\left(M^{-3}\right), \tag{25}$$

  where we use the superscript (1) to denote single trace states.

- For even M, the lowest energy of single trace states, $E_{\min}^{(1)}$, is achieved when $\eta_{M/2} = 0$ and all other $\eta_k = -1$; while the lowest energy state of the system is comprised of double trace states with each trace of $M/2$ bits (if $M/2$ is even, the two traces are of $M/2 - 1$ and $M/2 + 1$ bits). So we have

  $$\begin{aligned} E_{\min}^{(1)} &= -\frac{8M}{\pi} + \frac{2\pi}{3M} + 8 + \mathcal{O}\left(M^{-3}\right), \\ E_{\min} &= E_{\min}^{(2)} = -\frac{8M}{\pi} + \frac{4\pi}{3M} + \mathcal{O}\left(M^{-2}\right). \end{aligned}$$

  When $M/2$ is even, the lowest energy states have extra degeneracy, because the bosonic ground functions can be $\Phi_{M/2-1}^b \Phi_{M/2+1}^b$ and $\Phi_{M/2-1}^f \Phi_{M/2+1}^f$.

- For large M, the excitation energy is very small and the discrete energy levels become a continuous energy band. The difference of $E_{\min}^{(1)}$ between odd and even M is much large than the excitation energy, which implies only odd-bit chains participate in the low energy physics. Particularly, it also means a low energy odd-bit chain cannot decay into two chains.

Now let's consider the first excitation energy of odd M system. From above analysis, there are no double trace states in the low energy region, so we consider the triple trace states. From (25), the lowest energy of triple trace states is achieved when each trace has $M/3$ bits. Hence, we have

$$E_1 = -\frac{8M}{\pi} + \frac{16\pi}{M} + \mathcal{O}\left(M^{-2}\right),$$

from which it follows the energy gap between ground energy (25) and first excitation energy is $\frac{16\pi}{3M}$. If M is divisible by 3, the first excitation energy has no extra degeneracy. If $M = 3n \pm 1$, it has extra degeneracy: for $M = 3n + 1$, the bosonic ground function can be $\Phi^b_{n-1}\Phi^b_{n+1}\Phi^b_{n+1}$ and $\Phi^f_{n-1}\Phi^f_{n+1}\Phi^b_{n+1}$; for $M = 3n + 1$, the bosonic ground function can be $\Phi^b_{n-1}\Phi^b_{n-1}\Phi^b_{n+1}$ and $\Phi^b_{n-1}\Phi^f_{n-1}\Phi^f_{n+1}$.

Fig. 1 shows the energy spectrum at $N = \infty$ for M at 11, 21, 51, and 101. In the plot, energy states are represented by horizontal lines, with red color for single trace states, yellow color for triple trace states. The vertical coordinate is $M \times (E - E_{\min})$, the product of M with the difference between energy level and the lowest energy. The threshold for triple trace states is a blue line.

From the figure, it's clear that the energy gaps go smaller as M increase and the energy levels become continuous at large M. The energy of single trace states tend to distribute near multiples of $\frac{16\pi}{M}$ and the first excitation energy appears near $\frac{16\pi}{3M}$. The energy levels of triple trace states are even denser than single trace states. At $M = 101$, they almost filled the gap between consecutive single trace energy levels. All these behaviors illustrate the chains behave as continuous strings at large M.

4 Energy Spectrum at Finite N

In this section, we show numerically how the energy levels change with respect to N and the bit number M. We first introduce the methods to calculate the energy states of the system. We then analyze the result of the original Hamiltonian $H = H_0$, for which the $M = 3$ case has been investigated in [20]. Next, we move to the Hamiltonians of the form $H = H_0 + \xi\Delta H$ and investigate how the parameter ξ affects the energy levels. Finally, we explore the Hamiltonians of the form $H = -H_0 + \xi\Delta H$. For each case, we first analyze the change of energy levels with respect to N when M is fixed, and then with respect to M when N is fixed.

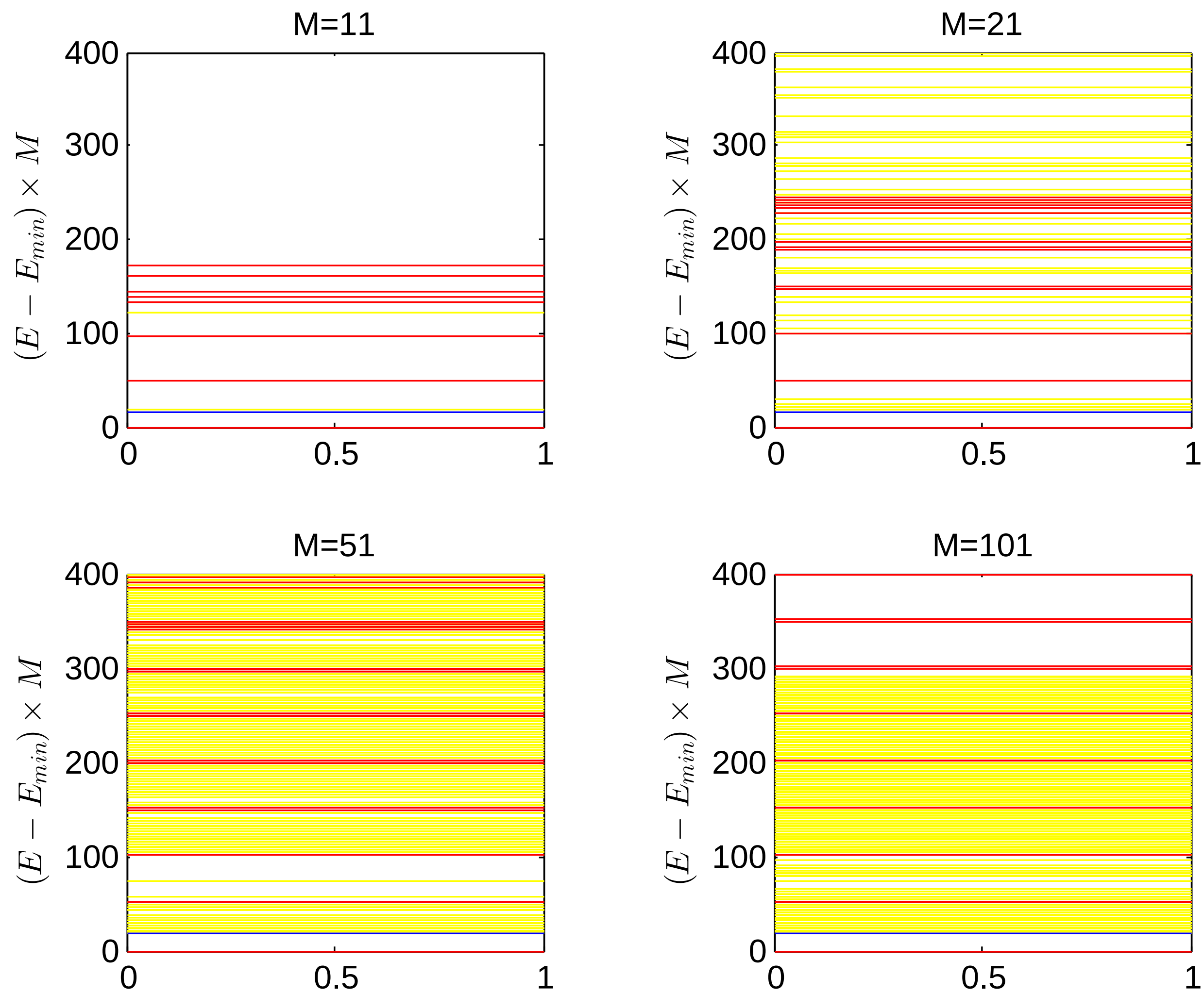

Figure 1: The energy levels of single trace states (red lines) and triple trace states (yellow lines) at $M = 11$, M=21, M=51 and M=101 and large N limit. Blue dash line is the threshold for multi-trace states energy when $M = \infty$.

4.1 The $\mathcal{H}$ Matrices

We have two methods to calculate the energy states of the system[3]. Both methods involve the $\mathcal{H}$ matrix defined as

$$H\,|i\rangle = \sum_j |j\rangle\,\mathcal{H}_{ji},$$

where $|i\rangle$ and $|j\rangle$ are M-bit trace states. Note that, since the trace state basis is not orthonormal, $\mathcal{H}$ is not the Hamiltonian matrix and even not Hermitian.

The first method, used in [20], is to calculate the eigenvalues of the $\mathcal{H}$ from equation

$$\mathcal{H}\,|E\rangle = E\,|E\rangle\,. \tag{26}$$

The relation between eigenvalues of $\mathcal{H}$ and of the Hamiltonian matrix is determined by the norm matrix, $G = \langle i|j\rangle$, as follows.

- If G is positive definite, *i.e.*, all its eigenvalues are positive, there is a one-to-one correspondence between the eigenvalues of $\mathcal{H}$ and the Hamiltonian. In this case, all the eigenstates of $\mathcal{H}$ are physical and have positive norm, which is defined as

  $$\langle E|E\rangle = \sum_{ij} v^{i*}\,\langle i|j\rangle\,v^j = v^\dagger G v$$

  for an eigenstate $|E\rangle = \sum_i |i\rangle\,v^i$. Our numerical calculation shows when $N \geq M$, the norm matrix G is always positive definite.

- When N is an integer and less than M, the norm matrix G is positive semidefinite, *i.e.*, some eigenvalues are zero and the others are positive. In this case, only those eigenstates of $\mathcal{H}$ with positive norm correspond to energy states of the Hamiltonian, while those eigenstates of $\mathcal{H}$ with zero norm are unphysical.

- When N is a non-integer and less than M, the norm matrix G is indefinite, *i.e.*, G has both positive and negative eigenvalues. There is a subtlety in this case. The eigenstates of $\mathcal{H}$ can be of positive norm, of zero norm, and of negative norm. The negative norm eigenstates of $\mathcal{H}$ stem from their coupling to ghost states, the eigenstates of G whose eigenvalues are negative. The zero and negative norm eigenstates are still unphysical. But positive norm eigenstates cannot be simply taken as energy states any more. A positive norm eigenstate is a physical energy state if it is orthogonal to every ghost state.

From above statements, we should treat positive norm eigenstates of $\mathcal{H}$ physical when N is large enough or small integral. Moreover, the eigenvalues of $\mathcal{H}$ can be non-real. This occurs

[3] In this subsection, we just state the properties of these two methods. The relevant mathematical proofs are provided in Appendix F.

for both positive-semidefinite and indefinite cases. For a non-real eigenvalue of $\mathcal{H}$, the norm of its eigenstate must be zero and its complex conjugate is also an eigenvalue of $\mathcal{H}$.

The second method is to solve a generalized eigenvalue problem,

$$(G\mathcal{H})\,|E\rangle = EG\,|E\rangle\,. \tag{27}$$

This method is helpful for filtering unphysical states when G is positive semidefinite. If G is a full-rank matrix, this is a regular generalized eigenvalue problem. If G is not a full-rank matrix, to solve the equation, we need to remove some rows and columns from G and $G\mathcal{H}$. If the rank of G is r, we can pick r independent rows and columns from G and $(G\mathcal{H})$ to form two $r \times r$ matrices as

$$\begin{aligned}
\tilde{G} &= \begin{pmatrix} G_{i_1i_1} & G_{i_1i_2} & \cdots & G_{i_1i_r} \\ G_{i_2i_2} & G_{i_2i_2} & \cdots & G_{i_2i_r} \\ \vdots & \vdots & \vdots & \vdots \\ G_{i_ri_1} & G_{i_ri_2} & \cdots & G_{i_ri_r} \end{pmatrix}, \\
\widetilde{G\mathcal{H}} &= \begin{pmatrix} (G\mathcal{H})_{i_1i_1} & (G\mathcal{H})_{i_1i_2} & \cdots & (G\mathcal{H})_{i_1i_r} \\ (G\mathcal{H})_{i_2i_2} & (G\mathcal{H})_{i_2i_2} & \cdots & (G\mathcal{H})_{i_2i_r} \\ \vdots & \vdots & \vdots & \vdots \\ (G\mathcal{H})_{i_ri_1} & (G\mathcal{H})_{i_ri_2} & \cdots & (G\mathcal{H})_{i_ri_r} \end{pmatrix}.
\end{aligned}$$

Then, Eq. (27) becomes

$$\left(\widetilde{G\mathcal{H}}\right)|E\rangle = E\tilde{G}\,|E\rangle\,,$$

whose eigenvalues and eigenstates are all physical.

The first method is used to investigate the change of eigenstates, including both physical and unphysical states, with respect to N for fixed M while the second one is for the change of physical energy levels with respect to M for fixed N. For different values of M, we calculated the $\mathcal{H}$ and G matrices, whose entries are expressed in terms of N. Then we solve Eq. (26) or (27) to find the their eigenstates. Since the number of trace states increases exponentially as M increases, it's only feasible to perform the calculation for small M. The highest value of M we study is 11, at which $\mathcal{H}$ and G are 1473×1473 matrices[4].

4.2 $H = H_0$

Let's first consider the case of odd M. Fig. 2 to 6 show the lowest five eigenvalues of $\mathcal{H}$ as a function of $1/N$ for odd M from 3 to 11. We use different line styles for different norm types: solid, dotted, and dash-dotted curves correspond to positive, negative, and zero norm eigenstates respectively. Dash-dotted curves actually associate with two complex eigenvalues which are conjugate to each other, and hence, represent only the real part of the eigenvalues.

[4]The source code of the project can be found in [22].

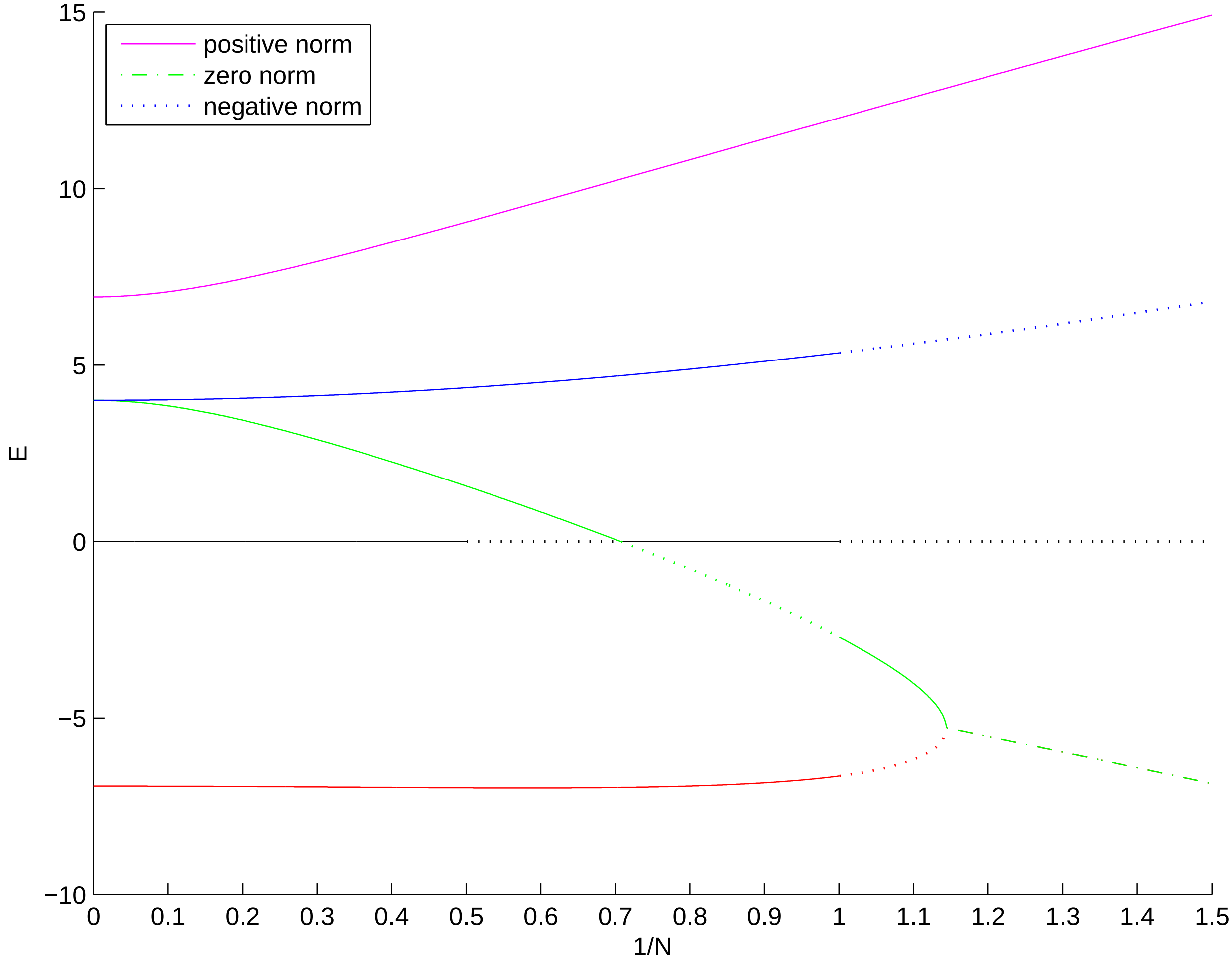

Figure 2: Lowest 5 energy states of the 3-bit system with Hamiltonian $H = H_0$.

For higher M, the eigenvalues decline dramatically in higher $1/N$, which would squeeze the lower $1/N$ part into small vertical size. To show more details in lower $1/N$, we split some plots into lower $1/N$ part and higher $1/N$ part, between which curves of the same color represent the same eigenstate. See Fig. 4 as an example.

