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# Virtuality and transverse momentum dependence of the pion distribution amplitude

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# Virtuality and Transverse Momentum Dependence of Pion Distribution Amplitude

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We describe basics of a new approach to transverse momentum dependence in hard exclusive processes. We develop it in application to the transition process  $\gamma^*\gamma \rightarrow \pi^0$  at the handbag level. Our starting point is coordinate representation for matrix elements of operators (in the simplest case, bilocal  $\mathcal{O}(0, z)$ ) describing a hadron with momentum  $p$ . Treated as functions of  $(pz)$  and  $z^2$ , they are parametrized through *virtuality distribution amplitudes* (VDA)  $\Phi(x, \sigma)$ , with  $x$  being Fourier-conjugate to  $(pz)$  and  $\sigma$  Laplace-conjugate to  $z^2$ . For intervals with  $z^+ = 0$ , we introduce the *transverse momentum distribution amplitude* (TMDA)  $\Psi(x, k_\perp)$ , and write it in terms of VDA  $\Phi(x, \sigma)$ . The results of covariant calculations, written in terms of  $\Phi(x, \sigma)$  are converted into expressions involving  $\Psi(x, k_\perp)$ . Starting with scalar toy models, we extend the analysis onto the case of spin-1/2 quarks and QCD. We propose simple models for soft VDAs/TMDAs, and use them for comparison of handbag results with experimental (BaBar and BELLE) data on the pion transition form factor. We also discuss how one can generate high- $k_\perp$  tails from primordial soft distributions.

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## I. INTRODUCTION

Analysis of effects due to parton transverse momentum is an important direction in modern studies of hadronic structure. The main effort is to use the transverse-momentum dependence of inclusive processes, such as semi-inclusive deep inelastic scattering (SIDIS) and Drell-Yan (DY) pair production, describing their cross sections in terms of *transverse momentum dependent distribution* (TMDs)  $F(x, k_\perp)$  [1].

TMDs are generalizations of the usual “longitudinal” parton densities  $f(x)$  that correspond to TMDs integrated over  $k_\perp$  (in a perturbative context, “unintegrated parton distributions” were used from the wake of QCD, see, e.g. Ref. [2]). Information about the transverse momentum structure is contained in the  $k_\perp^{2n}$  moments of  $F(x, k_\perp)$  for  $n \geq 1$ . Dealing with DY and SIDIS processes, one encounters  $\mathcal{O}(\langle k_\perp^{2n} \rangle)$  contributions that are not suppressed by inverse powers of  $Q^2$ , the high momentum probe. Such effects are combined into one function, TMD  $F(x, k_\perp)$ , the use of which in these cases is unavoidable.

However, in light-cone dominated processes, such as deep inelastic scattering in the limit of large  $Q^2$ , the higher moments of  $k_\perp^2$  generate just power corrections of  $\langle k_\perp^{2n} \rangle / Q^{2n}$  type to the leading power behavior described by the “collinear” parton distribution  $f(x)$ . Still, if one is interested in the region of moderately large  $Q^2$ ’s, the transverse momentum corrections may be rather important even for a light-cone dominated process. Then one may want to explicitly represent them as generated from a common TMD-type function. Effectively, this corresponds to a resummation of such power corrections.

A situation when accessible  $Q^2$  are not large enough to secure the dominance of the asymptotically leading collinear approximation, is very common in hard exclusive processes, such as pion and nucleon electromagnetic

form factors, where the overlap contributions of soft wave functions [3]  $\psi(x, k_\perp)$  are sufficient to describe existing data. These wave functions are apparent analogs of TMDs in case of exclusive processes. In particular, integrating  $\psi(x, k_\perp)$  over  $k_\perp$  gives [4] the distribution amplitude  $\varphi(x)$  [5], a basic object of the asymptotic perturbative QCD analysis for exclusive processes [4–8]. The latter is based on the operator product expansion (OPE) [9, 10], with  $x^n$  moments of  $\varphi(x)$  related to matrix elements of the lowest-twist operators.

One may expect that higher  $k_\perp^{2n}$  moments of  $\psi(x, k_\perp)$  should correspond to matrix elements of higher-twist operators of the OPE. However, a subtle point is that the standard OPE [9, 10] is constructed within a covariant 4-dimensional quantum field theory (QFT) framework, while the wave functions  $\psi(x, k_\perp)$  [3] mentioned in the context of the overlap contributions are the objects of a 3-dimensional light-front approach [11, 12].

Our goal in the present paper is to develop the basics of a formalism (its outline was given in Ref. [13]) that starts from a covariant 4-dimensional approach, but describes the structure of hadrons in hard exclusive processes in terms of functions  $\Psi(x, k_\perp)$  incorporating the dependence on the transverse momentum of its constituents. Just like in the light-front formalism, the organization of these functions has the structure of the Fock state decomposition, i.e. each function is characterized by the number of constituents involved.

The lowest, 2-body component is described by a function  $\Psi(x, k_\perp)$  that depends on a 3-dimensional variable  $x, k_\perp$ . To emphasize the distinction, we use “transverse momentum dependent distribution amplitude” (TMDA) as the name for the function  $\Psi(x, k_\perp)$  that appears in our approach. By construction,  $\Psi(x, k_\perp)$  has a direct connection with the operators that appear in the OPE of a covariant QFT. As a specific application, we choose the hard exclusive process of  $\gamma^*\gamma \rightarrow \pi^0$  transition that

involves just one hadron, and thus has the simplest structure.

The paper is organized as follows. Since the basic features of our approach are not sensitive to the spin of the particles, we begin in Sect. II with the discussion of the structure of the  $\gamma^*\gamma \rightarrow \pi^0$  transition form factor at the handbag(i.e., 2-body) level in a scalar model. With the goal of keeping the closest contact with the OPE, we start with a general analysis of the handbag diagram using the coordinate representation, in which the hadron structure is described through the  $z$ -dependence of a matrix element  $\langle p|\phi(0)\phi(z)\rangle \equiv \tilde{\chi}_p(z)$  involving two parton fields  $\phi$  ( $p$  being the hadron momentum).

By Lorentz invariance,  $\tilde{\chi}_p(z)$  depends on  $z$  through two variables,  $(pz)$  and  $z^2$ . A double Fourier transform of  $\tilde{\chi}_p(z)$  with respect to  $(pz)$  and  $z^2$  gives the *virtuality distribution amplitude* (VDA)  $\Phi(x, \sigma)$ , the basic object of our analysis. For any contributing Feynman diagram the support of the VDA is restricted by  $0 \leq x \leq 1$  and  $0 \leq \sigma \leq \infty$ . The variable  $x$  has the usual meaning of the fraction of the hadron momentum  $p$  carried by a parton, while the variable  $\sigma$  being conjugate to  $z^2$  may be interpreted as a generalized virtuality.

Projecting  $\tilde{\chi}_p(z)$  on the light front  $z_+ = 0$  results in the *impact parameter distribution amplitude* (IDA)  $\varphi(x, z_\perp)$ , whose further Fourier transform leads to the *transverse momentum dependent distribution amplitude* (TMDA)  $\Psi(x, k_\perp)$ . The properties of virtuality distributions, TMDAs and connections between them and related functions are discussed in Sect. III. A key point for subsequent applications is that  $\Psi(x, k_\perp)$  has a simple expression in terms of VDA  $\Phi(x, \sigma)$ . This observation may be used to rewrite the results of covariant calculations (initially given in terms of VDA  $\Phi(x, \sigma)$ ) through TMDA  $\Psi(x, k_\perp)$ , i.e., as a 3-dimensional integral over  $x$  and  $k_\perp$ . In this way we derive the expression for the transition form factor in terms of TMDA.

To emphasize a special role of the VDA representation, in Sect. IV we analyze the structure of the handbag amplitude in several other representations, namely, coordinate light-front variables, Sudakov and IMF parameterizations for the virtual momentum integration. We show that expressions for the form factor in all these cases are much more complicated than the VDA form (to which they are eventually equivalent), and one needs to resort to approximations in order to get a compact formula. Continuing to discuss the relation between the VDA approach and the method of operator product expansions, in Sect. IV D we outline the application of the VDA approach in the three-body distribution case.

Modifications that appear for spin-1/2 quarks and vector gluons are considered in Sect. V. In particular, we observe that the basic relations between the VDA-based distributions remain intact when matrix elements involve spin-1/2 quarks. Using the parameterization in terms of VDA  $\Phi(x, \sigma)$ , we calculate the handbag diagram and express the result in terms of the TMDA  $\Psi(x, k_\perp)$ . The change in the form factor formula reflects a more compli-

cated structure of the spin-1/2 hard propagator. Then we study the extension of our results onto gauge theories.

In Sect. VI, we formulate a few simple models for soft TMDAs, i.e. those that decrease faster than any inverse power of  $1/k_\perp^2$  for large  $k_\perp^2$ . In Sect. VIB we analyze the results of using these models to describe the data on the pion transition form factor.

Attaching perturbative propagators to the soft TMDA, as shown in Sect. VII, produces factors with  $1/k_\perp^2$  behavior. As a result, the quark-gluon interactions in QCD generate a hard  $\sim 1/k_\perp^2$  tail for TMDAs. The basic elements of generating hard tails from soft primordial TMDAs are illustrated in Sect. VII B. Finally, in Sect. VIII we formulate our conclusions and discuss directions of further applications of the VDA approach.

## II. TRANSITION FORM FACTOR IN SCALAR MODEL

### A. Handbag diagram in coordinate representation

Consider a general handbag diagram for a scalar analog of the  $\gamma^*\gamma \rightarrow \pi^0$  amplitude, see Fig. 1, with the hadronic

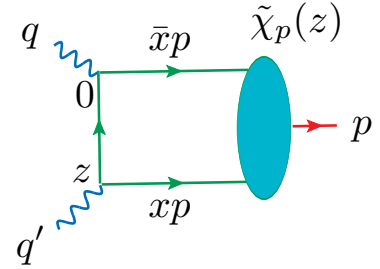


FIG. 1. Handbag diagram in the coordinate representation and parton momentum assignment.

blob being a matrix element connecting parton fields with the “pion”. In the coordinate representation we have

$$T(q, p) = \int d^4z e^{-i(q'z)} D^c(z) \langle p|\phi(0)\phi(z)|0\rangle, \quad (\text{II.1})$$

where  $D^c(z) = i/4\pi^2 z^2$  is the scalar massless propagator,  $q'$  is the momentum of the initial real “photon”,  $q'^2 = 0$  given by  $q' = p - q$ , with  $p$  being the momentum of the final “pion” and  $q$  is the momentum of the initial virtual “photon” ( $q^2 \equiv -Q^2$ ).

The pion structure is described by the matrix element  $\langle p|\phi(0)\phi(z)|0\rangle \equiv \tilde{\chi}_p(z)$  of the bilocal operator. To parametrize it, we start with a formal Taylor expansion

$$\phi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z_{\mu_1} \dots z_{\mu_n} \partial^{\mu_1} \dots \partial^{\mu_n} \phi(0). \quad (\text{II.2})$$

Then, information about the pion is contained in matrix elements  $\langle p|\phi(0)\partial^{\mu_1} \dots \partial^{\mu_n} \phi(0)|0\rangle$ . Due to Lorentz

invariance, they may be written as

$$\langle p|\phi(0)\partial^{\mu_1}\dots\partial^{\mu_n}\phi(0)|0\rangle = A_n^{(0)}i^n p^{\mu_1}\dots p^{\mu_n} + \text{terms containing } g^{\mu_i\mu_j}\text{'s} . \quad (\text{II.3})$$

Utilizing the fact that the  $\mu_k$  indices are symmetrized in the Taylor expansion by the  $z_{\mu_1}\dots z_{\mu_n}$  tensor, we may use a more organized expression

$$\langle p|\phi(0)(z\partial)^n\phi(0)|0\rangle = i^n \sum_{l=0}^{[n/2]} (z^2\Lambda^2)^l (pz)^{n-2l} A_n^{(l)} , \quad (\text{II.4})$$

with information about the pion structure accumulated now in constants  $A_n^{(l)}$ . The momentum scale  $\Lambda$  was introduced to secure that all  $A_n^{(l)}$ 's have the same dimension.

Take the lowest,  $l=0$  term. To perform summation over  $n$ , we may treat the coefficients  $A_n^{(0)}$  as moments of the *pion distribution amplitude* [5]  $\varphi(x)$

$$A_n^{(0)} = \int_0^1 \varphi(x) x^n dx \quad (\text{II.5})$$

(we want the quark at the  $z$  vertex to carry the momentum  $xp$ , see Fig.(1)). As a result,

$$\langle p|\phi(0)\phi(z)|0\rangle = \int_0^1 \varphi(x) e^{ix(pz)} dx + \mathcal{O}(z^2) . \quad (\text{II.6})$$

For the amplitude  $T(q,p)$  this gives

$$T(q,p) = \frac{i}{4\pi^2} \int_0^1 dx \varphi(x) \times \int \frac{d^4 z}{z^2} e^{-i(q'z)+ix(pz)} \left[ 1 + \mathcal{O}(z^2) \right] . \quad (\text{II.7})$$

The term given by  $l=0$  part in Eq. (II.4) produces

$$T^{(0)}(q,p) = - \int_0^1 \frac{\varphi(x)dx}{(q' - xp)^2 + i\epsilon} . \quad (\text{II.8})$$

An extra  $z^2$  factor in the  $l=1$  term of (II.4) cancels the  $1/z^2$  singularity of  $D^c(z)$ , and results in a contribution proportional to

$$\int d^4 z e^{-i(q'z)+ix(pz)} = (2\pi)^4 \delta^4(q' - xp) , \quad (\text{II.9})$$

which apparently should be treated as zero, because  $q'$  is not proportional to  $p$ . The same applies to terms with higher powers  $(z^2)^l$ , which produce integrals proportional to  $\square^{2l-2}\delta^4(q' - xp)$ . If one would use just a straightforward dimensional counting, one would expect that terms with higher powers of  $z^2$  result in contributions accompanied by powers of  $1/(q' - xp)^2$ , but as we see, they produce terms that are “invisible” in the  $1/(q' - xp)^2$  expansion. The actual power corrections appear when a  $\mathcal{O}[(z^2)^l]$  term in the matrix element  $\langle p|\phi(0)\phi(z)|0\rangle$  is accompanied by some nonzero power of  $\ln z^2$ .

Note also that since  $-(q' - xp)^2 = xQ^2 + x\bar{x}p^2$ , the  $1/(q' - xp)^2$  expansion does not coincide with the  $1/Q^2$  expansion when  $p^2 \neq 0$ . The things simplify when  $p^2 = 0$ . Then, for a massless propagator  $D^c(z)$ , a power correction of  $1/(Q^2)^{1+l}$  type can be obtained from  $(z^2)^l (\ln z^2)^{k \geq 1}$  terms only.

This simple analysis leads us to an important observation that, for a situation when  $p^2 = 0$  and the matrix element  $\langle p|\phi(0)(z\partial)^n\phi(0)|0\rangle$  is given by a regular power expansion in  $z^2$  (we shall say that one deals with a “soft” wave function in such a case), the whole amplitude  $T(q,p)$  with a massless hard propagator  $D^c(z)$  is *exactly* given by the first term  $T^{(0)}(q,p)$  only, namely, that

$$T^{\text{soft}}(q,p)|_{p^2=0} = \int_0^1 \frac{\varphi(x)dx}{xQ^2} \left[ 1 + \text{no powers of } 1/Q^2 \right] , \quad (\text{II.10})$$

with no higher  $1/Q^2$  corrections *under the  $x$ -integral*.

On the other hand, if the matrix element has a logarithmic singularity  $(z^2)^l \ln z^2$  with  $l \geq 1$ , the amplitude should have a power correction with  $(1/Q^2)^l$  behavior.

These observations may be used as a constraint (“OPE compatibility”) that should be required to hold in schemes that add transverse momentum dependence into the description of the pion structure. As we will see, some previously used approximate schemes that look natural otherwise, are not “OPE compatible”.

## B. Introducing virtuality distributions

As mentioned already, parametrizing matrix elements of local operators resulting from the Taylor expansion (II.2), one needs to deal with a set of parameters  $A_n^{(l)}$  for each  $n$ , with  $l$  being the number of metric tensors  $g^{\mu_i\mu_j}$  on the right hand side of Eq. (II.3), or power of  $z^2$  in Eq. (II.4). Collecting together terms with the same power of  $(pz)$ , we may write

$$\langle p|\phi(0)\phi(z)|0\rangle = \sum_{l=0}^{\infty} \frac{1}{l!} \left( \frac{z^2\Lambda^2}{4} \right)^l \sum_{N=0}^{\infty} i^N \frac{(pz)^N}{N!} B_N^{(l)} . \quad (\text{II.11})$$

By analogy with Eq. (II.5) which introduces the pion distribution amplitude  $\varphi(x)$  through the coefficients  $A_n^{(0)}$ , we define that the coefficients  $B_N^{(l)}$  are given by double moments of a function of two variables  $\Phi(x,\sigma)$ , which we call the *virtuality distribution amplitude* (VDA) :

$$B_N^{(l)} = (-i)^l \int_0^{\infty} d\sigma \sigma^l e^{-\epsilon\sigma/4} \int_0^1 dx x^N \Phi(x,\sigma) . \quad (\text{II.12})$$

Substituting this definition into Eq. (II.12) gives

$$\langle p|\phi(0)\phi(z)|0\rangle = \int_0^{\infty} d\sigma \int_0^1 dx \times \Phi(x,\sigma) e^{ix(pz) - i\sigma(z^2 - i\epsilon)/4} . \quad (\text{II.13})$$

We have derived this *VDA representation* from a formal Taylor expansion of the matrix element  $\langle p|\phi(0)\phi(z)|0\rangle$  of the bilocal operator. Such an expansion makes sense only if the matrix elements of local operators  $\langle p|\phi(0)(z\partial)^n\phi(0)|0\rangle$  are finite. In such a case we say that one deals with a *soft wave function*. However, Eq. (II.13) looks just like a double Fourier transform in two variables ( $zp$ ) and  $z^2$ . As such, it should hold for a very wide range of functions, including the functions that *are not* given by a convergent Taylor expansion in  $z^2$ .

This observation suggests that the VDA representation may be obtained under much weaker assumptions. One may also wonder why we have imposed particular limits of integration, namely,  $0 \leq x \leq 1$  and  $0 \leq \sigma < \infty$  on these Fourier integrals. Obviously, these limits do not appear automatically for any function of ( $pz$ ) and  $z^2$ . But it can be shown (see below) that any Feynman diagram contributing to  $\langle p|\phi(0)\phi(z)|0\rangle$  has the VDA representation with exactly these limits of integration. Also, it does not matter if the Feynman diagram has logarithmic singularities in  $z^2$ , the VDA representation (II.13) still holds, even though some of the moments (II.12) diverge in that case. It should be also noted that the logarithmic singularities in  $z^2$  come as  $\ln(z^2 - i\epsilon)$ , reflecting the causal structure of Feynman diagrams.

### C. VDA and $\alpha$ -representation

Using the  $\alpha$ -representation and techniques outlined in Refs. [14–16], it can be demonstrated that the VDA representation (II.13) holds for any Feynman diagram contributing to the relevant matrix element.

#### 1. Momentum space

Consider the momentum representation version of the matrix element

$$\int d^4 z e^{-i(kz)} \langle p|\phi(0)\phi(z)|0\rangle \equiv (4\pi i)^2 \chi_p(k), \quad (\text{II.14})$$

where  $k$  is the momentum of the quark going from the “ $z$ ” vertex, and  $\chi_p(k)$  is the Bethe-Salpeter wave function. Then, according to Refs. [14–16], the contribution of any Feynman diagram  $\mathcal{D}$  to  $\chi_p(k)$  can be represented as

$$\begin{aligned} \chi_p^{\mathcal{D}}(k) = & i^l \frac{P(\text{c.c.})}{(4\pi i)^{Ld/2}} \int_0^\infty \prod_{j=1}^l d\alpha_j [D(\alpha)]^{-d/2} \\ & \times \exp \left\{ ik_1^2 \frac{A(\alpha)}{D(\alpha)} + ik_2^2 \frac{B(\alpha)}{D(\alpha)} \right\} \\ & \times \exp \left\{ ip^2 \frac{C(\alpha)}{D(\alpha)} - i \sum_j \alpha_j (m_j^2 - i\epsilon) \right\}, \quad (\text{II.15}) \end{aligned}$$

where  $k_1 = k$ ,  $k_2 = p - k$ ,  $d$  is the space-time dimension,  $P(\text{c.c.})$  is the relevant product of coupling constants,  $L$  is the number of loops of the diagram, and  $l$

is the number of its internal lines. For our purposes, the most important property of this representation is that  $A(\alpha), B(\alpha), C(\alpha), D(\alpha)$  are positive functions (sums of products) of the  $\alpha_\sigma$ -parameters of a diagram. Thus, we have a general representation

$$\chi_p(k) = \int_0^1 dx \int_0^\infty d\lambda e^{i\lambda \bar{x}k^2 + i\lambda x(k-p)^2 - \epsilon\lambda} F(x, \lambda; p^2), \quad (\text{II.16})$$

where

$$\begin{aligned} F(x, \lambda; p^2) = & \sum_{\text{all diag}} i^l \frac{P(\text{c.c.})}{(4\pi i)^{Ld/2}} \int_0^\infty \prod_{j=1}^l d\alpha_j [D(\alpha)]^{-d/2} \\ & \times \delta \left( x - \frac{B(\alpha)}{A(\alpha) + B(\alpha)} \right) \delta \left( \lambda - \frac{A(\alpha) + B(\alpha)}{D(\alpha)} \right) \\ & \times \exp \left\{ ip^2 \frac{C(\alpha)}{D(\alpha)} - i \sum_j \alpha_j (m_j^2 - i\epsilon) \right\}. \quad (\text{II.17}) \end{aligned}$$

Eq. (II.16) may be also rewritten as

$$\chi_p(k) = \int_0^1 dx \int_0^\infty d\lambda e^{i\lambda(k-xp)^2 + i\lambda x \bar{p}^2 - \epsilon\lambda} F(x, \lambda; p^2). \quad (\text{II.18})$$

#### 2. Coordinate space

Making Fourier transform to the coordinate representation, we get

$$\langle p|\phi(0)\phi(z)|0\rangle = \int_0^\infty d\sigma \int_0^1 dx \Phi(x, \sigma) e^{ix(pz) - i\sigma(z^2 - i\epsilon)/4}. \quad (\text{II.19})$$

The functions  $F$  and  $\Phi$  are related by

$$e^{i\lambda x \bar{p}^2} F(x, \lambda; p^2) = \Phi(x, 1/\lambda) \quad (\text{II.20})$$

(the VDA  $\Phi$  also depends on  $p^2$ , i.e. in principle it should be written as  $\Phi(x, \sigma; p^2)$ , but we will not indicate this dependence explicitly, mainly because  $p^2$  is fixed for a given matrix element).

