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Top-quark pole mass in the tadpole-free $\overline{\text{MS}}$ scheme

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The complex pole mass of the top quark is presented at full two-loop order in the Standard Model, augmenting the known four-loop QCD contributions. The input parameters are the $\overline{\text{MS}}$ Yukawa and gauge couplings, the Higgs self-coupling, and the Higgs vacuum expectation value (VEV). Here, the VEV is defined as the minimum of the full effective potential in Landau gauge, so that tadpoles vanish. This is an alternative to earlier results that instead minimize the tree-level potential, resulting in a VEV that is gauge-fixing independent but accompanied by negative powers of the Higgs self-coupling in perturbative expansions. The effects of non-zero Goldstone boson mass are eliminated by resummation. I also study the renormalization scale dependence of the calculated pole mass.

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I. INTRODUCTION

The top-quark mass is one of the key parameters of the Standard Model of particle physics. It is important for precision electroweak fits, and for matching of the Standard Model to ultraviolet physics, including both stability of the electroweak vacuum and theories that attempt to address the hierarchy problem.

In this paper, I will consider the relation between the $\overline{\text{MS}}$ Lagrangian quantities and the complex pole squared mass, which formally is a physical observable [1–4] and does not depend on the choice of gauge fixing terms or on the renormalization scheme and the renormalization scale to all orders in perturbation theory, but is subject to non-perturbative renormalon ambiguities associated with the hadronization scale [5, 6]. The relationship between the pole mass and the top-quark mass as measured by hadron collider experiment collaborations is somewhat problematic and is the subject of continuing investigations [7]. In the approximation that the width of the top-quark is neglected, the real part of the complex pole squared mass coincides with the on-shell squared mass. There exist several other useful definitions of the top-quark mass, depending on the precise relation to experimental quantities. These include the potential-subtracted mass [8], the 1S mass [9, 10], and the running $\overline{\text{MS}}$ mass.

In this paper, I consider the relation between the top-quark pole mass and the $\overline{\text{MS}}$ Lagrangian quantities. Many previous works have contributed to this subject. First, the pure QCD contributions have been given at 1-loop order

[1], 2-loop order [11] (confirmed in [12, 13]), 3-loop order [14] (with previous approximate results in [15, 16], and a useful summary of formulas in [17]), and 4-loop order [18]. Besides these pure QCD contributions, the full 1-loop contributions to the pole mass have been given in ref. [19] (see also ref. [20] and eq. (B.5) of ref. [21]). The 2-loop mixed QCD contributions were found in [22], and confirmed in ref. [23]. The full 2-loop contributions have been studied in the gaugeless limit (with the electroweak vector boson masses neglected compared to the top-quark mass) in refs. [24–28]. Most recently the full 2-loop results were given in ref. [29].

The purpose of the present paper is to give an alternative calculation of the full 2-loop contributions to the top-quark pole mass, using a different organization of perturbation theory than in the above references. Note that the definition of the running $\overline{\text{MS}}$ top-quark mass is not unique for a given renormalization scale Q , because the mass is proportional to the Higgs VEV, which can be defined in more than one way. One way, called the “tree-level VEV” here, is

$$v_{\text{tree}} = \sqrt{-m^2/\lambda}, \quad (1.1)$$

where λ and m^2 are the Higgs self-coupling and squared mass parameter in the $\overline{\text{MS}}$ scheme, normalized so that the tree-level $\overline{\text{MS}}$ renormalized potential for the canonically normalized complex Higgs doublet field Φ is

$$V(\Phi, \Phi^\dagger) = m^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2. \quad (1.2)$$

An advantage of v_{tree} , emphasized for example in refs. [29, 30], is that it and the corresponding tree-level mass $m_t = y_t v_{\text{tree}}/\sqrt{2}$ are manifestly independent of the gauge-fixing procedure, due to the way that they are defined in terms of the $\overline{\text{MS}}$ Lagrangian parameters. A disadvantage is that, although there are no tree-level tadpoles, there are tadpole loop diagrams involving the Higgs field, which have to be included in any calculation based on v_{tree} or m_t . As a consequence, the perturbative loop expansion parameters include

$$N_c y_t^4 / (16\pi^2 \lambda) \quad (1.3)$$

rather than the usual $N_c y_t^2 / 16\pi^2$. The presence of powers of $\lambda \approx 0.126$ in the denominators of perturbative expansions is due to the tadpoles, and is indicative of the fact that the tree-level VEV is not a very good approximation for the true vacuum state of the theory after including loop corrections. For example, in ref. [30], it was noted that when using v_{tree} , the 1-loop non-QCD correction is surprisingly huge, almost canceling the 1-loop QCD effect, due to the $1/\lambda$ tadpole effects.

In this paper, I follow the alternative scheme of defining the running $\overline{\text{MS}}$ squared masses of the top quark, bottom quark, electroweak vector bosons, and the Higgs scalar boson by

$$t = y_t^2 v^2 / 2, \quad (1.4)$$

$$b = y_b^2 v^2 / 2, \quad (1.5)$$

$$W = g^2 v^2 / 4, \quad (1.6)$$

$$Z = (g^2 + g'^2) v^2 / 4, \quad (1.7)$$

$$h = 2\lambda v^2 \quad (1.8)$$

where the the VEV v of the Higgs field is defined to be the minimum of the full effective potential in Landau gauge. As a benefit of this definition, the sum of all Higgs tadpole graphs, including the tree-level Higgs tadpole, vanishes identically. The price to be paid for this is that the VEV v , and therefore also t and the other tree-level

masses, depend on the gauge-fixing method. Therefore, calculations based on v are restricted to Landau gauge in the electroweak sector (or any other gauge-fixing choice; Landau gauge is chosen only because the effective potential is simple). Although one therefore apparently loses the check of requiring independence of the gauge-fixing parameters, the checks obtained from the cancellation of the unphysical Landau gauge Goldstone boson degrees of freedom from the complex pole squared mass (and other observables) are just as powerful. A benefit of this definition of the VEV is that v is in some sense a more faithful description of the true vacuum state. Negative powers of λ are absent in perturbative expansions of pole masses and other physical quantities. Indeed, this provides another useful check.

As a practical matter, the Standard Model effective potential is now known at full 2-loop order [31], together with the 3-loop contributions in the approximation that QCD and top-Yukawa couplings are large compared to all other couplings [32], and the 4-loop contributions at leading order in QCD [33], and with resummation of the Goldstone boson contributions [34, 35] (see also [36]) to avoid spurious infrared singularities and imaginary parts. As a consequence, one can write a loop expansion for the relationship between the two VEVs, showing the tadpole contributions explicitly,

$$v_{\text{tree}}^2 = v^2 + \frac{1}{\lambda} \sum_{\ell=1}^{\infty} \frac{1}{(16\pi^2)^\ell} \hat{\Delta}_\ell, \quad (1.9)$$

where $\hat{\Delta}_1$ and $\hat{\Delta}_2$ are known exactly, and $\hat{\Delta}_3$ is known in the approximation that the QCD coupling and top Yukawa coupling are much larger than the other couplings. They are given in eqs. (4.19)-(4.21) of ref. [34]. Also, $\hat{\Delta}_4$ is known only at leading order in QCD; this is given in eq. (5.5) of ref. [33].

The methods and results of the present paper are designed to be compatible with similar full 2-loop calculations of the complex pole squared masses of the Higgs scalar (with leading 3-loop contributions) in ref. [37], the W boson in ref. [38], and the Z boson in ref. [39]. All of these use the same VEV definition v , as an alternative to similar results that are expressed in terms of v_{tree} , or parameterized in terms of other quantities such as the Fermi constant G_F . For important previous results on the electroweak vector boson masses and the Higgs mass in the Standard Model using other schemes, see [40]-[68].

The input parameters here are taken to be the $\overline{\text{MS}}$ top-quark and bottom-quark Yukawa couplings, the $SU(3)_c$, $SU(2)_L$ and $U(1)_Y$ gauge couplings, the Higgs self-coupling, and the VEV as discussed above:

$$y_t, y_b, g_3, g, g', \lambda, v, \quad (1.10)$$

all at a specified renormalization scale Q . In principle, the result also depends on the lighter quark and lepton Yukawa couplings (or masses), but their contributions are very small, with the largest contribution coming from the 2-loop QCD contribution of the bottom quark mass, as noted below. The effects of CKM mixing on the top-quark pole mass calculation are also negligible. The W , Z , and h physical masses, as well as quantities such as G_F and $\sin^2 \theta_W$, are all regarded as output quantities in this pure $\overline{\text{MS}}$ scheme adopted here.