From these figures, we see several features of the eigenstates of $\mathcal{H}$. At $N = \infty$, the ground states are nondegenerate, while the first excited states are nondegenerate for M divisible by 3 and degenerate otherwise. This is consistent with the analytic discussion of previous section. As $1/N$ increase, degeneracies are broken and the solid curves turn to dotted or dash-dotted curves, which implies the disappearance of physical states. If a physical state disappears at

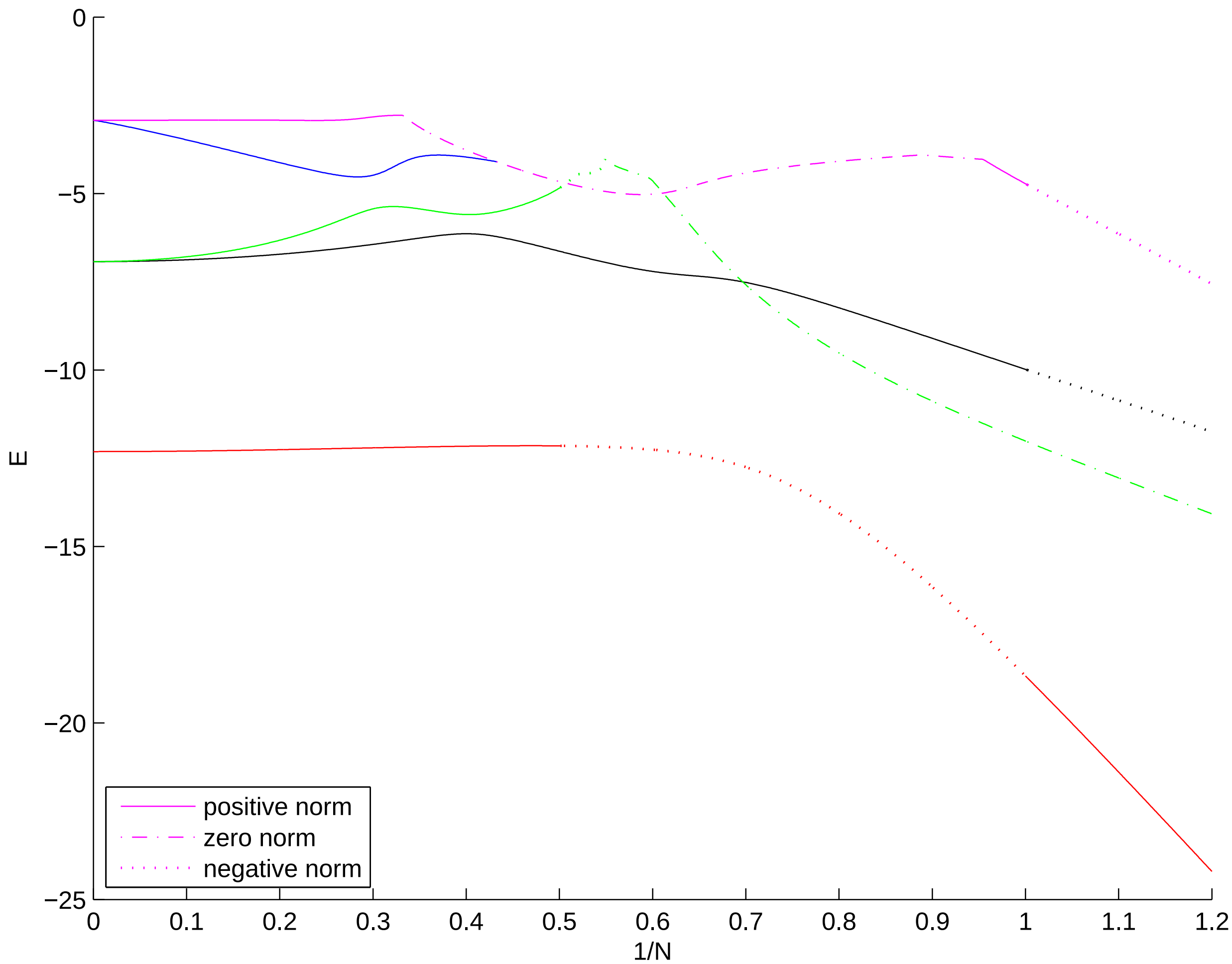

Figure 3: Lowest 5 eigenstates of $\mathcal{H}$ at $M = 5$ for $H = H_0$.

an integer value $N = n$, it also disappears at $N = n - 1$, $n - 2$, and *etc.* For convenience, we denote as N_M^* the maximum value of N where the first disappearance of ground state occurs for bit number M. From the figures, we see that $N_M^* = (M - 1)/2$ for $M \le 11$. If it's true for all M, it follows that, for ground states surviving, N must increase linearly as M increase. The eigenvalues drop dramatically at large $1/N$, as the right parts of Fig. 4 to 6 show. But it does not imply the decrease of energy levels, since all these eigenstates are actually unphysical.

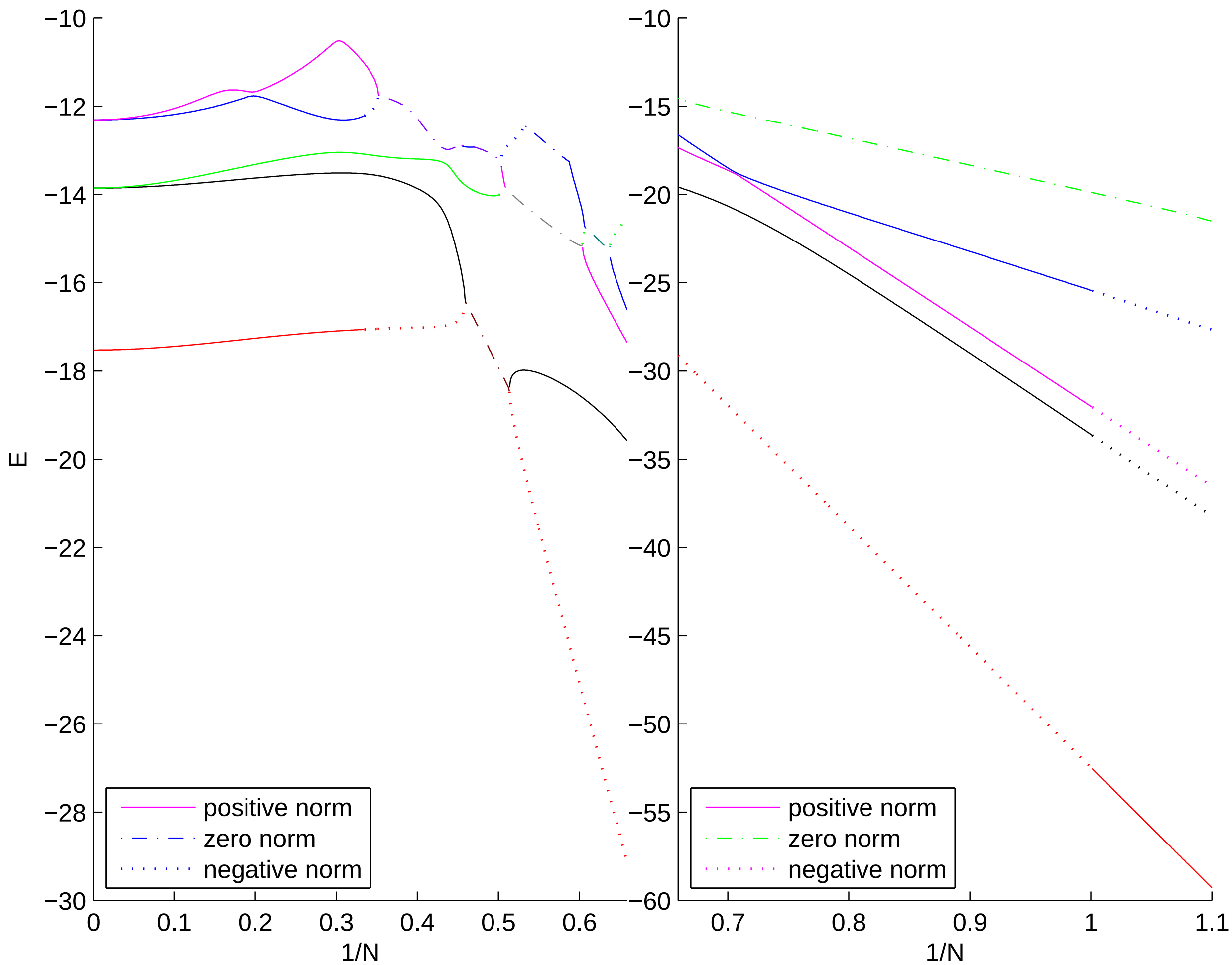

Figure 4: Lowest 5 eigenstates of $\mathcal{H}$ at $M = 7$ for $H = H_0$.

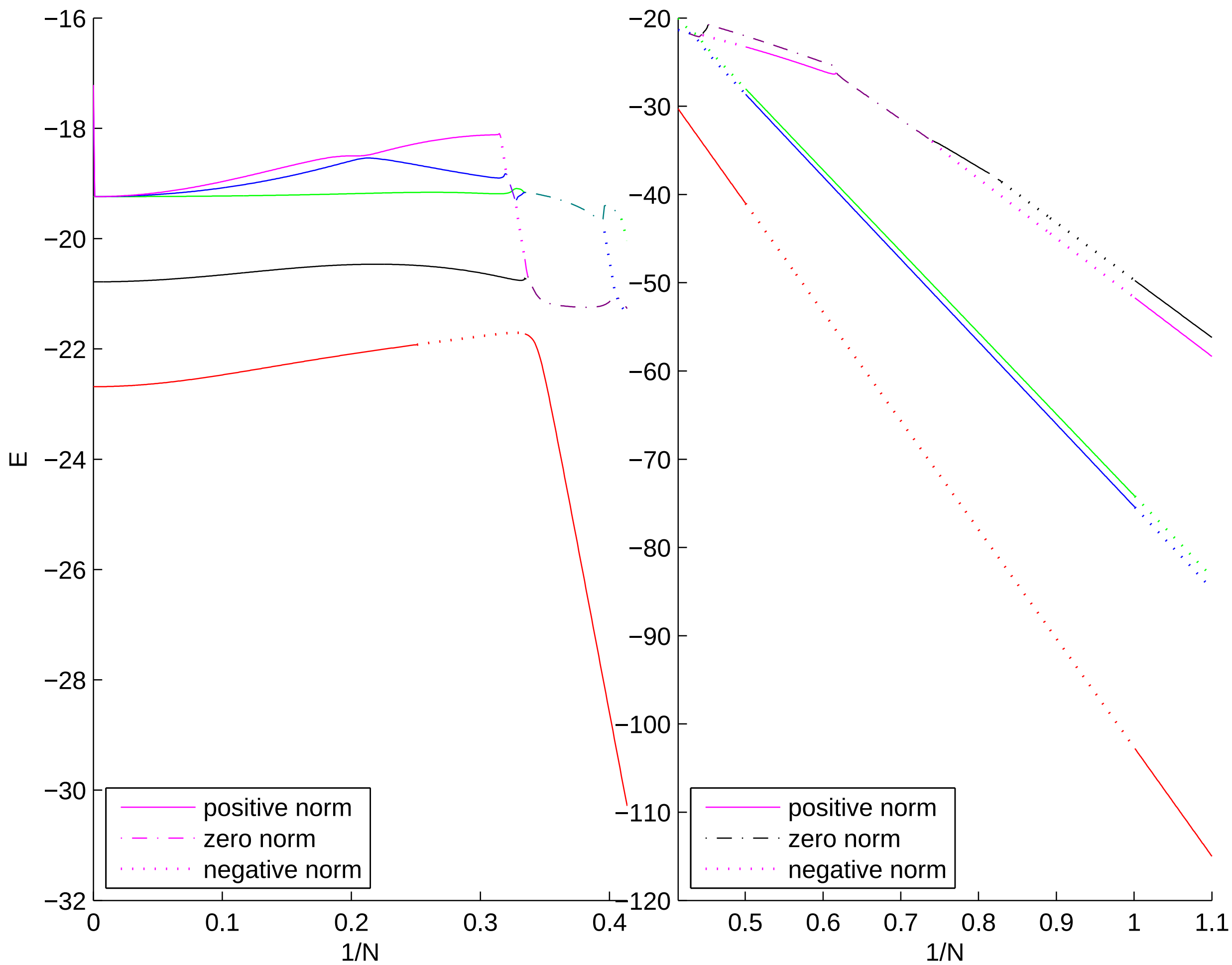

Figure 5: Lowest 5 eigenstates of $\mathcal{H}$ at $M = 9$ for $H = H_0$.

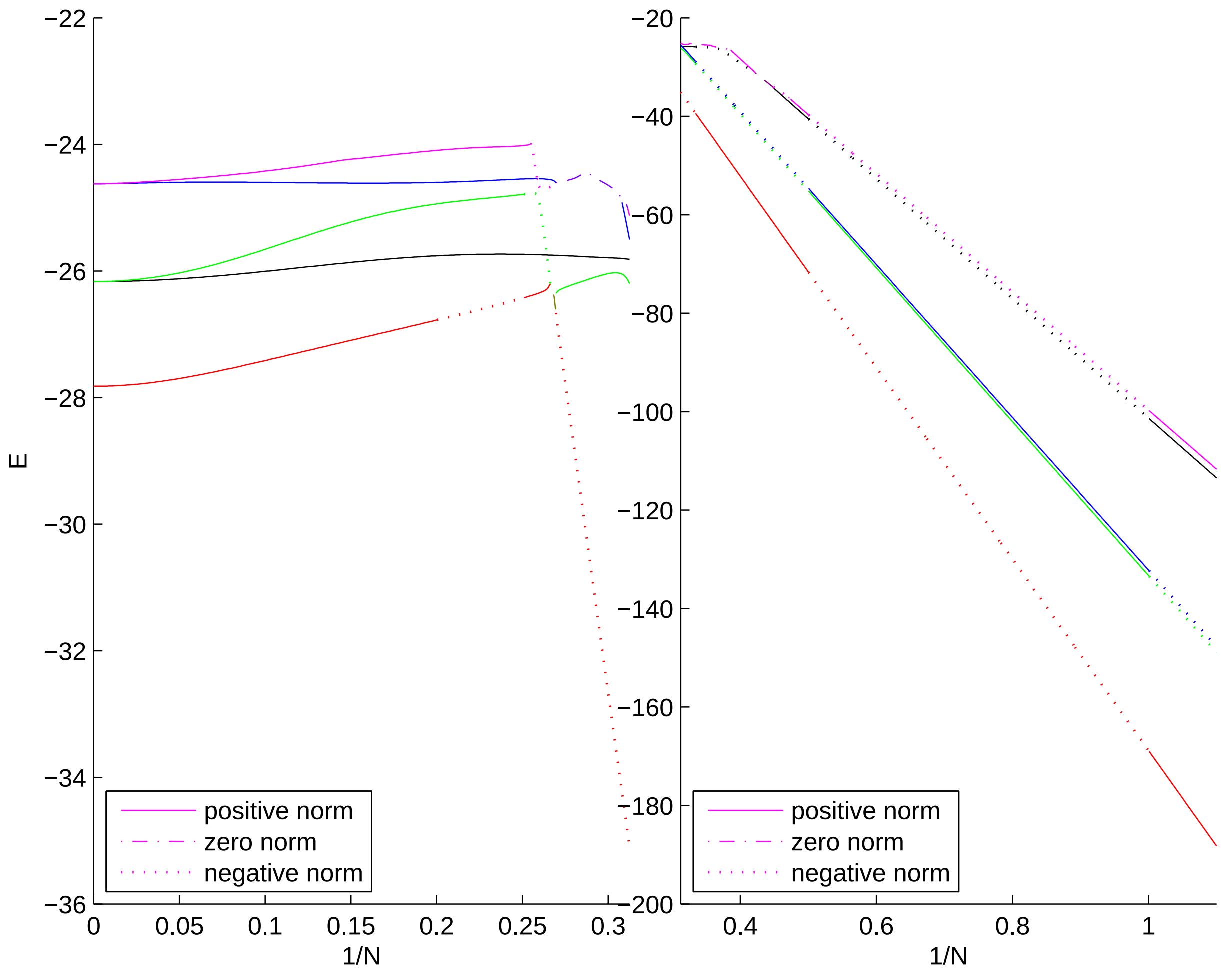

Figure 6: Lowest 5 eigenstates of $\mathcal{H}$ at $M = 11$ for $H = H_0$.

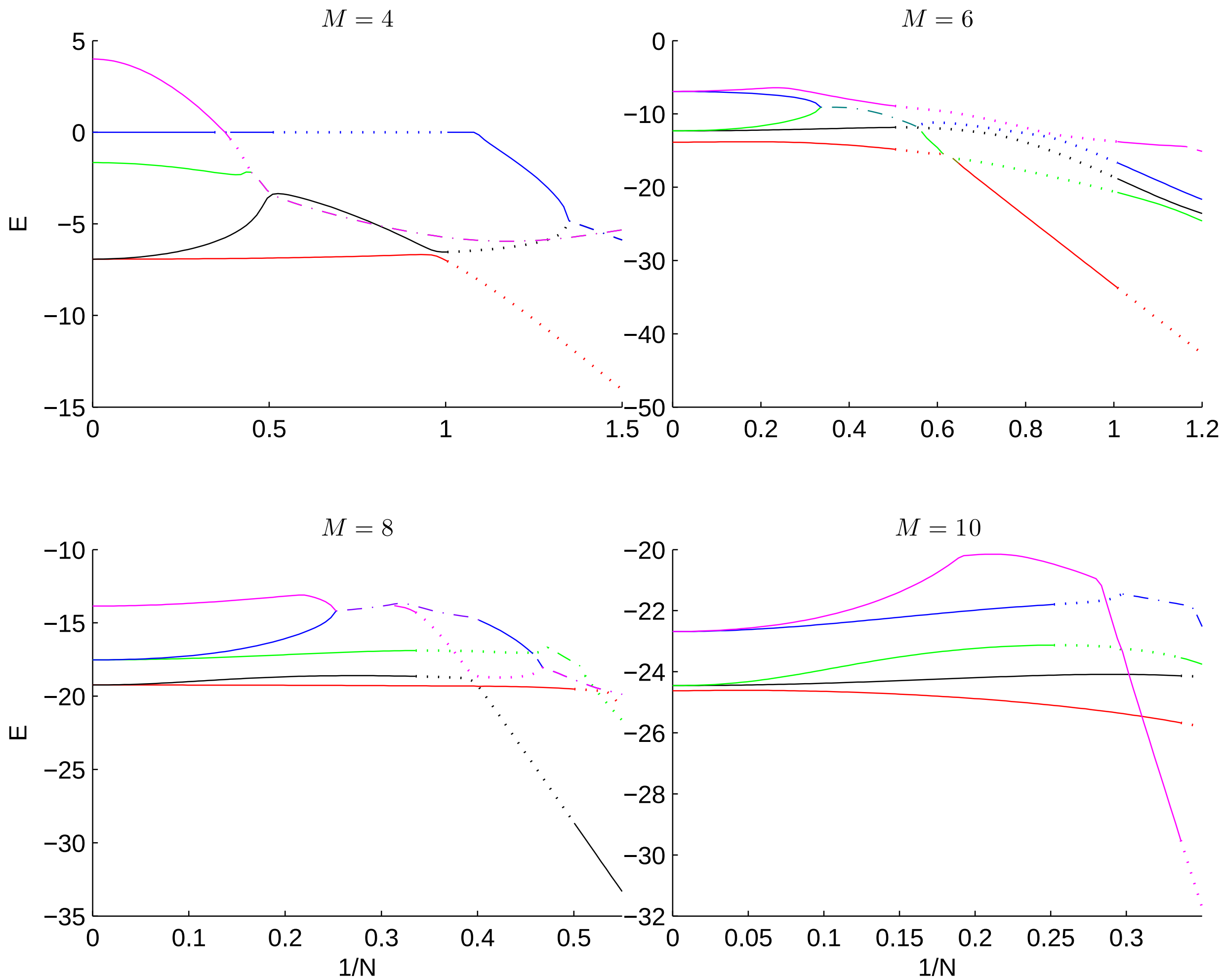

Figure 7: Lowest 5 eigenstates of $\mathcal{H}$ for $H = H_0$ at $M = 4, 6, 8, 10$

For even M, we have similar plots as Fig. 7. At $N = \infty$, the lowest eigenstates are degenerate for $M = 4$ and 8 and nondegenerate for $M = 6$ and 10. It is again consistent with our analysis in previous section. The lowest states also disappear when N is small. But unlike the odd M case, there is no simple formula to determine N_M^*. The reason is that the lowest energy of $E(\eta_i)$ in (22) is excluded by the cyclic constraint (24).

