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Study of the Renormalization-Group Evolution of $\mathcal{N}=1$ Supersymmetric Gauge Theories Using Padé Approximants

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We study asymptotically free SU(N_c) gauge theories with $\mathcal{N}=1$ supersymmetry, including the purely gluonic theory and theories with N_f copies of a pair of massless chiral superfields in the respective representations R and $\bar{R}$ of SU(N_c). The cases in which R is the fundamental representation and the symmetric and antisymmetric rank-2 tensor representation are considered. We calculate Padé approximants to the beta functions for these theories in the $\overline{\rm DR}$ scheme up to four-loop order for the gluonic theory and up to three-loop order for the theories with matter superfields and compare results for IR zeros and poles with results from the NSVZ beta function. Our calculations provide a quantitative measure, for these theories, of how well finite-order perturbative results calculated in one scheme reproduce properties of a known beta function calculated in a different scheme.

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I. INTRODUCTION

A fundamental issue in quantum field theories is the question of how accurately, the beta function, calculated to a finite loop order, describes the renormalization-group (RG) evolution of the theory when probed on different Euclidean energy/momentum scales μ. There are two related aspects to this question. First, higher-order terms modify the value of the beta function calculated to a given order, and this modification generically increases with the size of the interaction coupling. Second, although the one-loop and two-loop terms in a beta function are independent of the scheme used for regularization and renormalization, the three-loop and higher-loop terms depend on this scheme. Therefore, when studying a given property of a beta function at higher-loop order, it is necessary to ascertain how significant this scheme dependence is. A particularly appealing context in which to investigate properties of the beta function and their scheme dependence is a supersymmetric gauge theory, because of the strong constraints that the supersymmetry places on the properties of the theory.

In this paper we shall use Padé methods to study the beta function of vectorial, asymptotically free, $\mathcal{N}=1$ supersymmetric SU(N_c) gauge theories (at zero temperature and chemical potential). We investigate both the purely gluonic supersymmetric Yang-Mills (SYM) gauge theory and theories with matter content consisting of N_f copies of massless chiral superfields Φ_i and $\tilde{\Phi}_i$, $i=1,...,N_f$, which transform according to the respective representations R and $\bar{R}$ of SU(N_c). We consider the cases where R is the fundamental representation and where R is the symmetric and antisymmetric rank-2 tensor representation of SU(N_c). We denote the running gauge coupling of the theory by $g=g(\mu)$ and define $\alpha(\mu)=g(\mu)^2/(4\pi)$. (The argument μ will often be suppressed in the notation.) The beta function is $\beta_g=dg/dt$, where $dt=d\ln\mu$, or equivalently,

$$\beta_\alpha \equiv \frac{d\alpha}{dt} = \frac{g}{2\pi}\,\beta_g \ . \tag{1.1}$$

The beta function thus describes how the gauge coupling increases from the deep ultraviolet (UV) at large μ to the infrared (IR) at small μ. The asymptotic freedom property guarantees that in the deep UV, $\alpha(\mu)\ll 1$, so one can calculate the properties of the theory and its beta function reliably using perturbative methods. Depending on whether or not the theory contains matter chiral superfields, and if so, what R and N_f are, this UV to IR evolution may be governed by an IR zero of the beta function. If the theory has an IR zero and if it occurs at sufficiently small coupling, it may be an exact IR fixed point of the renormalization group; alternatively, if it occurs at sufficiently large coupling, spontaneous chiral symmetry breaking may occur, giving dynamical masses to some particles, so that these are integrated out in the low-energy effective field theory applicable below the scale of condensate formation. into the IR is governed by a different beta function. In our analysis, we will make use of the closed-form calculation of β_α by Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) in [1] (see also [2]), denoted β, which is exact within the scheme used for its calculation. This beta function exhibits a pole at a certain value of the coupling [1–3] (see Eq. (2.15) below). Furthermore, we will make use of a number of exact results that have been obtained using effective holomorphic action methods concerning the infrared properties of this theory [4, 5]. The beta function of an $\mathcal{N}=1$ supersymmetric gauge theory with general chiral superfield matter content has been calculated up to three-loop order [6, 7], and, for the pure gluonic SYM theory, up to four-loop order [8] in the dimensional reduction scheme with minimal subtraction [9], denoted $\overline{\rm DR}$. Using these results, we calculate Padé approximants to the beta function of the pure gluonic SYM theory up to four-loop order and to the beta functions of the theories with chiral superfields in the fundamental and rank-2 tensor representations up to three-loop order. The theories with sufficiently large matter superfield content have a perturbative IR zero in the beta function. In previous work in [10] with T. Ryttov and in [11]-[12], we have calculated properties of the beta function, including an IR zero, from the two and three-loop beta function. In the present work we extend these studies in several ways. Using our calculation of Padé approximants for the SYM theory and for theories with various matter superfield content, we address and answer several questions: (i) how the value of the IR zero in the Padé approximants compares with the IR zero in the $\overline{\rm DR}$ and NSVZ beta functions, for cases where such an IR zero is present; (ii) whether these Padé approximants to the $\overline{\rm DR}$ beta function exhibit a robust indication of a pole, as in the NSVZ beta function; (iii) if the answer to question (ii) is affirmative, whether this pole occurs at a value of α near to the value in the NSVZ beta function and, moreover, closer to the origin than an IR zero (if the latter is present) and hence dominates the UV to IR evolution. Our calculations and analysis provide a quantitative measure, for these various supersymmetric theories, of how well finite-order perturbative results calculated in the $\overline{\rm DR}$ scheme reproduce the properties of the NSVZ beta function. Some related work is in Refs. [13]-[18].

This paper is organized as follows. In Sect. II we discuss the beta function and exact results on the properties of the theory. In Sect. III we calculate and analyze Padé approximants for the pure gluonic supersymmetric Yang-Mills theory. Sect. IV is devoted to the corresponding calculation and analysis of Padé approximants for the theory with chiral superfields in the fundamental and conjugate fundamental representation. In Sect. V we investigate the theory with chiral superfields in the rank-2 tensor and conjugate tensor representations. Our conclusions are given in Sect. VI.

II. BETA FUNCTION AND EXACT RESULTS

A. Beta Function

In this section we review some basic results on the beta function and also some exact results that we will use for our analysis. The beta function (1.1) has the series expansion

$$\beta_\alpha = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ , \tag{2.1}$$

where

$$a \equiv \frac{g^2}{16\pi^2} = \frac{\alpha}{4\pi} \ , \tag{2.2}$$

b_ℓ is the ℓ-loop coefficient, and $\bar{b}_\ell = b_\ell/(4\pi)^\ell$ the reduced ℓ-loop coefficient. The first two coefficients in the expansion (2.1), which are scheme-independent, are [6, 19]

$$b_1 = 3C_A - 2T_f N_f \tag{2.3}$$

and

$$b_2 = 6C_A^2 - 4(C_A + 2C_f)T_f N_f \ . \tag{2.4}$$

In the commonly used $\overline{\rm DR}$ scheme, the three-loop coefficient is [7]

$$\begin{aligned} b_3 \ &= \ 21C_A^3 + 4(-5C_A^2 - 13C_A C_f + 4C_f^2)T_f N_f \\ &+ \ 4(C_A + 6C_f)(T_f N_f)^2 \ . \end{aligned} \tag{2.5}$$

For pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory (with no matter chiral superfields, i.e., $N_f = 0$), the four-loop coefficient, b_4, has also been calculated [8] and will be used in our analysis of this SYM theory below.

If $N_f = 0$, then $b_1 > 0$; as N_f increases from zero, b_1 decreases monotonically and passes through zero, reversing sign, at the value $N_f = N_{f,b1z}$, where [20]

$$N_{f,b1z} = \frac{3C_A}{2T_f} \tag{2.6}$$

(and the subscript $b1z$ stands for "b_1 zero"). If N_c and R are such that $N_{f,b1z}$ is an integer and if $N_f = N_{f,b1z}$, so that $b_1 = 0$, then b_2 has the negative value $-12C_A C_f$. Hence, the requirement of asymptotic freedom, which means $\beta < 0$ near the origin, is true (given the minus sign that we have extracted in Eq. (2.1)) if and only if $b_1 > 0$. Therefore, we restrict to

$$N_f < N_{f,b1z} \ . \tag{2.7}$$

Similarly, for $N_f = 0$, $b_2 > 0$, and as N_f increases from zero, b_2 decreases monotonically and passes through zero, reversing sign, at the value $N_f = N_{f,b2z}$, where

$$N_{f,b2z} = \frac{3C_A^2}{2T_f(C_A + 2C_f)} \ . \tag{2.8}$$

For an arbitrary fermion representation R, $N_{f,b2z} < N_{f,b1z}$, so there is always an interval I in N_f where $b_1 > 0$ and $b_2 < 0$. This interval is $N_{f,b2z} < N_f < N_{f,b1z}$, i.e.,

$$I: \quad \frac{3C_A^2}{2T_f(C_A + 2C_f)} < N_f < \frac{3C_A}{2T_f} \ . \tag{2.9}$$

For $N_f \in I$, the two-loop (2ℓ) beta function has an IR zero at

$$\begin{aligned} \alpha_{IR,2\ell} \ &= \ -\frac{\bar{b}_1}{\bar{b}_2} = -\frac{4\pi b_1}{b_2} \\ &= \ \frac{2\pi(3C_A - 2T_f N_f)}{2(C_A + 2C_f)T_f N_f - 3C_A^2} \ . \end{aligned} \tag{2.10}$$

Clearly, if N_f is too close to $N_{f,b2z}$, then b_2 is sufficiently small that $\alpha_{IR,2\ell}$ is too large for this perturbative two-loop result to be reliable. As noted, the two-loop beta function encodes the maximal scheme-independent perturbative information about the theory.

Given that $N_f \in I$ and that $\alpha_{IR,2\ell}$ is sufficiently small for the perturbative analysis of the beta function to be reasonable, a natural next step in the analysis of the UV to IR evolution of the theory is to examine the three-loop beta function. The three-loop beta function has two zeros away from the origin, given by the equation $b_1 + b_2 a + b_3 a^2 = 0$ or equivalently, $\bar{b}_1 + \bar{b}_2\alpha + \bar{b}_3\alpha^2 = 0$. The solutions are

$$\alpha = \frac{1}{2\bar{b}_3}\Big[-\bar{b}_2 \pm \sqrt{\bar{b}_2^2 - 4\bar{b}_1\bar{b}_3}\ \Big] \ . \qquad (2.11)$$

The smaller one of these two solutions is the one that will be relevant for our analysis, and we label it as $\alpha_{IR,3\ell}$.

B. NSVZ Beta Function

A closed-form expression for the beta function was derived by Novikov, Shifman, Vainshtein, and Zakharov in [1] and discussed further in [2]; this is

$$\beta_{\alpha,NSVZ} = \frac{d\alpha}{dt} = -\frac{\alpha^2}{2\pi}\left[\frac{b_1 - 2T_f N_f \gamma_m}{1 - 2C_A a}\right] \ , \qquad (2.12)$$

where γ_m is the anomalous dimension of the fermion bilinear bilinear product $\psi_i^T C\tilde{\psi}_i$, or equivalently, $\bar{\psi}_i\psi_i$, of component fermion fields in the quadratic superfield operator product $\Phi_i\tilde{\Phi}_i$ (no sum on the flavor index i). As noted above, this is an exact result within the scheme used in [1]. This anomalous dimension has the series expansion

$$\gamma_m = \sum_{\ell=1}^{\infty} c_\ell\, a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell\, \alpha^\ell \ , \qquad (2.13)$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop series coefficient. Only the one-loop coefficient c_1 is scheme-independent, and is

$$c_1 = 4C_f \ . \qquad (2.14)$$

Given our restriction to asymptotically free supersymmetric gauge theories, $\beta_{\alpha,NSVZ}$ has a UV zero at $\alpha = 0$. For the pure gluonic SYM theory, $\beta_{\alpha,NSVZ}$ has no IR zero; for theories with nonzero matter superfield content, it may or may not have an IR zero, depending on this content. This will be discussed further below. As is evident from Eq. (2.12), $\beta_{\alpha,NSVZ}$ has a pole at

$$a_{pole,NSVZ} = \frac{\alpha_{NSVZ}}{4\pi} = \frac{1}{2C_A} \ . \qquad (2.15)$$

An important property of this pole is that its position is independent of R and N_f.