In the following, the complex pole squared mass[†] is denoted by

$$s_{\text{pole}} = M_t^2 - i\Gamma_t M_t. \quad (1.11)$$

Methods for calculating the complex pole mass at higher orders in perturbation theory from the fermion self-energy

[†] The pole squared mass is sometimes parameterized instead as $s_{\text{pole}} = (M_t' - i\Gamma_t'/2)^2$, but M_t' exceeds M_t by less than 2 MeV, which is negligible compared to both experimental and theoretical uncertainties.

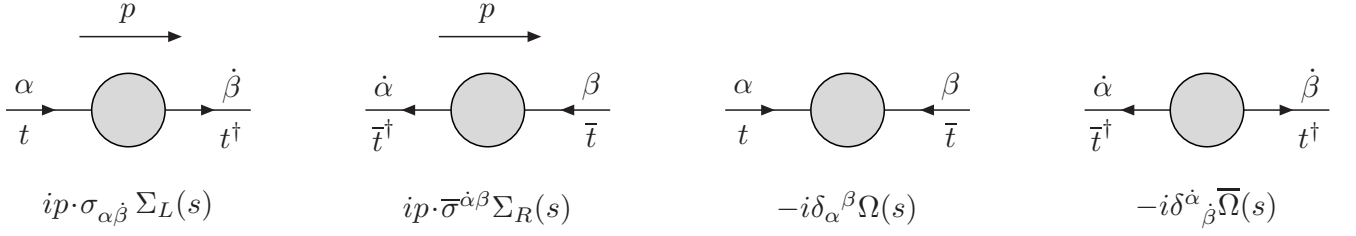


FIG. 1.1: Fermion one-particle-irreducible self-energy functions $\Sigma_L(s)$, $\Sigma_R(s)$, $\Omega(s)$, and $\bar{\Omega}(s)$, using 2-component notation following ref. [69]. Here the external momentum invariant is $s = -p^2$, using a metric signature $(-, +, +, +)$. The 2-component fields t and \bar{t} are the left-handed $SU(2)_L$ doublet quark and singlet antiquark, respectively, and t , t^\dagger , \bar{t} , and \bar{t}^\dagger are labeled as ingoing.

components are well-known, and given in various ways in several of the references mentioned above. In this paper, I use the 2-component fermion notation of ref. [69], and followed the procedure outlined in [27]. Defining the 2-component fermion self-energy functions as in figure 1.1, the complex pole mass is the solution of

$$0 = \text{Det} \left(s_{\text{pole}} - [1 - \Sigma_L(s_{\text{pole}})]^{-1} [m + \bar{\Omega}(s_{\text{pole}})] [1 - \Sigma_R(s_{\text{pole}})]^{-1} [m + \Omega(s_{\text{pole}})] \right), \quad (1.12)$$

where m is the tree-level quark mass. In the case of the top quark with the approximation of this paper that the CKM mixing is absent, the self-energy functions are numbers, not matrices in flavor space, and the absence of complex couplings implies that $\bar{\Omega} = \Omega$. The loop expansion of the self-energy functions is:

$$\Sigma_L = \frac{1}{16\pi^2} \Sigma_L^{(1)} + \frac{1}{(16\pi^2)^2} \Sigma_L^{(2)} + \dots \quad (1.13)$$

$$\Sigma_R = \frac{1}{16\pi^2} \Sigma_R^{(1)} + \frac{1}{(16\pi^2)^2} \Sigma_R^{(2)} + \dots \quad (1.14)$$

$$\Omega = \frac{1}{16\pi^2} \Omega^{(1)} + \frac{1}{(16\pi^2)^2} \Omega^{(2)} + \dots, \quad (1.15)$$

and it follows that the complex pole mass is the solution of

$$s_{\text{pole}} = m^2 + \frac{1}{16\pi^2} \Pi^{(1)}(s_{\text{pole}}) + \frac{1}{(16\pi^2)^2} \Pi^{(2)}(s_{\text{pole}}) + \dots, \quad (1.16)$$

where

$$\Pi^{(1)}(s) = m^2 [\Sigma_L^{(1)}(s) + \Sigma_R^{(1)}(s)] + 2m\Omega^{(1)}(s), \quad (1.17)$$

$$\begin{aligned} \Pi^{(2)}(s) = & m^2 [\Sigma_L^{(2)}(s) + \Sigma_R^{(2)}(s) + \Sigma_L^{(1)}(s)\Sigma_R^{(1)}(s) + (\Sigma_L^{(1)}(s))^2 + (\Sigma_R^{(1)}(s))^2] \\ & + 2m[\Omega^{(2)}(s) + \Omega^{(1)}(s)(\Sigma_L^{(1)}(s) + \Sigma_R^{(1)}(s))] + [\Omega^{(1)}(s)]^2. \end{aligned} \quad (1.18)$$

Now, expanding s_{pole} about m^2 , one obtains

$$s_{\text{pole}} = m^2 + \frac{1}{16\pi^2} \Pi^{(1)}(m^2) + \frac{1}{(16\pi^2)^2} \left[\Pi^{(2)}(m^2) + \Pi^{(1)}(m^2) \Pi^{(1)'}(m^2) \right] + \dots, \quad (1.19)$$

The remaining part of the calculation is very similar to the strategy used in refs. [37–39], so the reader is referred to those papers for more details, and only a brief outline will be given here. The fermion self-energy functions are computed in terms of bare couplings and masses in $d = 4 - 2\epsilon$ dimensions, and then the bare quantities are expanded in terms of renormalized $\overline{\text{MS}}$ quantities. This is more efficient than doing counterterm diagrams separately. The Tarasov algorithm [70] is then used to reduce the loop integrals to a set of basis functions, for which I use the

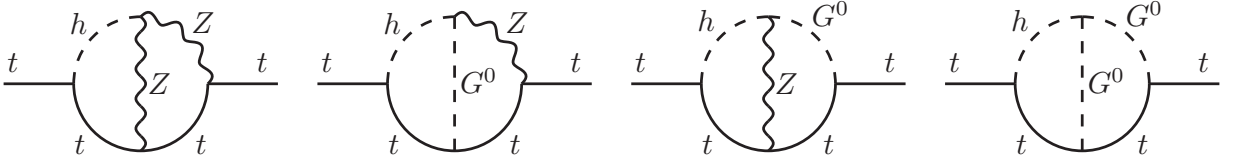


FIG. 1.2: Feynman diagrams that individually involve the basis integrals $M(h, Z, t, t, 0)$ and $M(h, 0, t, t, Z)$ and $M(h, 0, t, t, 0)$, coming from the unphysical degrees of freedom from the Z and the neutral Goldstone boson. In the total pole squared mass, the contributions proportional to each of those basis integrals cancel.

conventions and notations of refs. [71, 72], with 1-loop basis integrals $A(x) \equiv x\overline{\ln}(x) - x$ and $B(x, y)$ and 2-loop basis integrals $I(x, y, z)$, $S(x, y, z)$, $T(x, y, z)$, $\overline{T}(0, x, y)$, $U(x, y, z, u)$, and $M(x, y, z, u, v)$. Here x, y, z, \dots are squared mass arguments, and there are also implicit arguments for the external momentum invariant s and the renormalization scale Q , and

$$\overline{\ln}(x) \equiv \ln(x/Q^2). \quad (1.20)$$

The software package TSIL [72] is used to evaluate the basis integral functions. In some cases, the basis integrals can be evaluated analytically in terms of polylogarithms, which TSIL does using results from refs. [11, 13, 22, 71–79]. When this is not possible, TSIL instead computes the basis integrals by Runge-Kutta integration of the differential equations in the external momentum invariant s as found in ref. [71], using methods similar to those in refs. [80]. The 1-loop self-energy functions of s are then expanded around the tree-level squared mass t . Due to the presence of the massless gluon and photon, this expansion results in threshold logarithms $\overline{\ln}(t - s)$, which cancel against 2-loop contributions, providing a useful check. Another important and non-trivial check is the cancellation of poles in ϵ .