We now consider the physical ground energy as a function of M when N is fixed, shown as Fig. 8. The physical ground states have different trend at different values of N. For

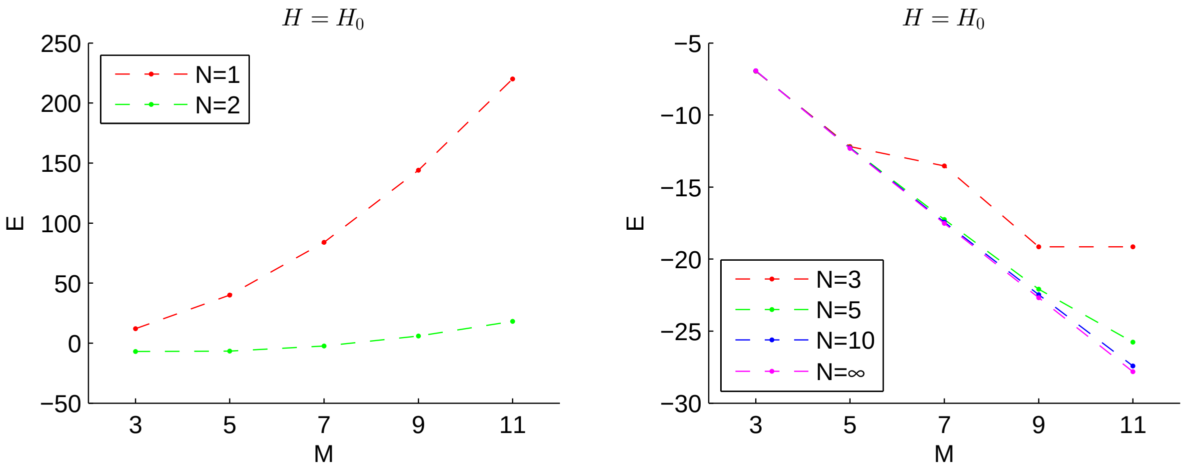

Figure 8: Change of physical ground states with respect to M for fixed N. Only the ground energies at odd M are sampled.

$N = 1$, the physical ground state climbs significantly. This is consistent with analytical calculation, which shows the ground state is a quadratic function of M when $N = 1$. For $N = 2$, the ground state only goes up slightly. When $N \geq 3$, it turns to be downwards. For large N, the physical ground energy drops almost linearly with respect to M at rate $-8/\pi$, as predicted by Eq. (25). This indicates the system become stringy when N is large enough.

Fig. 9 shows how the excitation energy changes with respect to M for fixed N. The vertical axis of Fig. 9 is $M \times (E_1 - E_0)$, where $E_1 - E_0$ is the gap between the first excited energy and lowest energy. For stringy behavior, $M \times (E_1 - E_0)$ should be constant for large M. Though we only calculate up to $M = 11$, we still see the trend that, for N large enough, $M \times (E_1 - E_0)$ is almost a constant between 15 and 20. As a reference, the analytic prediction of the gap at $N = \infty$ is $16\pi/3M$. That being said, there is no inconsistency between the numerical results and stringy behavior.

4.3 Variations of H

In this subsection, we will analyze the energy levels of two variations of the Hamiltonian, $H = H_0 + \xi \Delta H$ and $H = -H_0 + \xi \Delta H$.

Fig. 10 shows the eigenvalues of $\mathcal{H}$ as function of $1/N$ when $M = 3$ and Hamiltonian is of the form $H = H_0 + \xi \Delta H$. As ξ increase, the disappearance point of highest eigenstate moves in the small N direction: for $\xi = -1$, it's at $N = 2$; for $\xi = -0.6$, it's at $N = 1$;

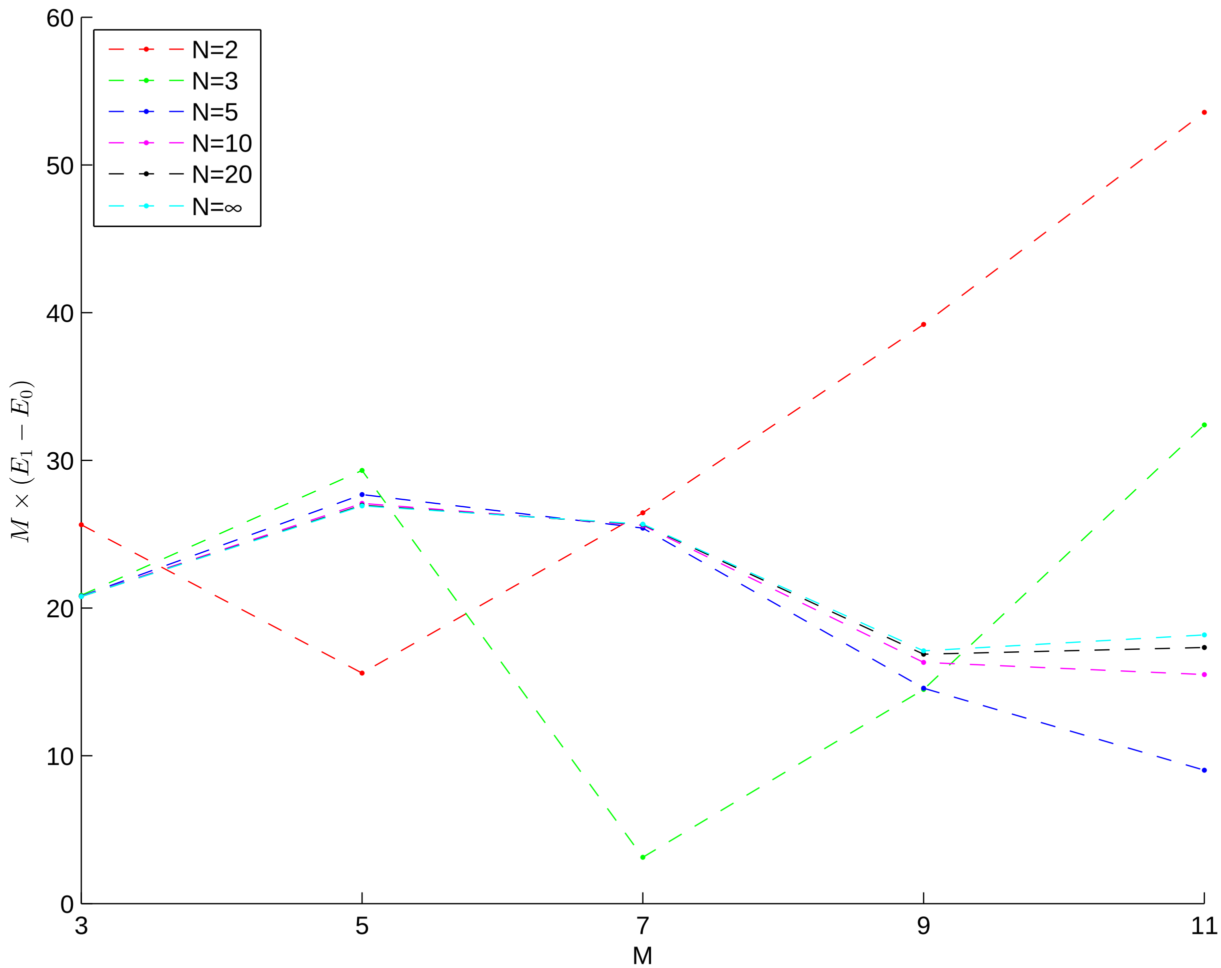

Figure 9: $(E_1 - E_0) \times M$ as a function of M

when $\xi \geq -0.1$, the disappearance point occurs after $N < 1/2$. The disappearance point of ground state, N_3^*, moves in the opposite direction: for $-1 \leq \xi \leq -0.1$, $N_3^* = 1$; for $\xi = 0.5$, $1 < N_3^* < 2$; for $\xi = 3$, $N_3^* = 2$.

Since all eigenstates of $\mathcal{H}$ is physical when $N \geq M$, the largest value of N_M^* is $M - 1$. Particularly, for $M \leq 11$, we find $N_M^* = M - 1$ can be achieved when $\xi \geq 2$. N_M^* is minimal when $\xi = -1$, the lower bound of ξ under the stabilization constraint. The $\xi = -1$ case is shown as Fig. 11. While $N_M^* = (M-1)/2$ still holds for $M = 5$ and 7, $N_9^* = 2$ and $N_{11}^* = 3$ spoil the pattern. We don't have results for $M > 11$, but it seems that M/N_M^* could be large for large M. If it's true, it means that the ground eigenstates could survive when M is large and $N \ll M$.

Fig. 12 shows the change of physical ground energy with respect to M for fixed value of N. Note that only ground energies at odd M are evaluated. The ground energies have different trend for $\xi < -1$, $\xi = -1$, and $\xi > -1$: when $\xi = -1$, the ground energies decrease almost linearly for all N; when $\xi < -1$, the ground energies decline faster than linearly, which implies the system is not stable; when $\xi > -1$, the ground energy first declines and then increase for small N, and it declines linearly for large N. It follows that the system has stringy behavior if $\xi \geq -1$ and N is not too small.

For $H = -H_0 + \xi\Delta H$, in large N limit, the maximum value of $E(\eta_i)$ in (22) is allowed for both odd and even M. Consequently, the ground eigenstates are nondegenerate for all M, as shown in Fig. 13 for $H = -H_0 + 1.5\Delta H$. From the figure, we see that $N_M^* = M - 1$.

ξ has a similar impact on N_M^* as the $H = H_0 + \xi\Delta H$ case. Fig. 14 plots the eigenstates of $\mathcal{H}$ for $\xi = 1$, when N_M^* is minimal. There is no simple patter for N_M^*: for odd M, $N_3^* = 2$, $N_5^* = 2$, $N_7^* = 3$, and $N_9^* = 3$; for even M, $N_4^* = 3$, $N_6^* = 2$, $N_8 = 2$, and $N_{10}^* = 3$. It seems to suggest that the ground eigenstate could survive when M is large and $N \ll M$.

Fig. 15 shows the change of physical ground energy with respect to M at fixed N for $H = -H_0 + \xi\Delta H$. It's similar to the $H = H_0 + \xi\Delta H$ case. When $\xi = 0.5$, the system is not stable at finite N as the curves decline faster than linearly. $\xi = 1$ is the marginal case, in which all the physical ground energies drop almost linearly. When $\xi = 1.5$ or $\xi = 3$, the curves for small N are zig-zag, and particularly, when $\xi = 3$ and $N = 3$, the trend is slightly upward. It implies that the system is stable for large M.

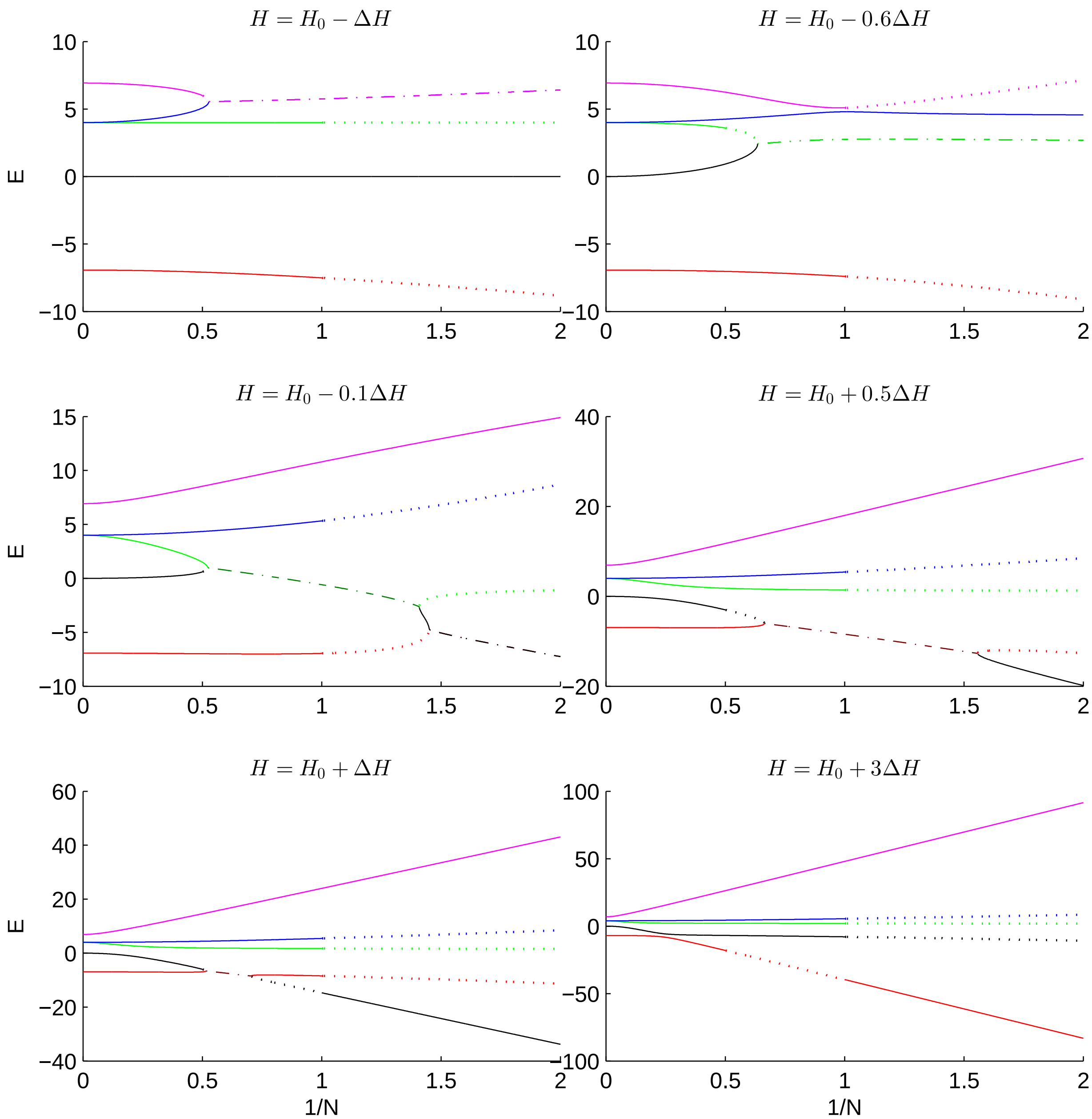

Figure 10: Eigenvalues of $\mathcal{H}$ matrices at $M = 3$ for Hamiltonian $H = H_0 + \xi\Delta H$, with $\xi = -1$, -0.6, -0.1, 0.5, 1, and 3.

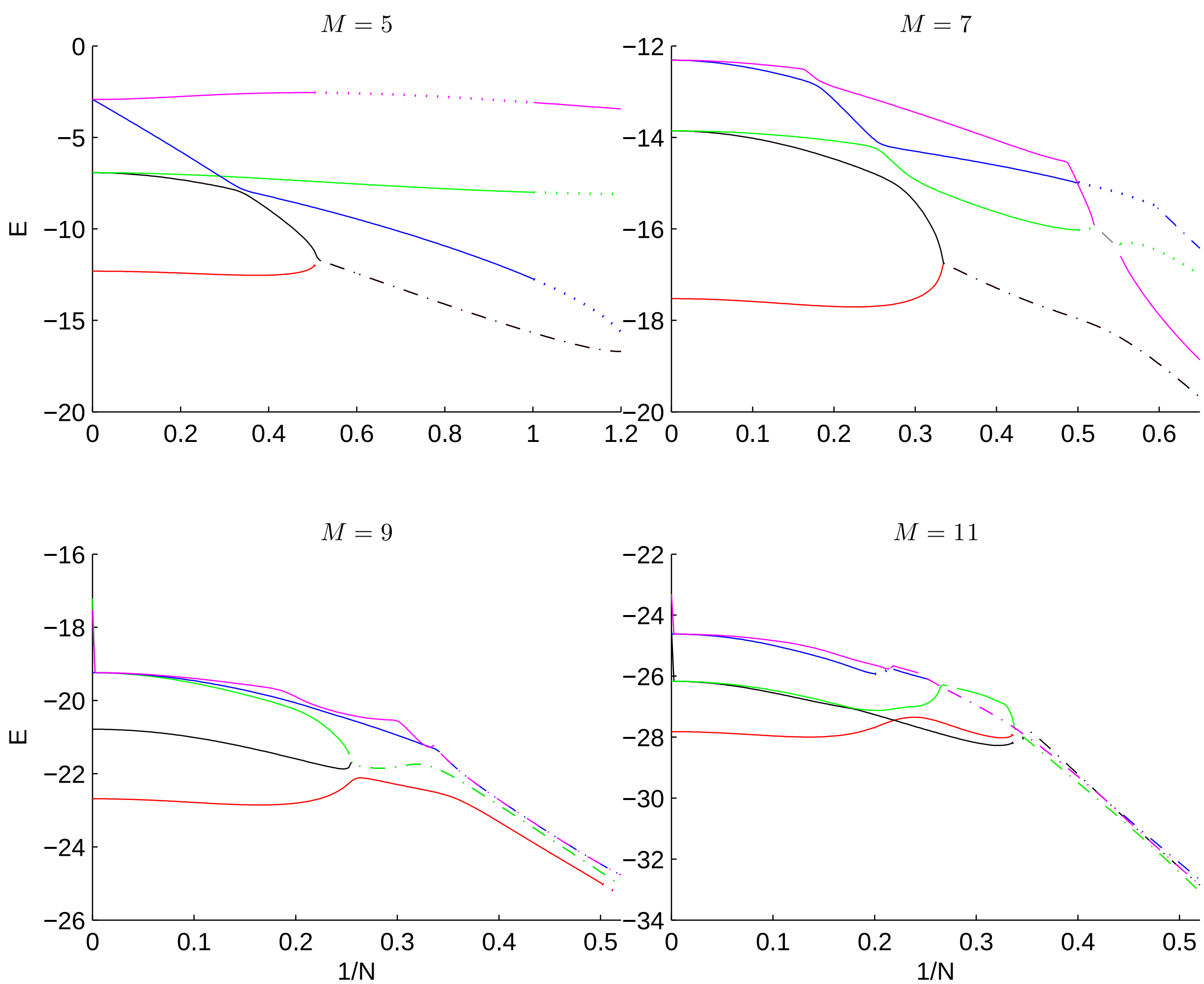

Figure 11: Lowest 5 eigenstates of $\mathcal{H}$ for $H = H_0 - \Delta H$ at $M = 5, 7, 9, 11$