#### 3. Important observations

Note that the momentum  $p$  in (II.20) is the *actual* momentum that appears in the matrix element. In this sense, a parton in the VDA picture carries the fraction  $xp$  of the *total* hadron momentum  $p$ , not just the fraction  $xp^+$  of its “plus” component  $p^+$ . In fact, all our discussion so far was absolutely Lorentz covariant, and there was no need to decompose momenta into any components, to project  $p$  on its “plus” part, etc. We also emphasize that there was no need to assume that  $p^2 = 0$ .

Another point is that the representation (II.19) has been obtained without any assumptions about *regularity* of the  $z^2 \rightarrow 0$  limit. This means that one can use the VDA representation even in cases when a formal Taylor expansion in  $z^2$  does not exist because of singularities in the  $z^2 \rightarrow 0$  limit. In other words, the matrix element  $\langle p|\phi(0)\phi(z)|0\rangle$  may be non-analytic for  $z^2 = 0$ , and still be given by a VDA representation.

#### D. Scalar handbag diagram in VDA representation

Using the VDA parametrization (II.13) we can take the  $z$ -integral in Eq. (II.1) to obtain

$$T(q, p) = \int_0^1 \frac{dx}{(q' - xp)^2 + i\epsilon} \int_0^\infty d\sigma \Phi(x, \sigma) \times \left\{ 1 - e^{i[(q' - xp)^2 + i\epsilon]/\sigma} \right\}. \quad (\text{II.21})$$

The first term in the brackets does not depend on  $\sigma$  and produces the integral

$$\int_0^\infty \Phi(x, \sigma) d\sigma \equiv \varphi(x), \quad (\text{II.22})$$

where  $\varphi(x)$  is the distribution amplitude defined by Eq. (II.5). Indeed, taking  $z^2 = 0$  in the VDA representation (II.13) and comparing the result with Eq. (II.6), we see that this is formally the case.

Of course, this reasoning assumes that the integral over  $\sigma$  in (II.22) converges at the upper limit, which happens when  $\Phi(x, \sigma)$  decreases faster than  $1/\sigma^{1+\epsilon}$  for large  $\sigma$ . If  $\Phi(x, \sigma) \sim 1/\sigma$  for large  $\sigma$ , then the integral (II.22) logarithmically diverges, which corresponds to a  $\ln z^2$  singularity for the matrix element. However, the  $\sigma$ -integral in Eq. (II.21) converges even in that case, because the sum of terms in the brackets behaves like  $1/\sigma$  for large  $\sigma$ .

As noted earlier, if the matrix element  $\langle p|\phi(0)\phi(z)|0\rangle$  can be expanded in a Taylor series in  $z^2$ , with finite coefficients, the higher ( $l \geq 1$ ) terms  $(z^2)^l$  of such an expansion cancel the singularity  $1/z^2$  of the massless scalar propagator  $D(z)$ . As a result, these terms produce terms proportional to  $\square_{q'}^l$  derivatives of the  $\delta^4(q' - xp)$  function. Taken separately, each of these terms is invisible in the  $T(q, p)$  amplitude, simply because  $q'$  is not proportional to  $p$ . Still, an infinite sum of delta-function derivatives in our case produces a non-trivial function given by the second term in Eq. (II.21). In other words, the “invisible” contributions are combined in the second term which, after integration over  $\sigma$ , results in a nontrivial function of  $(q' - xp)^2$ .

The pion structure is now described by the VDA  $\Phi(x, \sigma)$ , and by just modeling its  $\sigma$ -shape one can study the impact of higher  $l$  terms. However, it is very instructive to give an interpretation of these terms using the concept of parton transverse momentum.

### III. TRANSVERSE MOMENTUM DISTRIBUTIONS

#### A. Introducing TMDA

To bring in the transverse momentum dependence, we should decide, first, which directions are “transverse”. It is natural to define that the pion momentum  $p$  has only longitudinal components, becoming a purely “plus” vector in the  $p^2 = 0$  limit. Projecting the matrix element  $\langle p|\phi(0)\phi(z)|0\rangle$  on the light front  $z_+ = 0$ ,

$$\langle p|\phi(0)\phi(z)|0\rangle|_{z_+=0} = \int_0^1 dx \varphi(x, z_\perp) e^{ix(pz_-)}, \quad (\text{III.1})$$

we introduce the *impact parameter distribution amplitude* (IDA)  $\varphi(x, z_\perp)$ . It is related to VDA by

$$\varphi(x, z_\perp) = \int_0^\infty d\sigma \Phi(x, \sigma) e^{i\sigma(z_\perp^2 - i\epsilon)/4}. \quad (\text{III.2})$$

The next step is to treat the IDA function as a Fourier transform

$$\varphi(x, z_\perp) = \int \Psi(x, k_\perp) e^{i(k_\perp z_\perp)} d^2 k_\perp \quad (\text{III.3})$$

of the *transverse momentum dependent distribution amplitude* (TMDA)  $\Psi(x, k_\perp)$ . One can write the TMDA in terms of VDA as

$$\Psi(x, k_\perp) = \frac{i}{\pi} \int_0^\infty \frac{d\sigma}{\sigma} \Phi(x, \sigma) e^{-i(k_\perp^2 - i\epsilon)/\sigma}. \quad (\text{III.4})$$

The moments of TMDA  $\Psi(x, k_\perp)$  are formally given by

$$\int \Psi(x, k_\perp) k_\perp^{2l} d^2 k_\perp = \frac{l!}{i^l} \int_0^\infty \sigma^l \Phi(x, \sigma) d\sigma. \quad (\text{III.5})$$

They are proportional to the  $\sigma^l$  moments of the VDA  $\Phi(x, \sigma)$  and, hence, finite for a soft VDA. This means that a “soft” TMDA  $\Psi(x, k_\perp)$  should decrease faster than any power of  $1/k_\perp^2$  for large  $k_\perp$ .

#### B. Scalar handbag diagram in TMDA representation

Using the TMDA/VDA relation (III.4), one can rewrite Eq. (II.21) in terms of TMDA as

$$T(q, p) = - \int_0^1 \frac{dx}{(q' - xp)^2} \int_{k_\perp^2 \leq -(q' - xp)^2} d^2 k_\perp \Psi(x, k_\perp), \quad (\text{III.6})$$

which converts into

$$T(Q^2) = \int_0^1 \frac{dx}{xQ^2} \int_{k_\perp^2 \leq xQ^2} \Psi(x, k_\perp) d^2 k_\perp \quad (\text{III.7})$$

in the  $p^2 = 0$  case.

It should be emphasized that no Taylor expansions in  $z^2$  have been used in deriving (III.7). Still, if one deals with a soft TMDA, one can use Taylor expansion and separate the  $l = 0$  term in Eq. (II.21). Then, incorporating the reduction relation

$$\int \Psi(x, k_\perp) d^2 k_\perp = \varphi(x) \quad (\text{III.8})$$

one can write

$$T(Q^2) = \int_0^1 \frac{dx}{xQ^2} \left[ \varphi(x) - \int_{k_\perp^2 \geq xQ^2} \Psi(x, k_\perp) d^2 k_\perp \right]. \quad (\text{III.9})$$

As we have noted, a soft TMDA decreases for large  $k_\perp^2$  faster than any inverse power of  $k_\perp^2$ . As a result, the second term in Eq. (III.7) decreases for large  $Q^2$  faster than any power of  $1/Q^2$ , i.e. there are no  $1/Q^2$  power corrections to the  $\varphi(x)$  term under the  $x$ -integral in Eq. (III.7). This means that the VDA-based expression (III.7) in case of a soft VDA has an OPE-compliant form of Eq. (II.10).

Alternatively, if the matrix element  $\langle p | \phi(0) \phi(z) | 0 \rangle$  has a logarithmic singularity  $\ln z^2$  starting with  $(z^2)^l$  power, the  $\sigma$  moments of  $\Phi(x, \sigma)$  should diverge starting with  $\sigma^l$ , and  $k_\perp^2$  moments of  $\Psi(x, k_\perp)$  should diverge starting with  $k_\perp^{2l}$ . This means that  $\Psi(x, k_\perp)$  decreases as  $(1/k_\perp^2)^{l+1}$  for large  $k_\perp$ , i.e.,  $\Psi(x, k_\perp)$  has a power-like “hard tail”. Then the second term in Eq. (III.9) produces a  $\mathcal{O}((1/Q^2)^{l+1})$  contribution to  $T(Q^2)$ .

Thus, we see that the VDA-based formula (III.7) is in full compliance with the OPE approach.

### C. Impact parameter representation

Substituting the expression of TMDA in terms of IDA

$$\Psi(x, k_\perp) = \int \frac{d^2 b_\perp}{(2\pi)^2} \varphi(x, b_\perp) e^{-i(k_\perp b_\perp)} \quad (\text{III.10})$$

into the VDA-based formula (III.7) for  $T(Q^2)$  we obtain

$$T(Q^2) = \int_0^1 \frac{dx}{\sqrt{x}Q} \int_0^\infty db J_1(\sqrt{x}Qb) \varphi(x, b). \quad (\text{III.11})$$

Note that we intentionally use here the notation  $b_\perp$  for the impact parameter variable, to emphasize that it cannot be identified with the transverse part  $z_\perp$  of the integration variable  $z$  in the original coordinate representation integral (II.1).

Indeed, recall that our procedure has started with taking integral over  $d^4 z$  to obtain the result expressed by Eq. (II.21) in terms of VDA  $\Phi(x, \sigma)$  which was transformed then into Eq. (III.7) written in terms of TMDA  $\Psi(x, k_\perp)$ . After this starting integration, a connection of the final result with the  $z_\perp$ -integration has been completely lost. Then we have converted the TMDA result (III.7) into the expression (III.11) in terms of IDA  $\varphi(x, b_\perp)$ , in which  $b_\perp$  is a new auxiliary variable.

### D. Twist decomposition

So far, we did not mention the concept of twist, since ordering contributions by  $(z^2)^l$  power in Eq. (II.4) was sufficient for our purposes. But let us discuss now the twist expansion of the basic matrix element  $\langle p | \phi(0) \phi(z) | 0 \rangle$ .

#### 1. Traceless combinations

The operators  $\phi(0) \partial^{\mu_1} \dots \partial^{\mu_n} \phi(0)$  do not correspond to an irreducible representation. They are not traceless, and that is why their parametrization requires a set of numbers  $A_n^{(l)}$  rather than just one number. To get matrix elements corresponding to an irreducible representation one has to write the tensor  $z_{\mu_1} \dots z_{\mu_n}$  as a sum of products of powers of  $z^2$  and symmetric-traceless combinations  $\{\dots z_{\mu_i} \dots z_{\mu_j} \dots\}$  satisfying the irreducibility condition  $g^{\mu_i \mu_j} \{\dots z_{\mu_i} \dots z_{\mu_j} \dots\} = 0$ . Using the notation  $\{z\partial\}^n \equiv \{z_{\mu_1} \dots z_{\mu_n}\} \partial^{\mu_1} \dots \partial^{\mu_n}$  for products of traceless tensors, it is possible to derive [17]

$$\phi(z) = \sum_{l=0}^{\infty} \left( \frac{z^2}{4} \right)^l \sum_{N=0}^{\infty} \frac{N+1}{l!(N+l+1)!} \{z\partial\}^N (\partial^2)^l \phi(0). \quad (\text{III.12})$$

Now, parametrizing matrix elements of traceless operators

$$\begin{aligned} & \langle p | \phi(0) \{ \partial^{\mu_1} \dots \partial^{\mu_N} \} (\partial^2)^l \phi(0) | 0 \rangle \\ &= i^n C_N^{(l)} \Lambda^{2l} \{ p^{\mu_1} \dots p^{\mu_N} \} \end{aligned} \quad (\text{III.13})$$

one needs just one number  $C_N^{(l)}$  for each operator. A usual way to make a projection on a traceless combination is to multiply Eq. (III.13) by a product  $n^{\mu_1} \dots n^{\mu_N}$  built from an auxiliary lightlike vector  $n$ . Since  $n^2 = 0$ , one has a relation

$$\langle p | \phi(0) (n\partial)^N (\partial^2)^l \phi(0) | 0 \rangle = i^N C_N^{(l)} \Lambda^{2l} (pn)^N \quad (\text{III.14})$$

involving ordinary scalar products  $(n\partial)$  and  $(np)$ . Choosing  $n$  to be in the “minus” direction, we may rewrite Eq. (III.14) as

$$\langle p | \phi(0) \partial_+^N (\partial^2)^l \phi(0) | 0 \rangle = i^N C_N^{(l)} \Lambda^{2l} p_+^N, \quad (\text{III.15})$$

with clear separation of derivatives  $\partial_+$  probing the longitudinal structure of the hadron, and contracted derivatives  $\partial^2$  sensitive to distribution of quarks in virtuality. The operators containing powers of  $\partial^2$  have higher twist, and their contribution to the light-cone expansion is accompanied by powers of  $z^2$ .

#### 2. Twist expansion and target mass effects

However, trying to use the twist decomposition (III.12) for getting a closed expression for  $\langle p | \phi(0) \phi(z) | 0 \rangle$  similar

to a VDA representation, one needs to perform a summation over  $N$

$$\begin{aligned} \langle p | \phi(0) \phi(z) | 0 \rangle &= \sum_{l=0}^{\infty} \left( \frac{z^2 \Lambda^2}{4} \right)^l \sum_{N=0}^{\infty} \frac{N+1}{l!(N+l+1)!} \\ &\times i^N \{zp\}^N C_N^{(l)} \end{aligned} \quad (\text{III.16})$$

that involves structures  $\{zp\}^N$  built from traceless combinations. It is possible to write them in simple powers,

$$\{zp\}^N = \frac{[(zp) + r]^{N+1} - [(zp) - r]^{N+1}}{2^{N+1}r}, \quad (\text{III.17})$$

where  $r = \sqrt{(zp)^2 - z^2 p^2}$  (see, e.g., Ref. [17]). Since Eq. (III.17) expresses  $\{zp\}^N$  in terms of powers of  $[(zp) \pm r]$ , treating  $C_N^{(l)}$  coefficients as appropriately normalized  $x^N \sigma^l$  moments of VDA  $\Phi(x, \sigma)$ , one can explicitly perform summation over  $N$  and obtain formulas involving exponentials  $e^{ix[(zp) \pm r]/2}$  instead of  $e^{ix(zp)}$  (see Refs. [18, 19] for formulas including also the spin-1/2 cases). However, further integration over  $z$  is rather complicated because of the square root involved in  $r$ .

Another way is to use the inverse expansion

$$\{zp\}^N = (zp)^N - \frac{1}{4}(N-1)z^2 p^2 (zp)^{N-2} + \dots \quad (\text{III.18})$$

After the re-expansion of  $\{zp\}^N$ , one would get series in powers of  $(pz)$  and  $z^2$ , in which some of  $(z^2)^l$  terms are accompanied by  $\Lambda^{2l}$  factors having a dynamical origin (virtuality of quarks) and some  $(z^2)^k$  terms that are accompanied by  $(p^2)^k$  factors, which are purely kinematical (they come from the re-expansion of  $\{zp\}^N$ ) and reflect nonzero mass of the hadron.

### 3. VDA representation and target mass effects

Thus it looks simpler to use Eq. (III.18), which would give target mass corrections as a series in  $p^2/Q^2$ . In fact, the most simple way is to avoid the twist decomposition altogether. Note that the VDA representation, first, is valid without approximations and, second, involves the actual hadron momentum  $p$ . This means that it is sufficient to merely treat  $p$  “as is”, e.g. to use  $(q' - xp)^2 = -(xQ^2 + x\bar{x}p^2)$  for the combination present in our result (III.6) for the Compton amplitude.

Proceeding in this way, one can include, if necessary, the kinematical hadron mass effects that are analogous to Nachtmann [20, 21] corrections in deep inelastic scattering. However, since our primary goal is to concentrate on dynamical effects (and also given the smallness of the pion mass) we will simplify the things by just taking  $p^2 = 0$ , in which case  $\{zp\}^N = (zp)^N$ .

Still, the discussion of the twist decomposition has an important outcome, namely, the understanding that higher  $l$  terms in Eq. (II.4) correspond to local operators

with increasing powers of contracted derivatives  $\partial^2$  that probe the parton’s virtuality. It is for this reason why  $\Phi(x, \sigma)$  is referred to as a virtuality distribution.

### 4. Equal virtualities

There is a kinematics in which the summation over spin  $N$  is not necessary and only the  $l$ -sum remains. Consider a situation when both photons are virtual, and moreover, have equal virtualities,  $q_1^2 = q_2^2 \equiv -Q^2$ . For a lightlike  $p$ , we have in this case  $(pq_1) = (pq_2) = 0$ . Choosing  $p$  in the “plus” direction, we conclude that both  $q_1$  and  $q_2$  do not have “minus” components, and their virtualities  $q_i^2$  are given by the transverse component  $q_{\perp}$  only,  $q_i^2 = -q_{\perp}^2 = -Q^2$ . Similarly,  $(q_1 - xp)$  and  $(q_2 - \bar{x}p)$  do not have the minus component, and  $(q_1 - xp)^2 = (q_2 - \bar{x}p)^2 = -q_{\perp}^2 = -Q^2$ . Parametrizing the bilocal matrix element, as usual, by (III.31), we obtain

$$\begin{aligned} T(Q^2, Q^2) &= \frac{1}{Q^2} \int_0^{\infty} d\sigma \left\{ 1 - e^{-[iQ^2 + \epsilon]/\sigma} \right\} \\ &\times \int_0^1 dx \Phi(x, \sigma). \end{aligned} \quad (\text{III.19})$$

In this result, VDA  $\Phi(x, \sigma)$  enters only through the integrated distribution

$$\Sigma(\sigma) = \int_0^1 dx \Phi(x, \sigma), \quad (\text{III.20})$$

in terms of which we have

$$T(Q^2, Q^2) = \frac{1}{Q^2} \int_0^{\infty} d\sigma \left\{ 1 - e^{-[iQ^2 + \epsilon]/\sigma} \right\} \Sigma(\sigma). \quad (\text{III.21})$$

In the OPE language, this means that operators with nontrivial,  $N \geq 1$  traceless combinations  $\{z\partial\}^N$  do not contribute, simply because their matrix elements result in  $(pq)^N$  factors that vanish. As a result, only the expansion in  $\phi(\partial^2)^l \phi$  operators is left.

Switching to TMDA, we get

$$T(Q^2, Q^2) = \frac{1}{Q^2} \int_{k_{\perp}^2 \leq Q^2} d^2 k_{\perp} \int_0^1 dx \Psi(x, k_{\perp}). \quad (\text{III.22})$$

Again, TMDA enters integrated over  $x$ .

## E. Basic relations for VDAs and TMDAs

### 1. Analytic continuation of TMDA

The TMDA/VDA relation (III.4) tells us that  $\Psi(x, k_{\perp})$  is a function of  $k_{\perp}^2$ . In what follows, we will also use the



notation  $\psi(x, k_\perp^2) \equiv \pi \Psi(x, k_\perp)$  emphasizing that  $\psi$  is explicitly a function of  $k_\perp^2$ . In fact, the relation

$$\psi(x, \kappa^2) = i \int_0^\infty \frac{d\sigma}{\sigma} \Phi(x, \sigma) e^{-i(\kappa^2 - i\epsilon)/\sigma} \quad (\text{III.23})$$

defines  $\psi(x, \kappa^2)$  not only for positive  $\kappa^2$ , when it may be interpreted in terms of the transverse momentum squared  $k_\perp^2$ , but also for negative values of  $\kappa^2$ , in which case it is understood as a formal parameter. In other words, Eq. (III.23) provides an analytic continuation of  $\Psi(x, k_\perp)$  into the region of negative and complex  $k_\perp^2$ . Formally, the relation between  $\psi(x, \kappa^2)$  and  $\Phi(x, \sigma)$  may be inverted:

$$\Phi(x, \sigma) = \frac{1}{2\pi i \sigma} \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\kappa^2 e^{i\kappa^2/\sigma} \psi(x, \kappa^2). \quad (\text{III.24})$$

In practice, it is  $\Phi(x, \sigma)$  that is a primary function: it is extracted from explicit expressions for matrix elements (or their models), and then one obtains  $\psi(x, \kappa^2)$  using Eq. (III.23).

## 2. VDA representation for the Bethe-Salpeter function

Sometimes it is convenient to use the momentum representation version of the matrix element

$$\int d^4 z e^{-i(kz)} \langle p | \phi(0) \phi(z) | 0 \rangle \equiv (4\pi i)^2 \chi_p(k), \quad (\text{III.25})$$

where  $k$  is the momentum of the quark going from the “ $z$ ” vertex, and  $\chi_p(k)$  is the Bethe-Salpeter wave function. In the VDA representation,

$$\chi_p(k) = \int_0^\infty \frac{d\sigma}{\sigma^2} \int_0^1 dx \Phi(x, \sigma) e^{i(k-xp)^2/\sigma - \epsilon/\sigma}. \quad (\text{III.26})$$

Comparing (III.23) and (III.26) we may formally write

$$\chi_p(k) = \int_0^1 dx \left[ \frac{\partial}{\partial k_\perp^2} \psi(x, k_\perp^2) \right]_{k_\perp^2 = -(k-xp)^2}. \quad (\text{III.27})$$

In the regions, where  $(k-xp)^2$  is positive, one should understand  $\psi(x, k_\perp^2)$  through the analytic continuation specified by Eq. (III.23).