The resulting expression depends on the tree-level Goldstone boson squared mass $G = m^2 + \lambda v^2$ and the Higgs boson squared mass $H = m^2 + 3\lambda v^2$. Following the procedure in section IV of ref. [34], the Goldstone boson squared mass contributions are resummed, eliminating G completely, and eliminating H in favor of h defined in eq. (1.8), so that the Higgs boson squared mass parameter m^2 does not appear in the resulting expression. Another useful check is the absence of singularities as $G \rightarrow 0$. Finally, in the remaining expressions, yet another useful check is provided by the cancellation between the Goldstone contributions and the unphysical components of the electroweak vector bosons. For one example, consider the Feynman diagrams shown in figure 1.2. The contributions to s_{pole} from these individual diagrams from the neutral Goldstone boson G^0 and from the unphysical degrees of freedom of the Z boson involve the basis integrals $M(h, Z, t, t, 0)$ and $M(h, 0, t, t, Z)$ and $M(h, 0, t, t, 0)$. However, in the sum, those unphysical contributions cancel, and these basis integrals do not appear at all in the result for the pole squared mass. This, along with many other similar cases, illustrates how gauge invariance provides non-trivial checks, despite the calculation being restricted to Landau gauge.

The rest of this paper is organized as follows. In section II, results are presented for the full 2-loop complex pole mass of the top-quark, together with a review of the known 4-loop pure QCD results in a compatible form, using an expansion in which the external momentum invariant for the loop integrals is the tree-level squared mass t . In section III, the same results are presented after a re-expansion in which the external momentum invariant for the loop basis integrals is (the real part of) the top-quark pole squared mass. These two expansions differ by amounts that are formally of 5-loop order in pure QCD and 3-loop order for other terms. In section IV, I compare these two expansions for a realistic set of numerical input parameters, which provides a test of the unavoidable arbitrariness associated with the truncation of perturbation theory. I also study the renormalization scale dependence of the approximation.

Section V contains some concluding remarks. Some integral identities that were useful for the calculation are recorded in an Appendix.

II. COMPLEX POLE SQUARED MASS OF THE TOP-QUARK

In this section, the complex pole squared mass of the top-quark is written in the form:

$$s_{\text{pole}} = t + \frac{1}{16\pi^2} [g_3^2 \delta_{\text{QCD}}^{(1)} + \delta_{\text{non-QCD}}^{(1)}] + \frac{1}{(16\pi^2)^2} [g_3^4 \delta_{\text{QCD}}^{(2)} + g_3^2 \delta_{\text{mixed}}^{(2)} + \delta_{\text{non-QCD}}^{(2)}] \\ + \frac{1}{(16\pi^2)^3} g_3^6 \delta_{\text{QCD}}^{(3)} + \frac{1}{(16\pi^2)^4} g_3^8 \delta_{\text{QCD}}^{(4)} + \dots, \quad (2.1)$$

where $t = y_t^2 v^2/2$ is the tree-level squared mass, with v taken to minimize the full loop-corrected Landau gauge effective potential. In this section, all of the basis loop integrals are implicitly taken to be evaluated with the external squared momentum invariant set to $s = t$. I begin by reviewing the known pure QCD results. At 1-loop and 2-loop order, one has [1, 11]:

$$\delta_{\text{QCD}}^{(1)} = C_F t [8 - 6\overline{\ln}(t)], \quad (2.2)$$

$$\delta_{\text{QCD}}^{(2)} = C_F t \left[C_G \left(\frac{1111}{12} - \frac{8\pi^2}{3} + 8\pi^2 \ln(2) - 12\zeta_3 - \frac{185}{3} \overline{\ln}(t) + 11\overline{\ln}^2(t) \right) \right. \\ + C_F \left(-\frac{7}{4} + 10\pi^2 - 16\pi^2 \ln(2) + 24\zeta_3 - 15\overline{\ln}(t) + 18\overline{\ln}^2(t) \right) \\ + T_F \sum_{i=1}^{n_q} \left\{ 4(1 - q_i/t) [T(q_i, q_i, t) + \overline{\ln}(q_i) [2 - \overline{\ln}(t)]] \right. \\ \left. - 4(1 + q_i/t) U(t, 0, q_i, q_i) - 4(q_i/t) [2\overline{\ln}(q_i) + 1] + [4\overline{\ln}(t) + 1]/3 \right\} \Big], \quad (2.3)$$

where the QCD group theory quantities are

$$C_G = 3, \quad C_F = 4/3, \quad T_F = 1/2, \quad n_q = 6, \quad (2.4)$$

and the q_i are the quark squared masses. For $q_i = t$, one has (for $s = t$):

$$U(t, 0, t, t) = \frac{11}{2} - \frac{2\pi^2}{3} - 3\overline{\ln}(t) + \frac{1}{2}\overline{\ln}^2(t). \quad (2.5)$$

When q_i is a lighter quark squared mass, the integrals $T(q, q, t)$ and $U(t, 0, q, q)$ for $s = t$ are known [11, 72, 76, 77] in terms of dilogarithms, but it is practical to expand them for small quark masses:

$$T(q, q, t) = -\overline{\ln}(q) [2 - \overline{\ln}(t)] - \frac{1}{2} - \frac{\pi^2}{3} + \overline{\ln}(t) - \frac{1}{2}\overline{\ln}^2(t) + \mathcal{O}(q/t), \quad (2.6)$$

$$U(t, 0, q, q) = \frac{11}{2} + \frac{\pi^2}{3} - 3\overline{\ln}(t) + \frac{1}{2}\overline{\ln}^2(t) - 2\pi^2 \sqrt{q/t} + \mathcal{O}(q/t). \quad (2.7)$$

The leading correction due to small non-zero quark masses thus is of order m_q/m_t and comes only from the $U(t, 0, q, q)$ integral. Plugging in the group theory quantities, and keeping only the leading part in the light-quark mass expansion, one obtains:

$$\delta_{\text{QCD}}^{(2)} = t \left[2309/9 + 16\pi^2/9 + 32\pi^2 \ln(2)/9 - 16\zeta_3/3 - 204\overline{\ln}(t) + 60\overline{\ln}^2(t) \right. \\ \left. + \frac{16\pi^2}{3} \sum_{q=b,c,s,\dots} m_q/m_t + \mathcal{O}(m_q^2/m_t^2) \right]. \quad (2.8)$$

Here it is not clear whether it is best to use pole or running masses for m_q , since the resulting difference for s_{pole} between these choices is of the same parametric order as the presently unknown dependence of the 3-loop corrections on the lighter quark masses. However, even if one uses $\sum_q m_q = 7$ GeV, the net effect is to raise the top-quark pole mass by only about 14 MeV, which is small compared to both experimental and other theoretical uncertainties.

The 3-loop and 4-loop pure-QCD contributions can be written in the forms:

$$\delta_{\text{QCD}}^{(3)} = t \left[c_{3,0} + c_{3,1} \overline{\ln}(t) + c_{3,2} \overline{\ln}^2(t) + c_{3,3} \overline{\ln}^3(t) \right], \quad (2.9)$$

$$\delta_{\text{QCD}}^{(4)} = t \left[c_{4,0} + c_{4,1} \overline{\ln}(t) + c_{4,2} \overline{\ln}^2(t) + c_{4,3} \overline{\ln}^3(t) + c_{4,4} \overline{\ln}^4(t) \right], \quad (2.10)$$

where the 3-loop results from [14] are

$$\begin{aligned} c_{3,0} = & 551909/81 + 1589684\pi^2/1215 + 700\pi^4/81 - 42304\pi^2 \ln(2)/81 \\ & - 512\pi^2 \ln^2(2)/27 - 320 \ln^4(2)/9 - (2560/3)\text{Li}_4(1/2) - 13328\zeta_3/9 \\ & - 11512\pi^2\zeta_3/27 + 31600\zeta_5/27, \end{aligned} \quad (2.11)$$

$$c_{3,1} = -53696/9 - 352\pi^2/9 - 704\pi^2 \ln(2)/9 + 2272\zeta_3/3, \quad (2.12)$$

$$c_{3,2} = 2300, \quad (2.13)$$

$$c_{3,3} = -440, \quad (2.14)$$

while the 4-loop coefficients are [18]