C. General Result on IR Phase Properties

A number of exact results have been established for this asymptotically free supersymmetric gauge theory [1, 2, 4] (see also [5, 17]). We recall one property that is particularly relevant to our present work: if N_f is in the interval $N_{f,cr} < N_f < N_{f,b1z}$, where

$$N_{f,cr} = \frac{3C_A}{4T_f} = \frac{N_{b1z}}{2} \ , \qquad (2.16)$$

i.e., explicitly, the interval

$$I_{N_f,NACP}: \quad \frac{3C_A}{4T_f} < N_f < \frac{3C_A}{2T_f} \ , \qquad (2.17)$$

then the theory flows from weak coupling in the UV to a superconformal IR fixed point. The resultant theory is in a (deconfined) non-Abelian Coulomb phase (NACP) without any spontaneous chiral symmetry breaking. In Eqs. (2.16) and (2.17), it is understood that, physically, N_f must be an integer [20], so the actual values of N_f in the NACP are understood to be the integers that satisfy the inequality (2.17).

III. $\mathcal{N} = 1$ SUPERSYMMETRIC YANG-MILLS THEORY

In this section we study the case $N_f = 0$, i.e., supersymmetric Yang-Mills theory. We use Padé approximants to the n-loop beta function with $2 \le n \le 4$ calculated in the $\overline{\rm DR}$ scheme to investigate how the properties of this beta function compare with those of the NSVZ beta function. This comparison elucidates the question of how sensitive these properties are to the scheme used for the calculation.

In this SYM theory the beta function depends on a and C_A via the product

$$x \equiv C_A a \ . \qquad (3.1)$$

or equivalently, $\xi \equiv C_A\alpha$. Consequently, it is natural to re-express the beta function in terms of this product as the expansion variable. We thus define

$$\beta_x \equiv \frac{dx}{dt} = C_A\frac{da}{dt} = \frac{C_A}{4\pi}\,\beta_\alpha \ . \qquad (3.2)$$

Since $b_\ell \propto C_A^\ell$, we define

$$\hat{b}_\ell \equiv \frac{b_\ell}{C_A^\ell} \ . \qquad (3.3)$$

From Eqs. (2.3)-(2.5), one has the two scheme-independent coefficients

$$\hat{b}_1 = 3 \ , \quad \hat{b}_2 = 6 \qquad (3.4)$$

and, in the $\overline{\rm DR}$ scheme,

$$\hat{b}_3 = 21 \ . \qquad (3.5)$$

For this SYM theory, the four-loop coefficient has also been calculated in the $\overline{\rm DR}$ scheme [8], and it is

$$\hat{b}_4 = 102 \ . \qquad (3.6)$$

The beta function can be written as

$$\beta_x = -2x^2 \sum_{\ell=1}^{\infty} \hat{b}_\ell \, x^{\ell-1} \ . \tag{3.7}$$

The n-loop beta function $\beta_{x,n\ell}$ is defined by Eq. (3.7) with the upper limit on the sum given by $\ell = n$. Explicitly, using the $\overline{\rm DR}$ scheme for b_3 and b_4,

$$SYM: \quad \beta_{x,4\ell,\overline{\rm DR}} = -6x^2(1+2x+7x^2+34x^3) \ . \tag{3.8}$$

It will be convenient to define a reduced (rd) beta function, $\beta_{x,rd}$:

$$\beta_{x,rd} \equiv -\frac{\beta_x}{2x^2\,\hat{b}_1} = 1 + \frac{1}{\hat{b}_1} \sum_{\ell=2}^{\infty} \hat{b}_\ell \, x^{\ell-1} \ . \tag{3.9}$$

This separates off the factor that gives rise to a UV zero at $x=0$ so that we can concentrate on the region of interest, namely the IR behavior. The point here is that both $\beta_{x,NSVZ}$ and $\beta_{x,\overline{\rm DR}}$ are guaranteed to have the same UV behavior in the vicinity of the origin because of the asymptotic freedom of the theory and the fact that the first two orders in the loop expansion are scheme-independent. The question is how well they agree in the IR. As with the full beta function, we also define the n-loop truncation of $\beta_{x,rd}$, denoted $\beta_{x,rd,n\ell}$, as Eq. (3.9) with the upper limit on the sum given by $\ell = n$; this is thus a polynomial of degree $n-1$ in x. In the $\overline{\rm DR}$ scheme, the four-loop reduced beta function is

$$SYM: \quad \beta_{x,rd,4\ell,\overline{\rm DR}} = 1+2x+7x^2+34x^3 \ . \tag{3.10}$$

We first analyze the zeros of $\beta_{x,rd,n\ell}$ for $2 \le n \le 4$. Since each term is positive, it is clear that at the two-loop level and also, in the $\overline{\rm DR}$ scheme, at the $n=3, \ 4$ loop level, the respective n-loop reduced beta function has no physical zero. Specifically, the reduced two-loop beta function $\beta_{x,rd,2\ell}$ has only an unphysical zero away from the origin, at $x=-1/2$. With b_3 calculated in the $\overline{\rm DR}$ scheme, the reduced three-loop beta function $\beta_{x,rd,3\ell}$ has an unphysical pair of complex-conjugate zeros, at

$$x = \frac{1}{7}\,(-1 \pm \sqrt{6}\,i) = -0.14286 \pm 0.34993i \tag{3.11}$$

(Here and below, floating-point numbers are listed to the indicated accuracy.) The reduced four-loop beta function $\beta_{x,rd,4\ell,\overline{\rm DR}}$ has three unphysical roots, at

$$x = -0.3152, \quad x = 0.05466 \pm 0.3005i \ . \tag{3.12}$$

We now compare the properties of $\beta_{x,\overline{\rm DR}}$ and $\beta_{x,NSVZ}$ for the SYM theory. The series expansions of these two beta functions about $x=0$ are necessarily equal up to two-loop order inclusive, since the beta function is scheme-independent up to and including this order. Beyond two-loop order they differ, as is to be expected, since they are calculated in different schemes. An important question is whether, although they differ in detail, these two beta functions at least exhibit qualitatively similar physical properties. To answer this question, we first express the NSVZ beta function for the SYM theory in terms of β_x, obtaining

$$\beta_{x,NSVZ,SYM} = -\frac{6x^2}{1-2x} \ , \tag{3.13}$$

so that the reduced NSVZ beta function for the SYM theory is

$$\beta_{x,rd,NSVZ,SYM} = \frac{1}{1-2x} \ . \tag{3.14}$$

This $\beta_{x,rd,NSVZ,SYM} = 1/(1-2x)$ is in the exact form of a [0,1] Padé approximant, a property that will be used below. Clearly, $\beta_{x,rd,NSVZ,SYM}$ has no IR zero and, as is evident from Eq. (2.15) (or equivalently, Eq. (3.14)), it has a pole at $x=1/2$:

$$x_{pole,NSVZ} = \frac{1}{2} \ . \tag{3.15}$$

Interestingly, the $\beta_{x,rd,n\ell,\overline{\rm DR}}$ functions at the $n=2, \ 3, \ 4$ loop levels all share the same property as $\beta_{x,NSVZ}$ in having no (physical) IR zero. This property will be discussed further below.

We next carry out our Padé calculations and analysis. In general, the n-loop reduced beta function $\beta_{x,rd,n\ell}$ is a polynomial of degree $n-1$ in x. At loop order $n \ge 3$, the coefficients in this function depend on the scheme used for the calculation, and hence, where this is not obvious from context, we shall indicate the scheme with an additional subscript. Since all of the $[p,q]$ Padé approximants that we calculate will apply to the beta function in the $\overline{\rm DR}$ scheme, it is not necessary to indicate this. We can thus calculate $[p,q]$ Padé approximants of the form

$$[p,q]_{\beta_{x,rd,n\ell}} = \frac{1+\sum_{j=1}^p \, n_j x^j}{1+\sum_{k=1}^q d_k \, x^k} \tag{3.16}$$

with

$$p+q = n-1 \ , \tag{3.17}$$

where the n_j and d_j are x-independent coefficients of the respective polynomials in the numerator and denominator of $[p,q]_{\beta_{x,rd,n\ell}}$. (Our notation for Padé approximants follows the notation in, e.g., [21].) Thus, as with $\beta_{x,rd,n\ell}$ itself, each Padé approximant is normalized so that $[p,q]_{\beta_{x,rd,n\ell}} = 1$ at $x=0$. For a given $\beta_{x,rd,n\ell}$, there are thus n Padé approximants, namely the set

$$\{ \ [n-k,k-1]_{\beta_{x,rd,n\ell}} \ \} \quad \text{with } 1 \le k \le n \ . \tag{3.18}$$

We shall generically denote one of the p zeros of a $[p,q]]_{\beta_{x,rd,n\ell}}$ Padé approximant as $[p,q]_{zero}$ and one of the q poles of this approximant as $[p,q]_{pole}$; in each case, the value of n is given by Eq. (3.17) as $n = p+q+1$ and will sometimes be omitted for brevity.

Since

$$[n-1,0]_{\beta_{x,rd,n\ell}} = \beta_{x,rd,n\ell} \ , \tag{3.19}$$

i.e., the $[n-1,0]_{\beta_{x,rd,n\ell}}$ Padé approximant is identical to the n-loop reduced beta function itself, whose zeros we have already analyzed, we mainly restrict our consideration below to Padé approximants $[p,q]_{\beta_{x,rd,n\ell}}$ with $q \neq 0$. Since b_1 and b_2 are scheme-independent, it follows that for $n = 1, \ 2$, the corresponding Padé approximants are scheme-independent. Note that for an arbitrary polynomial $1+\sum_j f_j x^j$, the zero of the $[1,0]$ Padé approximant, $1 + f_1 x$, denoted as $[1,0]_{zero}$, occurs at minus the value of the pole in the $[0,1]$ approximant $1/(1-f_1x)$, denoted as $[0,1]_{pole}$, i.e.,

$$[1,0]_{zero} = -\frac{1}{f_1} = -[0,1]_{pole} \ . \tag{3.20}$$

We can also explore the correspondence between the pole at $x = 1/2$ in $\beta_{x,rd,NSVZ}$ and the structure of $\beta_{x,rd,n\ell,\overline{\rm DR}}$. Although $\beta_{x,rd,n\ell,\overline{\rm DR}}$ is a polynomial and hence obviously has no poles, we can investigate whether $[p,q]$ Padé approximants to $\beta_{x,rd,n\ell,\overline{\rm DR}}$ with $q \neq 0$ share properties in common with $\beta_{x,rd,NSVZ}$.

We shall address and answer the following specific questions concerning the $[p,q]$ Padé approximants:

1. Considering the $[p,q]$ Padé approximants to $\beta_{x,rd,n\ell,\overline{\rm DR}}$, do the $[p,q]$ approximants with $p \neq 0$ exhibit a physical IR zero?

2. Considering the $[p,q]$ Padé approximants to $\beta_{x,rd,n\ell,\overline{\rm DR}}$, do the $[p,q]$ approximants with $q \neq 0$ exhibit a physical pole?

3. If a given $[p,q]$ Padé approximant with $q \neq 0$ does exhibit a physical pole, does this pole dominate the UV to IR evolution? This is the case if and only if this pole occurs closer to the origin $x = 0$ than a physical IR zero.

4. If the answers to the previous two questions are affirmative, then is the value of the pole in the given $[p,q]$ Padé approximant with $q \neq 0$ close to the value $x_{IR,NSVZ} = 1/2$?

