CPT, CP, and C transformations of fermions, and their consequences, in theories with B-L violation

Susan Gardner and Xinshuai Yan

Phys. Rev. D **93**, 096008 — Published 20 May 2016

DOI: 10.1103/PhysRevD.93.096008

INT-PUB-15-074

CPT, CP, and C transformations of fermions, and their consequences, in theories with B-L violation

Susan Gardner* and Xinshuai Yan†

Department of Physics and Astronomy,

University of Kentucky, Lexington, Kentucky 40506-0055 USA

Abstract

We consider the transformation properties of fermions under the discrete symmetries CPT, CP, and C in the presence of B-L violation. We thus generalize the analysis of the known properties of Majorana neutrinos, probed via neutrinoless double beta decay, to include the case of Dirac fermions with B-L violation, which can be probed via neutron-antineutron oscillations. We show that the resulting CPT phase has implications for the interplay of neutron-antineutron oscillations with external fields and sources and consider the differences in the Majorana dynamics of neutrinos and neutrons in the context of theories with self-conjugate isospin $I = 0$ and $I = 1/2$ fields.

* gardner@pa.uky.edu

† xinshuai.yan@uky.edu

I. INTRODUCTION

In theories with B-L violation the possibility of Majorana fermions, which are particles that are their own antiparticles, emerges. Such particles, as long known, have special transformation properties under the discrete symmetries CPT and CP, as well as C [1–4]. Their observation would reveal the existence of dynamics beyond the Standard Model (SM). B-L violation can appear in theories of quarks, that carry baryon number B, and/or leptons, that carry lepton number L, though the possibility of Majorana neutrinos has had the most scrutiny. This is because a crisp dichotomy can arise in the theoretical description of a massive neutrino: it can be either a Dirac or a Majorana particle, in that its mass can emerge from either Dirac or Majorana mass terms. The neutrino mass could also emerge from mass terms of both types [5], though even if the neutrino were pseudo-Dirac [6], so that its Dirac mass would give a predominant contribution to its total mass, the mass eigenstates would be Majorana [7, 8]. Moreover, the observation of neutrinoless double beta decay [9] would establish the existence of the Majorana neutrino because the existence of B-L violation would generate an effective Majorana mass term even if such a mass term were not explicitly present [10].

The seminal papers of Kayser and Goldhaber [1] and Kayser [2] concern the analysis of the special CPT, CP, and C properties of Majorana fields and states and the implications of those properties for neutrinoless double beta decay. Earlier, Carruthers [3], as well as Feinberg and Weinberg [4], determined the existence of phase restrictions in the P and TC transformations, with Carruthers [3] analyzing the detailed properties of particle self-conjugate multiplets. These works contain the implicit assumption that phase restrictions are associated with particle self-conjugate fields, or, alternatively, that B-L symmetry is only broken through the appearance of a Majorana field. In this paper we generalize this earlier work to the treatment of Dirac fields with B-L violation. In order to preserve the symmetry restrictions found in the Majorana case, we find that the phases associated with the action of the discrete symmetries on fermion fields must be restricted in order to address the symmetry transformations of fermion interactions with B-L violation. In the absence of B-L violation, the phases and thus the phase restrictions we describe have no physical impact, so that our considerations are specific to theories with B-L violation.

Our analysis is pertinent to theories of both leptons and quarks with B-L violation, where we note that the possibility of B-L violation in the quark sector can be probed through neutron-antineutron (n-$\bar{n}$) oscillations. The n-$\bar{n}$ system with B-L violation bears direct comparison to the possibility of a pseudo-Dirac neutrino. We recall that, in the SM, the neutron and antineutron are Dirac fermions, as are the quarks that comprise them, because quantum chromodynamics (QCD), the accepted theory of the strong interactions, is a SU(3) gauge theory with a complex fundamental representation [11]. The empirical success of the quark model, which explains the significant magnetic moments of the neutron and proton, suggests that the Dirac mass of the neutron dominates its measured mass. Indeed, the current empirical limit on the free n-$\bar{n}$ oscillation time limits the Majorana mass to $\delta m = (\tau_{n\bar{n}})^{-1} \leq 6 \times 10^{-29}\,\mathrm{MeV}$ at 90% C.L. [12]. We will find that the phase restrictions on the discrete symmetry transformations in the presence of B-L violation have important implications for the interplay of n-$\bar{n}$ oscillations with external fields and sources; in particular, they resolve the conflict between Refs. [13, 14]. Generally, this interplay is key to improving the sensitivity of future experimental searches [15, 16].

Herewith we sketch an outline of the body of the paper. We begin, in Sec. II, by recapping the Majorana phase constraints [1, 2] before building a Majorana field from Dirac fields in order to study the discrete symmetry transformations of the Dirac fields in the presence of B-L violation. We find, as a result, constraints on the phases in the discrete symmetry transformations of fermion fields. With these in place we then turn, in Sec. III, to the CPT and CP transformation properties of B-L violating operators. Remarkably B-L violating operators can be constructed that are either even or odd under CPT, even though all the operators are explicitly Lorentz invariant. The CPT phase restriction we derive changes the sign of the B-L violating operators under CPT. With it in place, we find that the CPT-odd operators vanish upon use of fermion anticommutation relations, so that the CPT theorem is respected [17]. We consider the implications of these results in regards to the interplay of n-$\bar{n}$ oscillations with external fields and sources, as well as whether their observation can connote a breaking of CP symmetry, in Sec. IV. Noting the failure of locality in theories of self-conjugate fields with half-integral isospin [18–21], we consider the compatibility of the appearance of B-L violation with the SM in Sec. V, before offering a final summary.

II. MAJORANA PHASE CONSTRAINTS

To determine the phase-factor restrictions on the discrete symmetry transformations that emerge in the Majorana case, we follow Refs. [1, 2] and replace the Dirac field ψ in the discrete symmetry transformations of Eqs. (40,41,42) with a general Majorana field ψ_m, for which the plane-wave expansion is given by

$$\psi_m(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2E}} \sum_s \left\{ f(\mathbf{p},s)u(\mathbf{p},s)e^{-ip\cdot x} + \lambda f^\dagger(\mathbf{p},s)v(\mathbf{p},s)e^{ip\cdot x} \right\} . \tag{1}$$

We note that $f^\dagger$ and f denote the creation and annihilation operators for the Majorana particle of interest. The unimodular quantity λ is called a creation phase factor; it may be present, in general, and can be chosen arbitrarily. We refer the reader to Appendix A for a summary of our definitions, conventions, and other useful basic results.

Noting the C transformation

$$\mathbf{C}\psi_m(x)\mathbf{C}^{-1} = i\eta_c\gamma^2\psi_m^*(x) \tag{2}$$

and applying the Majorana relation,

$$i\gamma^2\psi_m^*(x) = \lambda^*\psi_m(x), \tag{3}$$

yields

$$\mathbf{C}\psi_m(x)\mathbf{C}^{-1} = \eta_c\lambda^*\psi_m(x) \tag{4}$$

and thus

$$\mathbf{C}f(\mathbf{p},s)\mathbf{C}^{-1} = \eta_c\lambda^* f(\mathbf{p},s)\,, \tag{5}$$

$$\mathbf{C}f^\dagger(\mathbf{p},s)\mathbf{C}^{-1} = \eta_c\lambda^* f^\dagger(\mathbf{p},s)\,. \tag{6}$$

Since $\mathbf{C}$ is a unitary operator, taking the Hermitian conjugate of either relation reveals that $\eta_c^*\lambda$ is real. Noting the CP transformation

$$\mathbf{CP}\psi_m(t,\mathbf{x})(\mathbf{CP})^{-1} = i\eta_p\eta_c\gamma^0\gamma^2\psi_m^*(t,-\mathbf{x}) \tag{7}$$

and Eq. (3) yields

$$\mathbf{CP}\psi_m(t,\mathbf{x})(\mathbf{CP})^{-1} = \eta_p\eta_c\lambda^*\gamma^0\psi_m(t,-\mathbf{x}) \tag{8}$$

and thus

$$\mathbf{CP}f(\mathbf{p},s)(\mathbf{CP})^{-1} = \eta_c\eta_p\lambda^* f(-\mathbf{p},s)\,, \tag{9}$$

$$\mathbf{CP}f^\dagger(\mathbf{p},s)(\mathbf{CP})^{-1} = -\eta_c\eta_p\lambda^* f^\dagger(-\mathbf{p},s)\,. \tag{10}$$