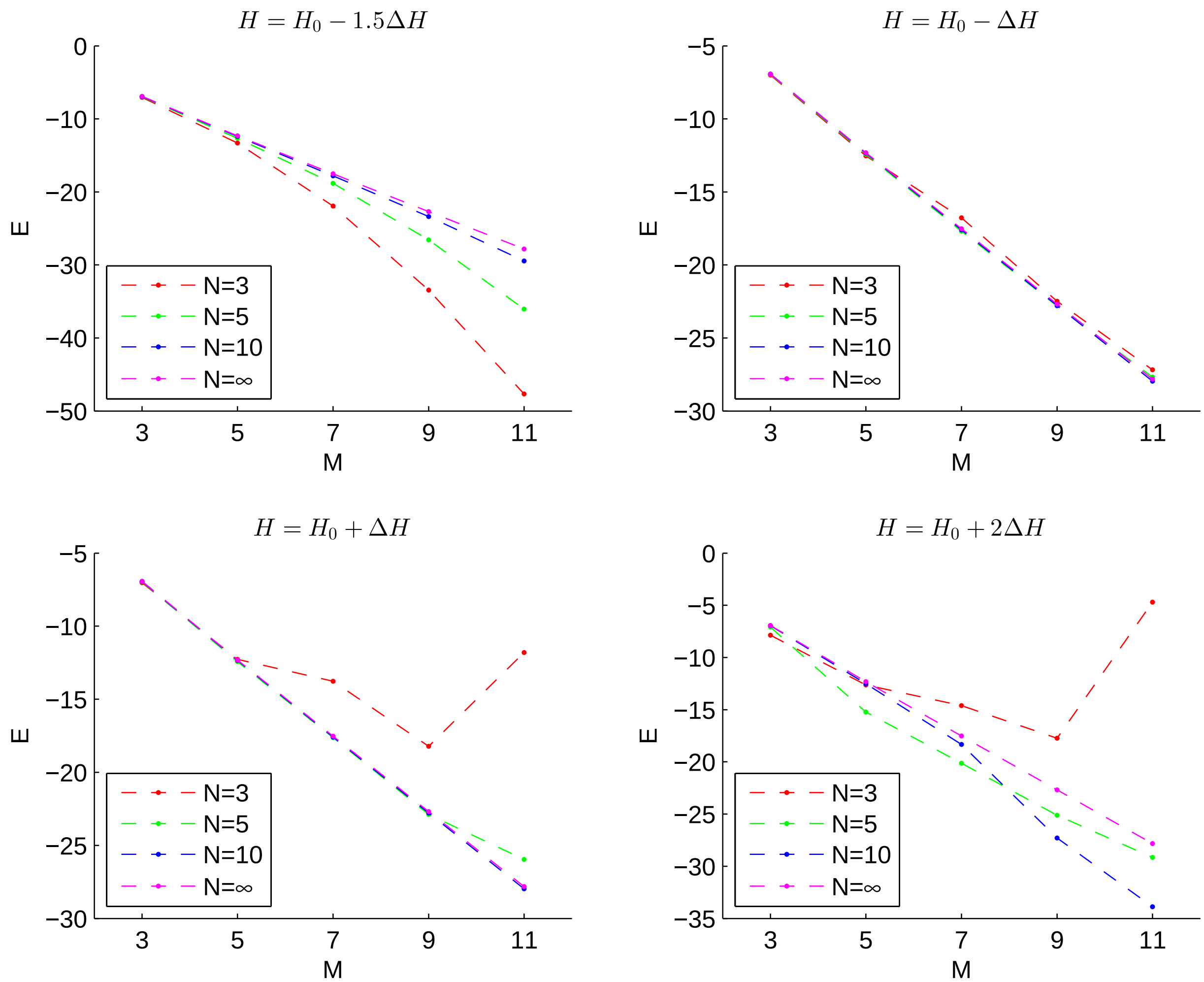

Figure 12: Change of physical ground energy with respect to M at fixed N

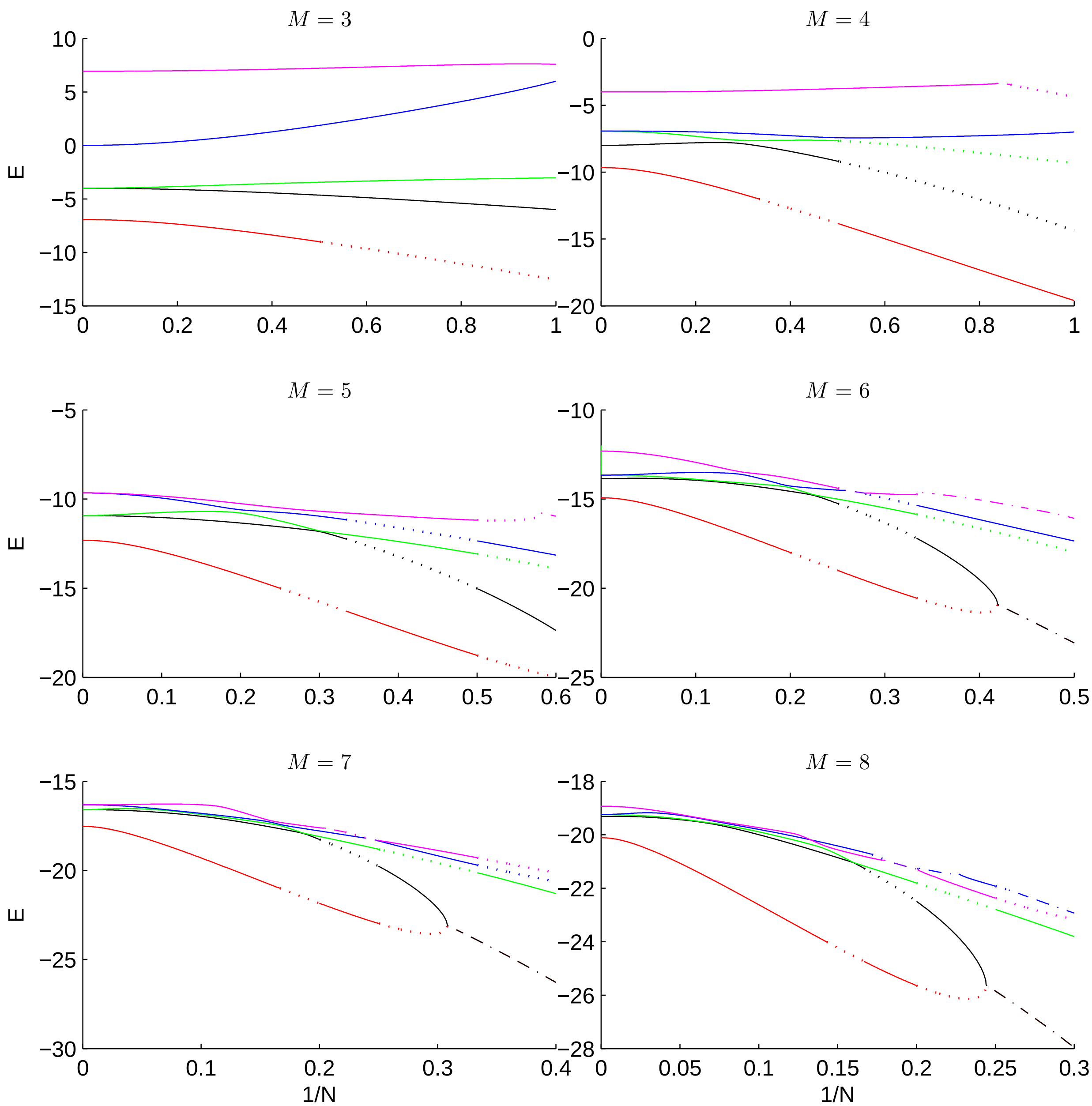

Figure 13: Eigenvalues of $\mathcal{H}$ for $H = -H_0 + 1.5\Delta H$. For each M, the ground state disappears at $N = M - 1$.

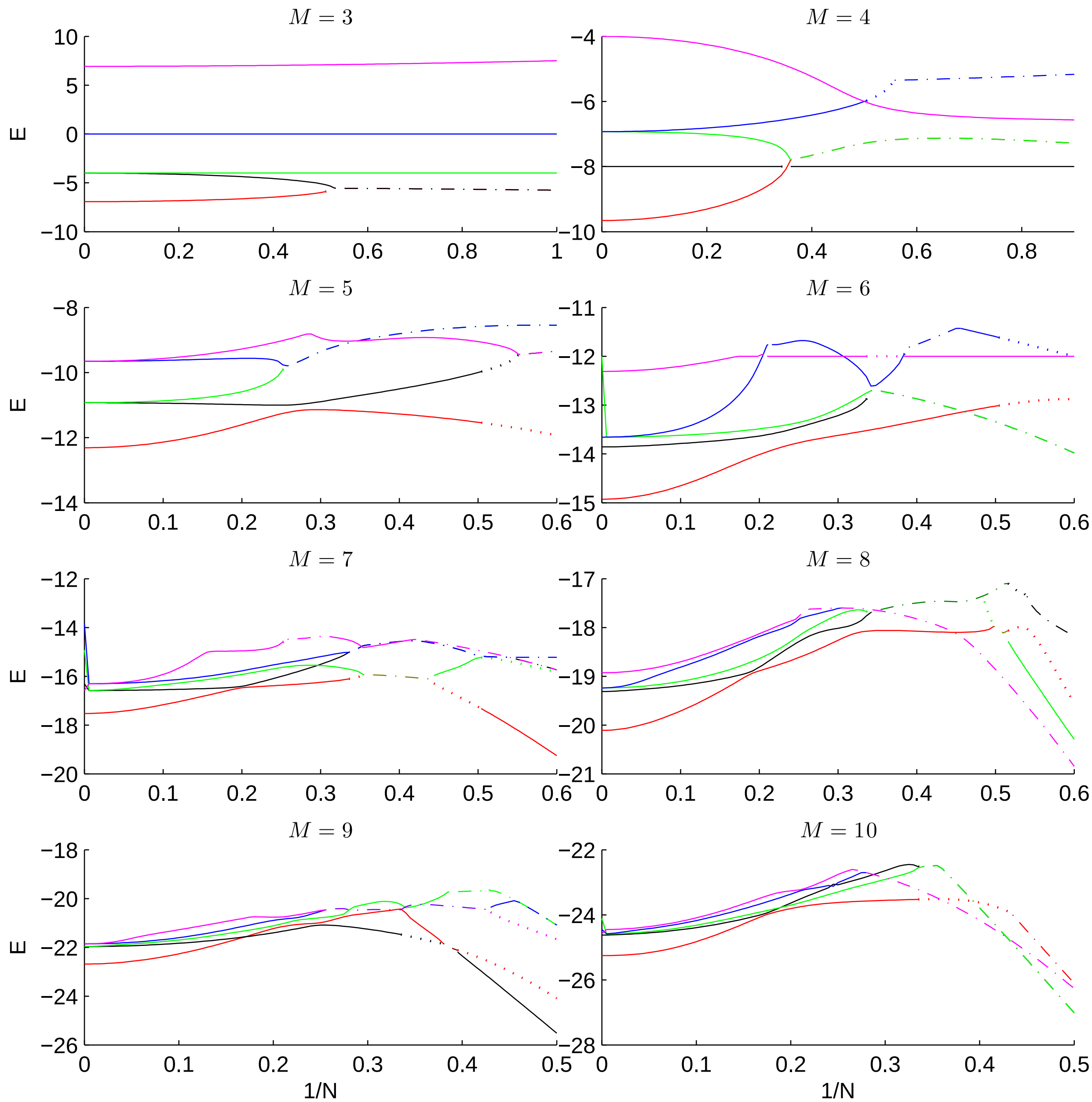

Figure 14: Eigenvalues of $\mathcal{H}$ for $H = -H_0 + \Delta H$ and $3 \leq M \leq 10$.

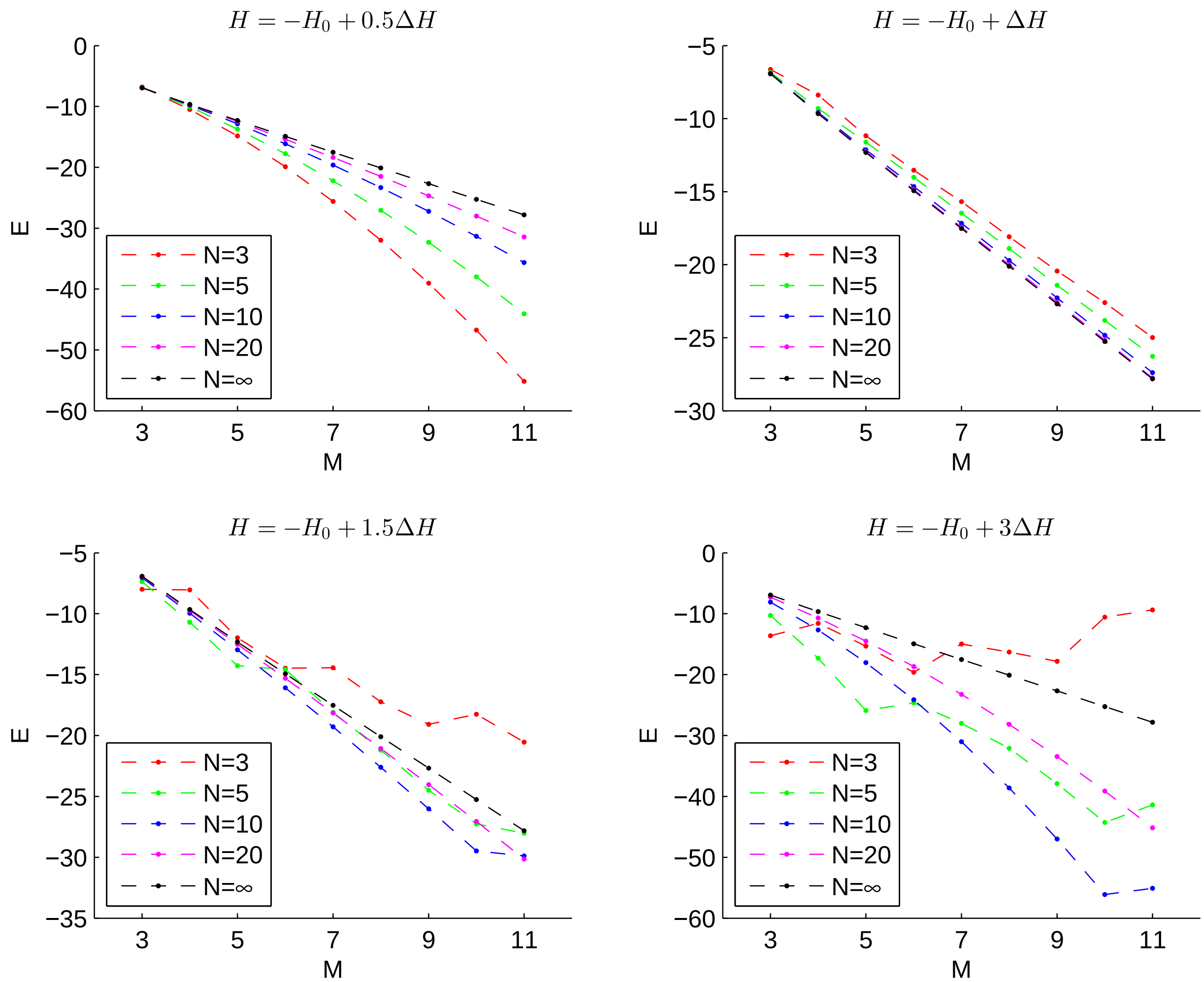

Figure 15: Physical ground energy of $-H_0 + \xi\Delta H$ at $\xi = 0.5, 1, 1.5, 3$ and $N = 3, 5, 10, 20, \infty$.

5 Summary and Conclusion

In this paper we have studied the string bit model with $s = 1$, $d = 0$. We studied possible forms of supersymmetric Hamiltonian and their excitation energies in the large N limit. We also performed numerical study of energy levels at finite N for Hamiltonians $H = \pm H_0 + \xi \Delta H$, where, at $N = \infty$, ΔH vanishes and H_0 produces the Green-Schwarz Hamiltonian.

We showed that the supersymmetry plays a crucial role in the model. The general Hamiltonian is chosen to be a linear combination of 8 single trace operators, which contain two consecutive creation operators followed by two annihilation operators. With the supersymmetry constraint, we reduce the number of parameters in the Hamiltonian to three. Another interesting consequence of supersymmetry is that, after imposing the supersymmetry constraint on the Hamiltonian, the excitation energy becomes of order M^{-1}, which implies the energy spectrum of the model is continuous when M is large.

In finite N, we numerically studied the energy spectrum of the model up to $M \leq 11$. There exists a maximal integer N_M^* that when $N \leq N_M^*$ the would-be ground energy eigenstate of M-bit system is unphysical. For $H = H_0$ and odd $M \leq 11$, the numerical computation shows $N_M^* = (M-1)/2$. If such a simple relation holds for all odd M, then, at large M, the surviving of ground state requires N to be large as well. For $H = \pm H_0 + \xi \Delta H$, N_M^* increases (decreases) as ξ increases (decreases). The maximum value of N_M^* is $(M-1)$. The minimum of N_M^* is achieved when $H = \pm H_0 \mp \Delta H$ because of the stabilization constraint $\xi \geq \mp 1$. In the minimum cases, one find that N_M^* is less than $(M-1)/2$ when $7 < M \leq 11$. If such a trend continues for $M > 11$, it means that the ground energy state might be able to survive at very large M and $M \gg N$.

For fixed finite N and $H = \pm H_0 + \xi \Delta H$, the system is stable only when $\xi \geq \mp 1$. The ground energy drops almost linearly with respect to M when $\xi \geq \mp 1$ and faster than linearly when $\xi < \mp 1$. The numerical computation also reveals the excitation energy is roughly proportional to M^{-1}. While we do not have data for $M > 11$, the trend is still evident. These properties indicate that the model has stringy behavior when $\xi \geq \mp 1$.

The numerical computation is performed up to $M = 11$. The bottleneck is the calculation of norm matrices. Our algorithm has $\mathcal{O}(M!)$ time complexity for computing each entry of the matrix. It needs significant improvement for numerical computation of higher M. This is one of the issues we need to address in future research.

We can also extend our work in other directions. Our strategy can be applied to the model with $s > 1, d = 0$ or $d > 0$ cases. We can also analytically calculate $1/N$ expansion of the model, in which some progress has been made by [23].

6 Acknowledgments

We thank Charles Thorn for his guidance on this work. This research was supported in part by the Department of Energy under Grant No. DE-SC0010296.

A Bosonic Trace States

1 bit

1 bosonic state:

$$|1\rangle = \mathrm{Tr}\bar{a}\,|0\rangle$$

2 bits

2 bosonic states:

$$|1\rangle = \mathrm{Tr}\bar{a}\bar{a}\,|0\rangle \qquad |2\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\,|0\rangle$$

3 bits

5 bosonic states:

$$|1\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\,|0\rangle \qquad |2\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\,|0\rangle \qquad |3\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\,|0\rangle$$

$$|4\rangle = \mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle \qquad |5\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\,|0\rangle$$

4 bits

10 bosonic states:

$$|1\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle \qquad |2\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\,|0\rangle \qquad |3\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\,|0\rangle$$

$$|4\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\,|0\rangle \qquad |5\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\,|0\rangle \qquad |6\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$

$$|7\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle \qquad |8\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\,|0\rangle \qquad |9\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\,|0\rangle$$

$$|10\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle$$

5 bits

21 bosonic states:

$$|1\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle \qquad |2\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle \qquad |3\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\,|0\rangle$$

$$|4\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\,|0\rangle \qquad |5\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\,|0\rangle \qquad |6\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\,|0\rangle$$

$$|7\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\,|0\rangle \qquad |8\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle \qquad |9\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{a}\bar{b}\,|0\rangle$$

$$|10\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle \qquad |11\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\,|0\rangle \qquad |12\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle$$

$$|13\rangle = \mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\,|0\rangle \qquad |14\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle \qquad |15\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\,|0\rangle$$

$$|16\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\,|0\rangle \qquad |17\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\,|0\rangle \qquad |18\rangle = \mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$$

$$|19\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle \qquad |20\rangle = \mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle \qquad |21\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle$$

6 bits

44 bosonic states:

$|1\rangle = \mathrm{Tr}\bar a\bar a\bar a\bar a\bar a\bar a\,|0\rangle$

$|2\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\bar a\bar a\bar a\,|0\rangle$

$|3\rangle = \mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar a\bar a\bar a\bar a\,|0\rangle$

$|4\rangle = \mathrm{Tr}\bar a\bar a\bar a\mathrm{Tr}\bar a\bar a\bar a\,|0\rangle$

$|5\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\bar a\bar a\,|0\rangle$

$|6\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar a\bar a\bar a\,|0\rangle$

$|7\rangle = \mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar a\bar a\,|0\rangle$

$|8\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\bar a\,|0\rangle$

$|9\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar a\bar a\,|0\rangle$

$|10\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\,|0\rangle$

$|11\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\,|0\rangle$

$|12\rangle = \mathrm{Tr}\bar a\bar a\bar a\bar a\bar b\bar b\,|0\rangle$

$|13\rangle = \mathrm{Tr}\bar a\bar a\bar a\bar b\bar a\bar b\,|0\rangle$

$|14\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\bar a\bar b\bar b\,|0\rangle$

$|15\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\bar b\bar a\bar b\,|0\rangle$

$|16\rangle = \mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar a\bar a\bar a\bar b\,|0\rangle$

$|17\rangle = \mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar a\bar a\bar b\bar b\,|0\rangle$

$|18\rangle = \mathrm{Tr}\bar a\bar b\mathrm{Tr}\bar a\bar a\bar a\bar b\,|0\rangle$