Thus, the function  $\chi_p(k)$  for all  $k$  may be obtained from the TMDA  $\psi(x, k_\perp^2)$  and its analytic continuation into the region of negative  $k_\perp^2$ . We can also say that the Bethe-Salpeter wave function in the coordinate representation

$$\tilde{\chi}_p(z) \equiv \langle p | \phi(0) \phi(z) | 0 \rangle \quad (\text{III.28})$$

is determined for all  $z$  by an analytic continuation from its values on the light-front  $z_+ = 0$ .

There is also a reduction relation from  $\chi_p(k)$  to  $\psi(x, k_\perp^2)$ . Taking for simplicity  $p^2 = 0$  (and normalization  $k^2 = k^+ k^- - k_\perp^2$ ), we have

$$\int_{-\infty}^\infty dk_- \chi_p(k) = -2\pi i \int_0^1 dx \delta(k^+ - xp^+) \psi(x, k_\perp^2). \quad (\text{III.29})$$

## 3. Bilocal function

In some cases, it is convenient to use an intermediate distribution  $B(x, z^2/4)$ , the *bilocal function* defined through

$$\langle p | \phi(0) \phi(z) | 0 \rangle \equiv \int_0^1 dx B(x, z^2/4) e^{ix(pz)}. \quad (\text{III.30})$$

Note that  $B(x, z^2/4)$  describes both positive and negative  $z^2$ , while IDA  $\varphi(x, z_\perp) = B(x, -z_\perp^2/4)$  corresponds to negative  $z^2$  only.

As we have seen, for any Feynman diagram of perturbation theory  $B(x, z^2/4)$  is a function of  $z^2 - i\epsilon$ . If the pion is in the initial state, the matrix element

$$\begin{aligned} \langle 0 | \phi(0) \phi(z) | p \rangle &= \int_0^\infty d\sigma \int_0^1 dx \\ &\times \Phi(x, \sigma) e^{-ix(pz) - i\sigma(z^2 - i\epsilon)/4}. \end{aligned} \quad (\text{III.31})$$

is still a function of  $z^2 - i\epsilon$ . This property is essential in the definition of VDA  $\Phi(x, \sigma)$  through

$$B(x, \beta) = \int_0^\infty d\sigma e^{-i\beta\sigma} \Phi(x, \sigma) d\sigma \quad (\text{III.32})$$

which has a form of a Laplace-type representation. Its formal inversion gives

$$\Phi(x, \sigma) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\beta e^{-i\beta\sigma} B(x, \beta - i\epsilon). \quad (\text{III.33})$$

In particular, combining (III.32) and (III.23) gives

$$B(x, z^2/4) = \frac{1}{\pi i} \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\kappa^2 K_0(\sqrt{\kappa^2 z^2}) \psi(x, \kappa^2) \quad (\text{III.34})$$

(for imaginary arguments,  $K_0$  should be understood as the Hankel function  $H_0^{(2)}$ ).

## 4. Moments of TMDA

The VDA/TMDA relation (III.4) is quite general in the sense that it holds even if the the matrix element  $\langle p | \phi(0) \phi(z) | 0 \rangle$  of the bilocal operator is non-analytic in the  $z^2 \rightarrow$  limit. However, if this limit is regular (which happens for a soft VDA  $\Phi(x, \sigma)$  that vanishes for large  $\sigma$  faster than any power of  $1/\sigma$ ), one can connect the  $\sigma$  moments of VDA  $\Phi(x, \sigma)$  and  $k_\perp^2$  moments of TMDA  $\Psi(x, k_\perp)$ ,

$$\int \Psi(x, k_\perp) k_\perp^{2n} d^2 k_\perp = \frac{n!}{i^n} \int_0^\infty \sigma^n \Phi(x, \sigma) d\sigma. \quad (\text{III.35})$$

This connection allows one to get the relation

$$B(x, \beta) = \int \Psi(x, k_\perp) J_0(2k_\perp \sqrt{-\beta}) d^2 k_\perp. \quad (\text{III.36})$$

For negative  $\beta = -z_\perp^2/4$ , this formula may be also obtained by performing the angular integration in Eq. (III.3), which means that Eq. (III.36) is valid even if the TMDA  $\Psi(x, k_\perp)$  is not soft. When  $\beta$  is positive (then we can write  $\beta = |z|^2/4$ ), one may understand Eq.(III.36) as

$$B(x, |z|^2/4) = \int \Psi(x, k_\perp) I_0(k_\perp |z|) d^2 k_\perp, \quad (\text{III.37})$$

where  $I_0$  is the modified Bessel function. This integral converges, e.g., for a Gaussian TMDA  $\Psi(x, k_\perp) \sim e^{-k_\perp^2/\Lambda^2}$ , and the basic bilocal function  $B(x, \beta)$  may be then expressed in terms of TMDA  $\Psi(x, k_\perp)$  both for spacelike and timelike values of  $\beta$ . For a TMDA with an exponential  $\sim e^{-\Lambda|k_\perp|}$  fall-off, the integral diverges for  $z^2 \geq \Lambda^2$ , which reflects a singularity of  $B(x, z^2/4)$  for time-like intervals with  $z^2 = \Lambda^2$ .

#### IV. SCALAR HANDBAG DIAGRAM IN TERMS OF VDA AND TMDA

##### A. Reducing handbag to a 3-dimensional integral

Our approach to get the TMDA expression for the scalar handbag diagram

$$T(q, p) = - \int_0^1 \frac{dx}{(q' - xp)^2} \int_{k_\perp^2 \leq -(q' - xp)^2} d^2 k_\perp \Psi(x, k_\perp), \quad (\text{IV.1})$$

by-passes the standard idea of starting with a 4-dimensional integral (see Fig. 2)

$$T(q, p) = \int \frac{\chi_p(k)}{(q' - k)^2} d^4 k, \quad (\text{IV.2})$$

decomposing the integration momentum  $k$  in the light-front components  $k = \{k_+, k_-, k_\perp\}$ , with the “plus” direction given by  $p$ , and then trying to integrate over  $k_-$ . An obvious difficulty of such an approach is that the  $k$ -dependence of  $\chi_p(k)$  is not explicit. The usual

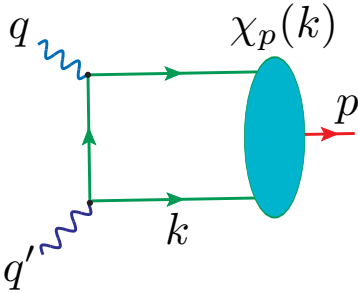


FIG. 2. Handbag diagram in momentum representation.

way out of this situation is to use some approximation

that eliminates the  $k_-$ -dependence of the hard propagator  $1/(q' - k)^2$ . After that, one deals with the function  $\chi_p(k)$  integrated over  $k_-$ , that depends on  $k_+$  and  $k_\perp$ . Below we consider two approximations of this kind.

##### 1. Neglecting $k^2$ in hard propagator

Since the  $q'$  photon is real,  $q'^2 = 0$ , we deal with

$$T(q, p) = \int \frac{\chi_p(k)}{2(q'k) - k^2} d^4 k. \quad (\text{IV.3})$$

As usual, for  $p^2 = 0$  one may choose  $p$  to define the plus direction and introduce  $x$  through  $k_+ = xp$ . Another light-like vector  $q'$  may be chosen to define the minus direction. Then  $2(q'k) = 2x(q'p) = xQ^2$ , and if we neglect  $k^2$  in the denominator we obtain

$$T^{(k^2 \rightarrow 0)}(q, p)|_{p^2=0} = \int_0^1 dx \frac{\varphi(x)}{xQ^2}, \quad (\text{IV.4})$$

where  $\varphi(x)$  is the distribution amplitude

$$\varphi(x) = \int d^4 k \delta(x - k_+/p_+) \chi_p(k). \quad (\text{IV.5})$$

Apparently, this formula gives the desired result (II.10) for soft wave functions. However, since it cannot produce any power correction in principle, it cannot be correct for wave functions corresponding to matrix elements that have logarithmic singularities for  $z^2 = 0$  in  $l \geq 1$  terms: in such cases we *should have*  $1/Q^2$  corrections.

Most importantly, neglecting virtuality  $k^2$  one also neglects transverse momentum effects altogether, while we want to keep track of them.

##### 2. Neglecting $k_-$ in hard propagator

To this end, we write a more detailed decomposition

$$k = xp + k_- + k_\perp \quad (\text{IV.6})$$

which gives  $k^2 = 2x(pk_-) - k_\perp^2$ , and the approximation is to neglect  $2x(pk_-)$ , while keeping the  $k_\perp^2$  part of  $k^2$  in the propagator. This gives

$$T^{(k_- \rightarrow 0)}(q, p)|_{p^2=0} = \int dx \int d^2 k_\perp \frac{\Psi(x, k_\perp)}{xQ^2 + k_\perp^2}, \quad (\text{IV.7})$$

where  $\Psi(x, k_\perp)$  is the transverse momentum dependent distribution amplitude,

$$\Psi(x, k_\perp) = \int dk_+ dk_- \delta(x - k_+/p_+) \chi_p(k). \quad (\text{IV.8})$$

Now, the formula (IV.7) always generates a tower of  $(k_\perp^2/Q^2)^n$  corrections, so it cannot be correct for soft wave functions. Furthermore, since taking  $k_- = 0$  in

the hard propagator is an approximation, it cannot be absolutely correct for “hard” wave functions as well.

To estimate the quality of this approximation, note that if  $k^2 = 0$ , we have  $2x(pk_-) = k_\perp^2$ , i.e. the term that is neglected has the same magnitude as the one that is kept. Assuming that  $k^2$  has some average value of  $\Lambda^2$  while  $k_\perp^2$  averages to  $\langle k_\perp^2 \rangle$ , we conclude that  $2x(pk_-)$  averages to  $\Lambda^2 + \langle k_\perp^2 \rangle$  which is not necessarily zero. In other words, neglecting  $2x(pk_-)$  is equivalent to assuming that a nonzero virtuality comes entirely from parton’s transverse momentum, which is a dynamical question, the answer to which is not clear *a priori*.

In general, the main idea of this procedure is based on neglecting  $k_-$  in the hard subprocess amplitude, and thus it creates an impression that the description of the hadron structure in terms of two variables  $x$  and  $k_\perp^2$  may be only obtained as a result of some approximation.

However, deriving our result (IV.1) we did not make any approximations. Such an outcome became possible because we were able to perform the 4-dimensional integration over  $k$  using the VDA representation which explicitly specified the dependence of  $\chi_p(k)$  on  $k$ . So, let us study what happens if we use a framework that involves a usual explicit decomposition of the integration variable (4-momentum  $k$  or 4-dimensional coordinate  $z$ ) into light-front components. When necessary, we will also incorporate the VDA representation adjusted to such a decomposition.

## B. Handbag diagram in coordinate light-front variables

To begin with, we try a Sudakov-type decomposition of the original coordinate space integral (II.1). Taking  $p$  in “+” direction, and  $q'$  in “-” direction, and using  $z = \{z_+, z_-, z_\perp\}$ , we have

$$T(q, p) = -\frac{i}{2(2\pi)^2} \int_0^1 dx \int d^2 z_\perp \int_{-\infty}^\infty dz_+ \int_{-\infty}^\infty dz_- \times e^{-iq'_- z_+ + ixp_+ z_-} \frac{B(x, z^2/4)}{z_+ z_- - z_\perp^2 - i\epsilon}. \quad (\text{IV.9})$$

The integrand has an explicit pole at  $z_+ = (z_\perp^2 + i\epsilon)/z_-$ , which corresponds to  $z^2 = 0$ . One may wish to calculate the integral over  $z_+$  by taking residue at this location. Whether this is possible, depends on the analyticity properties of  $B(x, z^2/4)$ .

### 1. Soft wave function

Take first a soft wave function case when  $B(x, z^2/4)$  is given by a  $(z^2)^l$  Taylor expansion with finite coefficients. Now, if one treats this expansion term by term, then one should take  $z^2 = 0$  in  $B(x, z^2/4)$ , which amounts to keeping just the lowest  $l = 0$  contribution. Since  $q'_- > 0$ , the integral is nonzero for  $z_- < 0$  only, i.e., the “0” vertex

corresponding to the virtual photon is *later* in the light-cone “time”  $z_-$  than the real photon vertex located at  $z_-$ . The result is

$$T^{\{\text{soft}\}}(q, p) = \frac{1}{2\pi} \int_0^1 dx \int d^2 z_\perp \int_0^\infty \frac{dz_-}{z_-} \times e^{-iz_\perp^2 q'_- / z_- + ixp_+ z_-} B^{\{\text{soft}\}}(x, 0) = T^{\{\text{soft}, l=0\}}(q, p). \quad (\text{IV.10})$$

Taking  $z_-$  integral and using that  $B^{\{\text{soft}\}}(x, 0) = \varphi(x)$ , we obtain the representation

$$T^{\{\text{soft}\}}(q, p) = \int_0^1 dx \int z_\perp dz_\perp K_0(z_\perp \sqrt{xQ^2}) \varphi(x), \quad (\text{IV.11})$$

in which the distribution amplitude  $\varphi(x)$  has no  $z_\perp$  dependence. Then the integral over  $z_\perp$  is trivial, with the result

$$T^{\{\text{soft}\}}(q, p) = \int_0^1 \frac{dx}{xQ^2} \varphi(x) \quad (\text{IV.12})$$

that agrees with Eq. (II.10), as expected.

Note that the only thing that is needed from the  $z_\perp$ -dependent factor  $K_0(\sqrt{xz_\perp^2 Q^2})$  for this agreement is that its integral over  $z_\perp^2$  gives  $1/xQ^2$ . In other words, any function  $Z(a)$  of  $a = xz_\perp^2 Q^2$  producing 1 after the  $0 \leq a \leq \infty$  integration, would produce Eq. (IV.12). Thus one has little grounds to argue that the specific  $z_\perp$ -dependence of  $K_0(z_\perp \sqrt{xQ^2})$  should be present in general case when the  $z_\perp$  dependence is added to the IDA.

Note also that to get the  $z_\perp$ -dependent IDA  $\varphi(x, z_\perp^2)$  one should project the bilocal function  $B(x, z^2)$ , onto the light-front  $z_+ = 0$ . However, for a residue taken at  $z_+ = (z_\perp^2 + i\epsilon)/z_-$  this is not the case when  $z_\perp^2 \neq 0$ .

### 2. General case, preliminary steps and reproduction of VDA result

Furthermore, one cannot take the integral (IV.9) by a simple residue if  $B(x, z^2/4)$  has singularities, like  $\ln z^2$ , in the complex  $z^2$  plane. To analyze a general case, we write  $B(x, z^2/4)/z^2$  in terms of VDA,

$$T(q, p) = \frac{1}{2(2\pi)^2} \int_0^1 dx \int d^2 z_\perp \int_{-\infty}^\infty dz_+ \times \int_{-\infty}^\infty dz_- e^{-iq'_- z_+ + ixp_+ z_-} \int_0^\infty \sigma d\sigma \int_0^1 d\beta \times \Phi(x, \beta\sigma) e^{-i\sigma(z_+ z_- - z_\perp^2 - i\epsilon)/4}. \quad (\text{IV.13})$$

Integrating over  $z_+$  produces

$$T(q, p) = \frac{1}{4\pi} \int_0^1 dx \int d^2 z_\perp \int_0^\infty \sigma d\sigma \int_0^1 d\beta \Phi(x, \beta\sigma) \times e^{i\sigma(z_\perp^2 + i\epsilon)/4} \int_0^\infty dz_- e^{ixp_+ z_-} \delta(q'_- + \sigma z_-). \quad (\text{IV.14})$$

Since  $q'_- > 0$ , we have  $z_- < 0$ , just like when we took a residue. Integrating over  $z_-$  and changing  $\beta\sigma \rightarrow \sigma$  gives

$$T(q, p) = \frac{1}{4\pi} \int_0^1 dx \int d^2 z_\perp \int_0^\infty d\sigma \Phi(x, \sigma) \times \int_0^1 \frac{d\beta}{\beta} e^{i\sigma(z_\perp^2 + i\epsilon)/4\beta} e^{-i\beta x Q^2/\sigma} . \quad (\text{IV.15})$$

It is rather easy to convert this expression into the VDA formula for  $T(q, p)$ . Indeed, integrating over  $z_\perp$  results in

$$T(q, p) = i \int_0^1 dx \int_0^\infty \frac{d\sigma}{\sigma} \Phi(x, \sigma) \int_0^1 d\beta e^{-i\beta x Q^2/\sigma} , \quad (\text{IV.16})$$

which, after the  $\beta$  integration, gives

$$T(q, p) = \int_0^1 \frac{dx}{x Q^2} \int_0^\infty d\sigma \Phi(x, \sigma) \left\{ 1 - e^{-ix Q^2/\sigma} \right\} \quad (\text{IV.17})$$

that coincides with Eq. (II.21) in our case of  $p^2 = 0$ .

### 3. General case, keeping $z_\perp$ dependence

However, if we want to keep  $z_\perp$  variable, we need to take the integrals over  $\sigma$  or  $\beta$  instead. Changing  $\beta = \sigma\rho$  in (IV.15) gives

$$T(q, p) = \frac{1}{4\pi} \int_0^1 dx \int_0^\infty d\sigma \Phi(x, \sigma) \int d^2 z_\perp \times \int_0^{1/\sigma} \frac{d\rho}{\rho} e^{i(z_\perp^2 + i\epsilon)/4\rho} e^{-i\rho x Q^2} . \quad (\text{IV.18})$$

At this stage, it is instructive to return to Eq. (IV.13) and combine there the integrations over  $z_+$  and  $z_-$  into one 2-dimensional integration over  $z_\parallel$

$$T(q, p) = \frac{1}{2(2\pi)^2} \int_0^1 dx \int_0^\infty \sigma d\sigma \int_0^1 d\beta \Phi(x, \beta\sigma) \times \int d^2 z_\perp d^2 z_\parallel e^{i\tilde{q} z_\parallel} e^{-i\sigma(z_\parallel^2 - z_\perp^2 - i\epsilon)/4} , \quad (\text{IV.19})$$

where  $\tilde{q} \equiv q' - xp$  has longitudinal components only. Now it is clear that the factor  $e^{-i\rho x Q^2}$  in Eq.(IV.18) comes from the  $d^2 z_\parallel$  integration. Using the fact that  $\tilde{q}$  is space-like,  $\tilde{q}^2 = -x Q^2$ , we can represent

$$e^{-i\rho x Q^2} = \int \frac{d^2 \zeta_\perp}{4\pi i \rho} e^{i\zeta_\perp^2/4\rho - i(\kappa_\perp \zeta_\perp)} , \quad (\text{IV.20})$$

where  $\kappa_\perp$  is a two-dimensional vector satisfying  $\kappa_\perp^2 = -\tilde{q}^2 = x Q^2$ . This gives

$$T(q, p) = \frac{1}{(4\pi)^2 i} \int_0^1 dx \int_0^\infty d\sigma \Phi(x, \sigma) \int d^2 z_\perp \int d^2 \zeta_\perp \times \int_0^{1/\sigma} \frac{d\rho}{\rho^2} e^{i(z_\perp^2 + \zeta_\perp^2 + i\epsilon)/4\rho} e^{-i(\kappa_\perp \zeta_\perp)} . \quad (\text{IV.21})$$

Integrating over  $\rho$  and then over  $\sigma$  to switch to IDA gives

$$T(q, p) = \frac{1}{(2\pi)^2} \int_0^1 dx \int d^2 z_\perp \int d^2 \zeta_\perp \times \frac{\varphi(x, z_\perp^2 + \zeta_\perp^2)}{z_\perp^2 + \zeta_\perp^2} e^{-i(\kappa_\perp \zeta_\perp)} . \quad (\text{IV.22})$$

Integrating over the angle between  $\kappa_\perp$  and  $\zeta_\perp$  we have

$$T(q, p) = \frac{1}{2} \int_0^1 dx \int_0^\infty dz_\perp^2 \int_0^\infty d\zeta_\perp^2 \times \frac{\varphi(x, z_\perp^2 + \zeta_\perp^2)}{z_\perp^2 + \zeta_\perp^2} J_0\left(\sqrt{x Q^2 \zeta_\perp^2}\right) . \quad (\text{IV.23})$$

Thus, the total impact parameter variable  $b^2 \equiv z_\perp^2 + \zeta_\perp^2$  of the IDA  $\varphi(x, b^2)$  comes from the transverse  $z_\perp$  part of the original  $d^4 z$  integration, and from an additional term  $\zeta_\perp^2$  reflecting the result of the integration over the longitudinal  $z_\parallel$  part of  $z$ . In other words, the additional term  $\zeta_\perp^2$  is associated with the virtuality  $x Q^2$  of the hard quark propagator.