Since $\mathbf{CP}$ is a unitary operator, taking the Hermitian conjugate of either relation shows that $\eta_p^*\eta_c^*\lambda$ must be imaginary. We have already established that $\eta_c^*\lambda$ is real, so that η_p^* itself must be imaginary. Under T we have

$$\mathbf{T}\psi_m(t,\mathbf{x})\mathbf{T}^{-1} = \eta_t\gamma^1\gamma^3\psi_m(-t,\mathbf{x}), \tag{11}$$

which yields

$$\mathbf{T}f(\mathbf{p},s)\mathbf{T}^{-1} = s\eta_t f(-\mathbf{p},-s), \tag{12}$$

$$\mathbf{T}f^\dagger(\mathbf{p},s)\mathbf{T}^{-1} = s\eta_t\lambda^2 f^\dagger(-\mathbf{p},-s). \tag{13}$$

Since $\mathbf{T}$ is an antiunitary operator, we write $\mathbf{T} = KU_t$, where U_t is a unitarity operator and K denotes complex conjugation. Then taking the Hermitian conjugate of either relation shows that $\eta_t\lambda$ must be real. Finally we note the CPT transformation of ψ_m

$$\mathbf{CPT}\psi_m(x)(\mathbf{CPT})^{-1} = -\eta_c\eta_p\eta_t\gamma^5\psi_m^*(-x)\,, \tag{14}$$

with $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, which yields,

$$\xi f(\mathbf{p},s)\xi^{-1} = s\lambda^*\eta_c\eta_p\eta_t f(\mathbf{p},-s)\,, \tag{15}$$

$$\xi f^\dagger(\mathbf{p},s)\xi^{-1} = -s\lambda\eta_c\eta_p\eta_t f^\dagger(\mathbf{p},-s)\,, \tag{16}$$

where we employ $\mathbf{CPT} \equiv \xi$. Since ξ is an antiunitary operator, we write $\xi = KU_{cpt}$, where U_{cpt} denotes a unitarity operator. Consequently, taking the Hermitian conjugate of either relation reveals that $\eta_c\eta_p\eta_t$ is pure imaginary. Since we have already established that η_p is imaginary, we see that $\eta_c\eta_t$ must also be real — and note that just this emerges from the analysis of the TC transformation as well. In contrast, the combination $\eta_c\eta_p$ itself is unconstrained. In summary, we have found all the restrictions on the phases that appear in C, P, T, and combinations thereof, and our results are equivalent to those in Refs. [1–3].

We now turn to the particular case of a Majorana field that is constructed from Dirac fields. Given Eqs. (40,41,42), the existence of phase restrictions in the application of C, P, and T to Dirac fields themselves may already be self-evident. However, we confirm this through explicit calculation. Thus we build ψ_m from the linear combination $a\psi + b\mathbf{C}\psi\mathbf{C}^{-1}$ in which a and b are complex numbers to be determined. Under C, ψ_m becomes

$$\mathbf{C}\psi_m\mathbf{C}^{-1} = \frac{b}{a}(a\psi + \frac{a^2}{b}\mathbf{C}\psi\mathbf{C}^{-1})\,.$$

Since ψ_m is a Majorana field, $\mathbf{C}\psi_m\mathbf{C}^{-1} \propto \psi_m$, yielding the condition $a^2 = b^2$, i.e., $a = \pm b$. After imposing a normalization condition on ψ_m, we find

$$\psi_{m\pm}(x) = \frac{1}{\sqrt{2}}(\psi(x) \pm \mathbf{C}\psi(x)\mathbf{C}^{-1})\,, \tag{17}$$

which has the plane-wave expansion

$$\begin{aligned}\psi_{m\pm} = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2E}} \sum_s \{ &\frac{1}{\sqrt{2}}[b(\mathbf{p},s) \pm \eta_c d(\mathbf{p},s)]u(\mathbf{p},s)e^{-ip\cdot x} \\ &+ \frac{1}{\sqrt{2}}[d^\dagger(\mathbf{p},s) \pm \eta_c b^\dagger(\mathbf{p},s)]v(\mathbf{p},s)e^{ip\cdot x}\}.\end{aligned} \tag{18}$$

Comparing with Eq. (1), we define

$$w_{m\pm}(\mathbf{p},s) \equiv \frac{1}{\sqrt{2}}[b(\mathbf{p},s) \pm \eta_c d(\mathbf{p},s)], \tag{19}$$

and observe that the second term can be written as

$$\frac{1}{\sqrt{2}}(d_s^\dagger(\mathbf{p}) \pm \eta_c b_s^\dagger(\mathbf{p})) = \pm\eta_c w_{m\pm}^\dagger(s,\mathbf{p}), \tag{20}$$

so that we can rewrite $\psi_{m\pm}$ in a simple way

$$\psi_{m\pm}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2E}} \sum_s \left\{ w_\pm(\mathbf{p},s)u(\mathbf{p},s)e^{-ip\cdot x} \pm \eta_c w_\pm^\dagger(\mathbf{p},s)v(\mathbf{p},s)e^{ip\cdot x} \right\}. \tag{21}$$

Comparing with Eq. (1), we find that λ is no longer arbitrary; rather, $\lambda = \pm\eta_c$. Since $\psi_{m\pm}$ is a Majorana field, our earlier lines of reasoning, as well as our conclusions, should still apply. Note, e.g., that applying the C transformation to $\psi_{m\pm}$ yields

$$\mathbf{C}\psi_{m\pm}(x)\mathbf{C}^{-1} = \frac{1}{\sqrt{2}}[(\eta_c i\gamma^2)\psi^*(x) \pm \psi(x)] = \pm\psi_{m\pm}\,, \tag{22}$$

which is automatically consistent with our earlier conclusion that $\eta_c^*\lambda$ is real, since $\lambda = \pm\eta_c$. Turning to the explicit CP and CPT transformation properties of $\psi_{m\pm}$ we confirm our earlier results that both $\eta_c^*\eta_p^*\lambda$ (or η_p) and $\eta_c\eta_p\eta_t$ are imaginary — and thus that $\eta_c\eta_t$ is real. Interestingly, the study of T and CT (or TC) transformations lead to no further phase restrictions. Under T, $\psi_{m\pm}$ becomes

$$\mathbf{T}\psi_{m\pm}(t,\mathbf{x})\mathbf{T}^{-1} = \frac{1}{\sqrt{2}}\{\eta_t\gamma^1\gamma^3\psi(-t,\mathbf{x}) \pm (\eta_c\eta_t)^*(i\gamma^2)^*\gamma^1\gamma^3\psi^*(-t,\mathbf{x})\}\,, \tag{23}$$

but noting Eq. (42) this should be equivalent to

$$\eta_t\gamma^1\gamma^3\psi_{m\pm}(-t,\mathbf{x}) = \frac{1}{\sqrt{2}}\{\eta_t\gamma^1\gamma^3\psi(-t,\mathbf{x}) \pm i\eta_t\eta_c\gamma^1\gamma^3\gamma^2\psi^*(-t,\mathbf{x})\}\,; \tag{24}$$

and we conclude that $\eta_c\eta_t$ is real. Upon applying CT (or TC) to $\psi_{m\pm}$ we find just the same constraint: that $\eta_c\eta_t$ must be real.