$|19\rangle = \mathrm{Tr}\bar a\bar a\bar a\mathrm{Tr}\bar a\bar b\bar b\,|0\rangle$

$|20\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\bar b\bar b\,|0\rangle$

$|21\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar a\bar a\bar b\,|0\rangle$

$|22\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar a\bar b\bar b\,|0\rangle$

$|23\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar b\mathrm{Tr}\bar a\bar a\bar b\,|0\rangle$

$|24\rangle = \mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar a\bar a\bar b\,|0\rangle$

$|25\rangle = \mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar b\mathrm{Tr}\bar a\bar a\bar a\,|0\rangle$

$|26\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar b\bar b\,|0\rangle$

$|27\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar a\bar b\,|0\rangle$

$|28\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar a\bar b\,|0\rangle$

$|29\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar b\,|0\rangle$

$|30\rangle = \mathrm{Tr}\bar a\bar a\bar b\bar b\bar b\bar b\,|0\rangle$

$|31\rangle = \mathrm{Tr}\bar a\bar b\bar a\bar b\bar b\bar b\,|0\rangle$

$|32\rangle = \mathrm{Tr}\bar a\bar b\bar b\bar a\bar b\bar b\,|0\rangle$

$|33\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar b\bar b\bar b\bar b\,|0\rangle$

$|34\rangle = \mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar a\bar b\bar b\bar b\,|0\rangle$

$|35\rangle = \mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar b\bar a\bar b\bar b\,|0\rangle$

$|36\rangle = \mathrm{Tr}\bar a\bar b\mathrm{Tr}\bar a\bar b\bar b\bar b\,|0\rangle$

$|37\rangle = \mathrm{Tr}\bar a\bar a\bar b\mathrm{Tr}\bar b\bar b\bar b\,|0\rangle$

$|38\rangle = \mathrm{Tr}\bar a\bar b\bar b\mathrm{Tr}\bar a\bar b\bar b\,|0\rangle$

$|39\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar b\bar b\bar b\,|0\rangle$

$|40\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\bar b\mathrm{Tr}\bar b\bar b\bar b\,|0\rangle$

$|41\rangle = \mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar a\mathrm{Tr}\bar b\bar b\bar b\,|0\rangle$

$|42\rangle = \mathrm{Tr}\bar b\mathrm{Tr}\bar a\bar b\mathrm{Tr}\bar a\bar b\bar b\,|0\rangle$

$|43\rangle = \mathrm{Tr}\bar a\mathrm{Tr}\bar a\mathrm{Tr}\bar b\mathrm{Tr}\bar b\bar b\bar b\,|0\rangle$

$|44\rangle = \mathrm{Tr}\bar b\mathrm{Tr}\bar b\bar b\bar b\bar b\bar b\,|0\rangle$

7 bits

89 bosonic states:

$$|1\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|2\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|3\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|4\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|5\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|6\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|7\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|8\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|9\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|10\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|11\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\,|0\rangle$$
$$|12\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|13\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\,|0\rangle$$
$$|14\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\,|0\rangle$$
$$|15\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\,|0\rangle$$
$$|16\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|17\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{b}\bar{a}\bar{b}\,|0\rangle$$
$$|18\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|19\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|20\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\bar{a}\bar{b}\,|0\rangle$$
$$|21\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|22\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|23\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{a}\bar{b}\,|0\rangle$$
$$|24\rangle = \mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|25\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|26\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|27\rangle = \mathrm{Tr}\bar{a}\bar{b}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|28\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|29\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{a}\bar{b}\,|0\rangle$$
$$|30\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|31\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|32\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|33\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|34\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|35\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|36\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|37\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|38\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|39\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|40\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|41\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|42\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|43\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|44\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\,|0\rangle$$
$$|45\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\,|0\rangle$$
$$|46\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|47\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|48\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\,|0\rangle$$
$$|49\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\,|0\rangle$$
$$|50\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|51\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|52\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|53\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\bar{b}\bar{a}\bar{b}\,|0\rangle$$
$$|54\rangle = \mathrm{Tr}\bar{a}\bar{b}\bar{a}\bar{b}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|55\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|56\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|57\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|58\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|59\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|60\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\bar{a}\bar{b}\,|0\rangle$$
$$|61\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{a}\bar{b}\bar{a}\bar{b}\,|0\rangle$$
$$|62\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|63\rangle = \mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|64\rangle = \mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|65\rangle = \mathrm{Tr}\bar{a}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|66\rangle = \mathrm{Tr}\bar{a}\bar{b}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|67\rangle = \mathrm{Tr}\bar{b}\bar{b}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\bar{b}\,|0\rangle$$
$$|68\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|69\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|70\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|71\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|72\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|73\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|74\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|75\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|76\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{a}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|77\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|78\rangle = \mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|79\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|80\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|81\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{a}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|82\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\,|0\rangle$$
$$|83\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|84\rangle = \mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|85\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|86\rangle = \mathrm{Tr}\bar{a}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|87\rangle = \mathrm{Tr}\bar{b}\bar{b}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|88\rangle = \mathrm{Tr}\bar{a}\mathrm{Tr}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$$
$$|89\rangle = \mathrm{Tr}\bar{b}\mathrm{Tr}\bar{a}\bar{b}\bar{b}\mathrm{Tr}\bar{b}\bar{b}\bar{b}\,|0\rangle$$

B Counting Problems on Trace States

How many trace states are there for a fixed bit number M? In this sections, we will first count the single trace states and then the trace states which includes both single and multiple trace states.

Counting Single Trace States

There are 2^M combinations of M-bit string consisting of $\bar{a}, \bar{b}$. By property of trace, a trace state is equivalent to its cyclic permutations. For example, $\mathrm{Tr}\,\bar{b}\bar{a}$ and $\mathrm{Tr}\,\bar{a}\bar{b}$ are equivalent states, so are $\mathrm{Tr}\,\bar{b}\bar{a}\bar{a}\bar{b}$ and $\mathrm{Tr}\,\bar{a}\bar{a}\bar{b}\bar{b}$. Actually, the latter case differs by a negative sign

$$\mathrm{Tr}\,\bar{a}\bar{a}\bar{b}\bar{b} = -\,\mathrm{Tr}\,\bar{b}\bar{a}\bar{a}\bar{b}.$$

The rule is that each swap of two $\bar{b}$ introduces a minus sign. It follows that some trace states is vanishing, for example, $\mathrm{Tr}\,\bar{b}\bar{b} = -\,\mathrm{Tr}\,\bar{b}\bar{b} = 0$.

To count the single trace states, we need the following definition and theorem [24].

Definition. Given a group G acting on a set X, the orbit of $x \in X$ is the set $Gx = \{g \cdot x | g \in G\}$. The set of orbits is denoted by X/G.

In our case, the cyclic group C_M is the group G. X is the 2^M combinations of M-bit operators, and x corresponds to one particular combination. X/G is the set of different combinations under the action of Cyclic group.

Theorem. *(Burnside's counting theorem). If G is a finite group acting on a finite set X, then*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}\,(g)|\,,$$

where $\mathrm{Fix}\,(g)$ *is the set of x that is invariant under action of g, i.e.,*

$$\mathrm{Fix}\,(g) = \{x \in X | g \cdot x = x\}\,.$$

To find the number of states, we need to find $|\mathrm{Fix}\,(g)|$ for each group member.

We first consider the odd M case. Let $c_k \in C_M$, $k = 1, 2, \cdots M$, be the group member that shifts k operators from the tail of the trace to the beginning. The identity of the group is $e = c_M$. Let (M, k) denote the greatest common divisor of M and k. For group member c_k, we equally partition the M bits into $M/\,(M, k)$ consecutive parts: the first part starts from bit 1 to bit (M, k), and the second part starts from bit $(M, k) + 1$ to bit $2\,(M, k)$, and *etc.* Under the action of c_k, the ith part transfer as

$$i\text{th part} \to \left(i + \frac{k}{(M,k)}\right)\text{th part}.$$

The trace is invariant under c_k if and only if all the parts are identical to each other. For bosonic trace states, each part need to bosonic, from which it follows

$$|\mathrm{Fix}\,(c_k)| = \sum_{\text{even } i} \binom{(M,k)}{i} = \frac{1}{2} 2^{(M,k)}. \tag{B.1}$$

Similarly, for fermionic single trace states, each part need to be fermionic,

$$|\mathrm{Fix}\,(c_k)| = \sum_{\text{odd } i} \binom{(M,k)}{i} = \frac{1}{2} 2^{(M,k)}, \tag{B.2}$$

which implies there are the same number of bosonic and fermionic single trace states for odd M. By Burnside's theorem, this number is given by

$$S_M = \frac{1}{2M} \sum_{k=1}^{M} 2^{(M,k)}. \tag{B.3}$$

For even M, let's first consider the fermionic states. For a group member c_k, $|\mathrm{Fix}\,(c_k)| = 0$ if $M/(M,k)$ is even. The reason is that odd number of $\bar{b}$ cannot be equally partitioned into even parts. Therefore, only odd $M/(M,k)$ contributes to $|\mathrm{Fix}\,(c_k)|$, which is still given by (B.2). And Eq. (B.3) becomes

$$S_M = \frac{1}{2M} \sum_{M/(M,k) \text{ is odd}} 2^{(M,k)}. \tag{B.4}$$

Let $i = M/(M,k)$, Eq. (B.4) can be written as

$$S_M = \frac{1}{2M} \sum_{\text{odd}\, i, i|M} \varphi(i)\, 2^{\frac{M}{i}}, \tag{B.5}$$

where $\varphi(i)$ is the Euler's totient function and $i|M$ means M is divisible by i. We see that Eq. (B.3) can also be written as Eq. (B.4) and (B.5).

For bosonic states, because there exists vanishing states, like $\mathrm{Tr}\,\bar{a}\bar{b}\bar{a}\bar{b} = -\,\mathrm{Tr}\,\bar{a}\bar{b}\bar{a}\bar{b} = 0$, the number of bosonic states equals to number of even-$\bar{b}$-state minus number of vanishing states. Consider the number of even-$\bar{b}$-states, which is denoted as B_M for convenience. For a group member c_k, we partition M bits equally into $M/(M,k)$ consecutive parts with each part (M,k) bits: if $M/(M,k)$ is odd, we need even number of $\bar{b}$ in each part; if $M/(M,k)$ is even, there can be any number of $\bar{b}$ in each part, from which it follows

$$\begin{aligned} B_M &= \frac{1}{M}\left(\sum_{M/(M,k) \text{ is odd}} 2^{(M,k)-1} + \sum_{M/(M,k) \text{ is even}} 2^{(M,k)} \right) \\ &= \frac{1}{2M}\left(\sum_{\text{odd}\, i, i|M} \varphi(i)\, 2^{\frac{M}{i}} + 2 \sum_{\text{even}\, i, i|M} \varphi(i)\, 2^{\frac{M}{i}} \right). \end{aligned} \tag{B.6}$$

Now consider the number of vanishing states, which is denoted as V_M. For each c_k, we again partition M bits into $M/(M,k)$ consecutive parts. If $M/(M,k)$ is even and all parts are identical with odd number of $\bar{b}$, then it's a vanishing state. But this does not cover all the possibilities. If (M,k) is even, we can perform finer partition: divide M-bits into $2M/(M,k)$ parts with each part of $(M,k)/2$ bits. If all the $2M/(M,k)$ parts are the same and contain odd number of $\bar{b}$, it's a vanishing state. We can continue perform the finer partition i times until $(M,k)/2^i$ is odd. There is a difference between odd $M/(M,k)$ and even $M/(M,k)$: it needs to perform at least one finer partition for odd $M/(M,k)$, while for even $M/(M,k)$ it doesn't. Therefore, the number of vanishing states reads

$$\begin{aligned} V_M &= \frac{1}{M}\sum_{\text{odd } M/(M,k)}\left(\sum_{i\geq 1 \text{ and } 2^i|(M,k)} 2^{\frac{(M,k)}{2^i}-1}\right) \\ &\quad +\frac{1}{M}\sum_{\text{even } M/(M,k)}\left(\sum_{i\geq 0 \text{ and } 2^i|(M,k)} 2^{\frac{(M,k)}{2^i}-1}\right) \\ &= \frac{1}{2M}\left[\sum_k\left(\sum_{i\geq 1, 2^i|(M,k)} 2^{\frac{(M,k)}{2^i}}\right)+\sum_{\text{even } M/(M,k)} 2^{(M,k)}\right] \\ &= \frac{1}{2M}\left[\sum_k\left(\sum_{i\geq 1, 2^i|(M,k)} 2^{\frac{(M,k)}{2^i}}\right)+\sum_{\text{even } i, i|M} \varphi(i)\, 2^{\frac{M}{i}}\right] \end{aligned} \tag{B.7}$$

Let $(M,k)/2^i = \frac{M}{j}$, then we have $2^i|j$ and $(M,k) = 2^i M/j$. The number of k satisfying $(M,k) = 2^i M/j$ equals to

$$\varphi\left(\frac{M}{2^i M/j}\right) = \varphi\left(\frac{j}{2^i}\right).$$

Now, the first term inside parenthesis of Eq. (B.7) can be written as

$$\sum_k\left(\sum_{i\geq 1, 2^i|(M,k)} 2^{\frac{(M,k)}{2^i}}\right) = \sum_{\text{even } j, j|M}\left[\sum_{i\geq 1, 2^i|j}\varphi\left(\frac{j}{2^i}\right)2^{\frac{M}{j}}\right]. \tag{B.8}$$

With the following property of the function φ,

$$\varphi(2m) = \begin{cases} 2\varphi(m) & \text{if } m \text{ is even} \\ \varphi(m) & \text{if } m \text{ is odd} \end{cases},$$

we see that

$$\sum_{i\geq 1, 2^i|j}\varphi\left(\frac{j}{2^i}\right) = \varphi(j), \quad \text{if } j \text{ is even,}$$

Now (B.8) becomes

$$\sum_k \left(\sum_{i \geq 1, 2^i | (M,k)} 2^{\frac{(M,k)}{2^i}} \right) = \sum_{\text{even } j, j|M} \varphi(j)\, 2^{\frac{M}{j}},$$

from which it follows

$$V_M = \frac{1}{M} \sum_{\text{even } i, i|M} \varphi(i)\, 2^{\frac{M}{i}}.$$

The difference of Eq. (B.6) and (B.7) is

$$S_M = B_M - V_M = \frac{1}{2M} \sum_{\text{odd } i, i|M} \varphi(i)\, 2^{\frac{M}{i}},$$

which is the same as the formula for fermionic states.

In sum, we conclude that there are equal number of bosonic and fermionic states for a given bit number M and both can be written as

$$S_M = \frac{1}{2M} \sum_{\text{odd } n, n|M} \varphi(n)\, 2^{\frac{M}{n}}. \tag{B.9}$$

Counting Trace States

Now consider the general trace states, including single and multiple trace states. Let $T^{(0)}_{m,r}$ be the number of r-bit bosonic trace states built out of single trace states of bits less than or equal to m. $T^{(1)}_{m,r}$ is defined similarly for fermionic trace states. We can build the recursive relation of $T^{(b)}_{m,r}$ as follows. Out of r string bits, we can assign $i \times m$ bits to i bosonic m-bit single trace states and $j \times m$ bits to j fermionic m-bit single trace states provided $(i+j)\,m \leq r$. There are $\binom{S_m}{i}$ ways to pick i fermionic m-bit single trace states, and $\binom{S_m+j-1}{j}$ ways to pick j bosonic m-bit single trace states. The rest $r-(i+j)\,m$ bits need to be built out of single trace states of bits less than m. Summation over all non-negative i, j yields

$$T^{(b)}_{m,r} = \sum_{(i+j)m \leq r} \binom{S_m}{i} \binom{S_m+j-1}{j} T^{((b+i) \mod 2)}_{m-1,r-(i+j)m}. \tag{B.10}$$

We can actually drop the superscript of T because $T^{(0)}_{m,r}$ equals to $T^{(0)}_{m,r}$ for all m, r. It can be proved by mathematical induction: for $m=1$, the only r-bit bosonic state is $(\mathrm{Tr}\,\bar{a})^r\,|0\rangle$ and the only r-bit fermionic state is $(\mathrm{Tr}\,\bar{a})^{r-1}\,\mathrm{Tr}\,\bar{b}\,|0\rangle$, which implies $T^{(0)}_{1,r} = T^{(1)}_{1,r}$. If $T^{(0)}_{m-1,r} = T^{(1)}_{m-1,r}$ holds for all r, then Eq. (B.10) gives the same result for $T^{(0)}_{m,r}$ and $T^{(0)}_{m,r}$, from which it follows $T^{(0)}_{m,r} = T^{(0)}_{m,r}$ holds for all values of m, r. Therefore, we can simply write (B.10) as

$$T_{m,r} = \sum_{(i+j)m \leq r} \binom{S_m}{i} \binom{S_m+j-1}{j} T_{m-1,r}. \tag{B.11}$$

M	S_M	T_M	$S_M \times {}^{M}/{}_{2^M}$	${}^{T_M}/{}_{2^M}$
1	1	1	0.500000000000	0.500000000000
2	1	2	0.500000000000	0.500000000000
3	2	5	0.750000000000	0.625000000000
4	2	10	0.500000000000	0.625000000000
5	4	21	0.625000000000	0.656250000000
6	6	44	0.562500000000	0.687500000000
7	10	89	0.546875000000	0.695312500000
8	16	180	0.500000000000	0.703125000000
9	30	365	0.527343750000	0.712890625000
10	52	734	0.507812500000	0.716796875000
11	94	1473	0.504882812500	0.719238281250
20	26216	761282	0.500030517578	0.726015090942
30	17895736	779724424	0.500001087785	0.726174958050
40	13743895360	798439834644	0.500000000466	0.726176799293
50	11258999068468	817602415099946	0.500000000001	0.726176820986
60	9607679205074672	837224873334502342	0.500000000001	0.726176821223

Table 1: Number of trace states

The number of M-bit bosonic or fermionic trace states is simply

$$T_M = T_{M,M}. \tag{B.12}$$

We use computer program to calculate the values of S_M and T_M, as shown in Table 1. The results reveal that when M is large

$$S_M \to \frac{2^{M-1}}{M}, \qquad T_M \to (0.7261768212\cdots) \times 2^M.$$

The limit of S_M shows that almost all the single trace states have M different cyclic permutations when M is large. This is not surprising: the density of the single trace with certain cyclic symmetry goes down as M increase. T_M increases as 2^M with a magic prefactor we don't understand, which could be an interesting mathematical problem to explore.

C Rank of Norm Matrix

The rank of norm matrix $G_{ij} = \langle i|j\rangle$ is the dimension of the trace state space, and also, the number of energy levels of the system. In this section, we show some interesting patterns of the rank of norm matrix. We only focus on the norm matrix of M-bit bosonic trace states, which is a $T_M \times T_M$ real symmetric matrix. By supersymmetry, the norm matrix of M-bit fermionic trace state space has the same rank as the one of M-bit bosonic trace state space.