$$c_{4,0} = (4.91 \pm 0.11) \times 10^5, \quad (2.15)$$

$$\begin{aligned} c_{4,1} = & -5110172/27 - 46066276\pi^2/1215 - 26348\pi^4/81 + 1231424\pi^2 \ln(2)/81 \\ & + 14848\pi^2 \ln^2(2)/27 + 9280 \ln^4(2)/9 + (74240/3)\text{Li}_4(1/2) \\ & + 1824176\zeta_3/27 + 333848\pi^2\zeta_3/27 - 1103600\zeta_5/27, \end{aligned} \quad (2.16)$$

$$c_{4,2} = 750374/9 + 5104\pi^2/9 + 320\pi^2 \ln(2)/9 + 3296\pi^2 \ln(2)/3 - 40624\zeta_3/3, \quad (2.17)$$

$$c_{4,3} = -65740/3, \quad (2.18)$$

$$c_{4,4} = 3190. \quad (2.19)$$

The above coefficients that are associated with logarithms of Q ($c_{3,n}$ and $c_{4,n}$ with $n \geq 1$) follow from the corresponding beta functions for g_3 at 2-loop [81] and 3-loop [82] order and for y_t at 2-loop [83], 3-loop [84, 85] and 4-loop [86, 87] order. Note that the 4-loop non-logarithmic contribution of eq. (2.15) is only known numerically at present, with an uncertainty from numerical integration [18, 88]. The results given above are equivalent to those found in refs. [1, 11, 14, 18], but are cosmetically different because the expansion in the present paper is for the pole squared mass written in terms of the tree-level squared mass. The actual difference is of 5-loop order.

The 1-loop non-QCD contribution to the top-quark pole-squared mass can be obtained straightforwardly as:

$$\begin{aligned} \delta_{\text{non-QCD}}^{(1)} = & Q_t^2 e^2 t [8 - 6\overline{\ln}(t)] + \frac{g^2}{4} \left\{ [2W - b - t - (b-t)^2/W] B(b, W) \right. \\ & + (2W - t + b) A(W)/W + 2t - 2A(b) \left. \right\} + [(I_3^t)^2 (g^2 + g'^2) \\ & - 2I_3^t Q_t g'^2 + 2Q_t^2 g'^4/(g^2 + g'^2)] [(Z - t) B(t, Z) + A(Z) - A(t) + t] \\ & + 2Q_t g'^2 [-I_3^t + Q_t g'^2/(g^2 + g'^2)] t [3B(t, Z) - 2] + \\ & + y_t^2 [(h - 4t) B(h, t) + A(h) - 2A(t) - A(b)]/2 - y_b^2 A(b)/2, \end{aligned} \quad (2.20)$$

where $e = gg'/\sqrt{g^2 + g'^2}$ and

$$Q_t = 2/3, \quad I_3^t = 1/2 \quad (2.21)$$

are the electric charge and third component of weak isospin for the top quark. In eq. (2.20), I have kept the contributions from a non-zero bottom-quark Yukawa coupling, but it is completely negligible in practice. This is partly because the leading dependence on b of eq. (2.20) is of order b/t , not $\sqrt{b/t}$ as in the 2-loop QCD contribution of eq. (2.8).

The 2-loop mixed and non-QCD contributions, $\delta_{\text{mixed}}^{(2)}$ and $\delta_{\text{non-QCD}}^{(2)}$ in eq. (2.1), both have the form:

$$\sum_j c_j^{(2)} I_j^{(2)} + \sum_{j \leq k} c_{j,k}^{(1,1)} I_j^{(1)} I_k^{(1)} + \sum_j c_j^{(1)} I_j^{(1)} + c^{(0)}, \quad (2.22)$$

where $I^{(1)}$ and $I^{(2)}$ are lists of 1-loop and 2-loop basis integrals defined in the conventions of [71, 72], and the coefficients $c_j^{(2)}$ and $c_{j,k}^{(1,1)}$ and $c_j^{(1)}$ and $c^{(0)}$ consist of rational functions of the tree-level squared masses t, Z, W, h , multiplied by a global factor of $1/v^2$ for $\delta_{\text{mixed}}^{(2)}$ and $1/v^4$ for $\delta_{\text{non-QCD}}^{(2)}$. These coefficients are the main new results of this paper. However, they are complicated, and in practice will be evaluated by computer, so these results are relegated to ancillary electronic files. For both the mixed and non-QCD 2-loop contributions, I set the lighter quark and lepton Yukawa couplings to 0, so the list of necessary 1-loop basis integrals is:

$$I^{(1)} = \{A(h), A(t), A(W), A(Z), B(h, t), B(t, Z), B(0, W)\}, \quad (2.23)$$

The list of 2-loop integrals needed for the mixed contributions is

$$\begin{aligned} I_{\text{mixed}}^{(2)} = & \{\zeta_2, I(0, t, W), I(h, t, t), I(t, t, Z), M(0, 0, t, W, 0), M(0, t, t, 0, t), \\ & M(0, t, t, h, t), M(0, t, t, Z, t), T(h, 0, t), T(W, 0, 0), T(Z, 0, t), \bar{T}(0, 0, W), \\ & \bar{T}(0, h, t), \bar{T}(0, t, Z), U(0, W, 0, t), U(t, h, t, t), U(t, Z, t, t)\}, \end{aligned} \quad (2.24)$$

while the 2-loop basis for the non-QCD case contains 49 additional integrals:

$$\begin{aligned} I_{\text{non-QCD}}^{(2)} = & I_{\text{mixed}}^{(2)} \cup \{I(0, h, W), I(0, h, Z), I(0, W, Z), I(h, h, h), I(h, W, W), \\ & I(h, Z, Z), I(W, W, Z), M(0, 0, W, W, 0), M(0, 0, W, W, Z), \\ & M(0, t, W, 0, W), M(0, t, W, h, W), M(0, t, W, Z, W), M(0, Z, W, t, 0), \\ & M(h, h, t, t, h), M(h, t, t, h, t), M(h, t, t, Z, t), M(h, Z, t, t, Z), \\ & M(t, t, Z, Z, h), M(t, Z, Z, t, t), S(0, 0, 0), S(0, h, W), T(h, 0, W), \\ & T(h, h, t), T(h, t, Z), T(t, h, Z), T(W, 0, h), T(W, 0, Z), T(W, t, W), \\ & T(Z, 0, W), T(Z, h, t), T(Z, t, Z), U(0, W, 0, 0), U(0, W, h, W), \\ & U(0, W, W, Z), U(h, t, 0, W), U(h, t, h, t), U(h, t, t, Z), U(t, 0, W, W), \\ & U(t, h, h, h), U(t, h, W, W), U(t, h, Z, Z), U(t, Z, 0, 0), U(t, Z, h, Z), \\ & U(t, Z, W, W), U(W, 0, 0, h), U(W, 0, 0, Z), U(Z, t, 0, W), \\ & U(Z, t, h, t), U(Z, t, t, Z)\}. \end{aligned} \quad (2.25)$$

The expressions for $\delta_{\text{mixed}}^{(2)}$ and for $\delta_{\text{non-QCD}}^{(2)}$ are provided in ancillary files called `delta2mixed_secII.txt` and `delta2nonQCD_secII.txt`, respectively. These files are available with the arXiv submission for this paper. It should

be noted that the presentation of these results is not unique, because of the existence of identities that hold between different basis integrals when the squared mass arguments are not generic. The relevant identities are listed in the Appendix.