In summary, we have found that in order to preserve the phase restrictions found in the Majorana case, the phases in the discrete symmetry transformations of fermion fields must themselves be restricted. Specifically we have found that η_p must be imaginary and that the combination $\eta_c\eta_t$ must be real. As a result, we find that $\mathbf{P}^2\psi(x)\mathbf{P}^{-2} = -\psi(x)$. Furthermore, we find that although $\eta_c\eta_p\eta_t$ is pure imaginary the combination $\eta_c\eta_p$ is unconstrained.

Before proceeding we note that the phase restrictions we have found are not restricted to our particular choice of gamma matrix representation and that certain aspects thereof apply to the transformations of two-component (Majorana) fields as well. For definiteness we consider representations in which $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ is satisfied, so that Eq. (40) holds [22]. This subset of possible representations includes the Weyl and Majorana representations as well, so that our choice spans all the commonly used ones. Moreover, unitary transformations exist that connect all the representations for which Eq. (40) holds [23]. For completeness, we present the particular phase restrictions associated with the discrete-symmetry transformations of two-component Majorana fields in Appendix B.

III. THEORIES OF DIRAC FERMIONS WITH B-L VIOLATION

We now turn to the implications of the phase constraints we have discussed and begin by considering the discrete symmetry transformations of various B-L violating operators with Dirac fields, for which the prototypical example is

$$\psi^T C\psi + \text{h.c.}\,, \tag{25}$$

where "h.c." denotes the Hermitian conjugate. Note that C satisfies $(\sigma^{\mu\nu})^{\mathrm{T}}C = -C\sigma^{\mu\nu}$, so that the construction of Eq. (25) is automatically Lorentz invariant. In what follows we work at energies far below the scale of B-L breaking; indeed, we work at sufficiently low-energy scales that we suppose the Dirac field ψ can be regarded as elementary. Moreover, since the primary use of such operators will be in theories of neutron-antineutron oscillations, or in theories of pseudo-Dirac neutrinos, we assume that the mass associated with the fermion field is dominated by its Dirac mass; this reduces the list of possible non-trivial operators that can appear. In what follows we enumerate all the lowest mass dimension B-L violating operators with Lorentz structures that span the possible bilinear covariants and discuss their transformation properties under CPT, as well as CP. We do not include operators with derivatives on the fermion field operators because the free-particle Dirac equation can be used to bring them to the form of those we do include. Thus we consider operators $\mathcal{O}_i$, namely,

$$\mathcal{O}_1 = \psi^T C \psi + \text{h.c.} \qquad \stackrel{\textbf{CPT}}{\Longrightarrow} -(\eta_c \eta_p \eta_t)^2 \,, \tag{26}$$

$$\mathcal{O}_2 = \psi^T C \gamma_5 \psi + \text{h.c.} \qquad \stackrel{\textbf{CPT}}{\Longrightarrow} -(\eta_c \eta_p \eta_t)^2 \,, \tag{27}$$

$$\mathcal{O}_3 = \psi^T C \gamma^\mu \psi \, \partial^\nu F_{\mu\nu} + \text{h.c.} \qquad \stackrel{\textbf{CPT}}{\Longrightarrow} +(\eta_c \eta_p \eta_t)^2 \,, \tag{28}$$

$$\mathcal{O}_4 = \psi^T C \gamma^\mu \gamma_5 \psi \, \partial^\nu F_{\mu\nu} + \text{h.c.} \qquad \stackrel{\textbf{CPT}}{\Longrightarrow} -(\eta_c \eta_p \eta_t)^2 \,, \tag{29}$$

$$\mathcal{O}_5 = \psi^T C \sigma_{\mu\nu} \psi \, F^{\mu\nu} + \text{h.c.} \qquad \stackrel{\textbf{CPT}}{\Longrightarrow} +(\eta_c \eta_p \eta_t)^2 \,, \tag{30}$$

$$\mathcal{O}_6 = \psi^T C \sigma_{\mu\nu} \gamma_5 \psi \, F^{\mu\nu} + \text{h.c.} \qquad \stackrel{\textbf{CPT}}{\Longrightarrow} +(\eta_c \eta_p \eta_t)^2 \,, \tag{31}$$

where we have included the axial tensor operator $\mathcal{O}_6$ even if not strictly necessary, and we have reported the phase factor for the transformation of each operator under CPT as well. Note that we have included the electromagnetic field strength tensor $F^{\mu\nu}$ and its source as needed to make the B-L violating operators transform as Lorentz scalars. Remarkably, the set of operators $\mathcal{O}_i$ do not transform under CPT with a definite sign, and the phase constraints we have derived in Sec. II, that $(\eta_c \eta_p \eta_t)^2 = -1$, only serves to flip the sign of each eigenvalue. The existence of CPT-odd operators that are Lorentz scalar is in apparent contradiction with the CPT theorem [17], which asserts that CPT breaking implies that Lorentz symmetry is broken also. Nevertheless, the theorem remains secure, because, as we shall show, the operators of Eqs. (28,30,31) vanish once the anticommuting nature of fermion fields is taken into account. This anticommuting behavior is implicit to the determination of the transformation of the Dirac bilinears under C and CPT and is not an additional assumption. That only the operators of Eqs. (28,30,31) vanish outright speaks to the key nature of the phase constraint $(\eta_c \eta_p \eta_t)^2 = -1$ in making theories with B-L violation consistent with the tenets of quantum field theory.

The idea that the operators in Eqs. (28,30,31) should have no effect has been discussed in particular contexts, though never from the viewpoint of their wrong CPT. For example, the vector, tensor, and axial tensor electromagnetic form factors of Majorana neutrinos have been shown to vanish [24–29], and we refer the reader to the succinct treatment of Ref. [23]. Similarly, in the phenomenology of flavor-spin neutrino oscillations, the flavor-diagonal ν transition magnetic moment has been noted to vanish [30–32]. We now establish that the operators of Eqs. (28,30,31) vanish regardless of whether Majorana or Dirac fields are employed.

A. CPT-odd operators with Majorana fields

In the case of Majorana fields, for which Eq. (3) holds, we can immediately show that the operators of Eqs. (28,30,31) — and only these of our list — vanish identically, and that this follows from the anticommuting nature of fermion fields. We note that Eq. (3) can be rewritten as any of $\psi_m^T C = \lambda\bar{\psi}_m$, $C^\dagger\psi_m^* = \lambda^*\gamma^0\psi_m$, $\psi_m^\dagger C^\dagger = -\lambda^*\psi_m^T\gamma^0$, and $C\psi_m = -\lambda\gamma^0\psi_m^*$. Thus $\mathcal{O}_1$, e.g., can be rewritten as $(\lambda+\lambda^*)\bar{\psi}_m\psi_m$ or $-(\lambda+\lambda^*)\psi_m^T\bar{\psi}_m^T$, but these are equal because $\bar{\psi}_m\psi_m = -\psi_m^T\bar{\psi}_m^T$. Therefore $\mathcal{O}_1$ need not vanish. Similarly for $\mathcal{O}_2$ we have $(\lambda-\lambda^*)\bar{\psi}_m\gamma_5\psi_m$, or $-(\lambda-\lambda^*)\psi_m^T\gamma_5\bar{\psi}_m^T$, and thus $\mathcal{O}_2$ also need not vanish. Noting that $C\gamma^\mu = -\gamma^{\mu\,T}C$ we see, however, that $\mathcal{O}_3 = (\lambda+\lambda^*)\bar{\psi}_m\gamma^\mu\psi_m j_\mu = (\lambda+\lambda^*)\psi_m^T\gamma^{\mu\,T}\bar{\psi}_m^T j_\mu$, with $j_\mu \equiv \partial^\nu F_{\mu\nu}$, and thus $\mathcal{O}_3$ vanishes. In contrast, we have that $\mathcal{O}_4 = (\lambda-\lambda^*)\bar{\psi}_m\gamma^\mu\gamma_5\psi_m j_\mu = -(\lambda-\lambda^*)\psi_m^T\gamma_5\gamma^{\mu\,T}\bar{\psi}_m^T j_\mu$, and we conclude that $\mathcal{O}_4$ can be nonzero. Finally, since $(\sigma^{\mu\nu})^T C\gamma^\mu = -C\sigma^{\mu\nu}$, we have that $\mathcal{O}_5 = (\lambda+\lambda^*)\bar{\psi}_m\sigma^{\mu\nu}\psi_m F_{\mu\nu} = (\lambda+\lambda^*)\psi_m^T(\sigma^{\mu\nu})^T\bar{\psi}_m^T F_{\mu\nu}$, as well as $\mathcal{O}_6 = (\lambda-\lambda^*)\bar{\psi}_m\sigma^{\mu\nu}\gamma_5\psi_m F_{\mu\nu} = (\lambda-\lambda^*)\psi_m^T\gamma_5(\sigma^{\mu\nu})^T\bar{\psi}_m^T F_{\mu\nu}$. We see that both $\mathcal{O}_5$ and $\mathcal{O}_6$ vanish as well. Thus we have proven what we set out to show.