We generate the norma matrices for $M \leq 11$ and calculate their ranks numerically. We find that when $N \geq M$ G has full rank and that when $N < M$ it is rank deficient. As N change from M to 1, rank of G changes from T_M to 1. We arrange the ranks of norm matrices for $M \leq 11$ and $N \leq M$ as a number triangle as below:

1										
1	2									
1	4	5								
1	6	9	10							
1	8	17	20	21						
1	10	31	40	43	44					
1	12	49	76	85	88	89				
1	14	75	140	167	176	179	180			
1	16	109	252	325	352	361	364	365		
1	18	147	436	621	694	721	730	733	734	
1	20	193	724	1165	1360	1433	1460	1469	1472	1473

The number at i-th row and j-th column is the rank of G for $M = i$ and $N = j$. For convenience, we denote it as $R_{i,j}$. We immediately see several patterns: $R_{M,M} = T_M$, $R_{M,M-1} = T_M - 1$, $R_{M,1} = 1$, and for M greater than 1, $R_{M,2} = 2M - 2$. If we define $R_{i,0} = 0$, then we can define new variables $D_{i,j} = R_{i,j} - R_{i,j-1}$, which represent the change of G's rank when $M = i$ and N change from j to $j - 1$. We arrange D_{ij} as another number triangle as below:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 3 & & 1 & & & \\
 & & 1 & & 5 & & 3 & & 1 & & \\
 & 1 & & 7 & & 9 & & 3 & & 1 & \\
1 & 9 & 21 & 9 & 3 & 1 & & & & & \\
1 & 11 & 37 & 27 & 9 & 3 & 1 & & & & \\
1 & 13 & 61 & 65 & 27 & 9 & 3 & 1 & & & \\
1 & 15 & 93 & 143 & 73 & 27 & 9 & 3 & 1 & & \\
1 & 17 & 129 & 289 & 185 & 73 & 27 & 9 & 3 & 1 & \\
1 & 19 & 173 & 531 & 441 & 195 & 73 & 27 & 9 & 3 & 1
\end{array}
$$

Going through each row from right to left, we find the following sequence:

$$1, \quad 3, \quad 9, \quad 27, \quad 73, \quad 195, \cdots .$$

For odd M, the sequence starts from $N = M$ and ends at $N = (M+1)/2$; for even M, the sequence starts from $N = M$ and ends at $N = M/2$. This means that, not matter what value M is, the changes of G's rank from N to $N-1$ for $N \geq M$ are the same.

Since we only obtain the norm matrices for $M \leq 11$, we don't know the next number of the sequence. Finding the pattern of the sequence is an interesting problem for future research.

D Calculation of $[H, Q]$

In this section, let's find the constraint of supersymmetric Hamiltonian, *i.e.*, the condition for $[H, Q] = 0$, where

$$Q = \exp\left(\frac{i\pi}{4}\right) \operatorname{Tr} \bar{a}b + \exp\left(-\frac{i\pi}{4}\right) \operatorname{Tr} \bar{b}a.$$

We first calculate the commutation between Q and each trace operators in (4). We have

$$\left[\operatorname{Tr}\bar{a}^2 b^2, \operatorname{Tr}\bar{a}b\right] = \operatorname{Tr}\bar{a}^2 b^2 \operatorname{Tr}\bar{a}b - \operatorname{Tr}\bar{a}b\operatorname{Tr}\bar{a}^2 b^2 = 0,$$

$$
\begin{aligned}
\left[\operatorname{Tr}\bar{a}^2 b^2, \operatorname{Tr}\bar{b}a\right] &= \operatorname{Tr}\bar{a}^2 b^2 \operatorname{Tr}\bar{b}a - \operatorname{Tr}\bar{b}a\operatorname{Tr}\bar{a}^2 b^2 \\
&= \operatorname{Tr}\bar{a}^2 ba - \operatorname{Tr}b\bar{a}^2 a + :\operatorname{Tr}\bar{a}^2 b^2 \operatorname{Tr}\bar{b}a: \\
&\quad - \left(\operatorname{Tr}\bar{b}\bar{a}b^2 + \operatorname{Tr}\bar{b}b^2\bar{a} + :\operatorname{Tr}\bar{a}^2 b^2 \operatorname{Tr}\bar{b}a:\right) \\
&= \operatorname{Tr}\bar{a}^2 (ba - ab) - \operatorname{Tr}\left(\bar{b}\bar{a} + \bar{a}\bar{b}\right) b^2,
\end{aligned}
$$

where : $\mathrm{Tr}\bar{a}^2b^2\mathrm{Tr}\bar{b}a$: denotes the normal ordering of $\mathrm{Tr}\bar{a}^2b^2\mathrm{Tr}\bar{b}a$. As we see, the normal ordering terms cancel out. This occurs for all the trace operators. So in the following calculation, we simply drop the normal ordering terms in most cases. From above two results, it follows

$$\begin{aligned}\left[\mathrm{Tr}\bar{a}^2b^2, Q\right] &= \exp\left(-i\frac{\pi}{4}\right)\left[\mathrm{Tr}\bar{a}^2b^2, \mathrm{Tr}\,\bar{b}a\right] \\ &= \exp\left(-i\frac{\pi}{4}\right)\left[\mathrm{Tr}\,\bar{a}^2\left(ba-ab\right) - \mathrm{Tr}\left(\bar{b}\bar{a}+\bar{a}\bar{b}\right)b^2\right].\end{aligned}$$

We repeat the calculation for the other trace operators as follows.

$$\begin{aligned}\left[\mathrm{Tr}\,\bar{b}^2a^2, \mathrm{Tr}\,\bar{a}b\right] &= \mathrm{Tr}\,\bar{b}^2a^2\,\mathrm{Tr}\,\bar{a}b - \mathrm{Tr}\,\bar{a}b\,\mathrm{Tr}\,\bar{b}^2a^2 \\ &= \mathrm{Tr}\,\bar{b}^2ab + \mathrm{Tr}\,a\bar{b}^2b - \mathrm{Tr}\,\bar{a}\bar{b}a^2 + \mathrm{Tr}\,\bar{a}a^2\bar{b} \\ &= \mathrm{Tr}\,\bar{b}^2\left(ab+ba\right) + \mathrm{Tr}\left(\bar{b}\bar{a}-\bar{a}\bar{b}\right)a^2,\end{aligned}$$

$$\left[\mathrm{Tr}\,\bar{b}^2a^2, \mathrm{Tr}\,\bar{b}a\right] = 0,$$

from which it follows

$$\begin{aligned}\left[\mathrm{Tr}\,\bar{b}^2a^2, Q\right] &= \exp\left(i\frac{\pi}{4}\right)\left[\mathrm{Tr}\,\bar{b}^2a^2, \mathrm{Tr}\,\bar{a}b\right] \\ &= \exp\left(i\frac{\pi}{4}\right)\left[\mathrm{Tr}\,\bar{b}^2\left(ab+ba\right) + \mathrm{Tr}\left(\bar{b}\bar{a}-\bar{a}\bar{b}\right)a^2\right]. \qquad (D.1)\end{aligned}$$

$$\begin{aligned}\left[\mathrm{Tr}\,\bar{a}^2a^2, \mathrm{Tr}\,\bar{a}b\right] &= \mathrm{Tr}\,\bar{a}^2a^2\,\mathrm{Tr}\,\bar{a}b - \mathrm{Tr}\,\bar{a}b\,\mathrm{Tr}\,\bar{a}^2a^2 \\ &= \mathrm{Tr}\,\bar{a}^2ab + \mathrm{Tr}\,\bar{a}^2ba + :\mathrm{Tr}\,\bar{a}^2a^2\,\mathrm{Tr}\,\bar{a}b: - \mathrm{Tr}\,\bar{a}b\,\mathrm{Tr}\,\bar{a}^2a^2 \\ &= \mathrm{Tr}\,\bar{a}^2\left(ab+ba\right),\end{aligned}$$

$$\begin{aligned}\left[\mathrm{Tr}\,\bar{a}^2a^2, \mathrm{Tr}\,\bar{b}a\right] &= \mathrm{Tr}\,\bar{a}^2a^2\,\mathrm{Tr}\,\bar{b}a - \mathrm{Tr}\,\bar{b}a\,\mathrm{Tr}\,\bar{a}^2a^2 \\ &= \mathrm{Tr}\,\bar{a}^2a^2\,\mathrm{Tr}\,\bar{b}a - \mathrm{Tr}\,\bar{b}\bar{a}a^2 - \mathrm{Tr}\,\bar{a}\bar{b}a^2 - :\mathrm{Tr}\,\bar{b}a\,\mathrm{Tr}\,\bar{a}^2a^2: \\ &= -\,\mathrm{Tr}\left(\bar{a}\bar{b}+\bar{b}\bar{a}\right)a^2,\end{aligned}$$

from which it follows

$$\begin{aligned}\left[\mathrm{Tr}\,\bar{a}^2a^2, Q\right] &= \exp\left(i\frac{\pi}{4}\right)\left[\mathrm{Tr}\,\bar{a}^2a^2, \mathrm{Tr}\,\bar{a}b\right] + \exp\left(-i\frac{\pi}{4}\right)\left[\mathrm{Tr}\,\bar{a}^2a^2, \mathrm{Tr}\,\bar{b}a\right] \\ &= \exp\left(i\frac{\pi}{4}\right)\mathrm{Tr}\,\bar{a}^2\left(ab+ba\right) - \exp\left(-i\frac{\pi}{4}\right)\mathrm{Tr}\left(\bar{a}\bar{b}+\bar{b}\bar{a}\right)a^2. \qquad (D.2)\end{aligned}$$

$$\begin{aligned}\left[\mathrm{Tr}\,\bar{b}^2b^2, \mathrm{Tr}\,\bar{a}b\right] &= \mathrm{Tr}\,\bar{b}^2b^2\,\mathrm{Tr}\,\bar{a}b - \mathrm{Tr}\,\bar{a}b\,\mathrm{Tr}\,\bar{b}^2b^2 \\ &= \mathrm{Tr}\,\bar{b}^2b^2\,\mathrm{Tr}\,\bar{a}b - \mathrm{Tr}\,\bar{a}\bar{b}b^2 + \mathrm{Tr}\,\bar{b}\bar{a}b^2 - :\mathrm{Tr}\,\bar{a}b\,\mathrm{Tr}\,\bar{b}^2b^2: \\ &= \mathrm{Tr}\left(\bar{b}\bar{a}-\bar{a}\bar{b}\right)b^2,\end{aligned}$$

$$\begin{aligned}\left[\operatorname{Tr}\bar{b}^2b^2,\operatorname{Tr}\bar{b}a\right] &= \operatorname{Tr}\bar{b}^2b^2\operatorname{Tr}\bar{b}a-\operatorname{Tr}\bar{b}a\operatorname{Tr}\bar{b}^2b^2\\ &= \operatorname{Tr}\bar{b}^2ba-\operatorname{Tr}\bar{b}^2ab+:\operatorname{Tr}\bar{b}^2b^2\operatorname{Tr}\bar{b}a:-\operatorname{Tr}\bar{b}a\operatorname{Tr}\bar{b}^2b^2\\ &= \operatorname{Tr}\bar{b}^2\left(ba-ab\right),\end{aligned}$$

from which it follows

$$\begin{aligned}\left[\operatorname{Tr}\bar{b}^2b^2,Q\right] &= \exp\left(i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{b}^2b^2,\operatorname{Tr}\bar{a}b\right]+\exp\left(-i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{b}^2b^2,\operatorname{Tr}\bar{b}a\right]\\ &= \exp\left(i\frac{\pi}{4}\right)\operatorname{Tr}\left(\bar{b}\bar{a}-\bar{a}\bar{b}\right)b^2+\exp\left(-i\frac{\pi}{4}\right)\operatorname{Tr}\bar{b}^2\left(ba-ab\right).\end{aligned}\tag{D.3}$$

$$\begin{aligned}\left[\operatorname{Tr}\bar{b}\bar{a}ba,\operatorname{Tr}\bar{a}b\right] &= \operatorname{Tr}\bar{b}\bar{a}ba\operatorname{Tr}\bar{a}b-\operatorname{Tr}\bar{a}b\operatorname{Tr}\bar{b}\bar{a}ba\\ &= \operatorname{Tr}\bar{b}\bar{a}bb+:\operatorname{Tr}\bar{b}\bar{a}ba\operatorname{Tr}\bar{a}b:-\operatorname{Tr}\bar{a}\bar{a}ba+:\operatorname{Tr}\bar{a}b\operatorname{Tr}\bar{b}\bar{a}ba:\\ &= \operatorname{Tr}\bar{b}\bar{a}bb+:\operatorname{Tr}\bar{b}\bar{a}ba\operatorname{Tr}\bar{a}b:-\operatorname{Tr}\bar{a}\bar{a}ba-:\operatorname{Tr}\bar{b}\bar{a}ba\operatorname{Tr}\bar{a}b:\\ &= \operatorname{Tr}\bar{b}\bar{a}bb-\operatorname{Tr}\bar{a}\bar{a}ba,\end{aligned}$$

$$\begin{aligned}\left[\operatorname{Tr}\bar{b}\bar{a}ba,\operatorname{Tr}\bar{b}a\right] &= \operatorname{Tr}\bar{b}\bar{a}ba\operatorname{Tr}\bar{b}a-\operatorname{Tr}\bar{b}a\operatorname{Tr}\bar{b}\bar{a}ba\\ &= \operatorname{Tr}\bar{b}\bar{a}a^2-:\operatorname{Tr}\bar{b}\bar{a}ba\operatorname{Tr}\bar{b}a:+\operatorname{Tr}\bar{b}\bar{b}ba-:\operatorname{Tr}\bar{b}a\operatorname{Tr}\bar{b}\bar{a}ba:\\ &= \operatorname{Tr}\bar{b}\bar{a}a^2-:\operatorname{Tr}\bar{b}\bar{a}ba\operatorname{Tr}\bar{b}a:+\operatorname{Tr}\bar{b}\bar{b}ba+:\operatorname{Tr}\bar{b}\bar{a}ba\operatorname{Tr}\bar{b}a:\\ &= \operatorname{Tr}\bar{b}\bar{a}a^2+\operatorname{Tr}\bar{b}\bar{b}ba,\end{aligned}$$

from which it follows

$$\begin{aligned}\left[\operatorname{Tr}\bar{b}\bar{a}ba,Q\right] &= \exp\left(i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{b}\bar{a}ba,\operatorname{Tr}\bar{a}b\right]+\exp\left(-i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{b}\bar{a}ba,\operatorname{Tr}\bar{b}a\right]\\ &= \exp\left(i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{b}\bar{a}bb-\operatorname{Tr}\bar{a}\bar{a}ba\right]\\ &\quad+\exp\left(-i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{b}\bar{a}a^2+\operatorname{Tr}\bar{b}\bar{b}ba\right].\end{aligned}\tag{D.4}$$

$$\begin{aligned}\left[\operatorname{Tr}\bar{a}\bar{b}ab,\operatorname{Tr}\bar{a}b\right] &= \operatorname{Tr}\bar{a}\bar{b}ab\operatorname{Tr}\bar{a}b-\operatorname{Tr}\bar{a}b\operatorname{Tr}\bar{a}\bar{b}ab\\ &= -\operatorname{Tr}\bar{a}\bar{b}b^2+:\operatorname{Tr}\bar{a}\bar{b}ab\operatorname{Tr}\bar{a}b:-\operatorname{Tr}\bar{a}^2ab+:\operatorname{Tr}\bar{a}b\operatorname{Tr}\bar{a}\bar{b}ab:\\ &= -\operatorname{Tr}\bar{a}\bar{b}b^2-\operatorname{Tr}\bar{a}^2ab,\end{aligned}$$

$$\begin{aligned}\left[\operatorname{Tr}\bar{a}\bar{b}ab,\operatorname{Tr}\bar{b}a\right] &= \operatorname{Tr}\bar{a}\bar{b}ab\operatorname{Tr}\bar{b}a-\operatorname{Tr}\bar{b}a\operatorname{Tr}\bar{a}\bar{b}ab\\ &= \operatorname{Tr}\bar{a}\bar{b}a^2-:\operatorname{Tr}\bar{a}\bar{b}ab\operatorname{Tr}\bar{b}a:-\operatorname{Tr}\bar{b}^2ab-:\operatorname{Tr}\bar{a}\bar{b}ab\operatorname{Tr}\bar{b}a:\\ &= \operatorname{Tr}\bar{a}\bar{b}a^2-\operatorname{Tr}\bar{b}^2ab,\end{aligned}$$

which follows

$$\begin{aligned}\left[\operatorname{Tr}\bar{a}\bar{b}ab,Q\right] &= \exp\left(i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{a}\bar{b}ab,\operatorname{Tr}\bar{b}a\right]+\exp\left(-i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{a}\bar{b}ab,\operatorname{Tr}\bar{a}b\right]\\ &= \exp\left(i\frac{\pi}{4}\right)\left[-\operatorname{Tr}\bar{a}\bar{b}b^2-\operatorname{Tr}\bar{a}^2ab\right]\\ &\quad+\exp\left(-i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{a}\bar{b}a^2-\operatorname{Tr}\bar{b}^2ab\right].\end{aligned}\tag{D.5}$$

$$\begin{aligned}
\left[\operatorname{Tr}\bar{a}\bar{b}ba, \operatorname{Tr}\bar{a}b\right] &= \operatorname{Tr}\bar{a}\bar{b}ba\operatorname{Tr}\bar{a}b - \operatorname{Tr}\bar{a}b\operatorname{Tr}\bar{a}\bar{b}ba \\
&= \operatorname{Tr}\bar{a}\bar{b}bb + :\operatorname{Tr}\bar{a}\bar{b}ba\operatorname{Tr}\bar{a}b: - \operatorname{Tr}\bar{a}^2ba + :\operatorname{Tr}\bar{a}b\operatorname{Tr}\bar{a}\bar{b}ba: \\
&= \operatorname{Tr}\bar{a}\bar{b}bb - \operatorname{Tr}\bar{a}^2ba,
\end{aligned}$$

$$\begin{aligned}
\left[\operatorname{Tr}\bar{a}\bar{b}ba, \operatorname{Tr}\bar{b}a\right] &= \operatorname{Tr}\bar{a}\bar{b}ba\operatorname{Tr}\bar{b}a - \operatorname{Tr}\bar{b}a\operatorname{Tr}\bar{a}\bar{b}ba \\
&= \operatorname{Tr}\bar{a}\bar{b}aa - :\operatorname{Tr}\bar{a}\bar{b}ba\operatorname{Tr}\bar{b}a: - \operatorname{Tr}\bar{b}\bar{b}ba - :\operatorname{Tr}\bar{a}\bar{b}ba\operatorname{Tr}\bar{b}a: \\
&= \operatorname{Tr}\bar{a}\bar{b}aa - \operatorname{Tr}\bar{b}^2ba,
\end{aligned}$$

from which it follows

$$\begin{aligned}
\left[\operatorname{Tr}\bar{a}\bar{b}ba, Q\right] &= \exp\left(i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{a}\bar{b}ba, \operatorname{Tr}\bar{a}b\right] + \exp\left(-i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{a}\bar{b}ba, \operatorname{Tr}\bar{b}a\right] \\
&= \exp\left(i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{a}\bar{b}bb - \operatorname{Tr}\bar{a}^2ba\right] \\
&\quad + \exp\left(-i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{a}\bar{b}aa - \operatorname{Tr}\bar{b}^2ba\right]. \qquad \text{(D.6)}
\end{aligned}$$

$$\begin{aligned}
\left[\operatorname{Tr}\bar{b}\bar{a}ab, \operatorname{Tr}\bar{a}b\right] &= \operatorname{Tr}\bar{b}\bar{a}ab\operatorname{Tr}\bar{a}b - \operatorname{Tr}\bar{a}b\operatorname{Tr}\bar{b}\bar{a}ab \\
&= -\operatorname{Tr}\bar{b}\bar{a}b^2 - \operatorname{Tr}\bar{a}^2ab,
\end{aligned}$$

$$\begin{aligned}
\left[\operatorname{Tr}\bar{b}\bar{a}ab, \operatorname{Tr}\bar{b}a\right] &= \operatorname{Tr}\bar{b}\bar{a}ab\operatorname{Tr}\bar{b}a - \operatorname{Tr}\bar{b}a\operatorname{Tr}\bar{b}\bar{a}ab \\
&= \operatorname{Tr}\bar{b}\bar{a}a^2 + \operatorname{Tr}\bar{b}^2ab,
\end{aligned}$$

from which it follows

$$\begin{aligned}
\left[\operatorname{Tr}\bar{b}\bar{a}ab, Q\right] &= \exp\left(i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{b}\bar{a}ab, \operatorname{Tr}\bar{a}b\right] + \exp\left(-i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{b}\bar{a}ab, \operatorname{Tr}\bar{b}a\right] \\
&= \exp\left(i\frac{\pi}{4}\right)\left[-\operatorname{Tr}\bar{b}\bar{a}b^2 - \operatorname{Tr}\bar{a}^2ab\right] \\
&\quad + \exp\left(-i\frac{\pi}{4}\right)\left[\operatorname{Tr}\bar{b}\bar{a}a^2 + \operatorname{Tr}\bar{b}^2ab\right]. \qquad \text{(D.7)}
\end{aligned}$$