### 4. Approximate expression

Due to weighting of  $\zeta_\perp^2$  by the Bessel function  $J_0\left(\sqrt{x Q^2 \zeta_\perp^2}\right)$  that rapidly decreases with  $\zeta_\perp^2$  for large  $x Q^2$ , one may estimate  $\zeta_\perp^2 \sim 1/x Q^2$  in this formula. Neglecting  $\zeta_\perp^2$  in the argument of IDA (but keeping it in the  $1/(z_\perp^2 + \zeta_\perp^2)$  factor) and using

$$\int_0^\infty \frac{\zeta_\perp d\zeta_\perp}{z_\perp^2 + \zeta_\perp^2} J_0\left(\sqrt{x Q^2 \zeta_\perp^2}\right) = K_0\left(|z_\perp| \sqrt{x Q^2}\right) , \quad (\text{IV.24})$$

we obtain the expression

$$T(q, p) = \int_0^1 dx \int z_\perp dz_\perp K_0(z_\perp \sqrt{x Q^2}) \varphi(x, z_\perp^2) + \dots . \quad (\text{IV.25})$$

Its explicit part coincides with a conjecture

$$T(q, p) \stackrel{?}{=} \int_0^1 dx \int z_\perp dz_\perp K_0(z_\perp \sqrt{x Q^2}) \varphi(x, z_\perp^2) , \quad (\text{IV.26})$$

that is used as an impact parameter representation for the pion transition form factor in many papers (see, e.g., Ref. [22]). However, Eq. (IV.26) is not OPE compliant for a soft wave function, since a  $(z_\perp^2)^l$  term from the expansion of  $\varphi(x, z_\perp^2)$  would produce an unwanted tower of  $(1/x Q^2)^{l+1}$  power corrections under the  $x$ -integral. As we see now, Eq. (IV.26) may be treated as an approximation only, based on the assumption that  $\zeta_\perp^2 \ll z_\perp^2$  in the argument of the IDA  $\varphi(x, z_\perp^2 + \zeta_\perp^2)$ .

In fact, due to a rapid decrease of the modified Bessel function  $K_0(y)$  with increasing  $y$ , the essential values of

$z_\perp^2$  in Eq. (IV.25) are restricted to  $z_\perp^2 \sim 1/xQ^2$ , which is of the same size as those for  $\zeta_\perp^2$  that were neglected compared to  $z_\perp^2$  in the argument of IDA. Thus, the correctness of the approximation leading to Eq. (IV.25) is very questionable.

The momentum space equivalent of Eq. (IV.25) is the formula (IV.7) obtained by neglecting the minus component  $k_-$  in the hard part, but keeping  $k_\perp^2$ . Thus, our coordinate space considerations suggests that the neglected  $k_-$  effects may be comparable in magnitude to those caused by  $k_\perp^2$ .

### 5. Comparison with exact result

To proceed without approximations, we change  $z_\perp^2 + \zeta_\perp^2 = b^2$ ,  $\zeta_\perp^2 = \gamma b^2$  in Eq. (IV.23) to get

$$T(q, p) = \frac{1}{2} \int_0^1 dx \int_0^\infty db^2 \int_0^1 d\gamma \times \varphi(x, b^2) J_0 \left( 2\sqrt{\gamma x Q^2 b^2} \right). \quad (\text{IV.27})$$

Integrating over  $\gamma$  gives the expression

$$T(q, p) = \int_0^1 \frac{dx}{\sqrt{x}Q} \int_0^\infty db \varphi(x, b^2) J_1(2\sqrt{x}Qb) \quad (\text{IV.28})$$

that coincides with Eq. (III.11), the VDA formula written in the impact parameter representation.

This discussion shows that the impact parameter  $b$  in the VDA/IDA formula (IV.28) differs from the transverse distance  $z_\perp$  in the original coordinate space integral (IV.9) for  $T$ , namely,  $b^2 = z_\perp^2 + \zeta_\perp^2$  with  $\zeta_\perp^2 = \mathcal{O}(1/xQ^2)$ . However, the essential  $z_\perp^2$  are also of the order of  $1/xQ^2$ .

### C. Calculation in momentum representation

The VDA result (III.7) for the handbag diagram can also be obtained using the momentum representation for the VDA. The integral now reads

$$T(q, p) = \int_0^1 dx \int \frac{d^4 k}{(q' - k)^2} \times \int_0^\infty d\alpha e^{i\alpha(k-xp)^2} \Phi(x, 1/\alpha). \quad (\text{IV.29})$$

Integrating over  $k$  gives

$$T(q, p) = \int_0^1 dx \int_0^\infty d\alpha \int \frac{d\alpha_1}{(\alpha + \alpha_1)^2} \times e^{i\alpha_1 \alpha \bar{q}^2 / (\alpha + \alpha_1)} \Phi(x, 1/\alpha). \quad (\text{IV.30})$$

Switching to  $\sigma_1 = 1/\alpha_1$ ,  $\sigma = 1/\alpha$  and integrating over  $\sigma_1$  gives the same VDA result as in Eq. (IV.17), which may be converted into

$$T(Q^2, p^2) = - \int_0^1 \frac{dx}{\bar{q}^2} \int_{\kappa_\perp^2 \leq -\bar{q}^2} d^2 \kappa_\perp \Psi(x, \kappa_\perp). \quad (\text{IV.31})$$

Note that the transverse momentum variable  $\kappa_\perp$  here has formally no direct connection with the momentum  $k$  of the starting integral (IV.29).

Alternatively, one may wish to choose a particular decomposition of  $k$ , say, the Sudakov parametrization, in which  $k$  is split into plus, minus and transverse components, and perform integration over the minus component, trying to get an expression in terms of the Sudakov transverse momentum  $k_\perp$ .

#### 1. Sudakov representation

Switching for simplicity to  $p^2 = 0$  case and using parametrization

$$k = \xi p + \eta q' + k_\perp, \quad (\text{IV.32})$$

gives

$$T(Q^2) = \int_0^1 dx \int_0^\infty d\alpha F(x, \alpha) \int_0^\infty d\alpha_1 \times \frac{Q^2}{2} \int_{-\infty}^\infty d\xi d\eta \int d^2 k_\perp e^{i\alpha(\xi-x)\eta Q^2} \times e^{i\alpha_1(\eta-1)\xi Q^2} e^{-i(\alpha+\alpha_1)k_\perp^2}. \quad (\text{IV.33})$$

The minus component of  $k$  is proportional to  $\eta$ . Integrating over it gives

$$T(q, p) = \int_{-\infty}^\infty d\xi \int d^2 k_\perp \int_0^\infty d\lambda \int_0^1 d\beta \int_0^1 dx F(x, \beta\lambda) \times e^{-i\beta\lambda\xi Q^2} \delta(\xi - \beta x) e^{-i\lambda k_\perp^2}. \quad (\text{IV.34})$$

Thus, the Sudakov variable  $\xi = \beta x$  is smaller than the VDA variable  $x$ . Integrating over  $\xi$  results in

$$T(q, p) = \int d^2 k_\perp \int_0^\infty d\lambda \int_0^1 d\beta \int_0^1 dx F(x, \beta\lambda) \times e^{-i\lambda\beta\bar{\beta}xQ^2} e^{-i\lambda k_\perp^2}. \quad (\text{IV.35})$$

Using integral over  $\lambda$  to introduce TMDA gives

$$T(Q^2) = - \int_0^\infty dk_\perp^2 \int_0^1 dx \int_0^1 d\beta \times \frac{\partial}{\partial k_\perp^2} \psi \left( x, \frac{k_\perp^2}{\beta} + \bar{\beta}xQ^2 \right). \quad (\text{IV.36})$$

Our intention is to keep  $k_\perp$ , but let us see first what happens if we integrate over it. Then

$$T(q, p) = \int_0^1 dx \int_0^1 d\beta \psi(x, \bar{\beta}xQ^2), \quad (\text{IV.37})$$

which leads to the VDA result

$$T(q, p) = \int_0^1 \frac{dx}{xQ^2} \int_0^{xQ^2} d\kappa^2 \psi(x, \kappa^2). \quad (\text{IV.38})$$

To keep the original Sudakov variable  $k_\perp$ , we will try to integrate over  $\beta$  in Eq. (IV.40). Using

$$-\frac{\partial}{\partial k_\perp^2} \psi(x, k_\perp^2/\beta + \bar{\beta}xQ^2) = \frac{\beta}{k_\perp^2 + \beta^2 xQ^2} \times \frac{\partial}{\partial \beta} \psi\left(x, \frac{k_\perp^2}{\beta} + \bar{\beta}xQ^2\right) \quad (\text{IV.39})$$

we obtain a rather long and complicated expression

$$T(q, p) = \int_0^1 dx \int_0^\infty dk_\perp^2 \left[ \frac{\psi(x, k_\perp^2)}{xQ^2 + k_\perp^2} + \int_0^1 d\beta \frac{\beta^2 xQ^2 - k_\perp^2}{(k_\perp^2 + \beta^2 xQ^2)^2} \psi\left(x, \frac{k_\perp^2}{\beta} + \bar{\beta}xQ^2\right) \right]. \quad (\text{IV.40})$$

Only the first term here is rather simple

$$T^{(1)}(q, p) = \int_0^1 dx \int dk_\perp^2 \frac{\psi(x, k_\perp^2)}{xQ^2 + k_\perp^2}, \quad (\text{IV.41})$$

and gives the expression used in many papers (see, e.g. Ref. [22]) based on the Sudakov parametrization. Note, however, that in our derivation it involves the VDA variable  $x$  rather than the Sudakov variable  $\xi$ .

As we discussed in Sec. IV A, one can get such a formula by neglecting the minus component of  $k$  in the hard propagator. We have also emphasized that this expression is not ‘‘OPE compliant’’. In particular, unlike the VDA approach expression (IV.38), it generates a tower of  $k_\perp^2/Q^2$  corrections which should be absent for soft wave functions  $\psi(x, k_\perp^2)$ . Still, it correctly reproduces in this case the leading power term (II.10). Also, for hard tails, when  $\psi^{\text{hard}}(x, k_\perp^2) \sim (\ln k_\perp^2)^m/k_\perp^2$ , it correctly reproduces the leading part of the resulting  $(\ln Q^2)^{m+1}/Q^2$  contribution to  $T(Q^2)$ . These observations justify, to some extent, the use of Eq. (IV.41). Nevertheless, it is just an approximation, while Eq. (IV.38) is an exact result.

The difference between them is given by the second integral in Eq. (IV.40). As one can see, it has a rather complicated form and contains TMDA in which the transverse momentum argument  $k_\perp^2$  is rescaled by  $1/\beta$  factor and then shifted by a  $\bar{\beta}$  fraction of the hard virtuality  $xQ^2$ . We can restore in this term the Sudakov variable  $\xi = \beta x$  to see that the first argument of TMDA here differs from the Sudakov variable  $\xi$  by the  $1/\beta$  factor.

In fact, since the  $\beta$  integration is present in the second term of (IV.41), we did not reach our goal of reducing the  $d^4k$  integral to integration over just the plus momentum fraction and transverse momentum. The only way we see to get rid of the  $\beta$  integral here is to return to the VDA result (IV.38).

## 2. Comparison in the impact parameter representation

The Sudakov variable  $k_\perp$  is Fourier-conjugate to the transverse coordinate  $z_\perp$  of the virtual photon vertex in

Eq. (IV.9), so we may write

$$\Psi(x, k_\perp) = \int \frac{d^2 z_\perp}{(2\pi)^2} \varphi(x, z_\perp) e^{-i(k_\perp z_\perp)}, \quad (\text{IV.42})$$

and present the approximation (IV.41) in the impact parameters space as

$$T^{(1)}(Q^2) = \int_0^1 dx \int_0^\infty dz_\perp z_\perp K_0(z_\perp \sqrt{xQ^2}) \varphi(x, z_\perp^2), \quad (\text{IV.43})$$

which coincides with the term explicitly written in Eq. (IV.25). It is instructive to compare this result with the impact parameter version (III.11) of the VDA formula,

$$T(Q^2) = \int_0^1 \frac{dx}{\sqrt{xQ}} \int_0^\infty db J_1(\sqrt{xQ}b) \varphi(x, b). \quad (\text{IV.44})$$

When  $\varphi(x, b) = \varphi(x)$  (which corresponds to the leading approximation of keeping only the  $l = 0$  term in Eq. (II.4)), both formulas give the same  $1/xQ^2$  result after integration over the impact parameter.

However, the two formulas have a different attitude with respect to further terms of the  $(b^2)^l$  expansion of a soft  $\varphi(x, b)$ . Indeed, the integrals of  $(b^2)^l$  with  $K_0(\sqrt{xQ}b)$  converge for all powers  $l$  (producing unwanted  $(1/Q^2)^{l+1}$  power corrections), while  $J_1(\sqrt{xQ}b)$  integrals diverge starting with  $l = 1$ . Because of oscillating nature of the  $J_1$  Bessel function, a proper regularization would set all these integrals to zero (or derivatives of  $\delta(xQ^2)$ , to be more precise), which corresponds to having no  $1/Q^2$  corrections under the  $x$ -integral.

Another evident difference between the two expressions is that  $J_1(\sqrt{xQ}b)/b$  is finite for  $b = 0$  while  $K_0(\sqrt{xQ}b)$  has a logarithmic singularity there. Moreover, if  $\Psi(x, k_\perp)$  is finite for  $k_\perp = 0$ , the exact formula (IV.1) gives a finite value for  $T(Q^2)$  in the  $Q^2 \rightarrow 0$  limit, namely,

$$T(Q^2 \rightarrow 0) = \pi \int_0^1 \Psi(x, k_\perp = 0) dx. \quad (\text{IV.45})$$

On the other hand, Eq. (IV.41) produces a logarithmically divergent result even if  $\Psi(x, k_\perp = 0)$  is finite. We now see that this well-known deficiency of Eq. (IV.41) (see, e.g., [23]) is just the result of approximations (equivalent to taking  $k_- = 0$  in the hard propagator) used in its derivation.

## 3. Calculation in the IMF variables

Another possibility is to write the  $d^4k$  integral in Eq. (IV.29) using a frame where  $p$  defines the ‘‘plus’’ direction, but the  $q_+ = 0$ . Defining the ‘‘minus’’ direction by a lightlike vector  $n$ , we have

$$q' = p + Q^2 n - q_\perp, \quad (\text{IV.46})$$

$$k = \xi p + \eta Q^2 n + k_\perp, \quad (\text{IV.47})$$

where  $n^2 = 0$  and  $2(pn) = 1$ . Then  $k^2 = \xi\eta Q^2 - k_\perp^2$  and

$$(q' - k)^2 = (1 - \xi)(1 - \eta)Q^2 - (k_\perp + q_\perp)^2. \quad (\text{IV.48})$$

This gives

$$\begin{aligned} T(q, p) &= \int_0^1 dx \int_0^\infty d\alpha F(x, \alpha) \int_0^\infty d\alpha_1 \\ &\times \frac{Q^2}{2} \int_{-\infty}^\infty d\xi d\eta \int d^2 k_\perp e^{i\alpha(\xi-x)\eta Q^2} \\ &\times e^{i\alpha_1(\xi-1)(\eta-1)Q^2} e^{-i\alpha k_\perp^2 - i\alpha_1(k_\perp + q_\perp)^2}. \end{aligned} \quad (\text{IV.49})$$

Performing integration over  $\eta$ , i.e., the minus component of  $k$ , and changing  $\alpha + \alpha_1 = \lambda$ ,  $\alpha = \beta\lambda$ , we obtain

$$\begin{aligned} T(q, p) &= \int d^2 k_\perp \int_0^1 dx \int_0^\infty d\lambda \\ &\times \int_0^1 d\beta F(x, \beta\lambda) \int_0^1 d\bar{\xi} \delta[\bar{\xi} - \beta\bar{x}] \\ &\times e^{i\lambda(\bar{\xi}-\beta)\bar{\beta}Q^2 - i\lambda(k_\perp + \bar{\beta}q_\perp)^2}. \end{aligned} \quad (\text{IV.50})$$

Thus, the IMF variable  $\bar{\xi} = \beta\bar{x}$  is smaller than the VDA variable  $\bar{x}$ . i.e.,  $\xi_{\text{IMF}}$  is larger than  $x$ . However, integrating over  $\xi$  and shifting integration variable  $k_\perp$  gives

$$\begin{aligned} T(q, p) &= \int d^2 k_\perp \int_0^1 dx \int_0^\infty d\lambda F(x, \beta\lambda) \\ &\times \int_0^1 d\beta e^{-i\lambda k_\perp^2 - i\lambda\beta\bar{\beta}xQ^2}, \end{aligned} \quad (\text{IV.51})$$

which coincides with the expression (IV.35) obtained using Sudakov variables, and further steps are the same.

#### 4. Summary

Thus, our examination did not reveal any advantages of using explicit decomposition of the integration momentum  $k$  in either Sudakov or IMF variables. To the contrary, their use results in the expression (IV.40) that is much more complicated than the VDA formula (IV.38). Moreover, in the exact expression (IV.41) we did not reach the goal of converting  $d^4k$  into  $d\xi d^2k_\perp$ : it contains an extra integration, which we managed to get rid of only by making approximations. Furthermore, the approximate expression (IV.41) is not satisfactory since it contains  $k_\perp$ -dependent terms in hard factors that produce towers of  $k_\perp^2/Q^2$  corrections. As a result, it is not OPE compatible for soft TMDAs.

Note that trying to keep the Sudakov or IMF transverse momentum variable, we have integrated over the plus-momentum variable  $\xi$  of these representations, thus using the VDA variable  $x$  in further expressions. As we have seen,  $\xi$  differs from  $x$  in both cases (though in opposite directions:  $\xi_{\text{Sud}} \leq x$ , while  $\xi_{\text{IMF}} \geq x$ ). If, instead, we would try to keep the  $\xi$  variables by integrating over  $x$ , we would get much more lengthy expressions, with

the  $\beta$  integration variable now entering both arguments of TMDA, making further simplifications virtually impossible. This is another argument in favor of using the VDA-based variables and formulas.

#### D. Three-body contributions

Using VDA we take into account the contributions of higher twist operators of  $\phi \dots (\partial^2)^n \phi(0)$  type. By equations of motion, like  $\partial^2 \phi = g\chi\phi$  in the case of  $g\phi^2\chi$  interaction of quarks  $\phi$  with gluons  $\chi$ , these operators may be converted into multi-body operators like  $\phi \dots \chi^n(0)\phi(0)$ . A distinctive feature of these operators is that the gluon field  $\chi(0)$  is taken at the same point as the quark field  $\phi(0)$ , so we still deal with an effectively bilocal operator. However, one may also wish to include configurations with three (or more) partons participating in the short-distance subprocess, which are described by multi-body operators with the gluon fields taken at locations different from those of the photon vertices.

Take a contribution with one gluon insertion (see Fig. 3). Then

$$\begin{aligned} T_3(q, p) &= \int d^4z e^{-i(qz)} \int d^4z_1 D^c(z - z_1) D^c(z_1) \\ &\times \langle p | \phi(z) \chi(z_1) \phi(0) | 0 \rangle. \end{aligned} \quad (\text{IV.52})$$

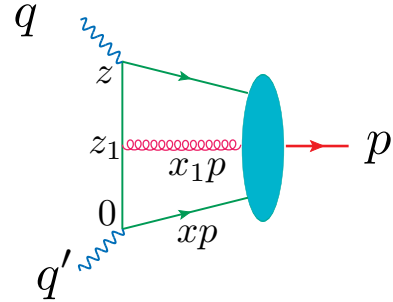


FIG. 3. Scalar diagram with a gluon insertion.

The tri-local matrix element depends in general on three intervals:  $z^2, z_1^2, (z - z_1)^2$  and two scalar products  $(pz), (pz_1)$ . Neglecting virtuality-related dependence on *all* intervals, we can parametrize

$$\begin{aligned} \langle p | \phi(z) \chi(z_1) \phi(0) | 0 \rangle &= \int_0^1 dx \int_0^{\bar{x}} dx_1 f(x, x_1) \\ &\times e^{ix_1(pz_1) + i(\bar{x} - x_1)(pz)} + \mathcal{O}(z_i^2). \end{aligned} \quad (\text{IV.53})$$

In this parametrization, the “gluon” has an (outgoing) momentum  $x_1p$ , and “quark” at 0 carries momentum  $xp$ . The quark at  $z$  has momentum  $(1 - x - x_1)p$ . The spectral property  $x \geq 0, x_1 \geq 0, x + x_1 \leq 1$  (we will denote this region as  $\Omega$ ) can be proven for any Feynman diagram contributing to  $f(x, x_1)$ , see Refs. [14, 15].

Taking  $z_1 = z$ , we get an effectively bilocal operator  $\phi(0)\chi(z)\phi(z)$ . Incorporating the equation of motion  $\partial^2\phi = g\chi\phi$ , we get a reduction relation

$$g \int_0^{\bar{x}} f(x, x_1) dx_1 = \Lambda^2 \varphi_1(x) \quad (\text{IV.54})$$

connecting  $f(x, x_1)$  with the distribution amplitude  $\varphi_1(x)$  corresponding to  $\phi\partial^2\phi$  operators in Eq. (III.13). This connection just states that average parton virtuality  $\Lambda^2$  is proportional to the average strength of the gluonic field  $\chi$  inside the hadron.

Since these two contributions are governed by the same scale, one may wonder if we should consider them together. However, there is an essential difference between the two. The virtuality correction is “invisible” when taken on its own, and contributes to a nontrivial function of  $Q^2$  only after summation through VDA with all other “invisible” contributions. On the other hand, the diagram with a gluon insertion is an explicit power correction to the handbag term. Its contribution is given by the parton formula

$$\begin{aligned} T_3(q, p) &= \int_{\Omega} \frac{f(x, x_1) dx dx_1}{[q' - xp]^2 [q' - (x + x_1)p]^2} + \mathcal{O}(1/Q^6) \\ &= \frac{1}{Q^4} \int_{\Omega} \frac{f(x, x_1) dx dx_1}{x(x + x_1)} + \mathcal{O}(1/Q^6) \end{aligned} \quad (\text{IV.55})$$

from which it is evident that the 3-parton term is suppressed by  $1/Q^2$  compared to the handbag diagram. This outcome is a consequence of the fact that the 3-parton amplitude is less singular

$$\int d^4 z_1 D^c(z - z_1) D^c(z_1) \sim \ln(z^2) \quad (\text{IV.56})$$

on the light cone than the handbag diagram, which has  $1/z^2$  singularity.