B. CPT-odd operators with Dirac fields

In the case of Dirac fields, for which Eq. (3) does not hold, a similarly ready proof that the operators of Eqs. (28,30,31) vanish is not available. In this case we evaluate the operators explicitly by postulating that the field operators satisfy equal-time anticommutation relations and expanding them in the free-particle, plane-wave expansion of Eq. (43). We then immediately find that $\mathcal{O}_5$ and $\mathcal{O}_6$ [13], as well as $\mathcal{O}_3$, vanish due to the anticommuting nature of fermion fields. Since our demonstration assumes that the fermion is both free and point-like, we now turn to ways in which we can make it more general, considering the conditions under which we can extend it to the case of bound particles, as well as to that of strongly bound composite particles. We would like our conclusions to be pertinent to $n-\bar{n}$ oscillations, for both free and bound neutrons.

In the case that the particle is loosely bound, e.g., the effect of the "wrong CPT" operators is still zero because the loosely bound state can be regarded as a linear superposition of free states of momentum $\mathbf{k}$, weighted by its wave function [33]. Since the wrong CPT operators vanish for free states, then the operators involving such loosely bound particles will also. We note that since the binding energies of neutrons in large nuclei are no more than ~ 8 MeV per particle, our argument should be sufficient to conclude that Eqs. (28,30,31) do not operate for bound neutrons.

An interesting question may be what happens if the fermion is actually a strongly bound composite particle, such as the neutron itself. We have explored this in the particular case of $n-\bar{n}$ oscillations using the M.I.T. bag model [34, 35], following the analysis of Ref. [36]. Since the quarks within the bag are free, an expansion of the quark fields in single-particle modes analogous to Eq. (43) exists [35], suggesting that the results of our earlier analysis at the nucleon level should be pertinent here as well. Indeed an explicit calculation of the transition matrix element $\langle\bar{n}|O_1|n\rangle$ using the O_1 operator of Ref. [36] with the substitution of $u_{\chi 1}^{T\alpha} C\sigma^{\mu\nu} u_{\chi 1}^{\beta} F_{\mu\nu}$ for $u_{\chi 1}^{T\alpha} C u_{\chi 1}^{\beta}$ yields zero. In what follows we assume that the operators of Eqs. (28,30,31) do indeed vanish if Lorentz symmetry is not broken. As an aside, we note that an explicit proof of the CPT theorem within confining theories is still lacking [37].

C. CP transformation properties

We now turn to the analysis of the CP properties of the surviving B-L violating operators, finding

$$\mathcal{O}_1 = \psi^T C\psi + \text{h.c.} \qquad \overset{\mathbf{CP}}{\Longrightarrow} -(\eta_c\eta_p)^2 \,, \tag{32}$$

$$\mathcal{O}_2 = \psi^T C\gamma_5\psi + \text{h.c.} \qquad \overset{\mathbf{CP}}{\Longrightarrow} -(\eta_c\eta_p)^2 \,, \tag{33}$$

$$\mathcal{O}_4 = \psi^T C\gamma^\mu\gamma_5\psi\, \partial^\nu F_{\mu\nu} + \text{h.c.} \qquad \overset{\mathbf{CP}}{\Longrightarrow} -(\eta_c\eta_p)^2 \,, \tag{34}$$

where we have left the phase dependence explicit. Noting our earlier determined phase constraint that $\eta_p^2 = -1$, we see, nevertheless, that the CP transformation properties of the operators are not definite — rather, they are given by η_c^2, where η_c is not determined.

Explicit examples of the indeterminate nature of the CP transformation, illustrated through the phase rotation $\psi \to \psi' = e^{i\theta}\psi$, can be found in Ref. [38]. The noted phase rotation has the effect of changing $\eta_c \to e^{2i\theta}\eta_c$, $\eta_t \to e^{-2i\theta}\eta_t$, with η_p unchanged, under $\psi \to \psi'$ in the C, T, and P transformations, respectively. We emphasize that the indeterminacy arises from that in η_c^2 and thus emerges generally for B-L violating operators. In Ref. [38] $\eta_c = \eta_p = 1$ and $\eta_t = i$ throughout, and although these choices are consistent with the phase constraint we have found for the CPT transformation, they are not consistent with the phase constraints we have found for P and TC, though this does not impact their conclusion regarding the indeterminacy of CP. If η_c^2 were set to -1, then Eq. (32) gives the result reported in Ref. [14]. We argue on physical grounds that the observation of $n - \bar{n}$ oscillations cannot itself constitute evidence of CP violation in the following section.

IV. IMPLICATIONS OF THE CPT AND CP PHASES

In this section we consider the consequences of the CPT and CP transformation properties we have determined in previous sections, particularly in regards to their implications for the interplay of the appearance of $n - \bar{n}$ oscillations with external magnetic fields. It has long been thought that experimental searches for free $n - \bar{n}$ oscillations must be performed in a high-vacuum, low-magnetic-field environment, because the energy of a neutron and antineutron generally ceases to be the same in the presence of matter or magnetic fields, suppressing $n - \bar{n}$ oscillations [39, 40]. However, if a $n - \bar{n}$ transition could connect a neutron and antineutron of opposite spin, then CPT invariance would guarantee that those states would be of the same energy in a magnetic field — and eliminating the magnetic field would no longer be necessary. In Ref. [13] it was argued that spin-dependent SM effects involving transverse magnetic fields could, in effect, realize $n - \bar{n}$ transitions in which the particle spin flips and thus accomplish this goal. However, this conclusion is sensitive to the CPT phase constraint we have discussed. To illustrate, we revisit the example analyzed in Ref. [13]: a neutron at rest that can oscillate to an antineutron is in a static magnetic field $\mathbf{B}_0$ and to which a static transverse field $\mathbf{B}_1$ is suddenly applied at $t = 0$. Noting that $\mathbf{B}_0$ fixes the spin quantization axis and defining $\omega_0 \equiv -\mu_n B_0$ and $\omega_1 \equiv -\mu_n B_1$, the Hamiltonian matrix in the $|n(+)\rangle$, $|\bar{n}(+)\rangle$, $|\bar{n}(-)\rangle$, $|\bar{n}(-)\rangle$ basis at $t > 0$ is of form

$$\mathcal{H} = \begin{pmatrix} M+\omega_0 & \delta & \omega_1 & 0 \\ \delta & M-\omega_0 & 0 & -\omega_1 \\ \omega_1 & 0 & M-\omega_0 & -\delta\eta_{cpt}^2 \\ 0 & -\omega_1 & -\delta\eta_{cpt}^2 & M+\omega_0 \end{pmatrix}, \tag{35}$$

where M is the neutron mass and δ, which is real in this example, denotes a $n(+) \to \bar{n}(+)$ transition matrix element. The other signs are fixed by Hermiticity and CPT invariance. We have now explicitly included the dependence of the B-L violating operator on the phase of the CPT transformation, namely, $\eta_{cpt} \equiv \eta_c\eta_p\eta_t$. In Ref. [13] the phase η_{cpt} was set to unity; in this work we have, rather, established that $\eta_{cpt}^2 = -1$.