As mentioned in the main text, the general form of Hermitian Hamiltonian is

$$\begin{aligned}
H &= \frac{1}{N}\Big[c_1\operatorname{Tr}\bar{a}^2a^2 + c_2\operatorname{Tr}\bar{b}^2b^2 + iz_1\operatorname{Tr}\bar{a}^2b^2 - iz_1^*\operatorname{Tr}\bar{b}^2a^2 \\
&\quad + c_3\operatorname{Tr}\bar{a}\bar{b}ba + c_4\operatorname{Tr}\bar{b}\bar{a}ab + z_2\operatorname{Tr}\bar{a}\bar{b}ab + z_2^*\operatorname{Tr}\bar{b}\bar{a}ba\Big].
\end{aligned}$$

With above calculation, we have

$$\begin{aligned}
N\exp\left(\frac{i\pi}{4}\right)[H,Q] &= c_1\left[i\,\mathrm{Tr}\,\bar{a}^2(ab+ba)-\mathrm{Tr}\left(\bar{a}\bar{b}+\bar{b}\bar{a}\right)a^2\right] \\
&\quad +c_2\left[i\,\mathrm{Tr}\left(\bar{b}\bar{a}-\bar{a}\bar{b}\right)b^2+\mathrm{Tr}\,\bar{b}^2(ba-ab)\right] \\
&\quad +iz_1\left[\mathrm{Tr}\,\bar{a}^2(ba-ab)-\mathrm{Tr}\left(\bar{b}\bar{a}+\bar{a}\bar{b}\right)b^2\right] \\
&\quad +z_1^*\left[\mathrm{Tr}\,\bar{b}^2(ab+ba)+\mathrm{Tr}\left(\bar{b}\bar{a}-\bar{a}\bar{b}\right)a^2\right] \\
&\quad +c_3\left[i\left(\mathrm{Tr}\,\bar{a}\bar{b}bb-\mathrm{Tr}\,\bar{a}^2ba\right)+\mathrm{Tr}\,\bar{a}\bar{b}aa-\mathrm{Tr}\,\bar{b}^2ba\right] \\
&\quad +c_4\left[i\left(-\mathrm{Tr}\,\bar{b}\bar{a}b^2-\mathrm{Tr}\,\bar{a}^2ab\right)+\mathrm{Tr}\,\bar{b}\bar{a}a^2+\mathrm{Tr}\,\bar{b}^2ab\right] \\
&\quad +z_2\left[i\left(-\mathrm{Tr}\,\bar{a}\bar{b}b^2-\mathrm{Tr}\,\bar{a}^2ab\right)+\mathrm{Tr}\,\bar{a}\bar{b}a^2-\mathrm{Tr}\,\bar{b}^2ab\right] \\
&\quad +z_2^*\left[i\left(\mathrm{Tr}\,\bar{b}\bar{a}bb-\mathrm{Tr}\,\bar{a}^2ba\right)+\mathrm{Tr}\,\bar{b}\bar{a}a^2+\mathrm{Tr}\,\bar{b}^2ba\right] \\
&= i\left(c_1-z_1-c_4-z_2\right)\mathrm{Tr}\,\bar{a}^2ab-\left(c_1-z_1^*-c_4-z_2^*\right)\mathrm{Tr}\,\bar{b}\bar{a}a^2 \\
&\quad +i\left(c_1+z_1-c_3-z_2^*\right)\mathrm{Tr}\,\bar{a}^2ba-\left(c_1+z_1^*-c_3-z_2\right)\mathrm{Tr}\,\bar{a}\bar{b}a^2 \\
&\quad -\left(c_2-z_1^*-c_4+z_2\right)\mathrm{Tr}\,\bar{b}^2ab+i\left(c_2-z_1-c_4+z_2^*\right)\mathrm{Tr}\,\bar{a}\bar{b}b^2 \\
&\quad +\left(c_2+z_1^*-c_3+z_2^*\right)\mathrm{Tr}\,\bar{b}^2ba-i\left(c_2+z_1-c_3+z_2\right)\mathrm{Tr}\,\bar{a}\bar{b}b^2.
\end{aligned}$$

Then $[H,Q]=0$ yields

$$\begin{cases}
c_1-z_1-c_4-z_2 &=0\\
c_1+z_1-c_3-z_2^* &=0\\
c_2-z_1^*-c_4+z_2 &=0\\
c_2+z_1^*-c_3+z_2^* &=0
\end{cases},$$

from which it follows (8).

E Proof of $(H'-\Delta H)\,|\text{Any trace state}\rangle=0$

ΔH and H' are defined as

$$\begin{aligned}
\Delta H &= \frac{2}{N}\,\mathrm{Tr}\left[\bar{a}\bar{b}ba+\bar{b}\bar{a}ab+\bar{a}^2a^2+\bar{b}^2b^2-\tilde{M}\right], \\
H' &= \frac{2}{N}\,\mathrm{Tr}\left(\bar{a}a\bar{a}a+\bar{b}b\bar{a}a-\bar{a}b\bar{b}a\right),
\end{aligned}$$

where

$$\tilde{M}=\mathrm{Tr}\left(\bar{a}a+\bar{b}b\right)-\frac{1}{N}\left(\mathrm{Tr}\,\bar{a}\,\mathrm{Tr}\,a+\mathrm{Tr}\,\bar{b}\,\mathrm{Tr}\,b\right).$$

We first prove that

$$N\left(H'-\Delta H\right)=\mathrm{Tr}\,G^2, \tag{E.1}$$

where the color operator G_α^β is defined as

$$G_\alpha^\beta=\left(\bar{a}a-:a\bar{a}:+\bar{b}b-:b\bar{b}:\right)_\alpha^\beta,$$

then it's sufficient to prove that

$$G_\alpha^\beta \left|\text{Any trace state}\right\rangle = 0. \tag{E.2}$$

Expanding $\operatorname{Tr} G^2$ yields

$$\operatorname{Tr} G^2 = \operatorname{Tr}\left(\bar{a}a - :a\bar{a}:\right)^2 + \operatorname{Tr}\left(\bar{b}b - :b\bar{b}:\right)^2 + 2\operatorname{Tr}\left(\bar{b}b - :b\bar{b}:\right)\left(\bar{a}a - :a\bar{a}:\right).$$

Expanding each term of the right hand side, we obtain

$$\begin{aligned}\operatorname{Tr}\left(\bar{a}a - :a\bar{a}:\right)^2 &= \operatorname{Tr}\bar{a}a\bar{a}a + \operatorname{Tr}\left(:a\bar{a}::a\bar{a}:\right) - \operatorname{Tr}\left(\bar{a}a{:}a\bar{a}{:} + :a\bar{a}{:}\bar{a}a\right)\\ &= 2\operatorname{Tr}\left(:\bar{a}a\bar{a}a:\right) + 2N\operatorname{Tr}\bar{a}a - \left(2\operatorname{Tr}\bar{a}^2a^2 + \operatorname{Tr}\bar{a}\operatorname{Tr}a\right),\end{aligned}$$

$$\begin{aligned}\operatorname{Tr}\left(\bar{b}b - :b\bar{b}:\right)^2 &= \operatorname{Tr}\left(\bar{b}b\bar{b}b + :b\bar{b}::b\bar{b}:\right) - \operatorname{Tr}\left(\bar{b}b{:}b\bar{b}{:} + :b\bar{b}{:}\bar{b}b\right)\\ &= 2N\operatorname{Tr}\bar{b}b - 2\left(\operatorname{Tr}\bar{b}^2b^2 + \operatorname{Tr}\bar{b}\operatorname{Tr}b\right),\end{aligned}$$

$$\operatorname{Tr}\left(\bar{b}b - :b\bar{b}:\right)\left(\bar{a}a - :a\bar{a}:\right) = \operatorname{Tr}\bar{b}b\bar{a}a + \operatorname{Tr}\left(:\bar{a}b\bar{b}a:\right) - \operatorname{Tr}\left(\bar{a}\bar{b}ba + \bar{b}\bar{a}ab\right).$$

It follows

$$\begin{aligned}\operatorname{Tr} G^2 &= 2\operatorname{Tr}\left(:\bar{a}a\bar{a}a: + \bar{b}b\bar{a}a + :\bar{a}b\bar{b}a:\right)\\ &\quad -2\operatorname{Tr}\left(\bar{a}\bar{b}ba + \bar{b}\bar{a}ab + \bar{a}^2a^2 + \bar{b}^2b^2\right)\\ &\quad +2N\operatorname{Tr}\left(\bar{a}a + \bar{b}b\right) - 2\operatorname{Tr}\bar{a}\operatorname{Tr}a - 2\operatorname{Tr}\bar{b}\operatorname{Tr}b\\ &= 2\operatorname{Tr}\left(\bar{a}a\bar{a}a + \bar{b}b\bar{a}a - \bar{a}b\bar{b}a\right)\\ &\quad -2\operatorname{Tr}\left(\bar{a}\bar{b}ba + \bar{b}\bar{a}ab + \bar{a}^2a^2 + \bar{b}^2b^2 - \tilde{M}\right)\\ &= N\left(H' - \Delta H\right).\end{aligned}$$

Now let's prove (E.2). Easy to check that

$$\begin{aligned}\left[\bar{a}_\alpha^\beta, G_\gamma^\delta\right] &= \bar{a}_\gamma^\beta\delta_\alpha^\delta - \delta_\gamma^\beta\bar{a}_\alpha^\delta,\\ \left[\bar{b}_\alpha^\beta, G_\gamma^\delta\right] &= \bar{b}_\gamma^\beta\delta_\alpha^\delta - \delta_\gamma^\beta\bar{b}_\alpha^\delta.\end{aligned}$$

Let X be an M-bit chain

$$X_\alpha^\beta = \left(\bar{x}_1\bar{x}_2\cdots\bar{x}_M\right)_\alpha^\beta, \quad \bar{x}_i = \bar{a}\,\text{or}\,\bar{b},$$

then

$$
\begin{aligned}
\left[X_\alpha^\beta, G_\gamma^\delta\right] &= \sum_{i=1}^{M} \left(\bar{x}_1 \cdots \bar{x}_{i-1}\right)_\sigma^\beta \left[\bar{x}_\rho^\sigma, G_\gamma^\delta\right] \left(\bar{x}_{i+1} \cdots \bar{x}_M\right)_\alpha^\rho \\
&= \sum_{i=1}^{M} \left(\bar{x}_1 \cdots \bar{x}_{i-1}\right)_\sigma^\beta \left(\left(\bar{x}_i\right)_\gamma^\sigma \delta_\rho^\delta - \delta_\gamma^\sigma \left(\bar{x}_i\right)_\rho^\delta\right) \left(\bar{x}_{i+1} \cdots \bar{x}_M\right)_\alpha^\rho \\
&= \sum_{i=1}^{M-1} \left(\bar{x}_1 \cdots \bar{x}_i\right)_\gamma^\beta \left(\bar{x}_{i+1} \cdots \bar{x}_M\right)_\alpha^\delta + \left(\bar{x}_1 \cdots \bar{x}_M\right)_\gamma^\beta \delta_\alpha^\delta \\
&\quad - \left(\sum_{i=2}^{M} \left(\bar{x}_1 \cdots \bar{x}_{i-1}\right)_\gamma^\beta \left(\bar{x}_i \cdots \bar{x}_M\right)_\alpha^\delta + \delta_\gamma^\beta \left(\bar{x}_1 \cdots \bar{x}_M\right)_\alpha^\delta\right) \\
&= \left(\bar{x}_1 \cdots \bar{x}_M\right)_\gamma^\beta \delta_\alpha^\delta - \delta_\gamma^\beta \left(\bar{x}_1 \cdots \bar{x}_M\right)_\alpha^\delta .
\end{aligned}
$$

On the other hand

$$
\left[X_\alpha^\beta, G_\gamma^\delta\right] |0\rangle = X_\alpha^\beta G_\gamma^\delta |0\rangle - G_\gamma^\delta X_\alpha^\beta |0\rangle = -G_\gamma^\delta X_\alpha^\beta |0\rangle ,
$$

from which it follows

$$
G_\gamma^\delta X_\alpha^\beta |0\rangle = \left(\delta_\gamma^\beta \left(\bar{x}_1 \cdots \bar{x}_M\right)_\alpha^\delta - \left(\bar{x}_1 \cdots \bar{x}_M\right)_\gamma^\beta \delta_\alpha^\delta\right) |0\rangle .
$$

Taking trace on indices of X yields

$$
G_\gamma^\delta \operatorname{Tr} X |0\rangle = 0.
$$

Therefore, we proved (E.2).

F Hamiltonian Eigenvalue Problem

This section proves several claims on the eigenvalue problems of $\mathcal{H}$,

$$
(\mathcal{H} - E) V = 0, \tag{F.1}
$$

where V is a vector and $\mathcal{H}$ is given by

$$
H |i\rangle = \sum_j |j\rangle \mathcal{H}_{ji}. \tag{F.2}
$$

First, let's prove the following two claims:

- If E is an eigenvalue of $\mathcal{H}$, its complex conjugate E^* is also an eigenvalue of $\mathcal{H}$.
- If E is not real, it must have $V^\dagger G V = 0$, where G is the norm matrix $G_{ij} = \langle i|j\rangle$.

Proof. Using (F.2), we have

$$\langle i| H |j\rangle = \sum_k \langle i|k\rangle \mathcal{H}_{kj} = (G\mathcal{H})_{ij} .$$

Since H is Hermitian, we also have

$$\langle i| H |j\rangle = \sum_k \mathcal{H}^\dagger_{ik} \langle k|j\rangle = \left(\mathcal{H}^\dagger G\right)_{ij} ,$$

which implies

$$G\mathcal{H} = \mathcal{H}^\dagger G. \tag{F.3}$$

Left multiplying Eq. (F.1) by G and taking complex conjugate yields

$$V^\dagger \left(\mathcal{H}^\dagger G - E^* G\right) = 0. \tag{F.4}$$

Using Eq. (F.3) and taking transpose of Eq. (F.4), we obtain

$$\left(\mathcal{H}^T - E^*\right) GV^* = 0.$$

Since $\mathcal{H}$ has the same eigenvalues as $\mathcal{H}^T$, E^* is an eigenvalue of $\mathcal{H}$.

Using (F.1), we have

$$\begin{aligned} EV^\dagger GV &= V^\dagger G\,(EV) = V^\dagger G\mathcal{H}V, \\ E^*V^\dagger GV &= \left(E^*V^\dagger\right) GV = V^\dagger \mathcal{H}^\dagger GV, \end{aligned}$$

from which it follows

$$(E - E^*)\, V^\dagger GV = V^\dagger \left(G\mathcal{H} - \mathcal{H}^\dagger G\right) V = 0.$$

Therefore, if E is not real, it must have $V^\dagger GV = 0$. □

The rest claims are related to whether G is positive-semidefinite. Let's discuss them case by case.

Positive-semidefinite G matrix

If G is a positive-semidefinite matrix, all its eigenvalues are non-negative. There exists a set of orthonormal basis spanning the trace state space. Suppose there are r trace states $|1\rangle, \cdots, |r\rangle$, with dimension $p \le r$. We can build a set of orthonormal basis $|i\}$ using a $p\times r$ matrix S,

$$\{i| = \sum_j S_{ij} \langle j| , \quad 1 \le i \le p,\, 1 \le j \le r, \tag{F.5}$$

where the basis and the matrix S satisfy

$$\{i|j\} = \sum_{k,l} S \langle k|l\rangle S = (SGS)_{ij} = \delta_{ij}.$$

In this basis, the $p \times p$ Hamiltonian matrix $\mathbf{H}$ is given by

$$\begin{aligned} \mathbf{H}_{ij} &\equiv \{i| H |j\} \\ &= \sum_{k,l} S_{ik} \langle k| H |l\rangle S^\dagger_{lj} \\ &= \sum_{k,l,m} S_{ik} \langle k|m\rangle \mathcal{H}_{ml} S^\dagger_{lj} \\ &= \left(SG\mathcal{H}S^\dagger\right)_{ij}. \end{aligned} \tag{F.6}$$

The eigenvalues of the Hamiltonian are given by the equation

$$(\mathbf{H} - E)\, W = 0, \tag{F.7}$$

where W is a p-dimensional vector. We claim:

- Every eigenvalue of $\mathbf{H}$ is an eigenvalue of $\mathcal{H}$.
- An eigenvalue E of $\mathcal{H}$ with an eigenvector V is also an eigenvalue of $\mathbf{H}$ if and only if $V^\dagger G V > 0$.

Proof. We extend the p basis vectors $|i\}$ to r vectors $|i\}'$ so that

$$\{i|j\}' = \begin{cases} \delta_{ij} & , \quad \text{if } i,j \le p \\ 0 & , \quad \text{if } i > p \text{ or } j > p \end{cases}.$$

This can be done by extending the $p \times r$ matrix S to a $r \times r$ invertible matrix R. The matrix R can be constructed as follows. We pick any invertible $r \times r$ matrix which contains S as the first p rows. For the $(p+1)$-th row vector, R_{p+1}, we calculate $R^\dagger_{p+1} G R_i$ for each $i \le p$. If $R^\dagger_{p+1} G R_i \neq 0$, we replace R_{p+1} with $R_{p+1} - \left(R^\dagger_{p+1} G R_i\right) R_i$. In this way, R_{p+1} will be orthogonal to all the first p row vectors, and since the dimension of the state space is p, $R^\dagger_{p+1} G R$ must be zero. Repeating this process for the rest rows, we obtain the invertible square matrix R.

The new basis are

$$\{i|' = R_{ij} \langle j|, \quad 1 \le i,j \le r,$$

which satisfy

$$\{i|j\}' = \left(RGR^\dagger\right)_{ij} = \left(\mathbf{I}_p \oplus \mathbf{O}_{r-p}\right)_{ij}, \tag{F.8}$$

where $\mathbf{I}_p$ is the $p \times p$ identity matrix and $\mathbf{O}_{r-p}$ the $(r-p)\times(r-p)$ zero matrix. In the new basis, we define a matrix,

$$\mathbb{H} = RG\mathcal{H}R^{\dagger} = \mathbf{H} \oplus \mathbf{O}_{r-p}. \tag{F.9}$$

Clearly, if E is an eigenvalue of $\mathbf{H}$ with eigenvector W, it is also an eigenvalue of $\mathbb{H}$,

$$(\mathbb{H} - E)\, W' = 0, \tag{F.10}$$

with the eigenvector W' satisfying

$$W'_i = \begin{cases} W_i & , \quad \text{if } 1 \le i \le p \\ 0 & , \quad \text{if } p < i \le r \end{cases}. \tag{F.11}$$

With relation (F.9) and (F.8), the left hand side of Eq. (F.10) can be expressed as

$$\begin{aligned} (\mathbb{H} - E)\, W' &= RG\mathcal{H}R^{\dagger}W' - E\left(\mathbf{I}_p \oplus \mathbf{O}_{r-p}\right) W' \\ &= RG\mathcal{H}R^{\dagger}W' - ERGR^{\dagger}W' \\ &= R\left(\mathcal{H}^{\dagger} - E\right) GR^{\dagger}W' \\ &= R\left(\mathcal{H}^{\dagger} - E\right) R^{-1}\left(\mathbf{I}_p \oplus \mathbf{O}_{r-p}\right) W' \\ &= R\left(\mathcal{H}^{\dagger} - E\right) R^{-1}W' \end{aligned} \tag{F.12}$$

Since R is invertible, we obtain

$$\left(\mathcal{H}^{\dagger} - E\right) R^{-1}W' = 0.$$

$R^{-1}W'$ cannot be zero as R^{-1} is invertible and $W' \neq 0$. As E is real, E is an eigenvalue of $\mathcal{H}^{\dagger}$ and $\mathcal{H}$.