5. If the answers to questions 1 and 2 are affirmative, then, independent of whether the poles in the $[p,q]$ approximants with $q \neq 0$ are close to $x_{IR,NSVZ} = 1/2$, do different approximants at least exhibit a stable physical pole? That is, do these $[p,q]$ Padé approximants with $q \neq 0$ exhibit a stable physical pole?

The answers to these questions elucidate how general and robust are the properties of the SYM beta function calculated in different schemes, in particular, the absence of an IR zero and the presence of a pole in the NSVZ beta function. We have already partially answered the first question, since we have shown that there is no IR zero in the two-loop beta function and also none in the three-loop or four-loop beta function in the $\overline{\rm DR}$ scheme, in agreement with the absence of an IR zero in the NSVZ beta function for this SYM theory. We complete this first answer by examining $[p,q]$ Padé approximants with both p and q nonzero and also address the questions pertaining to a pole.

From the two-loop reduced beta function $\beta_{x,rd,2\ell}$, we can calculate one Padé approximant with $q \neq 0$, namely

$$[0,1]_{\beta_{x,rd,2\ell}} = \frac{1}{1-2x} \ . \tag{3.21}$$

This is the same as the reduced NSVZ beta function, $\beta_{x,rd,NSVZ,SYM}$ in Eq. (3.14), and hence their poles are at the same location:

$$[0,1]_{pole,\beta_{x,rd,n\ell}} = \frac{1}{2} = x_{pole,NSVZ} \ . \tag{3.22}$$

Although $\beta_{x,rd,NSVZ,SYM} = 1/(1-2x)$ was obtained by a sum to infinite-loop order and hence is scheme-dependent, the pole in the [0,1] Padé approximant was derived from the two-loop beta function $\beta_{x,rd,2\ell}$ and hence is scheme-independent.

We proceed next to the comparison at the three-loop order. From the reduced three-loop reduced beta function, $\beta_{x,rd,3\ell,\overline{\rm DR}}$, we can calculate two Padé approximants with $q \neq 0$, namely

$$[1,1]_{\beta_{x,rd,3\ell}} = \frac{1-(3/2)x}{1-(7/2)x} \tag{3.23}$$

and

$$[0,2]_{\beta_{x,rd,3\ell}} = \frac{1}{(1+x)(1-3x)} \ . \tag{3.24}$$

As is evident from (3.23), the $[1,1]_{\beta_{x,rd,3\ell}}$ Padé has a pole at $x = 2/7 = 0.2857$ and a zero at $x = 2/3$. These are listed in Table I. As the theory flows from the UV to the IR, x increases from 0 and reaches the IR pole at $2/7$ before it reaches the zero, so the latter is not relevant to this UV to IR evolution from weak coupling. The $[0,2]_{\beta_{x,rd,3\ell}}$ Padé exhibits an unphysical pole at $x = -1$ and a physical pole at $x = 1/3$. Since this Padé has no zero, the pole at $x = 1/3$ again dominates the UV to IR evolution.

From the four-loop reduced beta function $\beta_{x,rd,4\ell,\overline{\rm DR}}$, we can calculate three $[p,q]$ Padé approximants with $q \neq 0$, namely

$$[2,1]_{\beta_{x,rd,4\ell}} = \frac{1-(20/7)x-(19/7)x^2}{1-(34/7)x} \ , \tag{3.25}$$

$$[1,2]_{\beta_{x,rd,4\ell}} = \frac{1-(14/3)x}{1-(20/3)x+(19/3)x^2} \ , \tag{3.26}$$

and

$$[0,3]_{\beta_{x,rd,4\ell}} = \frac{1}{1-2x-3x^2-14x^3} \ . \tag{3.27}$$

TABLE I: Values of zeros and poles, in the variable x, of various Padé approximants to $\beta_{x,rd,2\ell}$ and $\beta_{x,rd,n\ell,\overline{\rm DR}}$ with $n = 3,\ 4$ for $\mathcal{N} = 1$ supersymmetric Yang-Mills theory, SYM. Results are given to the indicated floating-point accuracy. The abbreviation NA means "not applicable".

n	$[p,q]$	zero(s)	pole(s)
2	[1,0]	$-1/2$	NA
2	[0,1]	NA	$1/2$
3	[2,0]	$0.143 \pm 0.350i$	NA
3	[1,1]	$2/3$	$2/7 = 0.286$
3	[0,2]	NA	$-1,\ 1/3$
4	[3,0]	$-0.315,\ \ 0.0547 \pm 0.3005i$	NA
4	[2,1]	$-1.330,\ \ 0.277$	$7/34 = 0.206$
4	[1,2]	$3/14 = 0.214$	$0.181,\ \ 0.871$
4	[0,3]	NA	$0.265,\ \ -0.240 \pm 0.461i$

The $[2,1]_{\beta_{x,rd,4\ell}}$ Padé has zeros at $x = (1/19)(-10 \pm \sqrt{233}\ )$, i.e, $x = 0.2771$ and $x = -1.3297$, and a pole at $x = 7/34 = 0.2059$. The $[1,2]_{\beta_{x,rd,4\ell}}$ Padé has a zero at $x = 3/14 = 0.2143$ and two poles, at $x = (1/19)(10 \pm \sqrt{43}\ )$, i.e., $x = 0.8714$ and $x = 0.1812$. Finally, the $[0,3]_{\beta_{x,rd,4\ell}}$ Padé has three poles, at $x = 0.26481$ and $x = -0.23955 \pm 0.4608i$. As with our other Padé results for the SYM theory, these are listed in Table I.

These results provide answers to the five questions that we posed above. Concerning the first question, the [1,0] approximant to $\beta_{x,rd,2\ell}$ and the [2,0], and [3,0] approximants to $\beta_{x,rd,n\ell,\overline{\rm DR}}$ for $n = 3,\ 4$ have no IR zero, in agreement with the NSVZ beta function. Although the [1,1] approximant to $\beta_{x,rd,3\ell\overline{\rm DR}}$ and the [2,1] and [1,2] approximants to $\beta_{x,rd,4\ell,\overline{\rm DR}}$ do have (physical) IR zeros, in each case, the IR zero occurs farther from the origin than the pole in the respective approximant and hence does not directly influence the UV to IR evolution. Thus, the results from all of these Padé approximants to $\beta_{x,rd,n\ell,\overline{\rm DR}}$ for $2 \le n \le 4$ are in agreement with the NSVZ beta function for this SYM theory as regards the absence of an IR zero affecting the UV to IR evolution.

The answer to the second question is yes; that is, all of the Padé approximants to the various n-loop reduced beta functions in the $\overline{\rm DR}$ scheme, $\beta_{x,rd,n\ell,\overline{\rm DR}}$, up to $n = 4$ loop order, agree with $\beta_{x,rd,NSVZ}$ as regards the property that they exhibit a physical IR pole. The answer to the third question is also yes; in each case where a $[p,q]$ Padé approximant to $\beta_{x,rd,n\ell,\overline{\rm DR}}$ exhibits a physical pole, this pole occurs closer to the origin than any physical zero(s) (if such a zero is present at all) and hence dominates the UV to IR evolution.

We come next to question 4, concerning the numerical agreement of the (physical) pole in the $[p,q]$ Padé approximants with $q \ne 0$ with the position of the pole at $x = 1/2$ in the NSVZ beta function. To answer this question, for each $[p,q]$ approximant to $\beta_{x,rd,n\ell,\overline{\rm DR}}$ with $q \ne 0$, one takes the pole among the q poles at a physical (positive real) value of x (if there is such a pole) closest to the origin. This is the IR pole for this approximant. As noted, this agreement is automatic in the two-loop case, so the question really applies at the three-loop and four-loop level. As is evident in Table I, the two Padé approximants with $q \ne 0$ formed from the three-loop beta function, namely [1,1] and [0,2], have poles at the respective values $x = 0.286$ and $x = 0.333$. At the four-loop level, the values of the poles closest to the origin in the [2,1], [1,2], and [0,3] Padé approximants are $x = 0.206,\ 0.181,\ 0.265$, respectively. None of these is particularly close to the value $x = 0.5$ of the pole in the NSVZ beta function. This is in contrast with the results that were obtained in [23]-[27] (see also [28]) concerning the scheme dependence of the IR zero in the beta function for a nonsupersymmetric asymptotically free non-Abelian gauge theory calculated up to four-loop level; in these studies, it was found that the position of this zero does not change very much as one applies various scheme transformations to the results calculated [23, 24] in the $\overline{\rm MS}$ scheme. Nevertheless, we can at least say that the values of the (physical) pole in the various $[p,q]$ approximants to $\beta_{x,rd,n\ell,\overline{DR}}$ with $n = 3,\ 4$ do not differ from the value $x = 1/2$ in $\beta_{x,rd,NSVZ}$ by more than a factor of about 2.8.

Finally, we address the fifth question. The importance of this question stems from the fact that when one switches schemes, one does not expect a pole (or zero) to occur at the same position as in another scheme, but at least different $[p,q]$ Padé approximants should yield a stable value of this pole, especially as one calculates to progressively higher-loop order. There are thus two categories of comparisons that one can make here, namely comparing the stability of the position of a pole appearing in $[p,q]$ Padé approximants for different loop orders n, and comparing this stability for a given n-loop order in $[p,q]$ Padé approximants with different p and q (satisfying $p + q = n - 1$), at a high-enough order so that there are several $[p,q]$ approximants with poles. Regarding the comparison among different loop orders, as is evident from Table I, the values of the poles range from the value from the two-loop result, which is automatically equal to $x = 0.5$, to a low of $x = 0.181$ for the physical pole in the $[1,2]_{\beta_{x,rd,4\ell}}$ Padé, a factor of 2.8 smaller. Regarding the range of values of physical pole positions from the Padé approximants at a given loop order, the range is given, at the three-loop order, by the ratio

$$\frac{[0,2]_{pole}}{[1,1]_{pole}} = \frac{7}{6} = 1.167 \ , \tag{3.28}$$

and, at the four-loop order, by two independent ratios, which may be taken to be

$$\frac{[0,3]_{pole}}{[1,2]_{pole}} = 1.462 \ , \tag{3.29}$$

and

$$\frac{[1,2]_{pole}}{[2,1]_{pole}} = 0.88005 \ . \tag{3.30}$$

One also has

$$\frac{[0,3]_{pole}}{[2,1]_{pole}} = \frac{[0,3]_{pole}}{[1,2]_{pole}} \frac{[1,2]_{pole}}{[2,1]_{pole}} = 1.286 \ . \tag{3.31}$$

Ideally, one would have hoped that this ratio, i.e., the scatter in the values of the IR pole positions, would decrease as the loop order increased, but, at least up to four-loop order, it does not.

Summarizing the findings from our Padé analysis for the SYM theory, the results show excellent agreement between the beta function, calculated up to four-loop order in the $\overline{\rm DR}$ scheme, and the NSVZ beta function, concerning the absence of an IR zero that affects the UV to IR evolution. Furthermore, the answers to questions 2 and 3 show that the Padé approximants to this beta function in the $\overline{\rm DR}$ scheme are consistent with the existence of an IR pole that dominates the UV to IR evolution, again in agreement with the NSVZ beta function. The answer to the fourth question can be interpreted as a consequence of the scheme-dependence of a pole in a beta function. The answer to the fifth question suggests that, assuming that the beta function in the $\overline{\rm DR}$ scheme does, indeed, encode evidence for a physical pole that dominates the UV to IR evolution in this SYM theory, one must calculate this beta function to higher than four-loop order in order for the $[p,q]$ Padé approximants with $q \neq 0$ to yield a stable value for the location of this zero.