In Ref. [13] the unpolarized n-$\bar{n}$ transition probability was found to be, noting $|\delta| \ll |\omega_0|, |\omega_1|$,

$$\begin{aligned}\mathcal{P}_{n\to\bar{n}}(t) = \delta^2 \Bigg[&\frac{\omega_1^2 t^2}{\omega_0^2+\omega_1^2} + \frac{\omega_0^2}{(\omega_0^2+\omega_1^2)^2}\sin^2(t\sqrt{\omega_0^2+\omega_1^2}) \\ &+ \frac{\omega_0^2\omega_1^2 t}{(\omega_0^2+\omega_1^2)^{5/2}}\left(1-\sin\left(2t\sqrt{\omega_0^2+\omega_1^2}\right)\right)\Bigg] + \mathcal{O}(\delta^3)\,,\end{aligned} \tag{36}$$

where if $|\omega_0| \sim |\omega_1|$ the first term is of $\mathcal{O}(1)$ in magnetic fields — and thus the quenching previously noted no longer appears. However, the exact eigenvalues at $t > 0$ are

$$\begin{aligned} E_1 &= M - \sqrt{\omega_0^2+(\delta-\omega_1)^2}\,, \\ E_2 &= M + \sqrt{\omega_0^2+(\delta-\omega_1)^2}\,, \\ E_3 &= M - \sqrt{\omega_0^2+(\delta+\omega_1)^2}\,, \\ E_4 &= M + \sqrt{\omega_0^2+(\delta+\omega_1)^2}\,. \end{aligned} \tag{37}$$

As pointed out in Refs. [14, 41], this is incompatible with rotational invariance because the eigenenergies do not depend on the magnitude of the total magnetic field $|\mathbf{B}|$ alone. However, once we have included the needed phase $\eta_{cpt}^2 = -1$, we then find that the energy eigenvalues at $t > 0$ do indeed depend on $|\mathbf{B}|$, as needed by rotational invariance [14, 41], recovering the form found in Ref. [14], and that $n(+) \to \bar{n}(-)$ and $n(-) \to \bar{n}(+)$ transitions no longer occur. As a result, $n\bar{n}$ transitions are quenched irrespective of the presence of transverse magnetic fields. We note that employing time-dependent magnetic fields in the manner familiar from the theory of magnetic resonance [42, 43], as discussed in Ref. [13], does not change this conclusion — the time-dependent case, upon a change of variable, resembles the static case we have already analyzed. Finally, then, the failure of rotational invariance in Eq. (37) [13] is a consequence of the inadvertent use of a Hamiltonian matrix in which the $n - \bar{n}$ transition operator broke CPT and hence Lorentz invariance; this is redressed through the inclusion of the phase η_{cpt}.

We now turn to the possibility of CP violation in free $n - \bar{n}$ oscillations in the absence of external fields, for which the $n - \bar{n}$ transition probability is controlled by $|\delta|^2$ [39]. Referring to Eqs. (32,33), though only Eq. (32) operates [13], we see that the probability transforms as $|\eta_c|^2 = 1$. Thus even if δ does not have definite CP its associated observable is CP even. Consequently the observation of free $n - \bar{n}$ oscillations cannot itself constitute a CP-violating effect. This is in contradistinction to the case of a permanent electric-dipole moment (EDM) d, for which the low-energy Hamiltonian for a particle with spin $\mathbf{S}$ is

$$\mathcal{H} = -\frac{\mu}{S}\mathbf{S} \cdot \mathbf{B} - \frac{d}{S}\mathbf{S} \cdot \mathbf{E}\,. \tag{38}$$

Here a nonzero value of d generates an observable CP-violating effect, even if it is generated by a single operator, because the spin-state energy splitting generated by the μ-term in a nonzero magnetic field changes upon the reversal of an applied electric field.

We conclude this section by noting that despite the failure of the specific method proposed in Ref. [13], spin-dependent effects could well prove key to realizing $n - \bar{n}$ oscillations. In particular, the $n - \bar{n}$ transition operator

$$\mathcal{O}_4 = \psi^T C \gamma^\mu \gamma_5 \psi \, \partial^\nu F_{\mu\nu} + \text{h.c.} \tag{39}$$

couples states of the same energy in a magnetic field, so that, in effect, $n(+) \to \bar{n}(-)$ can occur directly because the interaction with an external source, such as an electron beam, flips the spin. This is concomitant with the study of the crossed process $n(p_1, s_1) + n(p_2, s_2) \to \gamma^*(k)$, for which only $L = 1$ and $S = 1$ is allowed in the initial state via angular momentum conservation and Fermi statistics [14]. As a result, this particular operator does not require the eradication of magnetic fields to engender an observable effect. The experimental concept in this case would be completely different from those considered thus far, engendering $e + n \to \bar{n} + e$, e.g. Nuclear stability should also set limits on this source of B-L violation [14].

V. B-L VIOLATION AND THEORIES OF SELF-CONJUGATE FERMIONS

In our study of B-L violating operators, we have found that it is possible to write down operators which are odd under CPT but yet are also Lorentz invariant. These operators do vanish once the anticommuting nature of fermion fields is taken into account, though the precise stature of the results depends on whether the fermion fields are Majorana or Dirac. In the Majorana case, the demonstration is immediate, following from the definition of the Majorana field, Eq. (3), and the anticommuting nature of fermion fields, whereas in the Dirac case it is not. In the latter case canonical quantization and a Fourier expansion of the fermion field is required, though fermion antisymmetry still plays a crucial role. In this section we consider the roots of these differences and indeed why it should be possible to write down a CPT-odd, Lorentz-invariant operator, even if it does ultimately vanish. To do this, we recall theories of self-conjugate particles with half-integer isospin, which are non-local [18–21] and have anomalous CPT properties [44–50].

In attempting to rationalize the spectral pattern of the low-lying, light hadrons, Carruthers discovered a class of theories for which the CPT theorem does not hold [18]. We note the pions form a self-conjugate isospin multiplet (π^+, π^0, π^-), whereas the kaons form pair-conjugate multiplets (K^+, K^0) and $(\bar{K}^0, K^-)$, so that the particle and antiparticle appear in distinct isospin multiplets. Carruthers discovered that free theories of self-conjugate bosons with half-integer isospin are nonlocal, that the commutator of two self-conjugate fields with opposite isospin components do not vanish at space-like separations [18], rendering the theory noncausal and hence physically unacceptable. Moreover, since weak local communitivity fails [48], CPT symmetry is no longer expected to hold [51], nor should the theorem of Ref. [17] apply. These results were quickly generalized, and apply to theories of arbitrary spin [19–21]. Consequently it is possible to have self-conjugate theories of isospin $I = 0$, but it is not possible to have self-conjugate theories of $I = 1/2$. These developments are pertinent to the findings in this paper, because a Majorana fermion is a self-conjugate particle of $I = 0$, whereas the neutron and antineutron are members of pair-conjugate $I = 1/2$ multiplets. Since $p - \bar{p}$ oscillations are forbidden by electric charge conservation, a theory of $n - \bar{n}$ oscillations need not be a theory of self-conjugate isofermions. We note, however, that the very quark-level operators that generate $n - \bar{n}$ oscillations [36] would also produce $p - \bar{p}$ oscillations under the isospin transformation $u \leftrightarrow d$. Since QCD is symmetric under $u \leftrightarrow d$ exchange in its chiral limit, the admissible B-L violating operators in that case must then necessarily break isospin symmetry, so that self-conjugate isofermions do not appear. Since isospin symmetry is broken in the SM by quark mass and electric charge differences, the SM itself is compatible with the appearance of B-L violating operators in the quark sector.

VI. SUMMARY

In this paper we have determined the restrictions on the phases associated with the discrete symmetry transformations C, P, and T of fermion fields that appear in theories of B-L violation, generalizing the earlier work of Refs. [1, 2]. These phase constraints do not impact B-L conserving theories because the phases are unimodular, but they are key to determining the behavior of B-L violating operators under discrete symmetry transformations because they enter as the phase squared. As a result, they have important implications for the interplay of B-L violating dynamics with the SM.