One may also wish to improve the precision and include the virtuality effects by keeping  $\chi(z_1)$  in the above integral. Combining the denominators  $1/z_1^2$  and  $1/(z - z_1)^2$  through Feynman parameter  $u$  and shifting the integration variable  $z_1 \rightarrow z_1 + uz$ , we arrive at (cf. Ref. [24])

$$\int_0^1 du \int \frac{\chi(uz + z_1) d^4 z_1}{[z_1^2 + u(1 - u)z^2]^2} . \quad (\text{IV.57})$$

Note that  $\chi(uz + z_1)$  is integrated over  $z_1$  with a function that depends on  $z_1$  through  $z_1^2$  only. Hence, if we expand  $\chi(uz + z_1)$  around the point  $uz$  using analog of Eq.(II.2), all terms containing traceless combination  $\{z_1\partial\}^N$  with  $N \geq 1$  give zero after integration over  $z_1$ , so we can use

$$\begin{aligned} \chi(uz + z_1) &= \sum_{l=0}^{\infty} \left(\frac{z_1^2}{4}\right)^l \frac{(\partial^2)^l \chi(uz)}{l!(l+1)!} + \text{traceless } \{z_1\partial\}^N \\ &\equiv \int_0^{\infty} d\sigma_1 \tilde{\chi}(uz, \sigma_1) e^{-i\sigma_1(z_1^2 - i\epsilon)/4} + \dots \end{aligned} \quad (\text{IV.58})$$

The  $\sigma_1$ -dependence of the field  $\tilde{\chi}(uz, \sigma_1)$  takes care of the effects due to the virtuality (off-shellness) of the scalar gluon field  $\chi$ . Thus, we end up with the integral

$$\begin{aligned} T_3(q, p) &= \int_0^1 du \int d^4 z e^{-i(qz)} \\ &\times \int_0^{\infty} d\sigma_1 \langle p | \phi(z) \tilde{\chi}(uz, \sigma_1) \phi(0) | 0 \rangle \\ &\times \int \frac{d^4 z_1}{[z_1^2 + u(1 - u)z^2]^2} e^{-i\sigma_1(z_1^2 - i\epsilon)/4} . \end{aligned} \quad (\text{IV.59})$$

The  $d^4 z_1$  integral can be calculated in terms of incomplete gamma function. The result depends on the combination  $u\bar{u}z^2\sigma_1$  and has a logarithmic dependence on it for small  $z^2$ . One can in principle keep this dependence, but as the first step we may neglect the gluon virtuality effects given by  $(z_1^2\partial^2)^l\phi$  in Eq. (IV.58), and look at the virtuality effects due to non-zero value of  $z^2$ . Then one deals with

$$\begin{aligned} T_3(q, p) &= \int_0^1 du \int d^4 z e^{-i(qz)} \ln(u\bar{u}z^2 M^2) \\ &\times \langle p | \phi(z) \chi(uz) \phi(0) | 0 \rangle . \end{aligned} \quad (\text{IV.60})$$

One can see that all the fields involved in the matrix element here are located on the straight line connecting 0 and  $z$ , i.e. we deal with a “string” operator [24]. With respect to  $z$ , the matrix element is a function of  $(pz)$  and  $z^2$  (note that it has been never assumed that  $z$  is on the light cone, so we can take  $z^2 \neq 0$ ). Then we can represent this matrix element using a VDA-type parametrization like

$$\begin{aligned} \langle p | \phi(z) \tilde{\chi}(uz) \phi(0) | 0 \rangle &= \int_0^{\infty} d\sigma \int_{\Omega} dx dx_1 \Phi(x, x_1, \sigma) \\ &\times e^{ix_1 u(pz) + i(\bar{x} - x_1)(pz) - i\sigma(z^2 - i\epsilon)/4} . \end{aligned} \quad (\text{IV.61})$$

In the present paper, we will not proceed further with the analysis of the 3-body and higher Fock components, leaving it to future investigations. However, some elements of the technique used above are helpful in the analysis of gauge theories.

## V. QCD CASE

The justification for spending time on scalar models is that the same construction may be built in QCD.

### A. Spin-1/2 quarks

A realistic case is when quarks have spin 1/2. Then the handbag diagram for the pion transition form factor is given by

$$\begin{aligned} \int d^4 z e^{-i(q'z)} \langle p | \bar{\psi}(0) \gamma^\nu S^c(-z) \gamma^\mu \psi(z) | 0 \rangle \\ = i\epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta F(Q^2) , \end{aligned} \quad (\text{V.1})$$



where  $S^c(z) = \not{z}/2\pi^2(z^2)^2$  is the propagator for a massless fermion. Writing the antisymmetric part of  $\gamma^\nu \not{z} \gamma^\mu$  as  $i z_\beta \epsilon^{\mu\nu\alpha\beta} \gamma_5 \gamma_\alpha$  we parametrize the resulting matrix element as

$$\langle p | \bar{\psi}(0) \gamma_5 \gamma_\alpha \psi(z) | 0 \rangle = i \int_0^\infty d\sigma \int_0^1 dx \times [p_\alpha \Phi(x, \sigma) + z_\alpha \Phi_2(x, \sigma)] e^{ix(pz) - i\sigma(z^2 - i\epsilon)/4}. \quad (\text{V.2})$$

Here we have two VDAs that are associated with the two possibilities  $p_\alpha$  and  $z_\alpha$  for a vector carrying the Lorentz index  $\alpha$ . In fact, the term proportional to  $z_\alpha$  does not contribute to the  $\gamma^* \gamma \rightarrow \pi^0$  amplitude at the handbag level, since it disappears after convolution with  $z_\beta \epsilon^{\mu\nu\alpha\beta}$ .

All the formulas relating VDA with TMDA and IDA that were derived in the scalar case are valid without changes. However, the spinor propagator has a different functional form. Using it, we obtain

$$F(Q^2) = \int_0^\infty d\sigma \int_0^1 \Phi(x, \sigma) \frac{dx}{xQ^2} \times \left\{ 1 + \frac{i\sigma}{xQ^2} \left[ 1 - e^{-[ixQ^2 + \epsilon]/\sigma} \right] \right\}. \quad (\text{V.3})$$

The first two terms are given by two lowest moments of VDA  $\Phi(x, \sigma)$ . Assuming that they exist, we may write

$$F(Q^2) = \int_0^1 \frac{dx}{xQ^2} \left\{ \varphi(x) - \varphi_1(x) \frac{\Lambda^2}{xQ^2} - \int_0^\infty d\sigma \Phi(x, \sigma) \frac{i\sigma}{xQ^2} e^{-[ixQ^2 + \epsilon]/\sigma} \right\}. \quad (\text{V.4})$$

For large  $Q^2$ , Eq. (V.4) shows a correction with a power-like  $\Lambda^2/Q^4$  behavior that corresponds to the twist-4  $\bar{\psi} \gamma_5 \gamma_\alpha D^2 \psi$  operator. Though it is accompanied by a  $z^2$  factor, the latter does not completely cancel the  $1/z^4$  singularity of the spinor propagator, and as a result this contribution has a “visible”  $1/Q^4$  behavior. The remaining term corresponds to contributions “invisible” in the OPE.

Note that, taken separately, the twist-4 contribution results in a very singular  $\varphi_1(x)/x^2$  integral. If the pion “daughter” DA  $\varphi_1(x)$  does not vanish like  $x^{1+\alpha}$  with a positive  $\alpha$  in the end-point region, the purely twist-4 contribution diverges. However, it is easy to check that when the “invisible” terms are added, the  $x$ -integral in the original Eq. (V.3) converges. Indeed, writing it in terms of TMDA

$$F(Q^2) = \int_0^1 \frac{dx}{xQ^2} \int_0^{xQ^2} \frac{dk_\perp^2}{xQ^2} \int_{k_\perp'^2 \leq k_\perp^2} \Psi(x, k'_\perp) d^2 k'_\perp, \quad (\text{V.5})$$

and then in terms of IDA

$$F(Q^2) = 2 \int_0^1 \frac{dx}{xQ^2} \int_0^\infty \frac{db}{b} J_2(\sqrt{x}Qb) \varphi(x, b), \quad (\text{V.6})$$

we see that  $J_2(\sqrt{x}Qb)/x \rightarrow \text{const}$  for small  $x$ , so that  $x$ -integral converges if  $\varphi(x, b) \sim x^{-1+\alpha}$  with however small positive  $\alpha$ , i.e.  $\varphi(x, b)$  may be even singular for small  $x$ . Taking the  $Q^2 \rightarrow 0$  limit under the integral, we get

$$F(Q^2 = 0) = \frac{1}{4} \int_0^1 dx \int_0^\infty b db \varphi(x, b), \quad (\text{V.7})$$

assuming that the  $b$ -integral is finite. This is the case if  $\Psi(x, k_\perp)$  is finite for  $k_\perp = 0$ . Then we have

$$F(Q^2 \rightarrow 0) = \frac{\pi}{2} \int_0^1 \Psi(x, k_\perp = 0) dx. \quad (\text{V.8})$$

This result also follows directly from Eq. (V.5) if  $\Psi(x, k_\perp = 0)$  is finite. It also coincides with the IMF light-front approach result of Ref.[12].

In fact, Eq. (V.5) can be re-written in the form involving just one transverse momentum integration

$$F(Q^2) = \int_0^1 \frac{dx}{xQ^2} \int_{k_\perp^2 \leq xQ^2} \Psi(x, k_\perp) \left[ 1 - \frac{k_\perp^2}{xQ^2} \right] d^2 k_\perp \quad (\text{V.9})$$

and explicitly showing the twist-4 correction given by the  $k_\perp^2$  moment of TMDA  $\Psi(x, k_\perp)$ .

## B. Adding a gluon

In gauge theories, the handbag contribution in a covariant gauge should be complemented by diagrams corresponding to operators  $\bar{\psi}(z) \dots \mathcal{A}(z_i) \dots \psi(0)$  containing twist-0 gluonic field  $A_{\mu_i}(z_i)$  inserted into the fermion line between the points  $z$  and  $0$  (see Fig. 4).

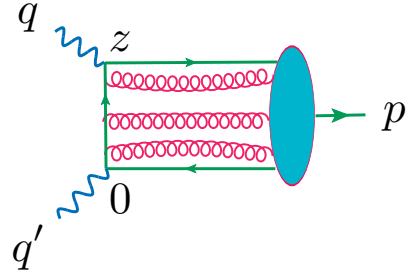


FIG. 4. Structure of the handbag contribution in QCD.

Consider an insertion of the gluon field  $A^\alpha(z_1)$  into the quark propagator connecting  $z$  and  $0$  vertices. Taking the gluon with momentum  $k$  we have

$$\int d^4 z_1 S^c(z - z_1) \gamma^\alpha S^c(z_1) e^{i(kz_1)} \quad (\text{V.10})$$

or

$$\int d^4 z_1 \int_0^\infty \sigma_1 d\sigma_1 \int_0^\infty \sigma_2 d\sigma_2 (\not{z} - \not{z}_1) \gamma^\alpha \not{z}_1 \times e^{-i\sigma_1(z - z_1)^2/4 - i\sigma_2 z_1^2/4} e^{i(kz_1)} \quad (\text{V.11})$$

Integrating over  $z_1$  and changing  $\sigma_1 + \sigma_2 = \sigma$ ,  $\sigma_1 = t\sigma$  gives

$$\int_0^\infty \sigma d\sigma \int_0^1 dt \bar{t} \exp \left\{ -i t \bar{t} \frac{z^2}{4} \sigma + it(kz) + i \frac{k^2}{\sigma} \right\} \times [(\bar{t} \not{z} - 2 \not{k}/\sigma) \gamma^\alpha (t \not{z} + 2 \not{k}/\sigma) - 4i \gamma^\alpha / \sigma] . \quad (\text{V.12})$$

Changing further  $\sigma \rightarrow \sigma/t\bar{t}$  produces the integral

$$\int_0^\infty \sigma d\sigma \int_0^1 dt \exp \left\{ -i \frac{z^2}{4} \sigma + it(kz) + i t \bar{t} \frac{k^2}{\sigma} \right\} \times [\text{Trace}] , \quad (\text{V.13})$$

with the trace given by

$$[\text{Trace}] = 2z^\alpha \not{z} - \gamma^\alpha (z^2 + 4i\sigma) + 2k_\beta [\bar{t} \not{z} \gamma^\alpha \gamma^\beta - t \gamma^\beta \gamma^\alpha \not{z}] / \sigma - 4t \bar{t} \not{k} \gamma^\alpha \not{k} / \sigma^2 . \quad (\text{V.14})$$

Writing

$$e^{it \bar{t} k^2 / \sigma} = 1 + \left[ e^{it \bar{t} k^2 / \sigma} - 1 \right] , \quad (\text{V.15})$$

we can integrate over  $\sigma$  the term corresponding to “1”. In particular,

$$\int_0^\infty \sigma d\sigma \exp \left\{ -i \frac{z^2}{4} \sigma \right\} = -\frac{16}{(z^2)^2} , \quad (\text{V.16})$$

while

$$\int_0^\infty \sigma d\sigma \exp \left\{ -i \frac{z^2}{4} \sigma \right\} \left[ z^2 + \frac{4i}{\sigma} \right] = 0 \quad (\text{V.17})$$

and

$$\int_0^\infty d\sigma \exp \left\{ -i \frac{z^2}{4} \sigma \right\} = -\frac{4i}{z^2} . \quad (\text{V.18})$$

Thus, the leading singularity  $1/(z^2)^2$  is accompanied by the same gamma-matrix factor  $\not{z}$  as in the original quark propagator. The factor  $z^\alpha$  means that the field  $A^\alpha$  appears as  $(zA)$ , while the exponential  $e^{it(kz)}$  shows that the field is taken at the running argument  $tz$ , with integral over  $t$  from 0 to 1. This means that it corresponds to the linear in  $A$  part

$$ig \int_0^1 dt z_\alpha A^\alpha(tz) \quad (\text{V.19})$$

of the gauge link

$$E(0, z; A) \equiv P \exp \left[ ig \int_0^1 dt z_\alpha A^\alpha(tz) \right] . \quad (\text{V.20})$$

(Our derivation given above is inspired by that given in Ref. [24]).

Furthermore, terms corresponding to the sub-leading singularity  $1/z^2$  are proportional to  $k$ , i.e. the derivative of the gluon field, which in fact combines into the field-strength tensor  $F_{\alpha\beta}$ . Indeed, if one takes the  $\alpha \leftrightarrow \beta$

symmetric part of the  $\gamma$ -matrix terms in  $k_\beta \gamma^\alpha \gamma^\beta$  and  $k_\beta \gamma^\beta \gamma^\alpha$ , it gives  $k^\alpha$  which results in a vanishing contribution since  $(kA) = 0$ , and only terms proportional to  $k_\beta \sigma^{\alpha\beta}$ , i.e.,  $k_\beta A_\alpha - k_\alpha A_\beta$  remain, which corresponds to the field-strength tensor  $F_{\alpha\beta}$ .

One can also take the  $\sigma$  integral in its original form:

$$\begin{aligned} \int_0^\infty \sigma d\sigma \exp \left\{ -i \frac{z^2}{4} \sigma + i t \bar{t} \frac{k^2}{\sigma} \right\} &= -\frac{8t \bar{t} k^2}{z^2} K_2(\sqrt{t \bar{t} k^2 z^2}) \\ &= -\frac{16}{(z^2)^2} + \frac{4t \bar{t} k^2}{z^2} + \dots , \end{aligned} \quad (\text{V.21})$$

while

$$\begin{aligned} \int_0^\infty \sigma d\sigma \exp \left\{ -i \frac{z^2}{4} \sigma + i t \bar{t} \frac{k^2}{\sigma} \right\} \left[ z^2 + \frac{4i}{\sigma} \right] &= -8t \bar{t} k^2 K_0(\sqrt{t \bar{t} k^2 z^2}) \\ &= \mathcal{O}(k^2 \ln(k^2 z^2)) , \end{aligned} \quad (\text{V.22})$$

and

$$\begin{aligned} \int_0^\infty d\sigma \exp \left\{ -i \frac{z^2}{4} \sigma + i t \bar{t} \frac{k^2}{\sigma} \right\} &= -4i \sqrt{\frac{t \bar{t} k^2}{z^2}} K_1(\sqrt{t \bar{t} k^2 z^2}) \\ &= -\frac{4i}{z^2} + \mathcal{O}(k^2 \ln(k^2 z^2)) , \end{aligned} \quad (\text{V.23})$$

which produces the results discussed above, provided that one neglects  $\mathcal{O}(k^2/z^2)$  and  $\mathcal{O}(k^2 \ln(k^2 z^2))$  terms.

### C. Quark propagator in external gluon field

To see how the  $E$ -factor (V.20) emerges to all orders in the external field  $gA$  [7, 25], we observe that the sum of gluon insertions is equivalent to substituting the free propagator  $S^c(z_1 - z_2)$  by a propagator  $\mathcal{S}^c(z_1, z_2; A)$  of a fermion in an external gluonic field  $A$ . This propagator satisfies the Dirac equation

$$i \left[ \frac{\not{\partial}}{\partial z_1} - ig \not{A}(z_1) \right] \mathcal{S}^c(z_1, z_2; A) = -\delta^4(z_1 - z_2) . \quad (\text{V.24})$$

Looking for a solution of this equation in the form

$$\mathcal{S}^c(z_1, z_2; A) = E(z_1, z_2; A) \mathfrak{S}_{\text{FS}}^c(z_1, z_2; A) \quad (\text{V.25})$$

involving the straight-line exponential (V.20), one can see that the factor  $\mathfrak{S}_{\text{FS}}^c$  should satisfy the Dirac equation

$$i \left[ \frac{\not{\partial}}{\partial z_1} - ig \not{A}(z_1) \right] \mathfrak{S}_{\text{FS}}^c(z_1, z_2; A) = -\delta^4(z_1 - z_2) , \quad (\text{V.26})$$

with the field [7, 25]

$$\mathfrak{A}^\mu(z; z_1) = (z - z_1)_\nu \int_0^1 s G^{\mu\nu}(z_1 + s(z - z_1)) ds \quad (\text{V.27})$$

being the vector potential in the Fock-Schwinger (FS) gauge [26, 27],

$$(z - z_1)_\mu \mathfrak{A}^\mu(z, z_1) = 0. \quad (\text{V.28})$$

Here,  $z$  denotes an arbitrary position in space while  $z_1$  specifies the “fixed point” of the gauge and in our case refers to an end-point in the Compton amplitude.

Since the field-strength tensor  $G^{\mu\nu}$  has twist equal to (at least) 1, the insertion of this field into the free propagator results in power  $(\Lambda^2 Q^2)^l$  corrections to the Compton amplitude. Thus, we can write

$$\mathcal{S}^c(0, z; A) = E(0, z; A) \{S_c(z) + \mathcal{O}(G)\}. \quad (\text{V.29})$$

### 1. Bilocal operator in gauge theories

Keeping the first term in Eq. (V.29) we need to deal with the gauge-invariant bilocal operator

$$\mathcal{O}^\alpha(0, z; A) \equiv \bar{\psi}(0) \gamma_5 \gamma^\alpha E(0, z; A) \psi(z). \quad (\text{V.30})$$

Its important property is that the Taylor expansion for  $\mathcal{O}^\alpha(0, z; A)$  has the same structure as that for the original  $\bar{\psi}(0) \gamma_5 \gamma^\alpha \psi(z)$  operator, with the only change that one should use covariant derivatives  $D^\mu = \partial^\mu - igA^\mu$  instead of the ordinary  $\partial^\mu$  ones:

$$E(0, z; A) \psi(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (zD)^n \psi(0). \quad (\text{V.31})$$

This result follows from the relation

$$\frac{d}{dz_\alpha} E(0, z; A) = E(0, z; A) [D^\alpha + ig\mathfrak{A}^\alpha(z, 0)] \quad (\text{V.32})$$

and the property  $z_\alpha \mathfrak{A}^\alpha(z, 0) = 0$  of the Fock-Schwinger field  $\mathfrak{A}^\alpha(z, 0)$ . As we have seen in Sect. II B, the Taylor expansion for a matrix element may be written in the form of the VDA parametrization

$$\begin{aligned} \langle p | \mathcal{O}^\alpha(0, z; A) | 0 \rangle &= i \int_0^\infty d\sigma \int_0^1 dx [p_\alpha \Phi(x, \sigma) \\ &+ z_\alpha \Phi_z(x, \sigma)] e^{ix(pz) - i\sigma(z^2 - i\epsilon)/4}. \end{aligned} \quad (\text{V.33})$$

Again, since the matrix element  $\langle p | \mathcal{O}^\alpha(0, z; A) | 0 \rangle$  is just a function of  $(pz)$  and  $z^2$ , one can treat the VDA representation as a particular case of the double Fourier representation with specific constraints on the limits of  $x$  and  $\sigma$  integration. These limits, in turn, reflect only the positivity of the functions  $A, B, C, D$  in the exponential of the  $\alpha$ -representation (II.15), which are completely determined by the denominators of the momentum space propagators of the relevant Feynman diagram and are the same in any theory. This observation justifies the use of the VDA representation (V.33) in QCD in general case.

## 2. Multilocal operators

Insertions of the nonzero-twist FS field  $\mathfrak{A}^\mu$  result in matrix elements of  $\bar{\psi}(0) \dots G(sz) \dots \psi(z)$  operators, which should be parametrized in terms of trilocal, etc. VDAs. These terms are analogous to  $\phi(0) \dots \chi(z_1) \dots \phi(z)$ , etc. operators of the scalar model. The technology of how to work with insertions of the Fock-Schwinger field  $\mathfrak{A}^\mu$  is well-developed, see e.g. Refs. [24, 28–30]. For the Compton amplitude, the contribution due to a single insertion of  $\mathfrak{A}^\mu$  was calculated by Balitsky and Braun [24] and shown to produce a  $1/Q^2$  correction to the leading term.