IV. SUPERSYMMETRIC SU(N_c) QUANTUM CHROMODYNAMICS

In this section we investigate an asymptotically free vectorial gauge theory with $\mathcal{N}=1$ supersymmetry, gauge group SU(N_c), and N_f copies (flavors) of massless chiral superfields Φ_i and $\tilde{\Phi}_i$, $i=1,...,N_f$, transforming according to the fundamental and conjugate fundamental representations of SU(N_c), with Young tableaux $\square$ and $\overline{\square}$:

$$\Phi_i : \ \square; \quad \tilde{\Phi}_i : \ \overline{\square} \ , \quad \text{with } i=1,...,N_f \ . \tag{4.1}$$

This theory is often called supersymmetric quantum chromodynamics (SQCD), and we shall also use this nomenclature, keeping in mind that the gauge group is generalized from the actual SU(3) color group of real-world QCD to SU(N_c). We restrict our consideration to values $N_f \neq 0$ here, since if $N_f = 0$, the present theory reduces to a pure supersymmetric Yang-Mills gauge theory, which we have also discussed above. As we did with the SYM theory, we shall use Padé approximants to investigate the question of the extent to which the beta function for this theory, as calculated in the $\overline{DR}$ scheme, exhibits properties in agreement with the properties of the NSVZ beta function, (2.12).

A. Some General Properties

We recall some basic well-known properties of this theory, many of which follow as special cases of the general discussion in Sect. II for $R=\square$. For this case, the upper bound on N_f imposed by the condition of asymptotic freedom, Eq. (2.7), reads

$$N_f < 3N_c \ . \tag{4.2}$$

The exact result (2.16) on the value of N_f at the lower boundary of the IR non-Abelian Coulomb phase reads [20]

$$N_{f,cr} = \frac{3N_c}{2} \ , \tag{4.3}$$

If N_c is odd, this is only a formal result, since $N_{f,cr}$ must be integral. Thus, the (chirally symmetric, deconfined) IR non-Abelian Coulomb phase is specified, from Eq. (4.4), by N_f in the interval

$$\frac{3N_c}{2} < N_f < 3N_c \ . \tag{4.4}$$

For our present case with R being the fundamental representation, Eq. (2.8) specializes to

$$N_{f,b2z} = \frac{3N_c}{2 - N_c^{-2}} \ . \tag{4.5}$$

Hence, the range of values of N_f in Eq. (2.9) where the two-loop beta function has an IR zero is [20]

$$\frac{3N_c}{2 - N_c^{-2}} < N_f < 3N_c \ . \tag{4.6}$$

Numerical values of $N_{f,cr}$, $N_{f,b2z}$, $N_{f,b3z}$, and $N_{f,b1z} = N_{f,max}$ were listed for $2 \le N_c \le 5$ in Table II of Ref. [10]. As was noted in [10], the value of $N_{f,b2z}$ in Eq. (4.5) is greater (for all finite N_c) than the exactly known lower boundary of the non-Abelian Coulomb phase in Eq. (4.3). Results for the values of the IR zero in the two-loop and three-loop beta function, $\alpha_{IR,2\ell}$ and $\alpha_{IR,3\ell}$, were given in [10].

B. Calculations of Padé Approximants

We now proceed to calculate and analyze the Padé approximants to the n-loop beta function for this SQCD theory. As before with the SYM theory, since our analysis concerns the behavior away from the UV fixed point at $\alpha=0$, it is convenient to deal with the reduced beta function defined by (3.9). Because the beta function β_α is known up to three-loop order, the reduced beta function has the form

$$\beta_{\alpha,rd,3\ell} = 1 + (b_2/b_1)a + (b_3/b_1)a^2 \ . \tag{4.7}$$

Since $G = \mathrm{SU}(N_c)$, it follows that $C_A = N_c$ and the variable x in Eq. (3.1) has the explicit form

$$x \equiv a\,N_c \equiv \frac{\xi}{4\pi} \ . \tag{4.8}$$

In general, the beta function and hence the Padé approximants to it depend on the two parameters N_c and N_f. It is natural to apply the Padé analysis to address the question of how the properties of the beta function calculated in the $\overline{\rm DR}$ scheme compare with those of the NSVZ beta function in the simplest context, namely the limit where the (appropriately scaled) beta function depends only one one variable. This is the 't Hooft-Veneziano or LNN (Large N_c and N_f) limit

$$LNN: \quad N_c \to \infty, \quad N_f \to \infty,$$
$$\text{with } r \equiv \frac{N_f}{N_c} \text{ fixed and finite} \tag{4.9}$$

with $x(\mu) = a(\mu)\,N_c$ a finite function of the Euclidean scale μ.

Our constraint of asymptotic freedom implies $r < 3$. We divide our analysis into two parts corresponding to two subdivisions of this interval, namely the NACP interval

$$I_{r,NACP}: \quad \frac{3}{2} < r < 3 \ , \tag{4.10}$$

where the UV to IR evolution leads to a non-Abelian Coulomb phase without any spontaneous chiral symmetry breaking, and the remaining interval $0 < r < 3/2$. In addition to simplifying the analysis of the beta function from dependence on variables to two to one variable, the LNN limit has the appeal that the interval (4.10) in which the two-loop beta function has an IR zero coincides with the interval leading to a non-Abelian Coulomb phase. This is in contrast to the situation for general N_c and N_f, in which b_2 vanishes in the interior of the NACP.

In the LNN limit, one focuses on the scaled beta function, which is finite in this limit. For this we use the same notation, β_x, as in Eq. (3.2), with it being understood that the LNN limit is taken, so that

$$\beta_x = \lim_{LNN} \frac{dx}{dt} \ , \tag{4.11}$$

equivalent to $\beta_\xi = \frac{d\xi}{dt}$. The function β_x has the expansion (3.7) with

$$\hat b_\ell \equiv \lim_{LNN} \frac{b_\ell}{N_c^\ell} \ . \tag{4.12}$$

As before, we denote the n-loop truncation of Eq. (4.11) as $\beta_{x,n\ell}$, and, where appropriate, we indicate the scheme used for loop order $n \ge 3$ by a subscript, as $\beta_{x,n\ell,\overline{\rm DR}}$. From Eqs. (2.3) and (2.4), it follows that the scheme-independent scaled coefficients are

$$\hat b_1 = 3 - r \tag{4.13}$$

and

$$\hat b_2 = 2(3-2r) \ . \tag{4.14}$$

From the expression (2.5) for b_3 calculated in the $\overline{\rm DR}$ scheme, one has

$$\hat b_3 = 21 - 21r + 4r^2 \ . \tag{4.15}$$

Thus,

$$\beta_{x,3\ell,\overline{\rm DR}} = -2x^2\Big[(3-r)+2(3-2r)x+(21-21r+4r^2)x^2\Big] \ , \tag{4.16}$$

and hence

$$\begin{aligned}\beta_{x,rd,3\ell,\overline{\rm DR}} &= 1 + \frac{\hat b_2}{\hat b_1}\,x + \frac{\hat b_3}{\hat b_1}\,x^2 \\ &= 1 + 2\Big(\frac{3-2r}{3-r}\Big)\,x + \Big(\frac{21-21r+4r^2}{3-r}\Big)\,x^2 \ .\end{aligned} \tag{4.17}$$

It will be convenient to define

$$D_s = -\hat b_3 = -21 + 21r - 4r^2 \ . \tag{4.18}$$

This polynomial D_s has the property that

$$D_s > 0 \quad \text{for} \quad \frac{1}{8}(21-\sqrt{105}\,) < r < \frac{1}{8}(21+\sqrt{105}\,) \ , \tag{4.19}$$

i.e., for $1.3441 < r < 3.9059$, which includes all of the interval $I_{r,NACP}$. (If r lies outside of the interval in Eq. (4.19), then $D_s < 0$.)

We recall from [11] that if the two-loop beta function has an IR zero, then, since this is a scheme-independent property, one may require a physically acceptable scheme to maintain the existence of this IR zero at loop level $n \ge 3$ and the condition that it should maintain it at the three-loop level implies that $b_3 < 0$ (see the proof in Section II.E of [11]). This condition is thus satisfied by the $\overline{\rm DR}$ scheme, since $\hat b_3 < 0$ for the interval $I_{r,NACP}$, where $\beta_{x,2\ell}$ has an IR zero.

C. Analysis for Interval $r \in I_{r,NACP}$

1. IR Zero of $\beta_{x,rd,2\ell}$

At the two-loop level, $\beta_{x,rd,2\ell}$ has a (scheme-independent) IR zero at

$$x_{IR,2\ell} = \frac{\xi_{IR,2\ell}}{4\pi} = \frac{(3-r)}{2(2r-3)} \ , \tag{4.20}$$

which is physical for $3/2 < r < 3$, i.e., for $r \in I_{r,NACP}$. The value of $x_{IR,2\ell}$ increases monotonically from 0 to arbitrarily large values as r decreases from 3 to 3/2 in the interval I_{NACP}. Clearly, for the values of r in the lower part of this interval, where $x_{IR,2\ell}$ becomes large, the perturbative calculation that yielded the expression for $x_{IR,2\ell}$ cannot be reliably applied.

2. IR Zero of $\beta_{x,rd,3\ell,\overline{\rm DR}}$

At the three-loop level, $\beta_{x,rd,3\ell,\overline{\rm DR}}$ has an IR zero at [10, 12]

$$x_{IR,3\ell} = \frac{-(2r-3)+\sqrt{C_s}}{D_s} \ , \tag{4.21}$$

where

$$C_s = -54 + 72r - 29r^2 + 4r^3 \ . \tag{4.22}$$

The polynomial C_s has only one real zero, at $r = 1.3380$, and is positive (negative) for r greater (less) than this value. Thus, C_s is positive for all $r \in I_{r,NACP}$. Since $\hat{b}_3 < 0$ for $r \in I_{r,NACP}$, it follows that

$$x_{IR,3\ell} \le x_{IR,2\ell} \ , \tag{4.23}$$

as a special case of an inequality that was proved in [11] (see Eq. (2.29) of [11]). The inequality (4.23) is a strict inequality except at the upper end of $I_{r,NACP}$ at $r = 3$, where both $x_{IR,3\ell}$ and $x_{IR,2\ell}$ vanish.

3. Analysis of IR Zero Using Padé Approximants

For $r \in I_{r,NACP}$, where the beta function has an IR zero, we address and answer the following set of questions concerning the comparison of the three-loop beta function calculated in the $\overline{\rm DR}$ scheme, the Padé approximants to it, and the NSVZ beta function:

1. Considering (i) the n-loop beta function and (ii) the $[p,q]$ Padé approximants to this beta function with $p \ne 0$, do these exhibit a physical IR zero?

2. If (i) the n-loop beta function and (ii) the $[p,q]$ Padé approximant to this beta function with $p \ne 0$ do exhibit a physical IR zero, does this IR zero dominate the UV to IR evolution? This is the case if and only if this IR zero occurs closer to the origin $x = 0$ than a physical IR pole (if the latter is present in a $[p,q]$ Padé with $q \ne 0$).

3. In each of the cases (i) and (ii), if the answers to the previous two questions are affirmative, then is the value of the IR zero close to the value $x_{IR,cfs,NSVZ}$ in $\beta_{x,rd,NSVZ}$, given in Eq. (4.38)?

4. In each of the cases (i) and (ii), if the answers to questions 1 and 2 are affirmative, then, independent of the closeness of the IR zero to $x_{IR,cfs,NSVZ}$, are the values at least close to each other?

5. For the $[p,q]$ Padé approximants with $q \ne 0$, if there is a physical pole, is its location near to the value $x = 1/2$ in the NSVZ beta function?