We have found that the phase associated with the transformation of a fermion field under CPT, η_{cpt}, must always be imaginary and that the phase associated with P, η_p, must be imaginary for fermions for which a P transformation exists. Generally, however, the phase associated with CP, η_{cp}, is indeterminate for B-L violating operators. We find that the constraint on η_{cpt} reconciles the disagreement between Refs. [13, 14], to the end that magnetic fields do indeed quench n-$\bar{n}$ oscillations mediated by the operator $\psi^T C\psi$+h.c. [14]. However, spin dependence can still play a key role in n-$\bar{n}$ transitions, as proposed in Ref. [13], and in this paper we have noted the prospects associated with the operator $\psi^T C\gamma^\mu\gamma_5\psi j_\mu$+h.c. [14], for which $n(+) \to \bar{n}(-)$, e.g., is mediated by the external current j_μ. We note that $n(+)$ and $n(-)$ are of the same energy irrespective of the external magnetic fields. Moreover, we have shown that the appearance of n-$\bar{n}$ oscillations does not in itself break CP, in contradistinction to Ref. [14], and that this is true irrespective of η_{cp}.

We expect that CPT is an exact symmetry of a local, Lorentz invariant quantum field theory [51], and if CPT is broken, then Lorentz invariance fails also [17]. We have found that it is possible to construct B-L violating, Lorentz-invariant operators that are either CPT even or odd, but that one set vanishes due to the anticommuting nature of fermion fields. The CPT phase constraint we have found is essential to making the nonvanishing B-L operators CPT even. Our ability to prove that the CPT-odd operators vanish depends on whether the fermion fields are Majorana or Dirac, with additional assumptions needed in the Dirac case. We have explained this in connection to theories of self-conjugate isofermions, for which locality fails [18–21], and the CPT properties are anomalous [44–50]. In this regard Majorana neutrinos and neutrons are distinct, because only the latter carry $I = 1/2$. The conservation of electric charge saves a theory with $n-\bar{n}$ oscillations, in which p-$\bar{p}$ oscillations do not occur, from being a theory of self-conjugate isofermions; nevertheless, CPT-odd, Lorentz-invariant operators can appear, though they ultimately appear to vanish.

APPENDICES

A. Discrete symmetries — definitions and other essentials

In this appendix we collect the definitions and basic results that underlie the central arguments of the paper. The discrete-symmetry transformations of a four-component fermion field $\psi(x)$ are given by

$$\mathbf{C}\psi(x)\mathbf{C}^{-1} = \eta_c C\gamma^0\psi^*(x) \equiv \eta_c i\gamma^2\psi^*(x) \equiv \eta_c\psi^c(x)\,, \tag{40}$$

$$\mathbf{P}\psi(t,\mathbf{x})\mathbf{P}^{-1} = \eta_p\gamma^0\psi(t,-\mathbf{x})\,, \tag{41}$$

$$\mathbf{T}\psi(t,\mathbf{x})\mathbf{T}^{-1} = \eta_t\gamma^1\gamma^3\psi(-t,\mathbf{x})\,, \tag{42}$$

where η_c, η_p, and η_t are unimodular phase factors of the charge-conjugation C, parity P, and time-reversal T transformations, respectively, and we have chosen the Dirac-Pauli representation for the gamma matrices. Note that $\psi^c(x)$ is the conjugate field and that $\mathbf{C}^2\psi(x)\mathbf{C}^{-2} = \psi(x)$ and $\mathbf{T}^2\psi(x)\mathbf{T}^{-2} = -\psi(x)$, irrespective of arbitrary phases, but that $\mathbf{P}^2\psi(x)\mathbf{P}^{-2} = \eta_p^2\psi(x)$. Our choices and results conform with those of Ref. [13] if the arbitrary phases are set to unity — and with those of Ref. [22], though we have chosen a specific representation of the gamma matrices.

The plane-wave expansion of a Dirac field $\psi(x)$ is given by

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}\sqrt{2E}} \sum_{s=\pm} \left\{ b(\mathbf{p},s)u(\mathbf{p},s)e^{-ip\cdot x} + d^\dagger(\mathbf{p},s)v(\mathbf{p},s)e^{ip\cdot x} \right\}\,, \tag{43}$$

with spinors defined as

$$u(\mathbf{p},s) = \mathcal{N}\begin{pmatrix} \chi^{(s)} \\ \frac{\sigma\cdot\mathbf{p}}{E+M}\chi^{(s)} \end{pmatrix} \quad ; \quad v(\mathbf{p},s) = \mathcal{N}\begin{pmatrix} \frac{\sigma\cdot\mathbf{p}}{E+M}\chi'^{(s)} \\ \chi'^{(s)} \end{pmatrix}\,, \tag{44}$$

noting $\chi'^{(s)} = -i\sigma^2\chi^{(s)}$, $\chi^+ = \binom{1}{0}$, $\chi^- = \binom{0}{1}$, and $\mathcal{N} = \sqrt{E+M}$. Noting that $b(d)$ annihilates a particle (antiparticle), we find the following transformation properties:

$$\mathbf{C}b(\mathbf{p},s)\mathbf{C}^\dagger = \eta_c d(\mathbf{p},s)\ ;\ \mathbf{C}d^\dagger(\mathbf{p},s)\mathbf{C}^\dagger = \eta_c b^\dagger(\mathbf{p},s)\,,$$

$$\mathbf{C}b^\dagger(\mathbf{p},s)\mathbf{C}^\dagger = \eta_c^* d^\dagger(\mathbf{p},s)\ ;\ \mathbf{C}d(\mathbf{p},s)\mathbf{C}^\dagger = \eta_c^* b(\mathbf{p},s)\,, \tag{45}$$

$$\mathbf{P}b(\mathbf{p},s)\mathbf{P}^\dagger = \eta_p b(-\mathbf{p},s)\ ;\ \mathbf{P}d^\dagger(\mathbf{p},s)\mathbf{P}^\dagger = -\eta_p d^\dagger(-\mathbf{p},s)\,,$$

$$\mathbf{P}b^\dagger(\mathbf{p},s)\mathbf{P}^\dagger = \eta_p^* b^\dagger(-\mathbf{p},s)\ ;\ \mathbf{P}d(\mathbf{p},s)\mathbf{P}^\dagger = -\eta_p^* d(-\mathbf{p},s)\,, \tag{46}$$

$$\mathbf{T}b(\mathbf{p},s)\mathbf{T}^{-1} = s\eta_t b(-\mathbf{p},-s)\ ;\ \mathbf{T}d^\dagger(\mathbf{p},s)\mathbf{T}^{-1} = s\eta_t d^\dagger(-\mathbf{p},-s)\,,$$

$$\mathbf{T}b^\dagger(\mathbf{p},s)\mathbf{T}^{-1} = s\eta_t^* b^\dagger(-\mathbf{p},-s)\ ;\ \mathbf{T}d(\mathbf{p},s)\mathbf{T}^{-1} = s\eta_t^* d(-\mathbf{p},-s)\,, \tag{47}$$

where, for convenience, we note that

$$\gamma^0 u(\mathbf{p},s) = u(-\mathbf{p},s) \quad ; \quad \gamma^0 v(\mathbf{p},s) = -v(-\mathbf{p},s)\,, \tag{48}$$

$$u(\mathbf{p},s) = i\gamma^2 v^*(\mathbf{p},s)\,, \tag{49}$$

$$u^*(\mathbf{p},s) = s\gamma^1\gamma^3 u(-\mathbf{p},-s) \quad ; \quad v^*(\mathbf{p},s) = s\gamma^1\gamma^3 v(-\mathbf{p},-s)\,, \tag{50}$$

$$\gamma^5 u(\mathbf{p},s) = -sv(\mathbf{p},-s)\,. \tag{51}$$

B. Phase restrictions for two-component fields

In this section we develop the phase restrictions associated with the discrete-symmetry transformations of two-component Majorana fields. We develop these in two different ways: the first by connecting Dirac fields, and our earlier phase constraints, with two-component Majorana fields and the second by analyzing the transformation properties of two-component Majorana fields directly.