III. RE-EXPANSION OF THE POLE SQUARED MASS

The results of the previous section can be rewritten by self-consistently re-expanding the loop integrals that depend on t , writing them instead in terms of the real part of the pole squared mass,

$$T \equiv \text{Re}[s_{\text{pole}}]. \quad (3.1)$$

The resulting expression is written as

$$\begin{aligned} s_{\text{pole}} = & t + \frac{1}{16\pi^2} [g_3^2 \Delta_{\text{QCD}}^{(1)} + \Delta_{\text{non-QCD}}^{(1)}] + \frac{1}{(16\pi^2)^2} [g_3^4 \Delta_{\text{QCD}}^{(2)} + g_3^2 \Delta_{\text{mixed}}^{(2)} + \Delta_{\text{non-QCD}}^{(2)}] \\ & + \frac{1}{(16\pi^2)^3} g_3^6 \Delta_{\text{QCD}}^{(3)} + \frac{1}{(16\pi^2)^4} g_3^8 \Delta_{\text{QCD}}^{(4)} + \dots \end{aligned} \quad (3.2)$$

and differs from the results of the preceding section by amounts of higher order, namely 5-loop order in the pure QCD part, and 3-loop order in the other parts. The pure QCD contributions are easily obtained from the results of the preceding section, or directly from refs. [1, 11, 14, 18]:

$$\Delta_{\text{QCD}}^{(1)} = T[32/3 - 8\overline{\ln}(T)], \quad (3.3)$$

$$\begin{aligned} \Delta_{\text{QCD}}^{(2)} = & T \left[2053/9 + 16\pi^2/9 + 32\pi^2 \ln(2)/9 - 16\zeta_3/3 - (292/3)\overline{\ln}(T) - 4\overline{\ln}^2(T) \right. \\ & \left. + \frac{16\pi^2}{3} \sum_{q=b,c,s,\dots} M_q/M_t + \mathcal{O}(M_q^2/M_t^2) \right], \end{aligned} \quad (3.4)$$

$$\Delta_{\text{QCD}}^{(3)} = T \left[a_{3,0} + a_{3,1}\overline{\ln}(T) + a_{3,2}\overline{\ln}^2(T) + a_{3,3}\overline{\ln}^3(T) \right], \quad (3.5)$$

$$\Delta_{\text{QCD}}^{(4)} = T \left[a_{4,0} + a_{4,1}\overline{\ln}(T) + a_{4,2}\overline{\ln}^2(T) + a_{4,3}\overline{\ln}^3(T) + a_{4,4}\overline{\ln}^4(T) \right], \quad (3.6)$$

where the 3-loop coefficients are:

$$\begin{aligned} a_{3,0} = & 420365/81 + 1560884\pi^2/1215 + 700\pi^4/81 - 46144\pi^2 \ln(2)/81 \\ & - 512\pi^2 \ln^2(2)/27 - 320 \ln^4(2)/9 - (2560/3)\text{Li}_4(1/2) - 12688\zeta_3/9 \\ & - 11512\pi^2\zeta_3/27 + 31600\zeta_5/27, \end{aligned} \quad (3.7)$$

$$a_{3,1} = -5648/3 - 32\pi^2/3 - 64\pi^2 \ln(2)/3 + 672\zeta_3, \quad (3.8)$$

$$a_{3,2} = -36, \quad (3.9)$$

$$a_{3,3} = 8, \quad (3.10)$$

and the 4-loop coefficients are:

$$a_{4,0} = (3.64 \pm 0.11) \times 10^5, \quad (3.11)$$

$$\begin{aligned} a_{4,1} = & -4581172/81 - 20170532\pi^2/1215 - 15148\pi^4/81 + 616000\pi^2 \ln(2)/81 \\ & + 6656\pi^2 \ln^2(2)/27 + 4160 \ln^4(2)/9 + (33280/3)\text{Li}_4(1/2) \\ & + 1061552\zeta_3/27 + 149656\pi^2\zeta_3/27 - 598000\zeta_5/27, \end{aligned} \quad (3.12)$$

$$a_{4,2} = 11482/3 + 208\pi^2/3 - 960\pi^2 \ln(2) + 3296\pi^2 \ln(2)/3 - 1808\zeta_3, \quad (3.13)$$

$$a_{4,3} = 244/3, \quad (3.14)$$

$$a_{4,4} = -26. \quad (3.15)$$

The 1-loop non-QCD part has the same form as eq. (2.20) with the replacement $t \rightarrow T$:

$$\begin{aligned} \Delta_{\text{non-QCD}}^{(1)} = & Q_t^2 e^2 T [8 - 6\overline{\ln}(T)] + \frac{g^2}{4} \left\{ [2W - b - T - (b - T)^2/W] B(b, W) \right. \\ & + (2W - T + b) A(W)/W + 2T - 2A(b) \left. \right\} + [(I_3^t)^2 (g^2 + g'^2) \\ & - 2I_3^t Q_t g'^2 + 2Q_t^2 g'^4/(g^2 + g'^2)] [(Z - T)B(T, Z) + A(Z) - A(T) + T] \\ & + 2Q_t g'^2 [-I_3^t + Q_t g'^2/(g^2 + g'^2)] T [3B(T, Z) - 2] + \\ & + y_t^2 [(h - 4T)B(h, T) + A(h) - 2A(T) - A(b)]/2 - y_b^2 A(b)/2. \end{aligned} \quad (3.16)$$

The 2-loop parts below absorb the residual terms from the expansion of t about T ,

$$t = T - \frac{1}{16\pi^2} \left(g_3^2 \Delta_{\text{QCD}}^{(1)} + \text{Re}[\Delta_{\text{non-QCD}}^{(1)}] \right) + \dots \quad (3.17)$$

Note that in eq. (3.16) I have chosen to keep the vertex coupling y_t as it is; only the 1-loop t 's that come from propagators of loop integrals are re-expanded in terms of T . This choice affects the residual terms that are absorbed into the 2-loop parts, and is somewhat arbitrary, but is motivated by the idea that resumming the internal top-quark propagators in the diagram should result in poles close to the on-shell mass, but there is no reason why the vertex y_t 's should resum in the same way.

The resulting 2-loop mixed and non-QCD contributions, $\Delta_{\text{mixed}}^{(2)}$ and $\Delta_{\text{non-QCD}}^{(2)}$, have the same form as eq. (2.22) in the previous section, but now the lists of necessary basis integrals $I^{(1)}$ and $I_{\text{mixed}}^{(2)}$ and $I_{\text{non-QCD}}^{(2)}$ are obtained from those given in the previous section by replacing t by T everywhere, including as the implicit external momentum squared argument of the loop integral functions.[†] In addition, the list $I^{(1)}$ used in the 2-loop parts must be augmented to include the real part of $B(0, W)$:

$$I^{(1)} = \{A(h), A(T), A(W), A(Z), B(h, T), B(T, Z), B(0, W), \text{Re}[B(0, W)]\}. \quad (3.18)$$

The reason for this addition is that the re-expansion of t in terms of T , eq. (3.17), involves the real part of $\Delta_{\text{non-QCD}}^{(1)}$. The only complex part of $\Delta_{\text{non-QCD}}^{(1)}$ is proportional to the basis integral $B(0, W)$, which has an imaginary part corresponding to the 2-body decay $t \rightarrow bW$. The new coefficients $c_j^{(2)}$ and $c_{j,k}^{(1,1)}$ and $c_j^{(1)}$ and $c^{(0)}$ are now rational functions of T, W, Z, h , multiplied by a factor of $1/v^2$ for $\Delta_{\text{mixed}}^{(2)}$ and $1/v^4$ for $\Delta_{\text{non-QCD}}^{(2)}$. These results are given in ancillary electronic files, `Delta2mixed_secIII.txt` and `Delta2nonQCD_secIII.txt`, included with the arXiv submission for this paper. As in the previous section, the presentation of these results is not unique because of the basis loop integral identities given in the Appendix.

[†] However, note that in the 2-loop parts $\Delta_{\text{mixed}}^{(2)}$ and $\Delta_{\text{non-QCD}}^{(2)}$, one could justify using T and t interchangeably, because the difference is of higher order and thus formally comparable to other 3-loop non-pure-QCD terms that remain uncalculated at this time. Here I choose to use T , in solidarity with the 1-loop terms of eq. (3.16) and the pure QCD contributions of eqs. (3.3)-(3.6). This has the practical benefit that if T is given as an input, t can be extracted without having to re-compute the 2-loop integrals in iteration.