We recall that the $[p,0] = [n-1]$ Padé approximant is identical to the reduced n-loop beta function $\beta_{x,rd,n\ell}$, as noted above in Eq. (3.19). As a special case of this, the two-loop reduced beta function $\beta_{x,rd,2\ell}$ yields only one Padé approximant with a zero, namely [1,0], which coincides with $\beta_{x,rd,2\ell}$, itself, so no further analysis is necessary. The three-loop reduced beta function $\beta_{x,rd,3\ell,\overline{\rm DR}}$ yields two Padé approximants with $p \ne 0$, namely [2,0] and [1,1]. The [2,0] approximant coincides with $\beta_{x,rd,3\ell,\overline{\rm DR}}$, which has already been analyzed. We calculate the [1,1] approximant to be

$$[1,1]_{\beta_{x,rd,3\ell}} = \frac{1-\left[\frac{E_s}{2(3-r)(2r-3)}\right]x}{1-\left[\frac{D_s}{2(2r-3)}\right]x} \ , \tag{4.24}$$

where

$$E_s = -27 + 36r - 17r^2 + 4r^3 \ . \tag{4.25}$$

The polynomial E_s has only one real zero, at $r = 1.3118$ and is positive (negative) for r greater (less) than this value. Therefore, E_s is positive for all $r \in I_{r,NACP}$. As r decreases from 3 to 3/2, E_s decreases from 36 to 9/4. Thus, the $[1,1]_{\beta_{x,rd,3\ell,\overline{\rm DR}}}$ Padé approximant has a zero at

$$x_{[1,1],zero} = \frac{2(3-r)(2r-3)}{E_s} \ . \tag{4.26}$$

This is positive semidefinite for all $r \in I_{r,NACP}$; it vanishes at both ends of this interval and reaches a maximum at $r = 1.8321$ (a zero of the function $81 - 198r + 189r^2 - 72r^3 + 8r^4$), where it has the value $x_{[1,1],zero} = 0.23898$. In order for $x_{[1,1],zero}$ to be relevant for the UV to IR evolution of the theory (from weak coupling in the UV), it is necessary that if this Padé approximant has a pole at a physical value of x, then this pole must occur farther from the origin than the zero. Below, when we analyze poles of the various Padé approximants, we will show that this condition is satisfied (although the distance between the zero and the pole vanishes as $r \searrow 3/2$). We thus denote

$$x_{[1,1],zero} = x_{IR,3\ell,[1,1]} \ . \tag{4.27}$$

We prove two inequalities. First,

$$x_{IR,3\ell,[1,1]} \le x_{IR,2\ell} \quad \text{for } r \in I_{r,NACP} \ . \tag{4.28}$$

This is proved by computing the difference

$$x_{IR,2\ell} - x_{IR,3\ell,[1,1]} = \frac{(3-r)^2\, D_s}{2(2r-3)\, E_s} \ . \tag{4.29}$$

This difference is positive semidefinite for $r \in I_{r,NACP}$, vanishing only as $r \nearrow 3$ at the upper end of this interval.

Second, we obtain the stronger inequality

$$x_{IR,3\ell,[1,1]} \le x_{IR,3\ell} \quad \text{for } r \in I_{r,NACP} \ . \tag{4.30}$$

(This is a stronger inequality since $x_{IR,3\ell} \le x_{IR,2\ell}$, by (4.23).) We have proved the inequality (4.30) by calculating the difference, $x_{IR,3\ell} - x_{IR,3\ell,[1,1]}$ and showing that it is positive semidefinite for $r \in I_{r,NACP}$, vanishing only at $r = 3$. Combining these inequalities, we have

$$x_{IR,3\ell,[1,1]} \le x_{IR,3\ell} \le x_{IR,2\ell} \quad \text{for } r \in I_{r,NACP} \ , \tag{4.31}$$

with equality only at $r = 3$, where all three terms in the inequality vanish.

4. IR Zero from NSVZ Beta Function

Applying the LNN limit to the NSVZ beta function (2.12) and calculating the resultant β_x in Eq. (4.11), we obtain

$$\beta_{x,NSVZ,LNN} = -2x^2 \left[\frac{3 - r(1+\gamma_m)}{1-2x} \right] . \tag{4.32}$$

Here,

$$\gamma_m = \sum_{\ell=1}^{\infty} \hat{c}_\ell x^\ell \ , \tag{4.33}$$

where the maximal scheme-independent coefficient in γ_m is the one-loop coefficient

$$\hat{c}_1 = 2 \ . \tag{4.34}$$

In terms of the closed-form (cf) and series (s) functions defined in [22], this beta function can be expressed as

$$\beta_{x,NSVZ} = -2x^2 \hat{b}_1 \, f_{x,cf,NSVZ} \, f_{x,s,NSVZ} \ , \tag{4.35}$$

where

$$f_{x,cf,NSVZ} = \frac{1}{1-2x} \tag{4.36}$$

and

$$f_{x,s,NSVZ} = 1 - \frac{\gamma_m}{\hat{b}_1} = 1 - \frac{\gamma_m}{3-r} \ , \tag{4.37}$$

where here the subscript s connotes the dependence on the series (4.33) for γ_m.

There are different approaches to calculating the IR zero of $\beta_{x,NSVZ}$. If one expands it in a series in x around $x = 0$ and calculates the resultant zero, one necessarily reproduces the one-loop and two-loop results obtained starting from the original series expansion, since these are scheme-independent. This analysis was carried out in [10–12]. An alternate approach proposed and analyzed in [22] is to incorporate the information obtained from the summation to infinite-loop order that yields the structure in Eqs. (4.35)-(4.37). Since the factor $f_{x,s,NSVZ}$ has no zero, one thus calculates the IR zero as the zero in $f_{x,cf,NSVZ}$. Substituting the expansion of γ_m to its maximal scheme-independent order $\gamma_m = 2x$, one thus solves the equation $1 - 2rx/(3-r) = 0$, obtaining

$$x_{IR,NSVZ} = \frac{\xi_{IR,NSVZ}}{4\pi} = \frac{3-r}{2r} \ , \tag{4.38}$$

As r decreases from 3 to 3/2 in the interval $I_{r,NACP}$, this IR zero, $x_{IR,NSVZ}$, increases from 0 to 1/2. The IR zero $x_{IR,NSVZ}$ has much better behavior than $x_{IR,2\ell}$ in that it increases to a finite value as r decreases to the lower end of the interval $I_{r,NACP}$, while $x_{IR,2\ell}$ diverges at this lower boundary of $I_{r,NACP}$. (As noted above, this divergence is only formal, since the perturbative calculation that yielded the expression for $x_{IR,2\ell}$ ceases to apply when the value of x becomes too large.)

In order for the IR zero $x_{IR,NSVZ}$ to be relevant to the UV to IR evolution of the theory, it is necessary and sufficient that this IR zero of the beta function should occur closer to the origin than the pole in $\beta_{x,NSVZ}$, which occurs at $x = 1/2$, as given in Eq. (3.15). The requisite condition

$$x_{IR,NSVZ} \le x_{pole,NSVZ} \quad \text{for } \frac{3}{2} < r < 3 \tag{4.39}$$

is satisfied, since, as we have observed above, $x_{IR,NSVZ} < 1/2$ in this interval, $3/2 < r \le 3$. As r approaches the lower boundary of $I_{r,NACP}$ at $r = 3/2$, $x_{IR,NSVZ}$ approaches $x_{pole,NSVZ}$ from below. The inequality (4.39) is a strict inequality except at the single point $r = 3$, where both $x_{IR,NSVZ}$ and $x_{pole,NSVZ}$ vanish.

As r decreases below 3/2 in the interval $0 < r < 3/2$, $x_{IR,NSVZ}$ increases monotonically above 1/2. Thus, for $0 < r < 3/2$, this IR zero at $x_{IR,NSVZ}$ occurs farther from the origin $x = 0$ than the IR pole in $\beta_{x,NSVZ}$ at $x = 1/2$ and hence is not directly relevant to the UV to IR evolution of the theory from weak coupling.

We next prove some additional inequalities. First,

$$x_{IR,NSVZ} \le x_{IR,2\ell} \quad \text{for } r \in I_{r,NACP} \ . \tag{4.40}$$

This is proved by calculating the difference, which is

$$x_{IR,2\ell} - x_{IR,NSVZ} = \frac{(3-r)^2}{2r(2r-3)} \ . \tag{4.41}$$

This is evidently positive-semidefinite, vanishing only at the upper end of the interval $I_{r,NACP}$ at $r = 3$, where both $x_{IR,2\ell}$ and $x_{IR,NSVZ}$ vanish. Next, we obtain the stronger inequality,

$$x_{IR,NSVZ} \le x_{IR,3\ell} \quad \text{for } r \in I_{r,NACP} \ , \tag{4.42}$$

with equality only at $r = 3$, where both $x_{IR,NSVZ}$ and $x_{IR,3\ell}$ both vanish. This is again proved by calculating the difference:

$$x_{IR,3\ell} - x_{IR,NSVZ} = \frac{63 - 78r + 29r^2 - 4r^3 + 2r\sqrt{C_s}}{2rD_s} \ . \tag{4.43}$$

TABLE II: Values of the IR zero of the beta function in the LNN limit of the SQCD theory, as a function of $r \in I_{r,NACP}$. The entries in the columns are: (i) $x_{IR,2\ell}$, IR zero of the 2-loop beta function $\beta_{x,2\ell}$, (ii) $x_{IR,3\ell}$, IR zero of the 3-loop beta function $\beta_{x,3\ell,\overline{\rm DR}}$, (iii) the IR zero calculated from the Padé approximant [1,1] to $\beta_{x,3\ell,\overline{\rm DR}}$, and (iv) $x_{IR,cfs}$ obtained from $\beta_{x,NSVZ}$. Since $x_{IR,2\ell}$ formally diverges as $r \searrow 1.5$, the perturbative calculation is not applicable (NA) there.

r	$x_{IR,2\ell}$	$x_{IR,3\ell}$	$x_{IR,3\ell,[1,1]}$	$x_{IR,cfs,NSVZ}$
1.5	NA	1.000	0.500	0.000
1.6	3.500	0.690	0.4375	0.162
1.7	1.625	0.529	0.382	0.220
1.8	1.000	0.424	0.238	0.333
1.9	0.6875	0.349	0.236	0.289
2.0	0.500	0.290	0.222	0.250
2.1	0.375	0.242	0.202	0.214
2.2	0.286	0.201	0.179	0.182
2.3	0.219	0.166	0.154	0.152
2.4	0.167	0.135	0.129	0.125
2.5	0.125	0.107	0.104	0.100
2.6	0.0909	0.0811	0.0801	0.0769
2.7	0.0625	0.0579	0.0576	0.05555
2.8	0.0385	0.0368	0.0367	0.0367
2.9	0.0179	0.0175	0.0175	0.0172
3.0	0	0	0	0

The denominator of this expression is positive for $r \in I_{r,NACP}$. In the numerator, although the polynomial $63 - 78r + 29r^2 - 4r^3$ is negative for $r \in I_{r,NACP}$, it is smaller in magnitude than the second term, $2r\sqrt{C_s}$, so the numerator is positive semidefinite in this interval, vanishing only at the upper end, at $r = 3$.

Comparing $x_{IR,NSVZ}$ with $x_{IR,3\ell,[1,1]}$, we find that

$$x_{IR,3\ell,[1,1]} - x_{IR,NSVZ} = \frac{(3-r)^2(r-1)(4r-9)}{2rE_s} \ . \tag{4.44}$$

Therefore, the relative size of $x_{IR,NSVZ}$ and $x_{IR,3\ell,[1,1]}$ is reversed between upper and lower subsections of the interval $I_{r,NACP}$:

$$x_{IR,NSVZ} \le x_{IR,3\ell,[1,1]} \quad \text{if } 2.25 \le r \le 3 \tag{4.45}$$

(with equality only at $x = 2.25$ and $r = 3$), while

$$x_{IR,NSVZ} > x_{IR,3\ell,[1,1]} \quad \text{if } 1.5 < r < 2.25 \ . \tag{4.46}$$

We summarize these results in Table II. The entries in Table II for $x_{IR,n\ell}$ with $n = 2,\ 3$ are equivalent to the entries for $\xi_{IR,n\ell} = 4\pi x_{IR,n\ell}$ with $n = 2,\ 3$ given in Table VII of [12]; the entries for $x_{IR,3\ell,[1,1]}$ are new here. As is evident, the numerical results in Table II obey the general inequalities (4.28) and (4.30) that we have proved above, as well as the inequality (4.23) proved in [11].