In Weyl representation, a Dirac spinor can be written as

$$\psi = \begin{pmatrix} \xi^\alpha \\ \eta_{\dot\beta} \end{pmatrix}\,. \tag{52}$$

where α and β can be 1 or 2. Here we employ the undotted and dotted notation used by Refs. [52, 53]. The undotted contravariant spinor ξ^α and the covariant spinor ξ_α are in the $(\frac{1}{2},0)$ representation of the Lorentz group SO(3,1), whereas the dotted covariant spinor $\eta_{\dot\beta}$ and the contravariant spinor $\eta^{\dot\beta}$ are in the $(0,\frac{1}{2})$ representation. One can raise or lower the undotted indices using the metric of SL(2,C)

$$g_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2_{\alpha\beta}, \tag{53}$$

$$g^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma^2_{\alpha\beta}\,, \tag{54}$$

i.e.,

$$\xi^\alpha = g^{\alpha\beta}\xi_\beta = -i\sigma^2_{\alpha\beta}\xi_\beta\,, \tag{55}$$

and use the same metric for dotted indices.

Since the C and P transformations of Eqs. (40,41) connect the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the Lorentz group and thus the two two-component fields in Eq. (52), a particular two-component field cannot transform into itself under P or C. However, it can transform into itself under CP or CPT (or T) [52, 53], so that phase constraints may exist for these particular transformations. We will now determine them in two different ways.

In Sec. II, we found the phase constraints associated with the discrete-symmetry transformations of a Dirac field. Revisiting the CP and CPT transformations in Weyl representation, we find

$$\mathbf{CP}\begin{pmatrix} \xi^\alpha(\mathbf{x},t) \\ \eta_{\dot\beta}(\mathbf{x},t) \end{pmatrix}(\mathbf{CP})^{-1} = \eta_{cp} i\gamma^0\gamma^2 \begin{pmatrix} \xi^{\alpha\dagger}(-\mathbf{x},t) \\ \eta^\dagger_{\dot\beta}(-\mathbf{x},t) \end{pmatrix}\,, \tag{56}$$

$$\mathbf{CPT}\begin{pmatrix} \xi^\alpha(x) \\ \eta_{\dot\beta}(x) \end{pmatrix}(\mathbf{CPT})^{-1} = -\eta_{cpt}\gamma^5 \begin{pmatrix} \xi^{\alpha\dagger}(-x) \\ \eta^\dagger_{\dot\beta}(-x) \end{pmatrix}\,. \tag{57}$$

Since in Weyl representation

$$i\gamma^0\gamma^2 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{58}$$

we use Eq. (55), e.g., to find

$$\mathbf{CP}\xi^\alpha(\mathbf{x},t)(\mathbf{CP})^{-1} = -\eta_{cp}\xi^\dagger_\alpha(-\mathbf{x},t)\,, \tag{59}$$

$$\mathbf{CP}\eta_{\dot\alpha}(\mathbf{x},t)(\mathbf{CP})^{-1} = -\eta_{cp}\eta^{\dot\alpha\dagger}(-\mathbf{x},t)\,, \tag{60}$$

$$\mathbf{CPT}\xi^\alpha(x)(\mathbf{CPT})^{-1} = \eta_{cpt}\xi^{\alpha\dagger}(-x)\,, \tag{61}$$

$$\mathbf{CPT}\eta_{\dot\alpha}(x)(\mathbf{CPT})^{-1} = -\eta_{cpt}\eta^\dagger_{\dot\alpha}(-x)\,, \tag{62}$$

where we note, as per Sec. II, that $\eta_{cpt} \equiv \eta_c\eta_p\eta_t = \pm i$. Here we find no direct constraint on the phase $\eta_{cp} \equiv \eta_c\eta_p$, or η_t for that matter, because the analysis of Sec. II determined that the combinations $\eta_{cp}^*\lambda$ and $\eta_t\lambda$ were imaginary and real, respectively. Since the phase λ has no meaning in the current context, no conclusions on η_{cp} or η_t can follow.

An alternate path to these results comes from the analysis of the plane-wave expansion of the two-component Majorana field $\xi_a(x)$ [54, 55]:

$$\xi_\alpha(x) = \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}(2E_\mathbf{p})^{1/2}} [x_\alpha(\mathbf{p}, s)a(\mathbf{p}, s)e^{-ipx} + \lambda y_\alpha(\mathbf{p}, s)a^\dagger(\mathbf{p}, s)e^{ipx}] , \qquad (63)$$

where x_α and y_α are two-component spinors, whose definition and other pertinent details can be found in Ref. [55]. Note that we have included a phase factor λ in $\xi_a(x)$, in analogy to the analysis of Sec. II. It is trivial to check that the phase λ included here functions in the same way as in Eq. (1) and that it is forced to 1 when $\xi_\alpha(x)$ is used to constuct a Dirac field [1]. Using the CP transformation of $\xi_\alpha(\mathbf{x}, t)$ [52, 53]

$$\mathbf{CP}\xi_\alpha(\mathbf{x}, t)(\mathbf{CP})^{-1} = \eta_{cp}(\xi^\alpha)^\dagger(-\mathbf{x}, t) \qquad (64)$$

and the relations [55]

$$(x^\alpha)^\dagger(\mathbf{p}, s) = x^{\dagger\dot\alpha}(\mathbf{p}, s) = -y_\alpha(-\mathbf{p}, s) , \qquad (65)$$

$$(y^\alpha)^\dagger(\mathbf{p}, s) = y^{\dagger\dot\alpha}(\mathbf{p}, s) = x_\alpha(-\mathbf{p}, s) \qquad (66)$$

yield

$$\mathbf{CP}a(\mathbf{p}, s)(\mathbf{CP})^{-1} = \eta_{cp}\lambda^* a(-\mathbf{p}, s) , \qquad (67)$$

$$\mathbf{CP}a^\dagger(\mathbf{p}, s)(\mathbf{CP})^{-1} = -\eta_{cp}\lambda^* a^\dagger(-\mathbf{p}, s) . \qquad (68)$$

Since $\mathbf{CP}$ is a unitary operator, taking the Hermitian conjugate of either relation proves that $\eta_{cp}\lambda^*$ must be imaginary.

Under CPT, we have

$$\mathbf{CPT}\xi^\alpha(x)(\mathbf{CPT})^{-1} = \eta_{cpt}(\xi^\alpha)^\dagger(-x) . \qquad (69)$$

Using the relations [55]

$$x^{\dagger\dot\alpha}(\mathbf{p}, -s) = 2sy^{\dagger\dot\alpha}(\mathbf{p}, s) , \qquad (70)$$

$$y^{\dagger\dot\alpha}(\mathbf{p}, -s) = -\frac{1}{2s}x^{\dagger\dot\alpha}(\mathbf{p}, s) , \qquad (71)$$

[1] Although the notation for a Dirac field employed by Refs. [52, 53] and [55] differs, our results are unchanged.

we find

$$\mathbf{CPT}a(\mathbf{p},s)(\mathbf{CPT})^{-1} = -\frac{1}{2s}\lambda^*\eta_{cpt}a(\mathbf{p},-s)\,, \tag{72}$$

$$\mathbf{CPT}a^\dagger(\mathbf{p},s)(\mathbf{CPT})^{-1} = 2s\lambda\eta_{cpt}a^\dagger(\mathbf{p},-s)\,. \tag{73}$$

Noting that **CPT** is an antiunitary operator, as in Sec. II, we can take the Hermitian conjugate of either equation to show that η_{cpt} must be imaginary. Alternatively, after Ref. [2], we define $\mathbf{CPT}|0\rangle = |0\rangle$ and note

$$\begin{aligned} 1 &= \langle 0|a(\mathbf{p},s)a^\dagger(\mathbf{p},s)|0\rangle \\ &= \langle 0|\mathbf{CPT}a(\mathbf{p},s)\mathbf{CPT}^{-1}\mathbf{CPT}a^\dagger(\mathbf{p},s)\mathbf{CPT}^{-1}|0\rangle\,. \end{aligned} \tag{74}$$

Then using Eqs. (72,73) shows that $\eta_{cpt} = \pm i$.