IV. NUMERICAL RESULTS

For the purposes of a numerical illustration of the results obtained above, consider a set of Standard Model benchmark $\overline{\text{MS}}$ parameters

$$y_t(Q_0) = 0.93690, \quad (4.1)$$

$$g_3(Q_0) = 1.1666, \quad (4.2)$$

$$g(Q_0) = 0.647550, \quad (4.3)$$

$$g'(Q_0) = 0.358521, \quad (4.4)$$

$$\lambda(Q_0) = 0.12597, \quad (4.5)$$

$$v(Q_0) = 246.647 \text{ GeV}, \quad (4.6)$$

defined at the $\overline{\text{MS}}$ input renormalization scale

$$Q_0 = 173.34 \text{ GeV}. \quad (4.7)$$

These parameters are the same as used in ref. [39]. As mentioned there, the real parts of the pole masses of the Higgs, W , and Z bosons, as calculated in refs. [37], [38], and [39] respectively, are:

$$M_h = 125.09 \text{ GeV}, \quad (4.8)$$

$$M_W = 80.329 \text{ GeV}, \quad (4.9)$$

$$M_Z = 91.154 \text{ GeV}, \quad (4.10)$$

with the latter two corresponding to the experimental Breit-Wigner lineshape masses

$$M_W^{\text{exp}} = 80.356 \text{ GeV}, \quad (4.11)$$

$$M_Z^{\text{exp}} = 91.188 \text{ GeV}. \quad (4.12)$$

Also, although it will play no direct role in the following, I note for completeness that the running $\overline{\text{MS}}$ Higgs squared mass parameter is $m^2(Q_0) = -(92.890 \text{ GeV})^2$, found by using the full 2-loop effective potential [31] with the leading QCD and top-Yukawa corrections [32] and Goldstone boson resummation [34, 35] (see also [36]), while one finds that the value obtained by including the 4-loop pure QCD corrections to the effective potential [33] is only slightly different: $m^2(Q_0) = -(92.926 \text{ GeV})^2$. For simplicity, I set $y_b = 0$, because it has a very small effect, as noted above.

Using these input parameters, the computed top-quark pole mass M_t is shown as a function of the choice of Q in figure 4.1, in various approximations. The figure was made by first using the 3-loop Standard Model renormalization group equations, found in refs. [89–94] and as implemented in the program `SMH` [37], to run the input parameters from the input scale Q_0 to the scale Q . Then, the formulas of section III are applied to find s_{pole} , using `TSIL` [72]. In addition to the integrals that are analytically known in terms of polylogarithms, only 12 calls of the relatively time-consuming Runge-Kutta evaluation function `TSIL_Evaluate` are needed in the full 2-loop case, because multiple basis integrals stemming from the same master topology are evaluated simultaneously. The total time to compute all of the basis integrals is well under 1 second on modern desktop or laptop computer hardware. For each point, a few iterations are required to self-consistently evaluate the complex pole mass, updating T with each iteration. In practical applications, the process will be different; one might supply T as an experimental input, and derive t and

therefore y_t from it. In that case, as noted in the previous footnote, iteration of the 2-loop part is unnecessary when using the formulas of section III. In such applications, only the 1-loop part will require iteration, because of the explicit appearance of y_t in eq. (3.16).

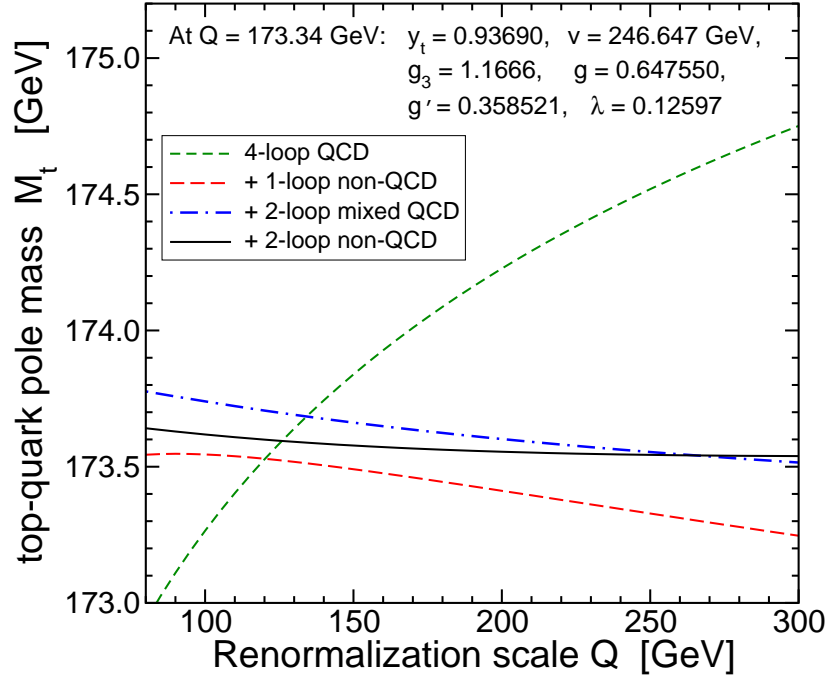


FIG. 4.1: The real part M_t of the top-quark pole mass as a function of the $\overline{\text{MS}}$ renormalization scale Q at which it is computed, in successive approximations from section III. The short-dashed (green) line is the result found in pure QCD at 4-loop order from using eqs. (3.3)-(3.15) in eq. (3.2). The long-dashed (red) line includes also the 1-loop non-QCD contributions $\Delta_{\text{non-QCD}}^{(1)}$ from eq. (3.16). The dot-dashed (blue) line adds in the 2-loop mixed QCD contributions $\Delta_{\text{mixed}}^{(2)}$ found in the ancillary file `Delta2mixed_secIII.txt`. The solid (black) line adds in the 2-loop non-QCD contributions $\Delta_{\text{non-QCD}}^{(2)}$ found in the ancillary file `Delta2nonQCD_secIII.txt`. The input parameters $y_t, v, g_3, g, g', \lambda$ at Q are obtained by 3-loop renormalization group running starting from eqs. (4.1)-(4.7).

The dashed line in figure 4.1 shows the result of the 4-loop pure QCD calculation as given in eqs. (3.3)-(3.15) above. The pure QCD result for M_t is seen to still have a significant scale dependence of more than 1.7 GeV for $80 \text{ GeV} < Q < 300 \text{ GeV}$, due to the effects of y_t and the electroweak couplings. This scale dependence is greatly reduced by including also the 1-loop non-QCD contributions from $\Delta_{\text{non-QCD}}^{(1)}$, as shown by the dashed (red) line. Further including the 2-loop mixed QCD corrections $\Delta_{\text{mixed}}^{(2)}$, as shown by the blue (dot-dashed) line, changes the result by less than 300 MeV for any choice of Q . Finally, using the full set of contributions in eq. (3.2) by including $\Delta_{\text{non-QCD}}^{(2)}$ as well, one obtains the solid (black) line with very little Q dependence. Note that with the choice $Q = M_t$, the most complete result given here for M_t is approximately 470 MeV lower than the 4-loop pure QCD result.

The complex pole mass also includes the parameter Γ_t , which corresponds to the total decay width of the top quark. This is shown in the same way as for M_t in Figure 4.2, again using the formulas in section III. In this case, the width is 0 as long as only pure QCD effects are included, so the first approximation shown includes the 4-loop pure QCD together with the 1-loop non-QCD contributions to the complex pole mass, $\Delta_{\text{non-QCD}}^{(1)}$, as the short-dashed (red) line. The long-dashed (blue) line includes also the $\Delta_{\text{mixed}}^{(2)}$ contribution, which lowers the prediction for the width by about 10%, but without a dramatic effect on the Q -dependence. Finally, including the full 2-loop effects (on top of the pure QCD 4-loop part) significantly reduces the Q -dependence. This is shown as the solid (black) line in figure 4.2. Of course, this result for Γ_t is not as useful or complete as a fully differential NNLO calculation of the decay width, as described for example in refs. [95, 96] and references therein. However, it is reassuring that the result found here is very nearly Q -independent.