5. *Poles of Padé Approximants*

Here we investigate the poles of the Padé approximants to $\beta_{x,rd,2\ell}$ and $\beta_{x,rd,3\ell,\overline{\rm DR}}$ in order to answer the questions posed above. At the two-loop level, from $\beta_{x,rd,2\ell}$ we can obtain one $[p,q]$ Padé approximant with $q \ne 0$, namely

$$[0,1]_{\beta_{x,rd,2\ell}} = \frac{1}{1 + \left[\frac{2(2r-3)}{3-r}\right]x} \ . \tag{4.47}$$

This has a pole at

$$x_{[0,1]_{pole}} = -\left[\frac{3-r}{2(2r-3)}\right] \ . \tag{4.48}$$

Since we are considering $r \in I_{NACP}$, i.e., $3/2 \le r \le 3$, this occurs at negative x and hence is unphysical. As a special case of the general result (3.20), we have $x_{[0,1]_{pole}} = -x_{IR,2\ell}$. So the fact that $x_{IR,2\ell}$ is physical guarantees that this pole is irrelevant.

We next proceed to the three-loop level. From $\beta_{x,rd,3\ell,\overline{\rm DR}}$ we can obtain two $[p,q]$ Padé approximants with $q \ne 0$. The first is the [1,1] approximant, given in Eq. (4.24). This has a pole at

$$x_{[1,1]_{pole}} = \frac{2(2r-3)}{D_s} \ . \tag{4.49}$$

We list values of $x_{[1,1]_{pole}}$ as a function of r in Table III. As r decreases from 3 to 3/2 in the interval $I_{r,NACP}$, the position of this pole decreases monotonically from 1 to 0. For a given $r \in I_{r,NACP}$, this pole occurs farther from the origin than the zero, i.e. $x_{[1,1]_{pole}} \ge x_{[1,1]_{zero}}$. We show this by calculating the difference,

$$x_{[1,1]_{pole}} - x_{[1,1]_{zero}} = \frac{8(2r-3)^3}{D_s\,E_s} \ . \tag{4.50}$$

The right-hand side of (4.50) is positive for $3/2 < r \le 3$ in I and vanishes as r decreases to 3/2 at the lower end of this interval I. Thus, the $[1,1]_{\beta_{x,rd,3\ell}}$ approximant exhibits a physical zero closer to the origin than the pole, and hence the pole is not relevant to the UV to IR evolution described by this Padé approximant. This irrelevance of the pole is similar to what we found for the $[0,1]_{\beta_{x,rd,2\ell}}$ Padé approximant; indeed in that case, the pole occurred at an unphysical, negative value of x. The confluence of the pole and the zero of the $[1,1]_{\beta_{x,rd,3\ell}}$ approximant as $r \searrow 3/2$ in I reflects the fact that as $r \searrow 3/2$, $[1,1]_{x,rd,3\ell} \to 1$, independent of x.

For the analysis of an IR zero, as was carried out in [10, 12, 22], the [0,2] Padé is not of interest, since it cannot reproduce an IR zero that is present in the analysis of $\beta_{x,rd,3\ell,\overline{\rm DR}}$. However, it is of interest for the questions that we address in this subsection. We calculate

$$[0,2]_{\beta_{x,rd,3\ell}} = \frac{1}{1 + \left[\frac{2(2r-3)}{3-r}\right]x + \left[\frac{E_s}{(3-r)^2}\right]x^2} \ .$$

TABLE III: Values of the IR pole(s) of the $[p,q]$ Padé approximants, with $q=1$ and $q=2$, to $\beta_{x,2\ell}$ and $\beta_{x,3\ell,\overline{\rm DR}}$, as functions of r, in the LNN limit of the SQCD theory. The entries in the columns are (i) $x_{[0,1]_{pole}}$, pole of [0,1] approximant to $\beta_{x,rd,2\ell}$; (ii) $x_{[1,1]_{pole}}$, pole of [1,1] approximant to $\beta_{x,rd,3\ell,\overline{\rm DR}}$; (iii) $\{x_{[0,2]_{pole}}\}$, poles of [0,2] approximant to $\beta_{x,rd,3\ell,\overline{\rm DR}}$.

r	$x_{[0,1]_{pole}}$	$x_{[1,1]_{pole}}$	$\{x_{[0,2]_{pole}}\}$
0.0	0.500	0.286	-1, 0.333
0.1	0.518	0.296	-1.034, 0.345
0.2	0.538	0.307	-1.070, 0.358
0.3	0.5625	0.319	-1.109, 0.373
0.4	0.591	0.332	-1.150, 0.390
0.5	0.625	0.348	-1.195, 0.410
0.6	0.667	0.366	-1.245, 0.434
0.7	0.719	0.387	-1.304, 0.463
0.8	0.786	0.414	-1.376, 0.500
0.9	0.875	0.449	-1.473, 0.549
1.0	1.000	0.500	-1.618, 0.618
1.1	1.1875	0.584	-1.876, 0.727
1.2	1.500	0.769	-2.519, 0.940
1.3	2.125	1.739	-11.368, 1.790
1.4	4.000	-0.7143	$0.303 \pm 1.527i$
1.5	∞	0	$\pm i$
1.6	-3.500	0.1695	$-0.0808 \pm 0.748i$
1.7	-1.625	0.255	$-0.110 \pm 0.588i$
1.8	-1.000	0.3125	$-0.119 \pm 0.473i$
1.9	-0.6875	0.359	$-0.118 \pm 0.385i$
2.0	-0.500	0.400	$-0.111 \pm 0.314i$
2.1	-0.375	0.440	$-0.101 \pm 0.256i$
2.2	-0.286	0.479	$-0.0895 \pm 0.208i$
2.3	-0.219	0.521	$-0.0770 \pm 0.167i$
2.4	-0.167	0.566	$-0.0644 \pm 0.132i$
2.5	-0.125	0.615	$-0.0519 \pm 0.101i$
2.6	-0.0909	0.671	$-0.0400 \pm 0.0753i$
2.7	-0.0625	0.734	$-0.0288 \pm 0.0526i$
2.8	-0.0385	0.807	$-0.0184 \pm 0.0328i$
2.9	-0.0179	0.895	$-0.00875 \pm 0.0154i$
3.0	0	1	0

(4.51)

Since the coefficients of both the x and x^2 terms in the denominator of $[0,2]_{\beta_{x,rd,3\ell}}$ are positive, this approximant clearly has no pole for physical (non-negative) x. Explicitly, the poles in $[0,2]_{\beta_{x,rd,3\ell}}$ occur at

$$x_{[0,2],pole} = \frac{(3-r)\Big[-(2r-3) \pm \sqrt{F_s} \ \Big]}{E_s} \ , \tag{4.52}$$

where

$$F_s = 36 - 48r + 21r^2 - 4r^3 \ . \tag{4.53}$$

The polynomial F_s has only one real zero, at $r = 1.3223$, and is positive (negative) for r less (greater) than this value. Hence, F_s is negative for all $r \in I_{r,NACP}$. We list values of $x_{[0,2],pole}$ as a function of r in Table III.

These results answer the five questions that we posed above. The answer to the first question is yes, the quantities $\beta_{x,rd,2\ell} = [1,0]_{\beta_{x,rd,2\ell}}$, $\beta_{x,rd,3\ell} = [2,0]_{\beta_{x,rd,3\ell,\overline{\rm DR}}}$, and $[1,1]_{\beta_{x,rd,3\ell,\overline{\rm DR}}}$ all exhibit (physical) IR zeros. This property is in agreement with existence of an IR in the NSVZ beta function for $r \in I_{r,NACP}$. Second, in each case, the respective IR zero controls the UV to IR evolution, and this again agrees with the NSVZ beta function. Actually, the only case to check is the $[1,1]_{\beta_{x,rd,3\ell,\overline{\rm DR}}}$ Padé approximant, for which we have proved that the pole is always farther from the origin than the IR zero. This is also evident from an inspection of the entries for zeros and poles in Tables II. and III.

Concerning the third and fourth questions, for r in the upper part of the interval $I_{r,NACP}$, where the IR zero in the n-loop beta function occurs at a rather small value, one expects that the values of this IR zero calculated from this beta function itself, from the Padé approximants to the (reduced) beta function, and from the NSVZ beta function should agree, and this expectation is borne out by the results, as listed in Table II. For example, for the illustrative value $r = 2.5$, $x_{IR,2\ell} = 0.125$, $x_{IR,3\ell} = 0.107$, $x_{IR,3\ell,[1,1]} = 0.104$, and $x_{IR,NSVZ} = 0.100$. Aside from the lowest-order, two-loop value, the last three values of the IR zero are quite close to each other. As r decreases in the interval $I_{r,NACP}$, the differences tend to grow somewhat. Thus, for $r = 2.0$, $x_{IR,2\ell} = 0.500$, $x_{IR,3\ell} = 0.290$, $x_{IR,3\ell,[1,1]} = 0.222$, and $x_{IR,NSVZ} = 0.250$. Again, aside from the lowest-order, two-loop value, the last three values are within about 10 % of each other.

Finally, concerning the fifth question, pertaining to the pole in the $[p,q]$ Padé approximants with $q \neq 0$, one should remark at the outset that since this pole occurs farther from the origin than the IR zero, it does not directly affect the evolution from weak coupling in the UV to the IR, so its precise value is not directly relevant for this evolution. The values of the pole position from the $[0,1]$, $[1,1]$ and $[0,2]$ Padé approximants are listed in Table III. As discussed in connection with Eq. (3.20), given the fact that the series expansion of the NSVZ beta function must agree with the $\overline{\rm DR}$ to two-loop order and given the identity (3.20), it follows that since the two-loop beta function (equivalently, the [1,0] Padé) has a physical IR zero, the pole in the [0,1] Padé approximant to $\beta_{x,r,2\ell}$ must occur at a negative and hence unphysical value of x. At the three-loop level, the two relevant two Pad'e approximants, [1,1] and [0,2] have poles at different values of x, and only one, namely [1,1], has a pole at a physical value of x. Furthermore, the position of this pole varies as a function of r, decreasing from 1 at $r = 3$ to 0 at $r = 3/2$, in contrast to the pole in $\beta_{x,NSVZ}$, which has a fixed value at $x_{pole,NSVZ} = 1/2$. More generally, even the unphysical poles in the [0,1] and [0,2] Padé approximants vary considerably as functions of r. Thus, these

Padé approximants do not exhibit evidence of a stable pole. Of course, these results do not preclude the possibility that if the beta function in the $\overline{\rm DR}$ scheme could be calculated to higher order, one might begin to see evidence of a stable pole in the $[p,q]$ Padé approximants with $q \neq 0$.

D. Analysis for $0 < r < 3/2$

In this interval of r, $\hat b_1$ and $\hat b_2$ have the same sign, so the two-loop beta function does not have any IR zero, and this is the maximum scheme-independent information that one has concerning the IR zero. Hence, using this two-loop beta function, one infers that as the scale μ decreases from the deep UV to the IR, the gauge coupling continuously increases, eventually exceeding the region where one can use perturbative methods to calculate it reliably. Formally, the three-loop beta function calculated in the $\overline{\rm DR}$ scheme continues to exhibit an IR zero, $x_{IR,3\ell,\overline{\rm DR}}$, given in Eq. (4.21), for a small interval of r below 3/2. However, one cannot take this to be a physically compelling result, in view of the fact that the maximal scheme-independent information available (from the two-loop beta function) does not exhibit any IR zero. Furthermore, as r decreases from 3/2 to the zero of D_s at $r = (21-\sqrt{105}\,)/8 = 1.3441$ (see Eq. (4.19)), $x_{IR,3\ell,\overline{\rm DR}}$ grows without bound, so that one can ignore it, since the value of the IR zero is beyond the regime where one would consider perturbation theory to be reliable. Indeed, as is evident from the expression for $[1,1]_{zero}$ in Eq. (4.27), the [1,1] Padé approximant to $\beta_{x,r,3\ell,\overline{\rm DR}}$ ceases to have a physical IR zero as r decreases through 3/2. The absence of a (physical) IR zero from a perturbatively reliable calculation using the beta function in the $\overline{\rm DR}$ scheme for $0 < r < 3/2$ is in agreement with the prediction from the NSVZ beta function; as discussed above, for r in the interval $0 < r < 3/2$, the formal zero at $x_{IR,NSVZ}$ lies farther from the origin than the pole at $x = 1/2$ and hence is not directly relevant to the UV to IR evolution of the theory.