## VI. MODELS OF SOFT VDAS

### A. Explicit models of soft transverse momentum dependence

Let us now discuss some explicit models of the  $k_\perp$  dependence of soft TMDAs  $\Psi(x, k_\perp)$ . In general, they are functions of two independent variables  $x$  and  $k_\perp^2$ . But it makes sense to start with a simple case of factorized models

$$\Psi(x, k_\perp) = \varphi(x) \psi(k_\perp^2), \quad (\text{VI.1})$$

in which  $x$ -dependence and  $k_\perp$ -dependence appear in separate factors. Since relations between VDAs, TMDAs and IDAs are the same in scalar and spinor cases, we will refer for simplicity to scalar operators.

### 1. Gaussian model

It is popular to assume a Gaussian dependence on  $k_\perp$ ,

$$\Psi_G(x, k_\perp) = \frac{\varphi(x)}{\pi\Lambda^2} e^{-k_\perp^2/\Lambda^2}. \quad (\text{VI.2})$$

In the impact parameter space, one gets IDA

$$\varphi_G(x, z_\perp) = \varphi(x) e^{-z_\perp^2 \Lambda^2/4} \quad (\text{VI.3})$$

that also has a Gaussian dependence on  $z_\perp$ . Writing

$$\varphi_G(x, z_\perp) = \frac{\varphi(x)}{2\pi} \int_{-\infty}^{\infty} \frac{i d\sigma}{\sigma + i\Lambda^2} e^{-iz_\perp^2 \sigma/4}, \quad (\text{VI.4})$$

we see that the integral here involves both positive and negative  $\sigma$ , i.e. formally  $\varphi_G(x, z_\perp)$  cannot be written in the VDA representation (III.2). This is a consequence of the fact that  $\psi_G(x, \kappa_\perp^2)$ , the analytic continuation of  $\Psi_G(x, k_\perp)$  into the time-like region of momenta, has an exponential increase for large negative  $\kappa^2$ .

However, the transition form factor for space-like virtual photons involves only the integral over positive  $\kappa^2$ , i.e. it is not sensitive to the behavior of  $\psi_G(x, \kappa_\perp^2)$  for

negative  $\kappa^2$ . One can see that the form factor formula in terms of TMDA (V.5) shows no peculiarities in case of the Gaussian ansatz. For this reason, we will use this model because of its calculational simplicity.

## 2. Simple non-Gaussian models

One may still argue that a Gaussian fall-off for large  $z_\perp$  is too fast. In particular, propagators  $D^c(z, m)$  of massive particles have an exponential  $e^{-m|z|}$  decrease for spacelike intervals  $z^2$ .

To build models for TMDAs that are more closely resembling perturbative propagators, we recall that the propagator of a scalar particle with mass  $m$  may be written as

$$D^c(z, m) = \frac{1}{(4\pi)^2} \int_0^\infty e^{-i\sigma z^2/4 - i(m^2 - i\epsilon)/\sigma} d\sigma. \quad (\text{VI.5})$$

The mass term assures that the propagator falls off exponentially  $\sim e^{-|z|m}$  for large space-like distances. At small intervals  $z^2$ , however, the free particle propagator has  $1/z^2$  singularity while we want the soft part of  $\langle p|\phi(0)\phi(z)|0\rangle$  to be finite at  $z = 0$ . The simplest way is to add a constant term  $(-4/\Lambda^2)$  to  $z^2$  in the VDA representation (II.19). So, we take

$$\Phi(x, \sigma) = \frac{\varphi(x)}{p(\Lambda, m)} e^{i\sigma/\Lambda^2 - im^2/\sigma - \epsilon\sigma} \quad (\text{VI.6})$$

as a model for VDA. The sign of the  $\Lambda^2$  term is fixed from the requirement that  $(4/\Lambda^2 - z^2)^{-1}$  should not have singularities for space-like  $z^2$ . The normalization factor  $p(\Lambda, m)$  is given by

$$\begin{aligned} p(\Lambda, m) &= \int_0^\infty e^{i\sigma/\Lambda^2 - im^2/\sigma - \epsilon\sigma} d\sigma \\ &= 2im\Lambda K_1(2m/\Lambda). \end{aligned} \quad (\text{VI.7})$$

*a.  $m = 0$  model.* To begin with, let us take  $m = 0$ , i.e.

$$\Phi(x, \sigma) = \varphi(x) \frac{e^{i\sigma/\Lambda^2 - \epsilon\sigma}}{i\Lambda^2}. \quad (\text{VI.8})$$

The bilocal matrix element in this case is given by

$$\langle p|\phi(0)\phi(z)|0\rangle = \frac{1}{1 - z^2\Lambda^2/4} \int_0^1 dx \varphi(x) e^{ix(pz)}, \quad (\text{VI.9})$$

which corresponds to

$$\varphi(x, z_\perp) = \frac{\varphi(x)}{1 + z_\perp^2\Lambda^2/4} \quad (\text{VI.10})$$

for IDA. One can see that the  $z_\perp^2$  term of the  $z_\perp$  expansion of  $\varphi(x, z_\perp)$  in this model was adjusted to coincide

with that of the exponential model, so that  $\Lambda^2$  has the same meaning of the scale of  $\phi\partial^2\phi$  operator. The TMDA for this Ansatz is given by

$$\Psi(x, k_\perp) = 2\varphi(x) \frac{K_0(2k_\perp/\Lambda)}{\pi\Lambda^2}. \quad (\text{VI.11})$$

It has a logarithmic singularity for small  $k_\perp$  that reflects a slow  $\sim 1/z_\perp^2$  fall-off of  $\varphi(x, z_\perp)$  for large  $z_\perp$ . The integrated TMDA that enters the form factor formula (V.5) is given by

$$\int_{k'^2_\perp \leq k_\perp^2} \Psi(x, k'_\perp) d^2k'_\perp = \varphi(x) \left[ 1 - \frac{2k_\perp}{\Lambda} K_1(2k_\perp/\Lambda) \right]. \quad (\text{VI.12})$$

It is also possible to calculate explicitly the next  $k_\perp$  integral involved there, see Eq. (VI.24) below.

For negative  $k_\perp^2 = -\kappa^2 - i\epsilon$ , the model gives

$$\begin{aligned} \psi(x, k_\perp^2 = -\kappa^2 - i\epsilon) &= \frac{2\varphi(x)}{\Lambda^2} K_0(-2i\kappa/\Lambda) \\ &= \pi \frac{\varphi(x)}{\Lambda^2} [-N_0(2\kappa/\Lambda) + iJ_0(2\kappa/\Lambda)], \end{aligned} \quad (\text{VI.13})$$

i.e. the analytic continuations of TMDA into the timelike region in this case generates an imaginary part.

*b.  $m \neq 0$  model.* The model with nonzero mass-like term

$$\Phi_m(x, \sigma) = \varphi(x) \frac{e^{i\sigma/\Lambda^2 - im^2/\sigma}}{2im\Lambda K_1(2m/\Lambda)} \quad (\text{VI.14})$$

corresponds to the function

$$\Psi_m(x, k_\perp) = \varphi(x) \frac{K_0\left(2\sqrt{k_\perp^2 + m^2}/\Lambda\right)}{\pi m\Lambda K_1(2m/\Lambda)}. \quad (\text{VI.15})$$

that is finite for  $k_\perp = 0$  in accordance with the fact that the impact parameter distribution amplitude in this case,

$$\varphi_m(x, z_\perp) = \varphi(x) \frac{K_1\left(m\sqrt{4/\Lambda^2 + z_\perp^2}\right)}{K_1(2m/\Lambda) \sqrt{1 + \Lambda^2 z_\perp^2/4}}, \quad (\text{VI.16})$$

has an exponential  $\sim e^{-m|z_\perp|}$  fall-off for large  $z_\perp$ .

## B. Modeling transition form factor by soft term

Let us now illustrate the impact of these models on the modification of the 2-body contribution to the transition form factor changed by the higher twist terms absorbed into the 2-body TMDA  $\Psi(x, k_\perp)$ .

Here we want to stress that a full analysis should also include 3-body  $\bar{\psi}G\psi$ , etc. TMDA contributions. One may expect that, compared to the 2-body term, the 3-body term will be suppressed for large  $Q^2$  by a factor of  $1/Q^2$  accompanied by a scale  $\sim \langle \bar{\psi}G\psi \rangle / \langle \bar{\psi}\psi \rangle \lesssim 1 \text{ GeV}^2$ . Thus, keeping just the 2-body TMDA is expected to be a good approximation for  $Q^2 \gtrsim 1 \text{ GeV}^2$ , however, for small  $Q^2$  the contribution of the many-body TMDAs may be of the same magnitude as the 2-body one.

### 1. Gaussian model

For the Gaussian model (VI.2), we have

$$F_G(Q^2) = \int_0^1 \frac{dx}{xQ^2} \varphi(x) \int_0^{xQ^2} \frac{dk_\perp^2}{xQ^2} \left[ 1 - e^{-k_\perp^2/\Lambda^2} \right], \quad (\text{VI.17})$$

which gives

$$F_G(Q^2) = \int_0^1 \frac{dx}{xQ^2} \varphi(x) \left[ 1 - \frac{\Lambda^2}{xQ^2} \left( 1 - e^{-xQ^2/\Lambda^2} \right) \right]. \quad (\text{VI.18})$$

If the pion DA  $\varphi$  vanishes as any positive power  $x^\alpha$  for  $x \rightarrow 0$ , the  $x$ -integral for the purely twist-2 contribution converges. For large  $Q^2$ , Eq. (VI.18) displays the power-like twist-4 contribution and the term that corresponds to contributions of “invisible” operators with twist 6 and higher. As we discussed, inclusion of virtuality corrections improves the convergence in the small- $x$  region. In particular, we obtain a finite value for  $Q^2 = 0$

$$F_G(Q^2 = 0) = \frac{f_\pi}{2\Lambda^2}, \quad (\text{VI.19})$$

where we have used the normalization condition

$$\int_0^1 \varphi(x) dx = f_\pi. \quad (\text{VI.20})$$

Restoring the overall normalization

$$F_{\gamma\gamma^* \rightarrow \pi^0}(Q^2) = \frac{\sqrt{2}}{3} F(Q^2). \quad (\text{VI.21})$$

we find that our interpolation of the 2-body TMDA contribution into small- $Q^2$  region gives

$$F_{\gamma\gamma^* \rightarrow \pi^0}^G(Q^2 = 0) = \frac{s_0}{6\Lambda^2} F_{\text{anomaly}}, \quad (\text{VI.22})$$

where  $s_0 = 4\pi^2 f_\pi^2 \approx 0.67 \text{ GeV}^2$ , and

$$F_{\text{anomaly}} = \frac{\sqrt{2}f_\pi}{s_0} \quad (\text{VI.23})$$

is the value of  $F_{\gamma\gamma^* \rightarrow \pi^0}(Q^2 = 0)$  given by the axial anomaly. If we take  $\Lambda^2 = 0.2 \text{ GeV}^2$ , the coefficient  $s_0/6\Lambda^2$  is about 0.53, i.e. the remaining 47% of the form factor magnitude for  $Q^2 = 0$  should be attributed to the many-body contributions. Note that in the light-front approach, as argued in Ref. [12], the 2-body  $\bar{q}q$  contribution produces precisely 1/2 of the  $Q^2 = 0$  magnitude, the rest coming from the many-body part. It should be emphasized, that both in the light-front approach and in our formalism, the magnitude of  $F_{\gamma\gamma^* \rightarrow \pi^0}(Q^2 = 0)$  is model-dependent. In other words, the value dictated by the anomaly is treated as a constraint on the parameters involved in the wave function or TMDA. In fact, as shown below, the  $m = 0$  non-Gaussian model gives a divergent result as  $Q^2 \rightarrow 0$ , and in this sense, one should not even try to extrapolate it to  $Q^2 = 0$ .

### 2. $m = 0$ model

Using the non-Gaussian model with  $m = 0$  (VI.8) gives

$$F(Q^2) = \int_0^1 \frac{dx}{xQ^2} \varphi(x) \left[ 1 - \frac{\Lambda^2}{xQ^2} + 2K_2(2\sqrt{x}Q/\Lambda) \right]. \quad (\text{VI.24})$$

Recall that the size of the twist-4 term is determined by the magnitude of the matrix element of the  $\bar{\psi}\gamma_5\gamma_\alpha D^2\psi$  operator. The fact that the twist-4 contribution in the expression above looks identical to that in the Gaussian model (VI.18) means that we use definitions in which the average parton virtuality in the two models is measured in the same  $\Lambda$  units.

Extracting the  $Q^2 \rightarrow 0$  limit from Eq. (VI.24), we observe that it contains logarithmically singular  $\ln(\Lambda/Q)$  terms:

$$F(Q^2) = \int_0^1 \frac{dx}{\Lambda^2} \varphi(x) \left\{ \ln \left( \frac{\Lambda}{\sqrt{x}Q} \right) + \frac{3}{4} - \gamma_E + \mathcal{O} \left( \frac{xQ^2}{\Lambda^2} \right) \right\}. \quad (\text{VI.25})$$

As established earlier, if the value of TMDA  $\Psi(x, k_\perp)$  at zero transverse momentum is finite, the  $Q^2 \rightarrow 0$  limit of  $F(Q^2)$  is given by Eq. (IV.45) that involves  $\Psi(x, k_\perp = 0)$ . In the  $m = 0$  model,  $\Psi(x, k_\perp)$  is proportional to  $K_0(2k_\perp/\Lambda)$  and is logarithmically divergent as  $k_\perp \rightarrow 0$ . Hence, a formal small- $Q^2$  expansion of Eq. (V.6) leading to Eq. (V.7) is not applicable in this case.

### 3. $m \neq 0$ model

Turning to the  $m \neq 0$  model, we have

$$F(Q^2) = \int_0^1 \frac{dx}{xQ^2} \varphi(x) \left\{ 1 - \frac{1}{xQ^2} \frac{\Lambda m K_2(2m/\Lambda)}{K_1(2m/\Lambda)} + \frac{\Lambda(xQ^2 + m^2)K_2(2\sqrt{xQ^2 + m^2}/\Lambda)}{xQ^2 m K_1(2m/\Lambda)} \right\}. \quad (\text{VI.26})$$

Again, the twist-4 power correction is explicitly displayed here. Note that the average quark virtuality understood as the scale that appears in the matrix element of the  $\bar{\psi}\gamma_5\gamma_\alpha D^2\psi$  operator now depends on the interplay of the confinement scale  $\Lambda$  and mass-scale  $m$ .

Using this expression or  $\Psi_m(x, k_\perp = 0)$  from Eq. (VI.15) we obtain

$$F_m(Q^2 = 0) = f_\pi \frac{K_0(2m/\Lambda)}{2m\Lambda K_1(2m/\Lambda)}. \quad (\text{VI.27})$$

Thus, we see again that the  $m = 0$  limit is logarithmically divergent. Writing

$$F_m(Q^2 = 0) = \frac{f_\pi}{\Lambda^2} \left[ \ln \left( \frac{\Lambda}{m} \right) - \gamma_E + \mathcal{O}(m^2/\Lambda^2) \right], \quad (\text{VI.28})$$

we may say that the size of  $F(Q^2 = 0)$  is basically set by the confinement scale  $\Lambda$ , with a coefficient logarithmically dependent on the ratio of  $\Lambda$  and the mass-type scale  $m$ .

### C. Comparing the data with soft models

Turning to comparison with experimental data, we note that the two most recent sets of data presented by BABAR [31] and BELLE [32] collaborations are inconsistent with each other for highest  $Q^2$  values. Since our goal now is just to illustrate the curves produced by various TMDA models, we consider these two sets of data separately.

In QCD, the twist-2 approximation for  $F(Q^2)$  in the leading (zeroth) order in  $\alpha_s$  is

$$F^{\text{LOpQCD}}(Q^2) = \int_0^1 \frac{dx}{xQ^2} \varphi(x). \quad (\text{VI.29})$$

Thus, in the asymptotic region, the value of

$$I(Q^2) \equiv \frac{1}{f_\pi} Q^2 F(Q^2)$$

taken from the data gives information about the shape of the pion DA. In particular, for DAs of  $\varphi_r(x) \sim (x\bar{x})^r$  type, one has  $I_r = 1 + 2/r$ , i.e.  $I^{\text{as}}(Q^2) = 3$  for the “asymptotic” wave function  $\varphi^{\text{as}}(x) = 6f_\pi x\bar{x}$ .

The most recent data [31, 32] still show a  $Q^2$  variation of  $I(Q^2)$  (see Figs. 5, 6), especially in case of BABAR data [31] which contain several points with  $I(Q^2)$  values well above 3. It was argued [33, 34] that BaBar data indicate that the pion DA is close to a flat function  $\varphi^{\text{flat}}(x) = f_\pi$  (a similar conclusion was made in Ref. [35]; an earlier motivation for a flat pion DA was made within the Nambu-Jona-Lasinio model [36]; the flat DA was discussed, in fact, as early as in 1980 [37, 38]).

The flat DA corresponds to  $r = 0$ , and pQCD gives  $I^{\text{flat}} = \infty$ . As shown in Ref. [33], inclusion of transverse momentum dependence of the pion wave function in the light-front formula of Ref. [12] (see also [23]) eliminates the divergence at  $x = 0$ , and one can produce a curve that fits the BABAR data. Similar curves may be obtained within the VDA approach described in the present paper.

In Fig. 5, we compare BABAR data with model curves corresponding to flat DA  $\varphi(x) = f_\pi$  and two types of transverse momentum distributions. First, we take the Gaussian model of Eq. (VI.18). For large  $Q^2$ , it gives (for a flat distribution)

$$F_G^{\text{flat}}(Q^2) = \frac{f_\pi}{Q^2} \left[ \ln \frac{Q^2}{\Lambda^2} - 1 + \gamma_E + \frac{\Lambda^2}{Q^2} + \dots \right]. \quad (\text{VI.30})$$

The logarithmic divergence of the pQCD formula converts here into a logarithmic increase of the  $Q^2 F(Q^2)$  combination.

A curve closely following the data is obtained for a value of  $\Lambda^2 = 0.35 \text{ GeV}^2$  which is larger than the standard estimate  $\delta^2 = 0.2 \text{ GeV}^2$  [39] for the matrix element

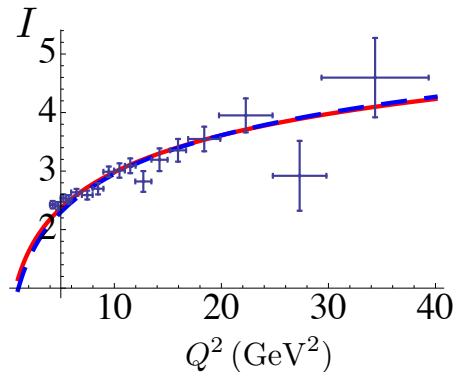


FIG. 5. BABAR data compared to model curves described in the text.

of the  $\bar{\psi}\gamma_5\gamma_\alpha D^2\psi$  operator. However, the higher-order pQCD corrections are known [40] to shrink the  $z_\perp$  width of the IDA  $\varphi(x, z_\perp)$ , effectively increasing the observed  $\Lambda^2$  compared to the primordial value of  $\Lambda^2$ . One should also take into account the correction due to the 3-body TMDA corresponding to  $\bar{\psi}G\psi$  operator generated by the insertion of the Fock-Schwinger field. Its magnitude is governed by the same scale  $\Lambda^2$  that appears in the  $\bar{\psi}D^2\psi$  operator. We plan to include this term in future studies. Our goal now is just to show that the VDA approach results in curves that are able to easily fit the data over a wide range of  $Q^2$  values.

For illustration, we also take the non-Gaussian  $m = 0$  model of Eq. (VI.24), to check what happens in case of unrealistically slow  $\sim 1/z_\perp^2$  decrease for large  $z_\perp$ . Still, if we take a larger value of  $\Lambda^2 = 0.6 \text{ GeV}^2$ , this model produces practically the same curve as the  $\Lambda^2 = 0.35 \text{ GeV}^2$  Gaussian model. The explanation is that in this case we have

$$F_{m=0}^{\text{flat}}(Q^2) = \frac{f_\pi}{Q^2} \left[ \ln \frac{Q^2}{\Lambda^2} - 1 + 2\gamma_E + \frac{\Lambda^2}{Q^2} + \dots \right] \quad (\text{VI.31})$$

for large  $Q^2$ , which, compared to Eq. (VI.30), amounts to adding the  $e^{\gamma_E} \approx 1.8$  factor in the argument of  $\ln(Q^2/\Lambda^2)$ .

Data from BELLE [32] give lower values for  $I$ , suggesting a non-flat DA. In Fig. 6, we show the curves corresponding to  $\varphi(x) \sim f_\pi(x\bar{x})^{0.4}$  DA. If we take the Gaussian model (VI.18), a good eye-ball fit to data is produced if we take  $\Lambda^2 = 0.3 \text{ GeV}^2$ . Practically the same curve is obtained in the non-Gaussian  $m = 0$  model of Eq. (VI.24) for  $\Lambda^2 = 0.4 \text{ GeV}^2$ . Again, a VDA-based analysis of the higher-order Sudakov effects [40] is needed to extract the value of  $\Lambda$  in the primordial TMDA. Note also that the curve is still well below the pQCD value  $I_{0.4} = 4.5$  for this DA.

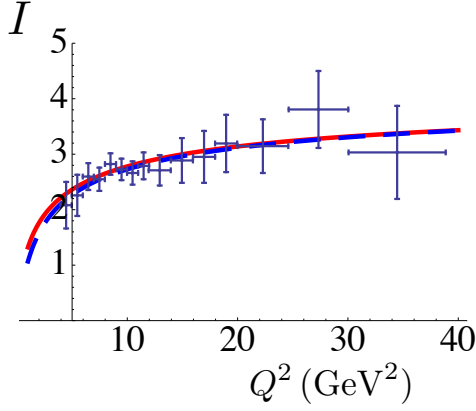


FIG. 6. BELLE data compared to model curves described in the text.