In presenting the results above, I chose to use the expansion of section III rather than that of section II. At least in

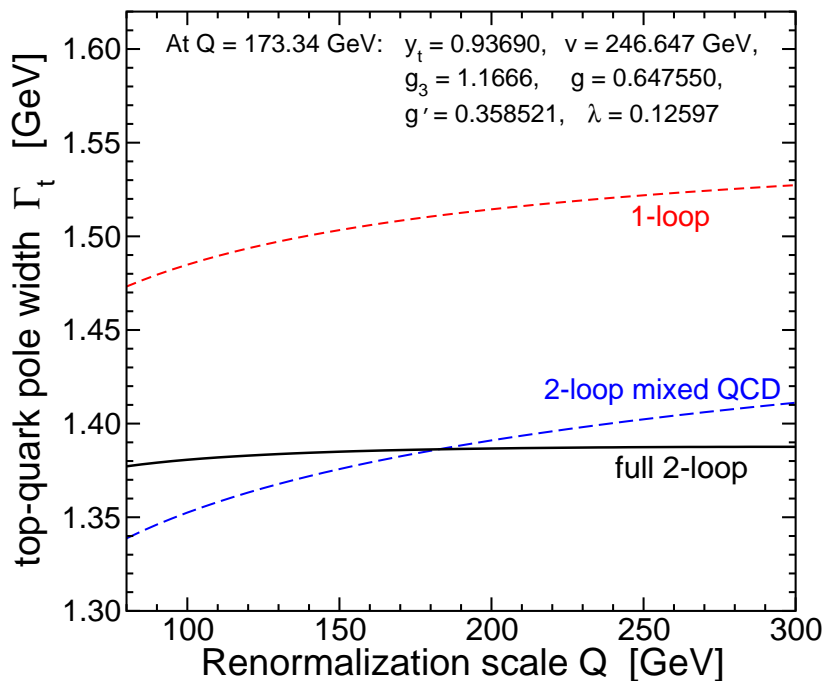


FIG. 4.2: The pole mass width Γ_t of the top quark, as a function of the $\overline{\text{MS}}$ renormalization scale Q at which it is computed, in successive approximations from section III. The short-dashed (red) line is the result found from the 1-loop non-QCD contributions $\Delta_{\text{non-QCD}}^{(1)}$ from eq. (3.16), in eq. (3.2), together with the pure QCD at 4-loop order from eqs. (3.3)-(3.15). The long-dashed (blue) line adds in the 2-loop mixed QCD contributions $\Delta_{\text{mixed}}^{(2)}$. The solid (black) line adds in the 2-loop non-QCD contributions $\Delta_{\text{non-QCD}}^{(2)}$. The input parameters $y_t, v, g_3, g, g', \lambda$ at Q are obtained by 3-loop renormalization group running starting from eqs. (4.1)-(4.7).

the case of the width Γ_t , it seems clear that the kinematics of the decay will be best captured by using the (real part of the) pole squared mass T rather than the running squared mass t as the external momentum squared in the loop integrals, since in general t is not close to the physical squared mass of the decaying top quark. However, the two expansions are formally equivalent within the approximations being used. The difference between them is a measure of the uncertainty introduced by the truncation of perturbation theory. This is illustrated in figure 4.3, which shows the results for M_t and Γ_t for the full 2-loop + pure QCD 4-loop expansions in terms of t from section II as dashed lines and in terms of T from section III as solid lines.

The expansion in terms of T in section III has a Q dependence that is slightly better for M_t , and significantly better for Γ_t , than the expansion in terms of t in section II. For Γ_t , this is in accord with the expectation that the expansion in terms of T should give a better approximation to the total decay width. For M_t , the total variation as Q is varied from 80 GeV to 300 GeV is only about 100 MeV. As usual, the Q dependence is only a lower bound on the theoretical error, but this seems to be reassuringly small compared to the experimental sources of error and uncertainty, at least for now. Of greater importance in the LHC era is the connection [7] between the experimental “Monte Carlo mass” determination and the pole mass or other physical versions of the top-quark mass to which it can be related by other calculations.

V. OUTLOOK

In this paper I have presented the complex top-quark pole mass at full 2-loop order, augmented by the known 4-loop QCD contributions, in the pure $\overline{\text{MS}}$ scheme. The VEV is defined to be the minimum of the full effective potential,

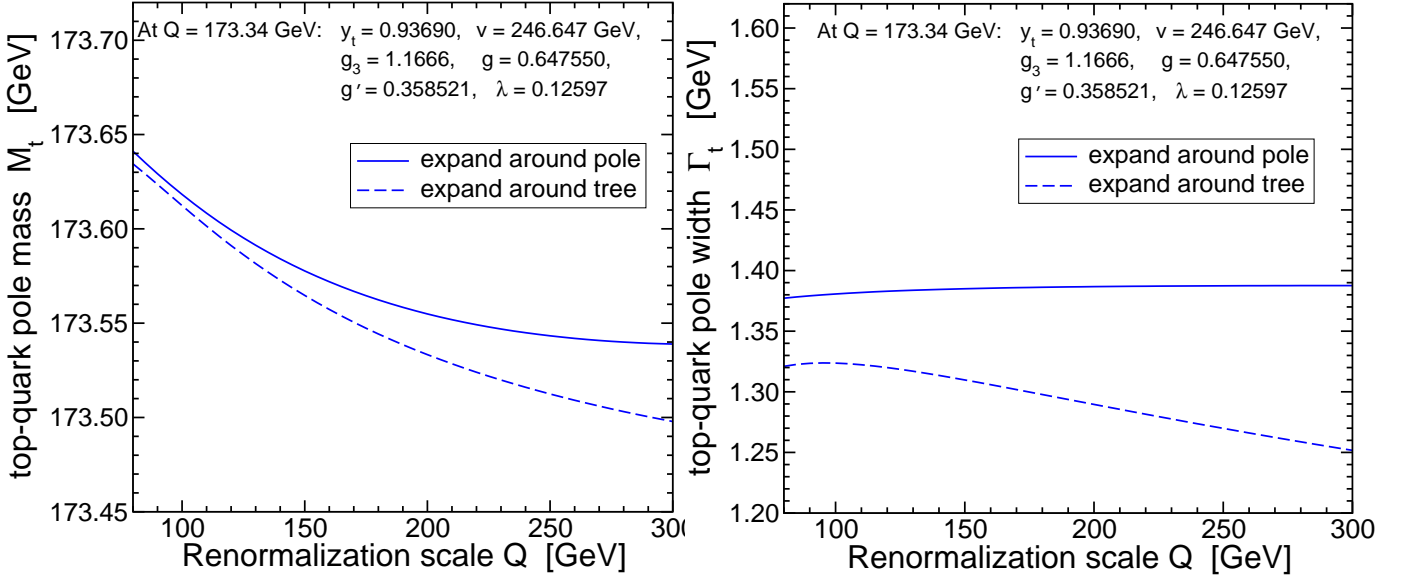


FIG. 4.3: Comparisons of the real (M_t , left panel) and imaginary (Γ_t , right panel) parts of the complex top-quark pole mass $M_t^2 - i\Gamma_t M_t$, computed using the “expansion around tree” method of section II (dashed lines), and the “expansion around pole” method of section III (solid lines). The input parameters $y_t, v, g_3, g, g', \lambda$ at the renormalization scale Q are obtained by 3-loop renormalization group running starting from eqs. (4.1)-(4.7). The differences between the methods are formally of 5-loop order in pure QCD and 3-loop order in the remaining contributions.

which makes it a specifically Landau gauge quantity, but avoids tadpole graphs. Since the VEV is dependent on the renormalization group scale, and therefore not a direct physical observable anyway, it should not be too worrisome that it is defined to be gauge-fixing dependent. The results found here are an alternative to the results of [29], which uses a tree-level definition of the running VEV that is independent of gauge-fixing but requires the presence of tadpole graphs that yield powers of $1/\lambda$ in perturbative expansions.

The results obtained in this paper differ in form, even at 1-loop order, from those found by other groups, due to the different definition of the VEV. However, one can check that at least the 1-loop contribution of eq. (2.20) is consistent with, for example, eq. (B.5) in ref. [21] or eqs. (60) and (70) of [29], after taking into account eq. (1.9) of the present paper. In ref. [29] it was noted that the $1/\lambda^\ell$ tadpole effects at loop order ℓ can all be absorbed into a running (Q -dependent) quantity $\Delta\bar{r}$, defined in terms of the Fermi constant and $\overline{\text{MS}}$ quantities, including the tree-level VEV, by

$$G_F = \frac{1 + \Delta\bar{r}}{\sqrt{2} v_{\text{tree}}^2}. \quad (5.1)$$

In view of eq. (1.9) above, one can write instead,

$$G_F = \frac{1 + \Delta\tilde{r}}{\sqrt{2} v^2}, \quad (5.2)$$

where the quantity $\Delta\tilde{r}$ is gauge-fixing dependent (because v is), but free of tadpoles, and in Landau gauge is related to $\Delta\bar{r}$ by the exact relation

$$1 + \Delta\bar{r} = (1 + \Delta\tilde{r}) \left(1 + \frac{1}{\lambda v^2} \sum_{\ell=1}^{\infty} \frac{1}{(16\pi^2)^\ell} \hat{\Delta}_\ell \right). \quad (5.3)$$

Some care must be taken in interpreting this, because the left side is implicitly a function of v_{tree} , and the right side a function of v , so that eq. (1.9) must be used again on the left side when making the equivalence beyond 1-loop order.