We proceed to address and answer the following questions for this interval $0 \le r < 3/2$:

1. Do the $[p,q]$ Padé approximants with $q \neq 0$ exhibit a physical pole?

2. If the answer to the first question is affirmative, then does this pole occur closer to the origin than any IR zero (if such a zero is present) and hence dominate the UV to IR evolution of the theory?

3. If the answers to the first two questions are affirmative, then is this IR pole close to the value $x_{pole,NSVZ} = 1/2$ of the pole in the NSVZ beta function?

4. If the answers to the first two questions are affirmative, then, independent of whether the poles in different $[p,q]$ Padé approximants with $q \neq 0$ are close to $x_{IR,NSVZ} = 1/2$, do these different approximants at least exhibit a stable (physical) pole?

In the present case, the relevant Padé approximants are [0,1], [1,1], and [0,2]. As is clear from the explicit expressions for these approximants and their poles given above, the poles do not occur at a fixed value of x and are not, in general, equal to $x_{IR,NSVZ}$. As is evident from (4.48), the pole in the [0,1] Padé approximant to $\beta_{x,r,2\ell}$ occurs at $x = 1/2$ only if $r = 0$ and increases monotonically without bound as r increases from 0 to 3/2. The pole in the [1,1] Padé approximant to $\beta_{x,r,3\ell,\overline{\rm DR}}$ increases monotonically as a function of r from 0.286 at $r = 0$ and diverges as r approaches the value $(21-\sqrt{105}\,)/8 = 1.3441$ from below. In the small interval $1.3441 < r < 3/2$, this pole occurs at negative x and hence is unphysical. Only at the value $r = 1$ is the position of this pole in the [1,1] Padé equal to 1/2. The [0,2] Padé approximant to $\beta_{x,r,3\ell,\overline{\rm DR}}$ has two poles, one of which is always unphysical. The other pole of the [0,2] Padé occurs at an r-dependent value that increases from 1/3 at $r = 0$ and diverges as r approaches the zero in E_s at $r = 1.3118$ from below. It is negative for r in the small interval $1.3118 < r < 1.3223$, where $r = 1.3223$ is the point where F_s has a zero; finally, in the small interval $1.3223 < r < 3/2$, it is complex. This pole in the [0,2] Padé is approximately equal to 1/2 for $r = 0.8$.

These results provide answers to the questions stated above. The answer to the first question is that over much of the interval $0 \le r < 3/2$, the [0,1], [1,1], and [0,2] Padé approximants do exhibit physical poles, but these all vary as functions of r and are not stable at any particular fixed value. Second, since there is no robust, scheme-independent IR zero in the beta function, these poles do dominate the UV to IR evolution. Third, the values of the poles in the various Padé approximants exhibit some scatter and are not, in general, equal to (the r-independent) value $x_{IR,NSVZ} = 1/2$. Concerning the fourth question, although the pole in the [0,1] Padé is not very close to the pole in the [1,1] Padé or the physical pole in the [0,2] Padé, the latter two poles are in fair agreement with each other. For example (see Table III) over much of the interval $0 \le r < 3/2$, the ratio of the physical pole in the [0,2] Padé divided by the pole in the [1,1] Padé is about 1.2, which is reasonably close to unity. One may interpret this as indicating that, at least for the three-loop beta function calculated in the $\overline{\rm DR}$ scheme for this theory, there is rough agreement, at about the 20 % level, between the pole in the [1,1] Padé approximant and the physical pole in the [0,2] Padé approximant.

V. CHIRAL SUPERFIELDS IN SYMMETRIC AND ANTISYMMETRIC RANK-2 TENSOR REPRESENTATIONS

A. Beta Function and IR Zeros

Here we consider a (vectorial, asymptotically free) $\mathcal{N} = 1$ supersymmetric SU(N_c) gauge theory with N_f copies of massless chiral superfields Φ_i and $\tilde{\Phi}_i$, $i = 1, ..., N_f$, transforming according to the symmetric and antisymmetric rank-2 tensor representations and their conjugates. We denote these symmetric and antisymmetric rank-2 tensor representations as S_2 and A_2, respectively, with corresponding Young tableaux $\square\!\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. While the S_2 theory is defined for all $N_c \geq 2$, the A_2 theory is defined for $N_c \geq 3$, since the A_2 representation is a singlet if $N_c = 2$. We restrict our consideration to values $N_f \neq 0$ here, since if $N_f = 0$, the present theory reduces to a pure supersymmetric Yang-Mills gauge theory, which we have already analyzed above. The one-loop and two-loop coefficients in the beta function are (e.g., [10])

$$b_1 = 3N_c - (N_c \pm 2)N_f \ , \tag{5.1}$$

and

$$b_2 = 2\Big[3N_c^2(1 - N_f) \mp 8(N_c - N_c^{-1})N_f\Big] \ . \tag{5.2}$$

where the upper and lower signs apply for the S_2 and A_2 theories, respectively. Evaluating Eq. (2.5), we obtain, for b_3 in the $\overline{\rm DR}$ scheme,

$$\begin{aligned} b_3 &= 7N_c^3(N_f - 1)(N_f - 3) \pm 2N_c^2 N_f(-33 + 17N_f) \\ &+ 8N_c N_f(1 + 5N_f) \mp 24N_f(N_f - 1) \\ &- 16N_c^{-1}N_f(2 + 3N_f) \pm 64N_c^{-2}N_f \ . \end{aligned} \tag{5.3}$$

It will often be convenient to refer to these two cases together as T_2 (standing for tensor, rank-2) with the above sign convention, and we shall do so. The one-loop coefficient decreases with increasing N_f and passes through zero with sign reversal for $N_f = N_{f,b1z,T_2}$, where

$$N_{f,b1z,T_2} = \frac{3N_c}{N_c \pm 2} \ . \tag{5.4}$$

In the S_2 theory, $N_{f,b1z,S_2}$ increases monotonically from $3/2$ for $N_c = 2$, approaching the limiting value 3 from below as $N_c \to \infty$, while in the A_2 theory, $N_{f,b1z,A_2}$ decreases monotonically from 9 for $N_c = 3$, approaching the limiting value 3 from above as $N_c \to \infty$. The two-loop coefficient also decreases with increasing N_f and passes through zero with sign reversal for $N_f = N_{f,b2z,T_2}$, where

$$N_{f,b2z,T_2} = \frac{3N_c^2}{3N_c^2 \pm 8(N_c - N_c^{-1})} \ . \tag{5.5}$$

From the exact results recalled above, it follows that the lower boundary of the IR non-Abelian Coulomb phase is

$$N_{f,cr,T_2} = \frac{N_{f,b1z,T_2}}{2} = \frac{3N_c}{2(N_c \pm 2)} \ . \tag{5.6}$$

In the S_2 theory, $N_{f,cr}$ increases monotonically from $3/4$ for $N_c = 2$, approaching $3/2$ from below as $N_c \to \infty$, while in the A_2 theory, $N_{f,cr}$ decreases monotonically from $9/2$ for $N_c = 3$, approaching $3/2$ from above as $N_c \to \infty$. Hence, the IR non-Abelian Coulomb phase exists for (integral) N_f values in the interval

$$I_{NACP,(S_2,A_2}: \quad \frac{3N_c}{2(N_c \pm 2)} < N_f < \frac{3N_c}{N_c \pm 2} \ . \tag{5.7}$$

In the S_2 theory, as noted in [10], $N_{f,b2z,S_2} < N_{f,cr,S_2}$, so $b_2 < 0$ if N_f is in the non-Abelian Coulomb interval (5.7). In the A_2 theory, $N_{f,b2z,A_2} < N_{f,cr,A_2}$ if $N_f = 1$ or $N_f = 2$, while $N_{f,b2z,A_2} > N_{f,cr,A_2}$ if $N_f = 3$.

Assuming that N_f is in the respective ranges where $b_2 < 0$ in the S_2 and A_2 theories, the theory has an IR zero in the beta function at the two-loop level, occuring at $a_{IR,2\ell,T_2} = -b_{1,T_2}/b_{2,T_2}$, i.e.,

$$a_{IR,2\ell,T_2} = \frac{3N_c - (N_c \pm 2)N_f}{2[3N_c^2(N_f - 1) \pm 8(N_c - N_c^{-1})N_f]} \ . \tag{5.8}$$

As noted above, this two-loop result is scheme-independent. To calculate the IR zero of the NSVZ beta function using the corresponding maximal scheme-independent information in γ_m, we use Eq. (2.14) for the one-loop term in γ_m and thus solve the equation $b_{1,T_2} - 2N_f T_f c_1 a = 0$, obtaining the result

$$a_{IR,NSVZ,T_2} = \frac{N_c[N_c(3 - N_f) \mp 2N_f]}{4N_f(N_c \pm 2)^2(N_c \mp 1)} \ . \tag{5.9}$$

To analyze the extent to which the perturbative beta functions of the S_2 and A_2 theories, calculated in the $\overline{\rm DR}$ scheme, exhibit similarities with the NSVZ beta function, we proceed to consider these functions at the three-loop level. To carry out this analysis, we focus on the case $N_c \to \infty$ and again work with the scaled beta function $\beta_x = dx/dt$. In this limit, the beta functions β_{x,S_2} and β_{x,A_2} and for the S_2 and A_2 theories become the same, and we shall denote the resulting beta function as β_{x,T_2}. Similarly, the intervals given in Eq. (5.7) also become the same, reducing to

$$I_{NACP,T_2}: \quad \frac{3}{2} < N_f < 3 \quad \text{for } N_c \to \infty \tag{5.10}$$

Hence, the interval I_{NACP,T_2} contains only one physical, integral value of N_f in this limit, namely $N_f = 2$.

With the definition (3.3), we have, for the (scheme-independent) one-loop and two-loop rescaled coefficients $\hat{b}_\ell$, the results

$$\hat{b}_{1,T_2} = 3 - N_f \ , \tag{5.11}$$

and

$$\hat{b}_{2,T_2} = -6(N_f - 1) \ . \tag{5.12}$$

Note that $\hat{b}_{2,T_2}$ vanishes for $N_f = 1$, so that the two-loop beta function has no IR zero for this value of N_f.

In general, the IR zero of the rescaled two-loop beta function β_{x,T_2} is $x_{IR,T_2} = -\hat{b}_{1,T_2}/\hat{b}_{2,T_2}$, i.e.,

$$x_{IR,2\ell,T_2} = \frac{3 - N_f}{6(N_f - 1)} \ . \tag{5.13}$$

The only value of N_f for which this has a finite, nonzero value is $N_f = 2$, and for $N_f = 2$, $x_{IR,2\ell,T_2} = 1/6$. In this limit, the rescaled NSVZ beta function $\beta_{x,NSVZ,T_2}$ has an IR zero at

$$x_{IR,NSVZ,T_2} = \frac{3 - N_f}{4N_f} \ . \tag{5.14}$$

For $N_f = 2$, this has the value

$$x_{IR,NSVZ,T_2} = \frac{1}{8} \quad \text{for } N_f = 2 \ . \tag{5.15}$$

In the $\overline{\rm DR}$ scheme, the three-loop coefficient in β_{x,T_2} is

$$\hat{b}_{3,T_2} = -7(N_f - 1)(3 - N_f) \ . \tag{5.16}$$

Hence, if $N_f = 1$, the three-loop beta function in the $\overline{\rm DR}$ scheme is the same as the (scheme-independent) two-loop beta function, in which the two-loop coefficient also vanishes. Thus, if $N_f = 1$ here, the beta function reduces simply to the one-loop term, which has no IR zero.