## VII. MODELING HARD TAIL

### A. Vertex models

#### 1. Hard vertex and two propagators

The simplest explicit example of a VDA-like object based on Feynman diagrams is given by a toy model in which the matrix element of the bilocal operator is given by a graph consisting of two perturbative propagators  $D^c(z_1, m)$  and  $D^c(z - z_1, m)$  ( $m$  being the relevant mass) joined at the point  $z_1$  in which the external momentum  $p$  enters (see Fig. 7). To derive the relevant VDA, let us

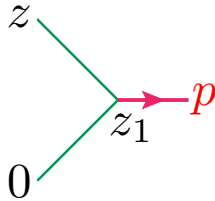


FIG. 7. Modeling VDA by a local current source.

use the momentum representation. Then we have

$$\begin{aligned} \chi(k, p) &= \frac{1}{(k^2 - m^2)[(p - k)^2 - m^2]} \\ &\equiv \int_0^\infty d\lambda \int_0^1 dx F(x, \lambda) e^{i\lambda \bar{x}k^2 + i\lambda x(k-p)^2 - i\lambda m^2 - \epsilon\lambda}, \end{aligned} \quad (\text{VII.1})$$

which gives the relation

$$e^{i\lambda x^2 p^2} F_2(x, \lambda) = \lambda e^{i\lambda x p^2 - i\lambda m^2}. \quad (\text{VII.2})$$

Using  $F(x, \lambda) = \Phi(x, 1/\lambda)$ , we find the VDA for this case

$$\Phi(x, \sigma) = \frac{1}{\sigma} e^{i(x\bar{x}p^2 - m^2)/\sigma}, \quad (\text{VII.3})$$

which yields the following result

$$\Psi(x, k_\perp) = \frac{1}{\pi} \frac{1}{k_\perp^2 + m^2 - x\bar{x}p^2} \quad (\text{VII.4})$$

for the TMDA. In the large transverse momentum region, it has a “hard” power-like  $\sim 1/k_\perp^2$  behavior. In the impact parameter space, we have

$$\varphi(x, z_\perp) = 2K_0(mz_\perp),$$

a function that has a logarithmic singularity for  $z_\perp = 0$ . This feature explains why, formally integrating  $\Psi(x, k_\perp)$  over  $d^2k_\perp$  to produce DA, one faces in this case a logarithmic divergence.

#### 2. Soft vertex and propagators

Instead of a point current, we can use a soft vertex, i.e. consider a model (see Fig. 8a)

$$\chi_2(k, p) = \frac{\chi_0(k, p)}{k^2(p - k)^2}, \quad (\text{VII.5})$$

where, for simplicity, we took massless propagators. Also, we will take  $p^2 = 0$  in this model. To proceed, we write the soft vertex in the VDA representation as

$$\chi_0(k, p) = \int_0^\infty d\alpha \int_0^1 dy F_0(y, \alpha) e^{i\alpha(k-yp)^2 - \epsilon\alpha}. \quad (\text{VII.6})$$

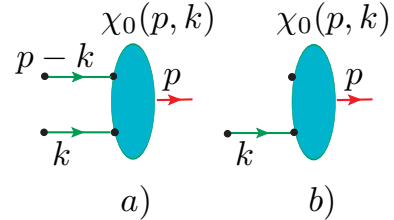


FIG. 8. Attaching propagators to a soft vertex.

*a. One perturbative propagator.* Consider first the case when just one propagator is added to the soft vertex (see Fig. 8b),

$$\chi_1(k, p) = -\frac{\chi_0(k, p)}{k^2}. \quad (\text{VII.7})$$

Then

$$\begin{aligned} \chi_1(k, p) &= i \int_0^1 dy \int_0^\infty d\alpha \int_0^\infty d\alpha_1 \\ &\times F_0(y, \alpha) e^{i\alpha(k-yp)^2 + i\alpha_1 k^2} \\ &= i \int_0^1 dy \int_0^\infty \lambda d\lambda \int_0^1 d\beta e^{i\lambda(k-\beta yp)^2} F_0(y, \beta\lambda), \end{aligned} \quad (\text{VII.8})$$

which gives

$$F_1(x, \alpha) = i\alpha \int_0^1 dy \int_0^1 d\beta \delta(x - \beta y) F_0(y, \beta\alpha) \quad (\text{VII.9})$$

or

$$F_1(x, \alpha) = i\alpha \int_x^1 \frac{dy}{y} F_0(y, \alpha x/y). \quad (\text{VII.10})$$

For the TMDA we have

$$\psi_1(x, k_\perp^2) = -\frac{\partial}{\partial k_\perp^2} \int_x^1 \frac{dy}{y} \psi_0(y, yk_\perp^2/x). \quad (\text{VII.11})$$

To illustrate the impact of adding a perturbative leg, let us take the simplest model when the soft distribution has no  $x$ -dependence (i.e. is “flat”),  $\psi_0(y, k_\perp^2) = \psi_0^F(y, k_\perp^2) \equiv \psi_0(k_\perp^2)$ . Despite the flatness of the soft distribution, we obtain a function

$$\psi_1^F(x, k_\perp^2) = \frac{\psi_0(k_\perp^2) - \psi_0(k_\perp^2/x)}{k_\perp^2} \quad (\text{VII.12})$$

with a nontrivial  $x$ -profile. For large  $k_\perp^2$ , and a fast-decreasing soft function  $\psi_0(k_\perp^2)$ , one can use a naive approximation  $\psi_1^F(x, k_\perp^2) \approx \psi_0(k_\perp^2)/k_\perp^2$  for some range of  $x$ -values that are not very close to 1. But eventually  $\psi_1^F(x, k_\perp^2)$  vanishes for  $x = 1$ , i.e. the parton corresponding to the “hard”  $1/k^2$  propagator cannot carry the whole momentum of the hadron, even though the soft vertex allowed this, and the purely soft  $(p - k)$  parton still can carry the  $\bar{x} = 1$  fraction.

*b. Two perturbative propagators.* Switching now to the soft vertex model with two perturbative propagators attached (VII.5), we obtain

$$\begin{aligned} F_1(x, \alpha) &= -\alpha^2 \int_0^1 dy \int_0^1 d\beta \int_0^{1-\beta} d\beta_2 F_0(y, \beta\alpha) \\ &\quad \times \delta(x - (\beta_2 + \beta)y) \\ &= -\alpha^2 \int_0^1 dy \int_0^{V_0(x,y)} F_0(y, \beta\alpha) d\beta, \end{aligned} \quad (\text{VII.13})$$

where

$$V_0(x, y) = \frac{x}{y} \theta(x < y) + \frac{\bar{x}}{y} \theta(x > y). \quad (\text{VII.14})$$

One may recognize in this function a part of the ERBL evolution kernel [4, 8]; we will turn to this point later. For TMDA in this case we have

$$\begin{aligned} \psi_2(x, k_\perp^2) &= \left( \frac{\partial}{\partial k_\perp^2} \right)^2 \int_0^1 dy \int_0^{V_0(x,y)} d\beta \psi_0(y, k_\perp^2/\beta) \\ &= -\frac{1}{k_\perp^2} \frac{\partial}{\partial k_\perp^2} \int_0^1 dy V_0(x, y) \psi_0(y, k_\perp^2/V_0(x, y)). \end{aligned} \quad (\text{VII.15})$$

For illustration, taking again a flat model  $\psi_0(y, k_\perp^2) = \psi_0(k_\perp^2)$ , we obtain

$$\psi_2^F(x, k_\perp^2) = \frac{\psi_0(k_\perp^2) - x\psi_0(k_\perp^2/x) - \bar{x}\psi_0(k_\perp^2/\bar{x})}{k_\perp^4}. \quad (\text{VII.16})$$

Thus, for large  $k_\perp^2$  and a fast-decreasing soft function  $\psi_0(k_\perp^2)$ , one can use a naive approximation  $\psi_2^F(x, k_\perp^2) \approx \psi_0(k_\perp^2)/k_\perp^4$  for some range of  $x$ -values that are not very close to 0 or 1. However, for non-zero  $k_\perp^2$ , the function  $\psi_1^F(x, k_\perp^2)$  vanishes both for  $x = 0$  and  $x = 1$ , i.e. neither parton can carry the whole momentum of the hadron.

For small  $k_\perp^2$ , we have

$$\psi_2^F(x, k_\perp^2 \rightarrow 0) = -\frac{1}{k_\perp^2} \psi_0'(k_\perp^2) + \dots, \quad (\text{VII.17})$$

i.e., the function  $\psi_2^F(x, k_\perp^2)$  behaves like  $\psi_0'(0)/k_\perp^2$  for small  $k_\perp^2$ , showing no dependence on  $x$  except for narrow regions of  $x$  close to 0 or 1.

### 3. Equations of motion

A situation, when one  $\chi$  function differs from another one by a perturbative propagator is encountered when one considers equations of motion. In particular, we have  $\partial^2 \phi = g\chi\phi$  in the case of  $g\phi^2\chi$  interaction of quarks  $\phi$  with gluons  $\chi$ . This imposes a relation

$$\partial^2 \langle p | \phi(0) \phi(z) | 0 \rangle = \langle p | \phi(0) g\chi(z) \phi(z) | 0 \rangle \quad (\text{VII.18})$$

between 2-body and 3-body matrix elements, which should be also satisfied by their VDA representations. In the momentum representation, the equation of motion imposes the restriction

$$-k^2 \chi(k, p) = \chi_1(k, p), \quad (\text{VII.19})$$

connecting the 2-body amplitude  $\chi(k, p)$  and the reduced 3-body amplitude  $\chi_1(k, p)$  in which the gluon field  $\chi$  is located at the same point with one of the  $\phi$  fields. Hence,  $\chi(k, p) = -\chi_1(k, p)/k^2$ , and we can use Eq. (VII.10) to write a relation

$$\Phi(x, \sigma) = \frac{i}{\sigma} \int_x^1 \frac{dy}{y} \Phi_1(y, \sigma y/x) \quad (\text{VII.20})$$

between the 2-body VDA  $\Phi(x, \sigma)$  corresponding to  $\chi(k, p)$  and the reduced 3-body VDA  $\Phi_1(x, \sigma)$  corresponding to  $\chi_1(k, p)$ .

However, for the purposes of VDA model-building, Eq. (VII.20) is not convenient since the basic 2-body VDA  $\Phi(x, \sigma)$  looks like generated from the 3-body VDA  $\Phi_1(x, \sigma)$  describing the matrix element  $\langle p | \phi(z_1) g\chi(z_3) \phi(z_2) | 0 \rangle$  in the  $z_3 \rightarrow z_1$  limit.

We would rather prefer to start with some model form for the 2-body VDA  $\Phi(x, \sigma)$  and then treat Eq. (VII.20)



as a constraint on the full 3-body VDA. Thus, we need an inverse relation in which  $\Phi_1(x, \sigma)$  is written in terms of  $\Phi(x, \sigma)$ . To this end, let us apply  $\partial^2$  to the VDA parametrization (II.19) of the 2-body matrix element:

$$\partial^2 \langle p | \phi(0) \phi(z) | 0 \rangle = \int_0^\infty d\sigma \int_0^1 dx \Phi(x, \sigma) e^{ix(pz) - i\sigma z^2/4} \times [x(pz)\sigma - \sigma^2 z^2/4 - 2i\sigma] \quad (\text{VII.21})$$

(we take  $p^2 = 0$  and  $z^2 \rightarrow z^2 - i\epsilon$  is implied here and below). Integrating the first term by parts over  $x$  assuming  $\Phi(0, \sigma) = \Phi(1, \sigma) = 0$ , while the second one over  $\sigma$  assuming  $\sigma^2 \Phi(x, \sigma) e^{-\epsilon\sigma} |_{\sigma \rightarrow \infty} = 0$ ,  $\sigma^2 \Phi(x, \sigma) |_{\sigma=0} = 0$ , we obtain

$$\partial^2 \langle p | \phi(0) \phi(z) | 0 \rangle = i \int_0^\infty d\sigma \int_0^1 dx e^{ix(pz) - i\sigma z^2/4} \times \left[ x \frac{\partial}{\partial x} + \sigma \frac{\partial}{\partial \sigma} \right] \sigma \Phi(x, \sigma) \quad (\text{VII.22})$$

Parametrizing  $\langle p | \phi(0) g \chi(z) \phi(z) | 0 \rangle$  by a VDA  $\Phi_1(x, \sigma)$  gives

$$\Phi_1(x, \sigma) = i \left[ x \frac{\partial}{\partial x} + \sigma \frac{\partial}{\partial \sigma} \right] \sigma \Phi(x, \sigma). \quad (\text{VII.23})$$

As a check, using Eq. (VII.23) in the right-hand side of Eq. (VII.20), one can verify that the outcome is indeed given by  $\Phi_1(x, \sigma)$ . Now, Eq. (VII.23) can be converted into a relation

$$\psi_1(x, k_\perp^2) = k_\perp^2 \psi(x, k_\perp^2) - x \frac{\partial}{\partial x} \int_{k_\perp^2}^\infty dk_\perp^2 \psi(x, k_\perp^2) \quad (\text{VII.24})$$

between TMDAs. As one might expect, the action of  $\partial^2$  on the 2-body function has resulted in a term containing the  $k_\perp^2$  factor. For the distribution amplitude

$$\psi_1(x) \equiv \int_0^\infty dk_\perp^2 \psi_1(x, k_\perp^2) \quad (\text{VII.25})$$

this gives

$$\psi_1(x) = \left[ 1 - x \frac{\partial}{\partial x} \right] \varphi^{(1)}(x), \quad (\text{VII.26})$$

where  $\varphi^{(1)}(x)$  is the  $k_\perp^2$  moment of the TMDA  $\psi(x, k_\perp^2)$ ,

$$\varphi^{(1)}(x) \equiv \int_0^\infty dk_\perp^2 k_\perp^2 \psi(x, k_\perp^2). \quad (\text{VII.27})$$

Note, that  $\psi_1(x)$  is not necessarily  $x \leftrightarrow \bar{x}$  symmetric even if  $\varphi_1(x)$  is. This is natural, since the fraction  $x$  in  $\psi_1(x, \sigma)$  corresponds to a “glued” field  $g\chi(z_1)\phi(z_1)$ , while the fraction  $\bar{x}$  is associated with a single field  $\phi(z_2)$ . To see the implications of the  $x \leftrightarrow \bar{x}$  symmetry on the VDA level, we apply the equation of motion to the second field in the matrix element  $\langle p | \phi(z_1) \phi(z_2) | 0 \rangle$ , which gives

$$\Phi_1(\bar{x}, \sigma) = i \left[ \bar{x} \frac{\partial}{\partial \bar{x}} + \sigma \frac{\partial}{\partial \sigma} \right] \sigma \Phi(x, \sigma). \quad (\text{VII.28})$$

Since  $\Phi(x, \sigma) = \Phi(\bar{x}, \sigma)$ , this equation is consistent with Eq. (VII.23). Still,  $\Phi_1(x, \sigma) \neq \Phi_1(\bar{x}, \sigma)$  in general.

## B. Generating hard tail from an initially soft VDA

### 1. Scalar fields

Our next model (see Fig. 9) involves two currents carrying momenta  $xp$  and  $(1-x)p \equiv \bar{x}p$  at locations  $z$  and  $0$ , respectively, connected by a perturbative propagator  $D^c(z, m)$ , and weighted with a function  $\varphi(x)$ . Then we

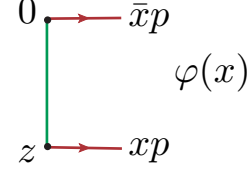


FIG. 9. Modeling VDA by a two-current state.

have

$$t_1(z, p) = g^2 \int_0^1 dx e^{ix(pz)} D^c(z, m) \varphi(x) \quad (\text{VII.29})$$

where  $g$  is the coupling constant. Using

$$D^c(z, m) = \frac{1}{(4\pi)^2} \int_0^\infty d\sigma e^{-i\sigma z^2/4 - im^2/\sigma} \quad (\text{VII.30})$$

we obtain

$$\Phi(x, \sigma) = \frac{\varphi(x)}{(4\pi)^2} e^{-im^2/\sigma}. \quad (\text{VII.31})$$

However, the integral for TMDA,

$$\Psi(x, k_\perp) = \frac{\varphi(x)}{(4\pi)^2} \int_0^\infty \frac{d\sigma}{\sigma} e^{-i(k_\perp^2 + m^2)/\sigma}, \quad (\text{VII.32})$$

in this case diverges at large  $\sigma$ . So, let us take a nontrivial primordial VDA  $\Phi_0(x, \sigma_0)$  instead of the above model corresponding to  $\varphi(x)\delta(\sigma_0)$ . Then

$$t_1(z, p) = g^2 D^c(z, m) \int_0^1 dx e^{ix(pz)} \times \int_0^\infty d\sigma_0 e^{-i\sigma_0 z^2/4} \Phi_0(x, \sigma_0) \quad (\text{VII.33})$$

and

$$\Phi_1(x, \sigma) = \frac{g^2}{(4\pi)^2} \sigma \int_0^1 d\beta e^{-im^2/\beta\sigma} \Phi_0(x, \beta\sigma), \quad (\text{VII.34})$$

which gives

$$\psi_1(x, k_\perp^2) = \frac{g^2}{(4\pi)^2} \int_{k_\perp^2}^\infty dk_\perp^2 \int_0^1 d\beta \psi_0(x, \beta(k_\perp^2 + m^2)) \quad (\text{VII.35})$$

or

$$\psi_1(x, k_\perp^2) = \frac{g^2}{(4\pi)^2} \int_{k_\perp^2}^\infty \frac{d\kappa^2}{\kappa^2 + m^2} \int_0^{\kappa^2 + m^2} d\kappa'^2 \psi_0(x, \kappa'^2) . \quad (\text{VII.36})$$

The next step is to take a model with two perturbative propagators attached. In other words, consider a model with vertices at locations  $z_1$  and  $z_2$  connected by a perturbative propagator  $D^c(z_1 - z_2, m)$  and two perturbative propagators connecting these points with points 0 and  $z$  (see Fig. 10).

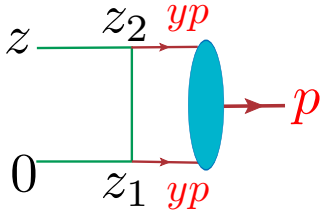


FIG. 10. Getting hard tail from a soft initial distribution.

Then we can use the formula (VII.15)

$$\psi_2(x, k_\perp^2) = -\frac{1}{k_\perp^2 + m^2} \frac{\partial}{\partial k_\perp^2} \int_0^1 dy \times \psi_1(y, (k_\perp^2 + m^2)/V_0(x, y)) \quad (\text{VII.37})$$

with  $\psi_1$  given by Eq. (VII.35). This gives

$$\psi_2(x, k_\perp) = \frac{g^2}{16\pi^2} \frac{1}{(k_\perp^2 + m^2)^2} \int_0^1 dy V_0(x, y) \times \left[ \int_0^{(k_\perp^2 + m^2)/V_0(x, y)} \psi_0(y, k_\perp'^2) dk_\perp'^2 \right] . \quad (\text{VII.38})$$

The part in square brackets may be written as

$$\left[ \dots \right] = \varphi_0(y) - \int_{(k_\perp^2 + m^2)/V_0(x, y)}^\infty \psi_0(y, k_\perp'^2) dk_\perp'^2 , \quad (\text{VII.39})$$

where  $\varphi_0(y)$  is the primordial distribution  $\Psi_0(y, k_\perp')$  integrated over all the transverse momentum plane. Hence, for large  $k_\perp$ , the leading  $1/k_\perp^4$  term is determined by the DA  $\varphi_0(y)$  only. A particular shape of the  $k_\perp$ -dependence of the soft TMDA  $\Psi_0(y, k_\perp)$  affects only the subleading  $\sim [V \otimes \psi_0](x, k_\perp^2)/k_\perp^2$  term. The form of  $k_\perp$  dependence of  $\Psi_0(y, k_\perp)$  is also essential for the behavior of  $\Psi^{B_0}(x, k_\perp)$  term at small  $k_\perp$ . In particular, we have

$$\left[ \dots \right]_{k_\perp=0} = \psi_0(y, k_\perp^2 = 0) , \quad (\text{VII.40})$$

which gives, e.g.,  $\varphi_0(y)/\Lambda^2$  in the Gaussian model (VI.2).