In summary, we have used two methods to find the phase constraints on **CP** and **CPT** for two-component fields, and have obtained the same results, which are that η_{cp} itself is unconstrained, though $\eta_{cp}\lambda^*$ must be imaginary, and η_{cpt} is always $\pm i$.

ACKNOWLEDGMENTS

We acknowledge partial support from the Department of Energy Office of Nuclear Physics under contract DE-FG02-96ER40989. The work of SG was performed, in part, at the Aspen Center for Physics, which is supported by the National Science Foundation grant PHY-1066293, and at the Institute of Nuclear Theory (INT). We thank M. Voloshin for comments concerning the interaction of $n-\bar{n}$ oscillations with electromagnetic fields and A. Vainshtein for extensive discussions of this topic and related issues in the context of an INT program “Intersections of BSM Phenomenology and QCD for New Physics Searches” in the fall of 2015. We are also grateful to R. Jaffe for helpful correspondence regarding the construction of antiquark states in the bag model, and we thank A. Nelson for a discussion concerning parity and two-component Majorana spinors, occurring within the context of the aforementioned INT program. We also thank T. Goldman for bringing early work on phase restrictions in discrete symmetry transformations to our attention.

[1] B. Kayser and A. S. Goldhaber, Phys. Rev. D **28**, 2341 (1983).

[2] B. Kayser, Phys. Rev. D **30**, 1023 (1984).

[3] P. A. Carruthers, *Spin and Isospin in Particle Physics* (Gordon and Breach, Science Publishers, Inc., New York, 1971). Note p. 133.

[4] G. Feinberg and S. Weinberg, Nuovo Cim. **14**, 571 (1959).

[5] G. Gelmini and E. Roulet, Rept. Prog. Phys. **58**, 1207 (1995) [hep-ph/9412278].

[6] L. Wolfenstein, Nucl. Phys. B **186**, 147 (1981).

[7] V. N. Gribov and B. Pontecorvo, Phys. Lett. B **28**, 493 (1969).

[8] S. M. Bilenky and B. Pontecorvo, Sov. J. Nucl. Phys. **38**, 248 (1983) [Lett. Nuovo Cim. **37**, 467 (1983)] [Yad. Fiz. **38**, 415 (1983)].

[9] S. R. Elliott and P. Vogel, Ann. Rev. Nucl. Part. Sci. **52**, 115 (2002) [hep-ph/0202264].

[10] J. Schechter and J. W. F. Valle, Phys. Rev. D **25**, 2951 (1982).

[11] N. Brambilla *et al.*, Eur. Phys. J. C **74**, 2981 (2014) [arXiv:1404.3723 [hep-ph]].

[12] M. Baldo-Ceolin *et al.*, Z. Phys. C **63**, 409 (1994).

[13] S. Gardner and E. Jafari, Phys. Rev. D **91**, 096010 (2015) [arXiv:1408.2264 [hep-ph]].

[14] Z. Berezhiani and A. Vainshtein, arXiv:1506.05096 [hep-ph].

[15] D. G. Phillips, II *et al.*, [arXiv:1410.1100 [hep-ex]].

[16] D. Milstead, arXiv:1510.01569 [physics.ins-det].

[17] O. W. Greenberg, Phys. Rev. Lett. **89**, 231602 (2002) [hep-ph/0201258].

[18] P. Carruthers, Phys. Rev. Lett. **18**, 353 (1967).

[19] Huan Lee, Phys. Rev. Lett. **18**, 1098 (1967).

[20] G. N. Fleming and E. Kazes, Phys. Rev. Lett. **18**, 764 (1967).

[21] Y. S. Jin, Phys. Lett. **24B**, 411 (1967).

[22] G. C. Branco, L. Lavoura, and J. P. Silva, *CP Violation* (Oxford University Press, New York, 1999).

[23] R. N. Mohapatra and P. B. Pal, *Massive neutrinos in physics and astrophysics* (World Scientific Publishing Company, Singapore, 2004).

[24] J. F. Nieves, Phys. Rev. D **26**, 3152 (1982).

[25] J. Schechter and J. W. F. Valle, Phys. Rev. D **24**, 1883 (1981) [Phys. Rev. D **25**, 283 (1982)].

[26] B. Kayser, Phys. Rev. D **26**, 1662 (1982).

[27] R. E. Shrock, Nucl. Phys. B **206**, 359 (1982).

[28] L. F. Li and F. Wilczek, Phys. Rev. D **25**, 143 (1982).

[29] S. Davidson, M. Gorbahn, and A. Santamaria, Phys. Lett. B **626**, 151 (2005) [hep-ph/0506085].

[30] L. B. Okun, M. B. Voloshin, and M. I. Vysotsky, Sov. Phys. JETP **64**, 446 (1986) [Zh. Eksp. Teor. Fiz. **91**, 754 (1986)].

[31] L. B. Okun, M. B. Voloshin, and M. I. Vysotsky, Sov. J. Nucl. Phys. **44**, 440 (1986) [Yad. Fiz. **44**, 677 (1986)].

[32] C. S. Lim and W. J. Marciano, Phys. Rev. D **37**, 1368 (1988).

[33] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley Publishing Company, USA, 1995).

[34] A. Chodos, R. L. Jaffe, K. Johnson, C. B. Thorn, and V. F. Weisskopf, Phys. Rev. D **9**, 3471 (1974).

[35] A. Chodos, R. L. Jaffe, K. Johnson, and C. B. Thorn, Phys. Rev. D **10**, 2599 (1974).

[36] S. Rao and R. Shrock, Phys. Lett. B **116** (1982) 238.

[37] V. A. Kostelecky, In *Santa Fe 1996, The state of physics at the end of the 20th century* 84-90 [hep-ph/9809542].

[38] K. Fujikawa and A. Tureanu, arXiv:1510.00868 [hep-ph].

[39] R. N. Mohapatra and R. E. Marshak, Phys. Lett. B **94**, 183 (1980).

[40] R. Cowsik and S. Nussinov, Phys. Lett. B **101**, 237 (1981).

[41] M. Voloshin, private communication, March, 2015.

[42] I. I. Rabi, Phys. Rev. **51**, 652 (1937).

[43] C. Cohen-Tannoudji *et al.*, *Quantum Mechanics* (Cambridge University Press, 2000).

[44] P. B. Kantor, Phys. Rev. Lett. **19**, 394 (1967).

[45] O. Steinmann, Phys. Lett. **25B**, 234 (1967).

[46] B. Zumino and D. Zwanziger, Phys. Rev. **164**, 1959 (1967).

[47] P. Carruthers, J. Math. Phys. **9**, 928 (1968).

[48] P. Carruthers, J. Math. Phys. **9**, 1835 (1968).

[49] P. Carruthers, Phys. Rev. **172**, 1406 (1968).

[50] P. Carruthers, Phys. Lett. **26B**, 158 (1968).

[51] R. F. Streater and A. S. Wightman, *PCT, spin and statistics, and all that* (Princeton University Press, Princeton, USA, 2000).

[52] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*

(Butterworth-Heinemann, Oxford, UK, 1982).

[53] Q. Duret and B. Machet, Annals Phys. **325**, 2041 (2010) [arXiv:0809.0431 [hep-ph]].

[54] K. M. Case, Phys. Rev. **107**, 307 (1957).

[55] H. K. Dreiner, H. E. Haber, and S. P. Martin, Phys. Rept. **494**, 1 (2010) [arXiv:0812.1594 [hep-ph]].