The three-loop beta function, $\beta_{x,3\ell,T_2}$, for the S_2 and A_2 theories in this limit, with $\hat{b}_3$ calculated in the $\overline{\rm DR}$ scheme, is

$$\beta_{x,3\ell,T_2} = -2x^2(\hat{b}_{1,T_2} + \hat{b}_{2,T_2}x + \hat{b}_{3,T_2}x^2) \ . \tag{5.17}$$

From Eq. (3.9), it follows that the reduced three-loop beta function $\beta_{x,rd,3\ell,T_2}$ is

$$\begin{aligned} \beta_{x,rd,3\ell,T_2} &= 1 + \frac{\hat{b}_{2,T_2}}{\hat{b}_{1,T_2}}x + \frac{\hat{b}_{3,T_2}}{\hat{b}_{1,T_2}}x^2 \\ &= 1 - \frac{6(N_f-1)}{3-N_f}\, x - 7(N_f-1)x^2 \ . \end{aligned} \tag{5.18}$$

If $N_f \neq 1$ (and $N_f \neq 3$), the equation $\beta_{x,rd,3\ell,T_2} = 0$ has, formally, two solutions, given by

$$x_{IR,3\ell,T_2,\pm} = \frac{1}{7(3-N_f)} \left[-3 \pm \sqrt{\frac{7N_f^2 - 33N_f + 54}{N_f - 1}} \ \right] \ . \tag{5.19}$$

For $N_f = 2$, the solution with the + sign in front of the square root is equal to $x_{IR,3\ell,T_2,+} = 1/7 = 0.14286$, which is reasonably close to the NSVZ result of 1/8 given in Eq. (5.15). (The solution with the minus sign in Eq. (5.19) has the unphysical negative value -1 and hence is not relevant.)

TABLE IV: Values of zeros and poles, in the variable x, of Padé approximants to $\beta_{x,rd,2\ell}$ and $\beta_{x,rd,3\ell,\overline{\rm DR}}$ for the $\mathcal{N}=1$ supersymmetric gauge theory with $N_f = 2$ copies of chiral superfields in the symmetric or antisymmetric rank-2 tensor representation and its conjugate, in the limit $N_c \to \infty$. Results are given to the indicated floating-point accuracy. The abbreviation NA means "not applicable".

n	$[p,q]$	zero(s)	pole(s)
2	[1,0]	$1/6 = 0.167$	NA
2	[0,1]	NA	$-1/6 = -0.167$
3	[2,0]	-1, $1/7 = 0.143$	NA
3	[1,1]	$6/43 = 0.1395$	$6/7 = 0.857$
3	[0,2]	NA	$-0.0698 \pm 0.1356i$

B. Padé Approximants

We next calculate the Padé approximants to the two-loop and three-loop reduced beta functions, $\beta_{x,rd,2\ell,T_2}$ and $\beta_{x,rd,3\ell,T_2}$, respectively. As before, these functions are identical to the [1,0] and [2,0] approximants, respectively, so our analysis given above of the zeros of $\beta_{x,rd,2\ell,T_2}$ and $\beta_{x,rd,3\ell,T_2}$ applies to these approximants. We list these results in Table IV and proceed to calculate and analyze the $[p,q]$ approximants with $q \neq 0$. With $\beta_{x,rd,2\ell,T_2}$ we can only calculate one $[p,q]$ approximant with $q \neq 0$, namely the [0,1] approximant. We find

$$[0,1]_{\beta_{x,rd,2\ell,T_2}} = \frac{1}{1 + \frac{6(N_f-1)}{3-N_f}x} \ . \tag{5.20}$$

Thus, as a special case of the relation (3.20), this Padé approximant has a pole at

$$\begin{aligned} [0,1]_{T_2,pole} &= -[1,0]_{T_2,zero} = -x_{IR,2\ell,T_2} \\ &= -\frac{(3-N_f)}{6(N_f-1)} \ . \end{aligned} \tag{5.21}$$

For $N_f = 2$, this pole occurs at $x = -1/6$.

At the three-loop level, assuming that $N_f \neq 1$ (where the three-loop and two-loop coefficients vanish), we calculate, for the [1,1] approximant, the result

$$[1,1]_{\beta_{x,rd,3\ell,T_2}} = \frac{1 - \frac{(7N_f^2 - 6N_f + 27)}{6(3-N_f)}x}{1 - \frac{7(3-N_f)}{6}x} \ . \tag{5.22}$$

The fact that $\hat{b}_2$ and $\hat{b}_3$ vanish at $N_f = 1$ is reflected in the property that this [1,1] Padé approximant reduces to unity at this value of N_f. For $N_f \neq 1$, this approximant has a IR zero at

$$[1,1]_{T_2,zero} = \frac{6(3-N_f)}{7N_f^2 - 6N_f + 27} \ . \tag{5.23}$$

and a pole at

$$[1,1]_{T_2,pole} = \frac{6}{7(3-N_f)} \ . \tag{5.24}$$

The polynomial $7N_f^2 - 6N_f + 27$ is positive for all (real) N_f. We note the inequality

$$[1,1]_{T_2,zero} < [1,1]_{T_2,pole} \ . \tag{5.25}$$

This is proved by calculating the difference,

$$[1,1]_{T_2,pole} - [1,1]_{T_2,zero} = \frac{6^3(3-N_f)}{7(N_f-1)(7N_f^2-6N_f+27)} \ . \tag{5.26}$$

Evidently, since $N_f \neq 0, \ 1$ and $N_f < 3$, the right-hand side of (5.26) is positive-definite. For $N_f = 2$, we have $[1,1]_{T_2,zero} = 6/43 = 0.1395$ and $[1,1]_{T_2,pole} = 6/7 = 0.8571$. These values are listed in Table IV.

We compute the [0,2] Padé approximant to $\beta_{x,rd,3\ell,T_2}$ to be

$$[0,2]_{\beta_x,rd,3\ell,T_2} = \frac{1}{1 + \frac{6(N_f-1)}{(3-N_f)}x + \frac{(N_f-1)(7N_f^2-6N_f+27)}{(3-N_f)^2}x^2} \ . \tag{5.27}$$

This has, formally, two poles, at the values

$$[0,2]_{T_2,pole} = \frac{(3-N_f)}{(7N_f^2-6N_f+27)}\bigg[-3 \pm \sqrt{\frac{-7N_f^2+15N_f-36}{N_f-1}}\ \bigg] \ . \tag{5.28}$$

However, recalling that $N_f \neq 0, \ 1$, one sees that these poles are both unphysical, because the polynomial in the square root, $-7N_f^2+15N_f-36$, is negative-definite. Explicitly, for $N_f = 2$, Eq. (5.28) yields the pole values $x = (1/43)(-3 \pm \sqrt{34}\,i\ ) = -0.0698 \pm 0.1356i$, as is listed in Table IV.

Let us summarize our results for these theories with a nonzero number, N_f, of copies of chiral superfields in the S_2 or A_2 representation and their respective conjugates, in the limit $N_c \to \infty$. In this limit, $N_f < 3$ for asymptotic freedom. For $N_f = 1$, the (scheme-independent) two-loop term in the beta function β_{x,T_2} vanishes, as does the three-loop term with the latter calculated in the $\overline{\rm DR}$ scheme, so that the beta function, calculated to these loop orders, does not contain any IR zero or any indication, via Padé approximants, of a pole.

For this theory with $N_f = 2$, our analysis of the two-loop and three-loop $\overline{\rm DR}$ beta functions, and the [1,1] Padé approximant to the latter all give evidence of an IR zero, with respective values 1/6, 1/7, and 6/43=0.1395, which decrease monotonically, approaching the the value $x_{IR,NSVZ,T_2} = 1/8$ from the NSVZ beta function. Thus, we find reasonably good agreement between these IR zeros calculated in different schemes. We come next to the questions of whether, for this theory, the $[p,q]$ Padé approximants with $q \neq 0$ to the two-loop and three-loop beta function give some indication of a (physical) pole and whether this pole occurs at a value close to the value $x = 1/2$ in the NSVZ beta function. Our result is that neither the [0,1] nor the [0,2] Padé approximants has any physical pole, while the [1,1] Padé does exhibit a pole, although it is roughly twice the value $x = 1/2$. As in the case of the theory containing chiral superfields in the $\square$ and $\overline{\square}$ representations, this suggests that it is necessary to calculate the beta function in the $\overline{\rm DR}$ scheme to higher-loop order and calculate higher-order Padé approximants to test for indications of a pole.

VI. CONCLUSIONS

In this paper we have studied asymptotically free vectorial SU(N_c) gauge theories with $\mathcal{N} = 1$ supersymmetry, including both pure gluonic supersymmetric Yang-Mills theory and theories with N_f copies of a pair of chiral superfields in respective representations R and $\bar{R}$, where R is the fundamental representation and the symmetric and antisymmetric rank-2 tensor representation of SU(N_c). We have calculated Padé approximants to the beta functions for these theories in the $\overline{\rm DR}$ scheme up to four-loop order for the SYM theory and up to three-loop order for the theories with matter superfields and have compared results for IR zeros and poles with results from the NSVZ beta function. For the pure supersymmetric YM theory, the Padé results for the four-loop beta function calculated in the $\overline{\rm DR}$ scheme show strong evidence for a pole, in good agreement with the NSVZ beta function.

For the theory with chiral superfields in the fundamental representation, in the large-N_c, large-N_f limit with $3/2 < r < 3$, where $r = N_f/N_c$, the theory evolves from weak coupling in the UV to a non-Abelian Coulomb phase in the IR and exhibits an IR fixed point in the renormalization group. Our analysis of the two-loop, three-loop beta function and the [1,1] Padé approximant to the latter yields robust evidence for the IR zero, with values in reasonable agreement with the value obtained from the NSVZ beta function, taking into account the difference between the $\overline{\rm DR}$ and NSVZ schemes. With regard to the question whether the Padé approximants of the beta function in the $\overline{\rm DR}$ scheme indicate a pole, as is present in the NSVZ beta function, we find that of the three Padé approximants analyzed here ([0,1], [1,1] and [0,2]), only one, namely the [1,1] approximant, yields a physical pole, and the x value of this pole varies considerably as a function of r, in contrast to the pole in the NSVZ beta function, which is a constant at $x = 1/2$, independent of r. With r in the lower interval $0 < r < 3/2$, the [0,1] and [1,1] approximants yield physical poles, and one of the two poles in the [0,2] approximant is physical, but again, the x values of these poles vary considerably as functions of r, in contrast to the fixed pole in the NSVZ beta function.

In the theory with chiral superfields in rank-2 tensor representations, in the $N_c \to \infty$ limit, if $N_f = 1$, both the two-loop and the $\overline{\rm DR}$ three-loop coefficients vanish, while if $N_f = 2$, the various Padé approximants yield an IR zero in the beta function in agreement with the NSVZ beta function, but only one, namely the [1,1] approximant, yields a physical pole.

Our calculations provide a quantitative measure, for these various supersymmetric theories, of how well finite-order perturbative results calculated in the $\overline{\text{DR}}$ scheme reproduce the properties of the NSVZ beta function. Our results indicate qualitative and rough quantitative agreement between the $\overline{\text{DR}}$ and NSVZ beta functions for the theories and matter superfield contents where these exhibit an IR zero. Our analysis of Padé approximants to the $\overline{\text{DR}}$ beta function for the pure gluonic Yang-Mills theory show qualitative consistency with the IR pole that is present in the NSVZ beta function. For the supersymmetric theories with matter superfields, our results suggest that it may be necessary to calculate the beta function to higher-loop order in the $\overline{\text{DR}}$ scheme in order to test consistency with the pole in the NSVZ beta function.

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[20] This and other expressions for special values of N_f involve an implicit generalization of N_f from nonzero integers to nonzero real numbers. It is understood implicitly that if a formal expression for a value of N_f evaluates to a non-integral (real) value, one infers an appropriate integral value of N_f from it, namely the closest integer larger or smaller than the expression, depending on the context.

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