I have checked that with this definition, $\Delta\tilde{r}$ is indeed tadpole-free through 2-loop order, at least in the gaugeless limit for $\Delta\tilde{r}$ that was presented explicitly in eqs. (37)-(39) of ref. [28].

The 2-loop mixed and non-QCD results found in this paper are too complicated to show in print, and not amenable to unassisted human estimate anyway, so they were provided explicitly in electronic form in four ancillary files. In the near future, they will be incorporated into a publicly available computer program library, together with the results for the pole masses of the Higgs scalar and the W and Z bosons, as found in refs. [37–39] using the same scheme as here. (For a recent program with similar aims but based on a different organization of perturbation theory, see [68].) The QCD coupling g_3 is determined from other measurements. In addition, the QED coupling combination $gg'/\sqrt{g^2 + g'^2}$, can be obtained from very low-energy experiments and renormalization group running, and the VEV can be related to the Fermi constant G_F through radiative corrections, in several different schemes. In the forthcoming program, one will be able to specify either the $\overline{\text{MS}}$ inputs $y_t, g_3, g, g', \lambda, v$ with the pole masses as outputs, or to specify the pole masses as inputs with the corresponding $\overline{\text{MS}}$ parameters as outputs, or various combinations thereof. This program will be an extension of the Higgs mass program **SMH** [37], and will also include the most advanced effective potential minimization and renormalization group running available.

Appendix A SOME TWO-LOOP INTEGRAL IDENTITIES

Listed below are some identities that hold between different 2-loop basis integrals in the notation of ref. [72], for one or more squared mass arguments equal to 0. The external momentum squared invariant is denoted s , and internal propagator squared masses are denoted x, y, z . In the results of section II, s is set equal to the tree-level top-quark squared mass t , while s is set equal to $T = \text{Re}[s_{\text{pole}}]$ in section III.

$$I(0, 0, x) = -x(1 + \pi^2/6) + A(x) - A(x)^2/2x, \quad (\text{A.1})$$

$$I(0, x, x) = -2x + 2A(x) - A(x)^2/x, \quad (\text{A.2})$$

$$\begin{aligned} S(0, x, y) = & 5s/8 - x - y + [x(x-s)T(x, 0, y) + y(y-s)T(y, 0, x) \\ & + (s-x-3y)A(x)/2 + (s-3x-y)A(y)/2 + A(x)A(y) \\ & + (s^2 - 2sx - 2sy + x^2 - 2xy + y^2)B(x, y)/2]/(s-x-y), \end{aligned} \quad (\text{A.3})$$

$$S(0, 0, x) = -xT(x, 0, 0) + (s-x)B(0, x)/2 + A(x)/2 - x + 5s/8, \quad (\text{A.4})$$

$$\begin{aligned} U(x, 0, y, z) = & [1/(z-y) + 1/(s-x)]yT(y, x, z) + [1/(y-z) + 1/(s-x)]zT(z, x, y) \\ & + [2xT(x, y, z) + 2S(x, y, z) - I(0, y, z) - A(x) - A(y) - A(z) \\ & + x + y + z - s/4]/(s-x) + B(0, x)[A(y) - A(z)]/(z-y), \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} U(x, 0, y, y) = & [(s-x-4y)T(y, y, x) - 4xT(x, y, y) - 4S(x, y, y) \\ & + 2I(0, y, y) + 2A(x) + 4A(y) - 3x - 4y + 3s/2]/(x-s) \\ & - [1 + A(y)/y]B(0, x), \end{aligned} \quad (\text{A.6})$$

$$U(x, y, 0, y) = 1 - T(y, 0, x) + [1 - A(y)/y]B(x, y), \quad (\text{A.7})$$

$$U(x, 0, 0, 0) = 1 - \overline{T}(0, 0, x) + 2B(0, x). \quad (\text{A.8})$$

The remaining identities below hold only with $s = t$, with t being one or more of the internal propagator squared masses, as indicated.

$$B(0, t) = 1 - A(t)/t, \quad (\text{A.9})$$

$$S(0, 0, t) = (5/8 - \pi^2/3)t + A(t)/2 - A(t)^2/2t, \quad (\text{A.10})$$

$$S(t, x, y) = -3t/8 - tT(t, x, y) + [A(t) + A(x) + A(y) - x - y \\ + I(0, x, y) - xT(x, t, y) - yT(y, t, x)]/2, \quad (\text{A.11})$$

$$S(t, t, t) = -3t/8 + 5A(t)/2 - 3A(t)^2/2t, \quad (\text{A.12})$$

$$T(t, 0, 0) = -1 + \pi^2/3 + A(t)^2/2t^2, \quad (\text{A.13})$$

$$T(t, 0, t) = -\pi^2/18 - A(t)/t + A(t)^2/2t^2 - B(t, t) \quad (\text{A.14})$$

$$T(t, 0, x) = [-(\pi^2/12)x - A(t) + A(t)A(x)/x - A(x)^2/4x]/t \\ - B(t, x) + (x/2t - 1)[T(x, 0, t) + B(t, x) + A(x)/x], \quad (\text{A.15})$$

$$T(t, x, x) = -2 - T(x, x, t) + A(t)/t - A(x)/x + A(t)A(x)/tx \quad (\text{A.16})$$

$$T(t, t, t) = -1 + A(t)^2/2t^2, \quad (\text{A.17})$$

$$\bar{T}(0, 0, t) = -\pi^2/3 - A(t)^2/2t^2, \quad (\text{A.18})$$

$$U(t, 0, x, y) = \{2[t - A(t)][A(y) - A(x)] + t[(y - x)T(t, x, y) \\ - 2xT(x, t, y) + 2yT(y, t, x)]\}/(t(x - y)), \quad (\text{A.19})$$

$$U(t, 0, 0, x) = (2 - x/2t)B(t, x) - (1 + x/2t)T(x, 0, t) + A(x)A(t)/tx \\ + A(x)^2/4tx - [1/x + 1/2t]A(x) + A(t)/t + (\pi^2/12)x/t, \quad (\text{A.20})$$

$$U(t, 0, 0, 0) = 3 + \pi^2/3 - 2A(t)/t + A(t)^2/2t^2, \quad (\text{A.21})$$

$$U(t, 0, t, t) = 3 - 2\pi^2/3 - 2A(t)/t + A(t)^2/2t^2, \quad (\text{A.22})$$

$$U(0, t, x, y) = [t - x - y - I(t, x, y)]/t + [2A(x)A(y) - 2yA(x) - 2xA(y) \\ - x(t - x + y)T(x, 0, y) - y(t + x - y)T(y, 0, x) \\ + (t^2 - 2tx - 2ty + x^2 - 2xy + y^2)B(x, y)]/(t(t - x - y)), \quad (\text{A.23})$$

$$U(0, t, t, x) = A(t)/t + A(x)/x - A(t)A(x)/tx + [(4t - x)B(t, x) - x \\ + I(0, 0, x) - 2I(t, t, x) + (2t - x)T(x, 0, t)]/2t, \quad (\text{A.24})$$

$$U(0, t, 0, x) = [t - x + (t - x)B(0, x) - I(0, t, x) - xT(x, 0, 0)]/t, \quad (\text{A.25})$$

$$M(0, t, t, 0, t) = (\pi^2 \ln(2) - 3\zeta_3/2)/t. \quad (\text{A.26})$$

Expansions in higher orders in $s - t$, which were needed in the calculations of this paper, are omitted for brevity but can be obtained straightforwardly from the above by using the differential equations in s listed in section IV of ref. [71]. Similarly, expansions in small Goldstone boson squared masses G can be obtained using the differential equations in section III of ref. [71]. These expansions include factors of $\overline{\ln}(t - s)$ and $\overline{\ln}(G)$, which cancel in the pole squared mass, providing useful checks.

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