Conversely, if E is a eigenvalue of $\mathcal{H}$ with eigenvector V, we have

$$RG\left(\mathcal{H} - E\right) V = 0.$$

The right hand side can be expressed as

$$\begin{aligned} RG\left(\mathcal{H} - E\right) V &= RG\left(\mathcal{H}R^{\dagger} - ER^{\dagger}\right) R^{\dagger -1}V \\ &= \left(\mathbb{H} - E\left(\mathbf{I}_p \oplus \mathbf{O}_{r-p}\right)\right) R^{\dagger -1}V \\ &= \left(\mathbb{H} - E\right)\left(\mathbf{I}_p \oplus \mathbf{O}_{r-p}\right) R^{\dagger -1}V, \end{aligned}$$

from which it follows

$$\left(\mathbb{H} - E\right)\left(\mathbf{I}_p \oplus \mathbf{O}_{r-p}\right) R^{\dagger -1}V = 0. \tag{F.13}$$

To let E be an eigenvalue of $\mathbb{H}$, we need $W' \equiv \left(\mathbf{I}_p \oplus \mathbf{O}_{r-p}\right) R^{\dagger -1}V$ to be a nonzero vector. By calculating the norm of W',

$$\begin{aligned} W'^{\dagger}W' &= V^{\dagger}R^{\dagger -1}\left(\mathbf{I}_p \oplus \mathbf{O}_{r-p}\right) R^{\dagger -1}V \\ &= V^{\dagger}GV, \end{aligned}$$

we find that E is eigenvalue of $\mathbb{H}$ if and only if $V^{\dagger}GV > 0$. Under this constraint, as $\mathbb{H} = \mathbf{H} \oplus \mathbf{O}_{r-p}$, E is also an eigenvalue of $\mathbf{H}$. □

Non-positive-semidefinite G

If G is not positive-semidefinite matrix, at least one of its eigenvalues are negative. There does not exist an orthonormal basis in trace state space. Suppose the $r \times r$ matrix G has p positive eigenvalues, q negative eigenvalues, and $s = r - p - q$ zero eigenvalues. We can properly choose a unitary matrix R so that the new basis $|i\}'$ satisfying

$$\{i|j\}' = \left(RGR^{\dagger}\right)_{ij} = \left(\mathbf{I}_p \oplus -\mathbf{I}_q \oplus \mathbf{O}_s\right)_{ij},$$

where $|1\}', \cdots, |p\}'$ are positive norm-square states, $|p+1\}', \cdots, |p+q\}'$ negative norm-square states, and $|p+q+1\}', \cdots, |r\}'$ zero norm states. The negative norm-square states are also called ghost states. The existence of ghost state implies the Hamiltonian is not unitary.

In analogy with (F.6) and (F.9), we define $\mathbf{H}$ and $\mathbb{H}$ by

$$\mathbf{H}_{ij} = \{i| H |j\}, \quad 1 \leq i, j \leq p + q,$$

and

$$\mathbb{H} = RG\mathcal{H}R^{\dagger} = \mathbf{H} \oplus \mathbf{O}_s.$$

We claim:

- If E is an eigenvalue of $\mathbf{H}$ with eigenvector W, it is an eigenvalue of $\mathcal{H}$ when W does not couple with any ghost state.

- If E is an eigenvalue of $\mathcal{H}$ with eigenvector V, it is an eigenvalue of $\mathbf{H}$ when $E = 0$ or $V^{\dagger}\text{abs}(G)\, V = V^{\dagger}GV > 0$, where the function abs is defined as

$$\text{abs}(G) = U^{\dagger} \begin{pmatrix} |g_1| & & & \\ & |g_2| & & \\ & & \ddots & \\ & & & |g_n| \end{pmatrix} U,$$

  with

$$G = U^{\dagger} \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_n \end{pmatrix} U$$

  be the eigendecomposition of G.

The condition $V^{\dagger}\text{abs}(G)\, V = V^{\dagger}GV$ implies, in the basis where G is diagonal, V does not couple with any ghost state. According to our numerical calculation, G is not positive-semidefinite only when $N < M$ and is not integer. The numerical calculation shows, except the $E = 0$ case, the condition $V^{\dagger}\text{abs}(G)\, V = V^{\dagger}GV$ is usually not satisfied when G is not positive-semidefinite. The proof of the claims is given as follows.

Proof. If E is an eigenvalue of $\mathbf{H}$ with eigenvector W, E is also an eigenvalue of $\mathbb{H}$ with eigenvector defined as

$$W_i' = \begin{cases} W_i & , \quad \text{if } 1 \leq i \leq p+q \\ 0 & , \quad \text{if } p+q < i \leq r \end{cases}. \tag{F.14}$$

In analogy with (F.12), we have

$$\begin{aligned}
(\mathbb{H} - E)\, W' &= RG\mathcal{H}R^\dagger W' - E\,(\mathbf{I}_{p+q} \oplus \mathbf{O}_s)\, W' \\
&= RG\mathcal{H}R^\dagger W' - E\left(RGR^\dagger + 2\mathbf{O}_p \oplus \mathbf{I}_q \oplus \mathbf{O}_s\right) W' \\
&= R\,(\mathcal{H} - E)\, GR^\dagger W' - 2E\,(\mathbf{O}_p \oplus \mathbf{I}_q \oplus \mathbf{O}_s)\, W' \\
&= R\,(\mathcal{H} - E)\, R^{-1}\,(\mathbf{I}_p \oplus -\mathbf{I}_q \oplus \mathbf{O}_s)\, W' - 2E\,(\mathbf{O}_p \oplus \mathbf{I}_q \oplus \mathbf{O}_s)\, W'.
\end{aligned}$$

If the following condition are satisfied, E is an eigenvalue of $\mathcal{H}$:

$$\begin{cases} (\mathbf{O}_p \oplus \mathbf{I}_q \oplus \mathbf{O}_s)\, W' & = 0 \\ (\mathbf{I}_p \oplus -\mathbf{I}_q \oplus \mathbf{O}_s)\, W' & \neq 0 \end{cases}.$$

With (F.14) and $W \neq 0$, it implies that if

$$(\mathbf{O}_p \oplus \mathbf{I}_q)\, W = 0, \tag{F.15}$$

E is an eigenvalue of $\mathcal{H}$. Eq. (F.15) is a constraint under which the eigenvector does not couple with the ghost states.

Conversely, if E is an eigenvalue of $\mathcal{H}$ with eigenvector V,

$$RG\,(\mathcal{H} - E)\, V = 0.$$

The left hand side of the equation can be expressed as

$$\begin{aligned}
RG\,(\mathcal{H} - E)\, V &= RG\mathcal{H}R^\dagger R^{\dagger -1} V - ERGR^\dagger R^{\dagger -1} V \\
&= \mathbb{H}R^{\dagger -1} V - E\,(\mathbf{I}_p \oplus -\mathbf{I}_q \oplus \mathbf{O}_s)\, R^{\dagger -1} V \\
&= (\mathbb{H} - E)\,(\mathbf{I}_{p+q} \oplus \mathbf{O}_s)\, R^{\dagger -1} V + 2E\,(\mathbf{O}_p \oplus \mathbf{I}_q \oplus \mathbf{O}_s)\, R^{\dagger -1} V \\
&= (\mathbb{H} - E)\, W' + 2EW'',
\end{aligned}$$

where we have defined

$$\begin{aligned}
W' &\equiv (\mathbf{I}_{p+q} \oplus \mathbf{O}_s)\, R^{\dagger -1} V, \\
W'' &\equiv (\mathbf{O}_p \oplus \mathbf{I}_q \oplus \mathbf{O}_s)\, R^{\dagger -1} V.
\end{aligned}$$

If $E = 0$, or $W' \neq 0$ and $W'' = 0$, E is an eigenvalue of $\mathbb{H}$. $W' \neq 0$ implies

$$\begin{aligned}
W'^\dagger W' &= V^\dagger R^{-1}\,(\mathbf{I}_{p+q} \oplus \mathbf{O}_s)\, R^{\dagger -1} V \\
&= V^\dagger R^{-1}\,(\mathbf{I}_p \oplus -\mathbf{I}_q \oplus \mathbf{O}_s)\, R^{\dagger -1} V \\
&= V^\dagger R^{-1} RGR^\dagger R^{\dagger -1} V \\
&= V^\dagger G V > 0,
\end{aligned} \tag{F.16}$$

where we use the constraint $W'' = 0$ in the second equality. With the equation

$$2\left(\mathbf{O}_p \oplus \mathbf{I}_q \oplus \mathbf{O}_s\right) = \left(RGR^\dagger\right)^2 - RGR^\dagger,$$

$W'' = 0$ is equivalent to

$$V^\dagger \left(GR^\dagger RG - G\right) V = 0. \tag{F.17}$$

Combining constraints (F.16) and (F.17), we find if

$$V^\dagger GR^\dagger RGV = V^\dagger GV > 0,$$

E is an eigenvalue of $\mathbf{H}$.

The matrix $GR^\dagger RG$ seems to be dependent on R, but actually, it only depends on G. Indeed, any unitary transformation $R \to UR$ does not change $GR^\dagger RG$. In general, if the eigendecomposition of G is

$$G = U^\dagger \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_n \end{pmatrix} U, \quad U^\dagger U = \mathbf{I},$$

we can choose R as

$$R_i = \begin{cases} \frac{1}{\sqrt{|g_i|}} U_i, & \text{if } g_i \neq 0 \\ U_i, & \text{if } g_i = 0 \end{cases}.$$

Then, we obtain

$$GR^\dagger RG = U^\dagger \begin{pmatrix} |g_1| & & & \\ & |g_2| & & \\ & & \ddots & \\ & & & |g_n| \end{pmatrix} U,$$

which clearly only depends on G. □

G Algorithms

The numerical computation is performed by C++ and matlab program. We use C++ program to generate the norm matrices and $\mathcal{H}$ matrices, then use matlab to find eigenvalues and eigenstates. Here we introduce the algorithms for generating trace states, calculating norm matrices, and building $\mathcal{H}$ matrices.

Generate trace states Trace states are represented by integer numbers. The bosonic and fermionic creation operators are mapped to 0 and 1 respectively. Then an M-bit single trace state is mapped an M-bit binary number and a multiple trace state is an array of integers. Because of the cyclic symmetry, a single trace state corresponds to several integers. Among these integers we choose the smallest integer. For example, $\mathrm{Tr}\,\bar{a}\bar{b}\bar{b}\,|0\rangle$ is mapped to $(011)_2 = 3$ rather than $(110)_2 = 6$. We then go through all integers between 0 and $2^M - 1$. A number is a single trace state only when it meets two conditions:

- There is no cyclic rotation on this integer producing a smaller integer.
- The corresponding trace state is nonvanishing. A trace state is vanishing if it can be partitioned into even number of identical consecutive parts each of which has odd number of $\bar{b}$. For example, $\mathrm{Tr}\,\bar{b}\bar{b}\bar{b}\bar{b}\,|0\rangle$ vanishes as it can be partitioned into four $\bar{b}$s.

After generating all single trace states, we can build multiple trace states out of single trace states. The procedure is similar to the recursive relation (B.11) for calculating the number of trace states.

Calculate norm matrices To build norm matrix, we need to calculate $\langle i|j\rangle$ for each pair of states i, j. The norm can be calculated as follows. If two M-bit states i, j do not have the same number of $\bar{b}$, then $\langle i|j\rangle = 0$. Otherwise, if both have n fermionic operators, there are $n!\,(M-n)!$ ways to contract their color indices. Take $\mathrm{Tr}\,\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle$ and $\mathrm{Tr}\,\bar{a}\,\mathrm{Tr}\,\bar{a}\bar{b}\bar{b}\,|0\rangle$ as an example. We first write the states as

$$\begin{aligned}\mathrm{Tr}\,\bar{a}\bar{a}\bar{b}\bar{b}\,|0\rangle &= \bar{a}^{\beta}_{\alpha}\bar{a}^{\gamma}_{\beta}\bar{b}^{\rho}_{\gamma}\bar{b}^{\alpha}_{\rho}\,|0\rangle\,,\\ \mathrm{Tr}\,\bar{a}\,\mathrm{Tr}\,\bar{a}\bar{b}\bar{b}\,|0\rangle &= \bar{a}^{i}_{i}\bar{a}^{k}_{j}\bar{b}^{l}_{k}\bar{b}^{j}_{l}\,|0\rangle\,.\end{aligned}$$

Using the commutation and anticommutation relations, we can expand the norm into $2!\times 2! = 4$ terms,

$$\begin{aligned}\langle 0|\,\mathrm{Tr}\,bbaa\,\mathrm{Tr}\,\bar{b}\,\mathrm{Tr}\,\bar{a}\bar{a}\bar{b}\,|0\rangle &= \langle 0|\,b^{\rho}_{\alpha}b^{\gamma}_{\rho}a^{\beta}_{\gamma}a^{\alpha}_{\beta}\bar{a}^{i}_{i}\bar{a}^{k}_{j}\bar{b}^{l}_{k}\bar{b}^{j}_{l}\,|0\rangle\\ &= \delta^{\beta}_{j}\delta^{k}_{\gamma}\delta^{\alpha}_{i}\delta^{i}_{\beta}\left(\delta^{\rho}_{l}\delta^{j}_{\alpha}\delta^{\gamma}_{k}\delta^{l}_{\rho} - \delta^{\rho}_{k}\delta^{l}_{\alpha}\delta^{\gamma}_{l}\delta^{j}_{\rho}\right)\\ &\quad+\delta^{\beta}_{i}\delta^{i}_{\gamma}\delta^{\alpha}_{j}\delta^{k}_{\beta}\left(\delta^{\rho}_{l}\delta^{j}_{\alpha}\delta^{\gamma}_{k}\delta^{l}_{\rho} - \delta^{\rho}_{k}\delta^{l}_{\alpha}\delta^{\gamma}_{l}\delta^{j}_{\rho}\right)\,.\end{aligned}$$

The sign of each term is determined by how many times a swap occurs among b and $\bar{b}$: odd (even) number of swaps produces a negative (positive) sign. The first term can be written as

$$\delta^{\beta}_{j}\delta^{k}_{\gamma}\delta^{\alpha}_{i}\delta^{i}_{\beta}\delta^{\rho}_{l}\delta^{j}_{\alpha}\delta^{\gamma}_{k}\delta^{l}_{\rho} = \left(\delta^{\beta}_{j}\delta^{i}_{\beta}\delta^{\alpha}_{i}\delta^{j}_{\alpha}\right)\left(\delta^{k}_{\gamma}\delta^{\gamma}_{k}\right)\left(\delta^{\rho}_{l}\delta^{l}_{\rho}\right)\,,$$

where Kronecker delta functions are put into three groups. The contraction of the indices in each group producing a factor of N, which implies the first term equals to N^3. Repeat the procedure, we obtain

$$\langle 0|\,\mathrm{Tr}\,bbaa\,\mathrm{Tr}\,\bar{b}\,\mathrm{Tr}\,\bar{a}\bar{a}\bar{b}\,|0\rangle = 2N^3 - 2N.$$

Finally, the result is normalized by multiplying $1/N^4$, which yields $2/N - 2/N^3$.

Our algorithm simply simulates the procedure, hence, has $\mathcal{O}(M!)$ time complexity to calculate each entry of a norm matrix. For numerical computation of higher M, we need to improve time complexity significantly.

Build $\mathcal{H}$ matrices To build $\mathcal{H}$ matrices, we need to calculate the action of trace operators on trace states. Let's take an example that the trace operator is $\operatorname{Tr} Aab$, where A is any creation operator chain. To calculate $\operatorname{Tr} Aab \operatorname{Tr} S |0\rangle$, we need to find all possible ways to partition S into the forms $B\bar{a}C\bar{b}D$ or $B\bar{b}C\bar{a}D$, where B, C, D are any creation operator chains. Each partition corresponds to one way to contract the indices among annihilation and creation operators. The results of these two contraction schemes are

$$\operatorname{Tr} Aab \operatorname{Tr} B\bar{b}C\bar{a}D |0\rangle \rightarrow (-1)^{\pi\left(AB\bar{b}CD \rightarrow A\bar{b}DBC\right)} \operatorname{Tr} ADB \operatorname{Tr} C |0\rangle , \tag{G.1}$$

$$\operatorname{Tr} Aab \operatorname{Tr} B\bar{a}C\bar{b}D |0\rangle \rightarrow (-1)^{\pi\left(ABC\bar{b}D \rightarrow A\bar{b}CDB\right)} \operatorname{Tr} AC \operatorname{Tr} DB |0\rangle , \tag{G.2}$$

where $\pi\left(AB\bar{b}CD \rightarrow A\bar{b}DBC\right)$ denotes the number of swaps occurring among the fermionic operators as the chain being reordered from$AB\bar{b}CD$ to $A\bar{b}DBC$. Let $f(A)$ denote the number of $\bar{b}$ in A, then

$$\pi\left(AB\bar{b}CD \rightarrow A\bar{b}DBC\right) = f(B) + f(D)\, f(BC) \, .$$

The complete result of $\operatorname{Tr} Aab \operatorname{Tr} S |0\rangle$ can be written as

$$\begin{aligned} \operatorname{Tr} Aab \operatorname{Tr} S |0\rangle &= \sum_{B\bar{b}C\bar{a}D=S} (-1)^{\pi\left(AB\bar{b}CD \rightarrow A\bar{b}DBC\right)} \operatorname{Tr} ADB \operatorname{Tr} C |0\rangle \\ &\quad + \sum_{B\bar{a}C\bar{b}D=S} (-1)^{\pi\left(ABC\bar{b}D \rightarrow A\bar{b}CDB\right)} \operatorname{Tr} AC \operatorname{Tr} DB |0\rangle . \end{aligned}$$

In analogy with (G.1) and (G.2), for two trace states, we have

$$\begin{aligned} \operatorname{Tr} Aab \operatorname{Tr} B\bar{a}C \operatorname{Tr} D\bar{b}E |0\rangle &\rightarrow (-1)^{\pi\left(ABCD\bar{b}E \rightarrow A\bar{b}CBED\right)} \operatorname{Tr} ACBED |0\rangle , \\ \operatorname{Tr} Aab \operatorname{Tr} B\bar{b}C \operatorname{Tr} D\bar{a}E |0\rangle &\rightarrow (-1)^{\pi\left(AB\bar{b}CDE \rightarrow A\bar{b}EDCB\right)} \operatorname{Tr} AEDCB |0\rangle . \end{aligned}$$

The algorithm takes $\mathcal{O}(M^2)$ to calculate one row of the $\mathcal{H}$ matrix. Since there are about 2^M trace states, it takes $\mathcal{O}\left(M^2 2^M\right)$ to build a $\mathcal{H}$ matrix, which is much faster than building a norm matrix.

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