## 2. Spin-1/2 quarks and scalar gluons

In case of spin-1/2 quarks interacting *via* a (pseudo)scalar or vector gluon field (in Feynman gauge), the factor coming from  $k$  and  $p - k$  legs is given by

$$\frac{\text{Tr}\{\gamma_\alpha \not{k} \not{p} (\not{p} - \not{k})\}}{k^2(p - k)^2} . \quad (\text{VII.41})$$

Using  $\not{p} = (\not{p} - \not{k}) + \not{k}$ , we arrive at

$$\frac{\text{Tr}\{\gamma_\alpha \not{k}\}}{k^2} + \frac{\text{Tr}\{\gamma_\alpha (\not{p} - \not{k})\}}{(p - k)^2} . \quad (\text{VII.42})$$

Representing  $\not{k} = x\not{p} + (\not{k} - x\not{p})$  and noticing that  $(\not{k} - x\not{p})$  results in  $\not{z}$  in coordinate representation, we may treat the equation above as

$$\left( \frac{x}{k^2} + \frac{1-x}{(p-k)^2} \right) \text{Tr}\{\gamma_\alpha \not{p}\} + \mathcal{O}(\not{z}) . \quad (\text{VII.43})$$

Now, for the first term we need to find  $\Phi_1^{B_0}(x, 1/\alpha)$  satisfying

$$\int_0^\infty d\alpha \int_0^1 dx \frac{1}{x} \Phi_1^{B_0}(x, 1/\alpha) e^{i\alpha(k-xp)^2} = \frac{\pi^2 g^2}{k^2} \int_0^\infty \frac{d\alpha_2}{\alpha_2} \int_0^1 dy \int_0^1 d\xi \Phi_0(y, \xi/\alpha_2) e^{i\alpha_2(k-y p)^2} . \quad (\text{VII.44})$$

This gives (switching  $1/\alpha = \sigma$ )

$$\Phi_1^{B_0}(x, \sigma) = \alpha_g \int_x^1 dy \int_0^1 d\xi \Phi_0\left(y, \frac{\xi\sigma}{V_1(x, y)}\right) , \quad (\text{VII.45})$$

where  $V_1(x, y) = (x/y)\theta(x \leq y)$  and  $\alpha_g = g^2/(16\pi^2)$ . The second term in Eq. (VII.43) gives a similar contribution, with  $V_1(x, y) \rightarrow V_2(x, y) \equiv (\bar{x}/\bar{y})\theta(y \leq x)$ . As a result, the total contribution for hard TMDA generated in case of spinor quarks is given by

$$\Psi_{1/2}^{B_0}(x, k_\perp) = \frac{\alpha_g}{\pi} \int_0^1 dy \int_0^1 d\xi \psi_0\left(y, \frac{\xi k_\perp^2}{V_0(x, y)}\right) . \quad (\text{VII.46})$$

It differs from the scalar expression (VII.38) just by the absence of the overall  $1/k_\perp^2$  factor. One may also write

$$\Psi_{1/2}^{B_0}(x, k_\perp) = \frac{\alpha_g}{\pi k_\perp^2} \int_0^1 dy V_0(x, y) \times \int_0^{k_\perp^2/V_0(x, y)} \psi_0(y, k_\perp'^2) dk_\perp'^2 \quad (\text{VII.47})$$

which corresponds to  $1/(k_\perp^2 + m^2)$  dependence in the correction (VII.38) to the TMDA. It produces a logarithmically divergent term for DA, corresponding to a logarithmic evolution of DA's in such models.

For a soft TMDA  $\psi_0(x, k_\perp^2)$ , the integral over all  $k_\perp^2$  produces the initial distribution amplitude  $\varphi_0(x)$ , so we have

$$\Psi_{1/2}^{B_0}(x, k_\perp) = \frac{\alpha_g}{\pi k_\perp^2} \int_0^1 dy V_0(x, y) [\varphi_0(y) - \delta\varphi] \quad (\text{VII.48})$$

where the correction term

$$\delta\varphi = \int_{k_\perp^2/V_0(x, y)}^\infty \psi_0(y, k_\perp'^2) dk_\perp'^2, \quad (\text{VII.49})$$

quickly vanishes for large  $k_\perp^2$ . For an illustration, take a factorized Ansatz for the initial TMDA,  $\psi_0(x, k_\perp^2) = \varphi_0(x) K(k_\perp^2)$ . Then we have

$$\delta\varphi = \varphi_0(y) \int_{k_\perp^2/V_0(x, y)}^\infty K(k_\perp'^2) dk_\perp'^2, \quad (\text{VII.50})$$

which gives

$$\delta\varphi = \varphi_0(y) e^{-k_\perp^2/V_0(x, y)\Lambda^2} \quad (\text{VII.51})$$

for a Gaussian form  $K(k_\perp^2) = e^{-k_\perp^2/\Lambda^2}/\Lambda^2$ , and

$$\delta\varphi = \varphi_0(y) \frac{2k_\perp K_1(2k_\perp/\Lambda\sqrt{V_0(x, y)})}{\Lambda\sqrt{V_0(x, y)}} \quad (\text{VII.52})$$

for the  $m = 0$  model when  $K(k_\perp^2) = 2K_0(2k_\perp/\Lambda)/\Lambda^2$ .

### 3. Evolution in the impact parameter space

For initially collinear quarks, we have  $\varphi^{\text{conv}}(x, z_\perp) \sim K_0(mz_\perp) \delta\varphi(x)$  in the impact parameter space. The logarithmic divergence for  $z_\perp = 0$  of this outcome corresponds to evolution of the DA. In the  $B_0$  model, we have (switching to  $\varphi(x, z_\perp) \rightarrow \varphi(x, z_\perp^2)$  in our notations below)

$$\begin{aligned} \varphi_Y^{B_0}(x, z_\perp^2) &= \alpha_g \int_0^1 dy V_0(x, y) \\ &\times \int_1^\infty \frac{d\nu}{\nu} \varphi_0\left(y, \nu z_\perp^2 V_0(x, y)\right). \end{aligned} \quad (\text{VII.53})$$

Substituting formally  $\varphi_0(y, z_\perp^2)$  by  $\varphi_0(y)$  in the  $z_\perp^2 \rightarrow 0$  limit, we get a logarithmically divergent integral over  $\nu$ . However, for a function  $\varphi_0(y, z_\perp^2)$  that rapidly decreases when  $z_\perp^2 \gtrsim 1/\Lambda^2$ , one gets  $\ln(z_\perp^2 \Lambda^2)$  as a factor accompanying the convolution of  $V_0(x, y)$  and  $\varphi_0(y)$ . Hence, the pion size cut-off contained in the primordial distribution provides the scale in  $\log(z_\perp^2)$ , and we may keep the hard quark propagators massless.

For scalar gluons, this cut-off also results in a finite value of  $\Psi_Y^{B_0}$  in the formal  $k_\perp \rightarrow 0$  limit:

$$\Psi_Y^{B_0}(x, k_\perp = 0) = \alpha_g \int_0^1 dy \Psi_0(y, k_\perp = 0). \quad (\text{VII.54})$$

Thus, the  $\Psi_Y^{\text{conv}}(x, k_\perp) \sim 1/k_\perp^2$  singularity of the ‘‘collinear model’’  $\Psi_0(y, k_\perp) = \varphi_0(y) \delta(k_\perp^2)/\pi$  converts into a constant  $\sim 1/\Lambda^2$  in the Gaussian model. Note also that the overall factor in Eq. (VII.54) then contains the  $x$ -independent integral of  $\varphi_0(y)$ , i.e.  $f_\pi$ , rather than the convolution  $\delta\varphi(x)$  as in Eq. (VII.38).

It should be emphasized that the VDA approach provides an unambiguous prescription of generating hard-tail terms like  $\Psi^{B_0}(x, k_\perp)$  from a soft primordial distribution  $\Psi_0(y, k_\perp)$ .

### 4. Particular choices of the soft $B_0$

Assuming a factorized Ansatz for the initial IDA  $\varphi_0(y, z_\perp^2) = \varphi_0(y) Z_0(z_\perp^2 \Lambda^2)$ , we have

$$\varphi_Y^{B_0}(x, z_\perp^2) = \alpha_g \int_0^1 dy V_0(x, y) \varphi_0(y) Z_1[\zeta^2 V_0(x, y)] \quad (\text{VII.55})$$

where

$$Z_1(\zeta^2) \equiv \int_1^\infty \frac{d\nu}{\nu} Z_0(\nu\zeta^2), \quad (\text{VII.56})$$

and we can study the sensitivity of  $\varphi^{B_0}(x, z_\perp^2)$  to a particular choice of the soft factor  $Z_0(\zeta^2)$ . In particular, in a Gaussian model,  $Z_0^{\text{exp}}(\zeta^2) = e^{-\zeta^2}$ , we have

$$Z_1^{\text{exp}}(\zeta^2) = \Gamma[0, \zeta^2] = -\ln(\zeta^2) - \gamma_E + \mathcal{O}(\zeta^2), \quad (\text{VII.57})$$

where we have explicitly displayed the small- $\zeta^2$  behavior to extract the  $\sim \ln z_\perp^2$  evolution term, which gives

$$\begin{aligned} \varphi_{\text{exp}}^{B_0}(x, z_\perp^2) &= \alpha_g \left[ \ln\left(\frac{e^{-\gamma_E}}{z_\perp^2 \Lambda_{\text{exp}}^2}\right) \int_0^1 dy V_0(x, y) \varphi_0(y) \right. \\ &\quad \left. + \int_0^1 dy V_0(x, y) \ln(V_0(x, y)) \varphi_0(y) \right] + \mathcal{O}(z_\perp^2 \Lambda_{\text{exp}}^2). \end{aligned} \quad (\text{VII.58})$$

In Fig. 11 we show the  $z_\perp$ -dependence of the the function  $\varphi_Y^{B_0}(x, z_\perp^2)$  for particular value  $x = 1/2$ . For comparison, we also show the original soft function  $\varphi_0(x = 1/2, z_\perp^2)$  and the purely logarithmic part of  $\varphi_Y^{B_0}(x = 1/2, z_\perp^2)$  given by  $(\ln 2) \ln(4/z_\perp^2 \Lambda^2)$ .

In Fig. 12, assuming the value  $\alpha_g = 0.2$  and using the same model, we illustrate the effect of adding the hard part to the initially soft TMDA.

Similarly, for a slow-decrease Ansatz  $Z_{\text{slow}}(\zeta^2) = 1/(1 + \zeta^2)$ , we have

$$Z_1^{\text{slow}}(\zeta^2) = \ln(1/\zeta^2 + 1) = -\ln \zeta^2 + \mathcal{O}(\zeta^2), \quad (\text{VII.59})$$

so that the major change for small  $z_\perp^2$  is the absence of the  $\gamma_E$  term, which amounts to a change of the evolution scale. The evolution terms of the two Ansätze coincide when  $\Lambda_{\text{slow}}^2 = e^{\gamma_E} \Lambda_{\text{exp}}^2$ .

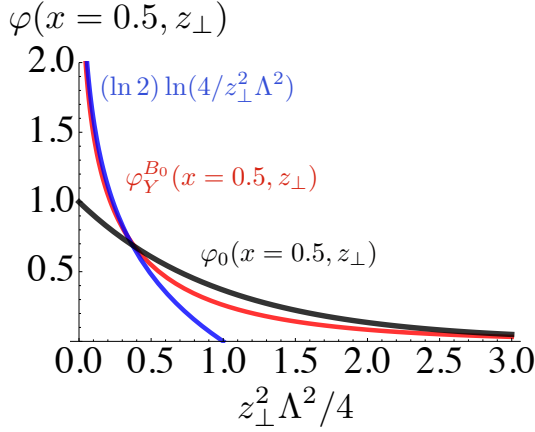


FIG. 11.  $z_\perp^2$ -dependence of the hard part (red online) for the Gaussian model with flat initial DA  $\varphi_0(x, z_\perp) = \exp[-z_\perp^2 \Lambda^2/4]$  (black online), in comparison with purely logarithmic part of the hard part (red online).

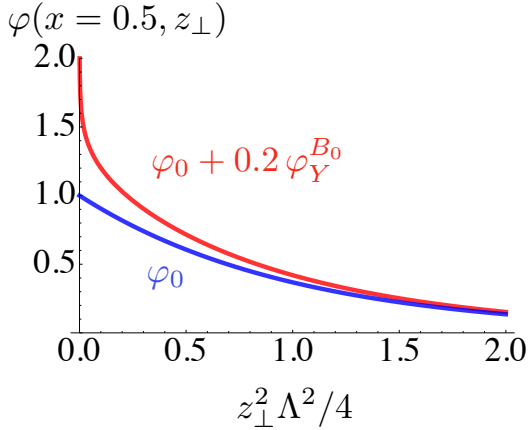


FIG. 12. Effect of adding the hard part to initially soft TMDA  $\varphi_0(x, z_\perp) = \exp[-z_\perp^2 \Lambda^2/4]$ .

In fact, the approximate  $\ln z_\perp^2 \Lambda^2$  form for the evolution factor is inconvenient, because it changes sign for  $z_\perp = 1/\Lambda$  and tends to infinity for large  $z_\perp^2$ , while the original expressions for  $Z_1(\zeta^2)$  are positive-definite and vanish in the  $\zeta \rightarrow \infty$  limit. To avoid this artifact of the small- $\zeta^2$  expansion for  $Z_1(\zeta^2)$ , one may represent, e.g.,

$$Z_1[\zeta^2 V_0(x, y)] = Z_1(\zeta^2) + [Z_1(\zeta^2 V_0(x, y)) - Z_1(\zeta^2)] .$$

Then  $Z_1(\zeta^2)$  gives a positive-definite evolution factor with a correct  $-\ln(\zeta^2)$  behavior for small  $\zeta^2$  and vanishes for large  $\zeta^2$ . The remainder is finite for  $\zeta^2 = 0$  and also vanishes for large  $\zeta^2$ . Since the logarithmic  $\ln(\zeta^2)$  part of all  $Z_1(\zeta^2)$ 's is universal, while the rest depends on the shape of  $Z_0(\zeta^2)$ , it make sense to declare one of  $Z_1(\zeta^2)$ 's to be a standard one, i.e. to represent

$$Z_1[\zeta^2 V_0(x, y)] = Z_1^{\text{stand}}(\zeta^2) + [Z_1(\zeta^2 V_0(x, y)) - Z_1^{\text{stand}}(\zeta^2)] ,$$

thus making the evolution part universal. Possible

choices for  $Z_1^{\text{stand}}(\zeta^2)$  are  $Z_1^{\text{exp}}(\zeta^2) = \Gamma[0, \zeta^2]$ , and  $Z_1^{\text{slow}}(\zeta^2) = \ln[1/\zeta^2 + 1]$ .

### 5. Ultraviolet-related addition to hard tail

Since the axial current is conserved, the full evolution kernel for the pion DA should have a plus-prescription form with respect to  $x$ :  $V_0(x, y) \rightarrow [V_0(x, y)]_+$ . In fact, calculating the  $\gamma^* \gamma \rightarrow \pi^0$  amplitude at  $\alpha_g$  order, we should include the one-loop vertex and self-energy corrections to the hard quark propagator  $S^c(z)$ . They produce, in particular, a factor of  $\ln(z^2)$  multiplying  $S^c(z)$ . We may treat it as a multiplicative modification of VDA convoluted with the original propagator  $S^c(z)$ . This corresponds to adding the  $\sim \alpha_g \ln(z_\perp^2) \varphi_0(x, z_\perp^2)$  term to  $\varphi_Y^{B0}(x, z_\perp^2)$  as another  $\mathcal{O}(\alpha_g)$  correction to IDA.

Since this term comes from an ultraviolet divergent contribution, we need to decide which UV renormalization prescription to use. While the  $\sim \ln(z_\perp^2)$  behavior for small  $z_\perp$  is not affected by the choice, the large  $z_\perp$  behavior depends on it and may be adjusted. It is convenient to take the Bessel function form  $K_0(z_\perp \mu)$ . Then the correction vanishes for large  $z_\perp$ , and never changes sign. Furthermore, its Fourier transform has a simple  $1/(k_\perp^2 + \mu^2)$  form that is finite for  $k_\perp = 0$ . Incorporating these considerations we fix the UV-related correction for IDA to be given by

$$\varphi_Y^{B0, \text{UV}}(x, z_\perp^2) = -\alpha_g K_0(z_\perp \mu) \varphi_0(x, z_\perp^2) . \quad (\text{VII.60})$$

For TMDA, this term gives

$$\Psi^{B0, \text{UV}}(x, k_\perp; \mu) = -\frac{\alpha_g}{2\pi} \int d^2 k'_\perp \frac{\Psi_0(x, k'_\perp)}{(k_\perp - k'_\perp)^2 + \mu^2} . \quad (\text{VII.61})$$

In actual calculations, it is convenient to use the formula

$$\psi^{B0, \text{UV}}(x, k_\perp^2; \mu) = -\frac{\alpha_g}{2} \int_0^1 \frac{d\xi}{1-\xi} \times \psi_0\left(x, \xi(k_\perp^2 + \mu^2/(1-\xi))\right) . \quad (\text{VII.62})$$

The leading large- $k_\perp$  term then comes from the  $\xi \sim 1/k_\perp^2$  region of integration.

Adding (VII.53) and self-energy part (with  $\mu = \Lambda/2$  and Bessel form for log singularity) gives

$$\delta\varphi_Y(x, z_\perp^2) = \alpha_g \left[ \int_0^1 dy V_0(x, y) \int_1^\infty \frac{d\nu}{\nu} \times \varphi_0\left(y, \nu z_\perp^2 V_0(x, y)\right) - K_0(z_\perp \Lambda/2) \varphi_0(x, z_\perp^2) \right] . \quad (\text{VII.63})$$

In Fig. 13 we show total IDA  $\varphi(x, z_\perp^2) = \varphi_0(x, z_\perp^2) + \delta\varphi_Y(x, z_\perp^2)$  taking for the soft IDA a Gaussian model for the  $z_\perp$ -dependence and flat DA  $\varphi_0(x) = 1$ . One can see

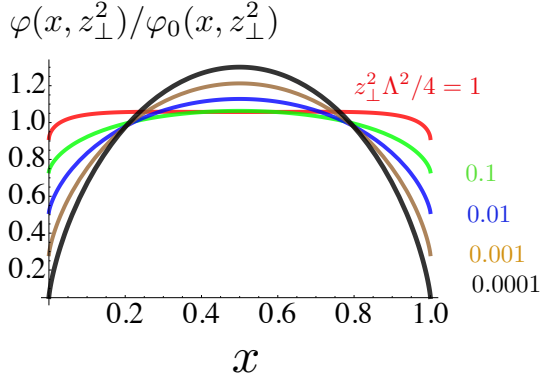


FIG. 13.  $z_\perp^2$ -evolution of the IDA shape in a Gaussian model with initially flat DA  $\varphi_0(x, z_\perp^2) = e^{-z_\perp^2/4\Lambda^2}$  and  $\alpha_g = 0.2$ .

the change of the  $x$ -profile from a flat form for large  $z_\perp$  to asymptotic  $\sim x\bar{x}$  for small  $z_\perp$ .

If the function  $\Psi_0(x, k'_\perp)$  rapidly decreases with growing  $k'^2_\perp$ , then the leading contribution for large  $k^2_\perp$  is obtained from the region of small  $k'_\perp$  which gives

$$\psi^{B_0, UV}(x, k_\perp; \mu) = -\frac{\alpha_g}{2} \frac{\varphi_0(x)}{k^2_\perp} + \dots \quad (\text{VII.64})$$

for large  $k^2_\perp$ . In the formal  $k_\perp \rightarrow 0$  limit, we have a finite result

$$\psi^{B_0, UV}(x, k^2_\perp = 0; \mu) = -\frac{\alpha_g}{2} \int_0^\infty \frac{\psi_0(x, k'^2_\perp)}{k'^2_\perp + \mu^2} d(k'^2_\perp). \quad (\text{VII.65})$$

#### 6. Hard tail contribution to the transition amplitude

The integral giving the transition form factor

$$F(Q^2) = \int_0^1 \frac{dx}{xQ^2} \int_{k_\perp^2 \leq xQ^2} \Psi(x, k_\perp) \left[1 - \frac{k_\perp^2}{xQ^2}\right] d^2k_\perp \quad (\text{VII.66})$$

in case of the hard exchange contribution may be written as

$$F^{B_0}(Q^2) = \alpha_g \int_0^1 \frac{dx}{xQ^2} \int_0^1 dy \int_0^{xQ^2} dk^2_\perp \left[1 - \frac{k^2_\perp}{xQ^2}\right] \times \int_0^1 d\xi \psi_0\left(y, \frac{\xi k^2_\perp}{V_0(x, y)}\right) \quad (\text{VII.67})$$

or

$$F^{B_0}(Q^2) = \alpha_g \int_0^1 dx \int_0^1 dy \int_0^1 (1 - \kappa) d\kappa \times \int_0^1 d\xi \psi_0\left(y, \frac{\xi \kappa x Q^2}{V_0(x, y)}\right). \quad (\text{VII.68})$$

Denoting  $\xi\kappa \equiv \lambda$ , we have

$$F^{B_0}(Q^2) = \alpha_g \int_0^1 dx \int_0^1 dy \int_0^1 d\lambda [\ln(1/\lambda) - 1 + \lambda] \times \psi_0\left(y, \frac{\lambda x Q^2}{V_0(x, y)}\right). \quad (\text{VII.69})$$

For large  $Q^2$ , one can separate here the terms with a power-like behavior of  $\ln(Q^2)/Q^2$ ,  $1/Q^2$  and  $\Lambda^2/Q^4$  type, and those which have (for a soft  $\psi_0$ ) a faster than an inverse power decrease with the increase of  $Q^2$ .

### C. Hard tail in QCD

#### 1. Yukawa-type contributions

In quantum chromodynamics, working in Feynman gauge, the only change in the box diagram and ultraviolet-divergent terms is in the overall factor, namely, one should take  $\alpha_g = C_F \alpha_s/(2\pi)$  in Eq. (VII.63), with  $C_F$  being the color factor. In particular the  $1/k^2_\perp$  hard tail generated by these ‘‘Yukawa-type’’ contributions is accompanied by the

$$V_Y(x, y) = \frac{\alpha_s}{2\pi} C_F \left[ \frac{x}{y} \theta(x < y) + \frac{\bar{x}}{\bar{y}} \theta(x > y) - \frac{1}{2} \delta(x - y) \right] = \frac{\alpha_s}{2\pi} C_F [V_0(x, y)]_+ \quad (\text{VII.70})$$

part of the ERBL evolution kernel, with ‘‘+’’ denoting the plus-prescription with respect to  $x$ :

$$[V_0(x, y)]_+ = V_0(x, y) - \delta(x - y) \int_0^1 dz V_0(z, y). \quad (\text{VII.71})$$

For vector gluons, one should also take into account contributions coming from the gauge link  $E(0, z; A)$ , which generate the remaining part of the QCD ERBL evolution kernel.

#### 2. Link contributions in case of collinear initial quarks

Let us start with a collinear initial state, i.e. take  $\Phi_0^{\text{coll}}(y, \sigma) = \varphi_0(y) \delta(\sigma)$ . There are two possibilities: gluon may be connected to  $yp$  (Fig. 14a) or  $(1 - y)p$  (Fig. 14b) quark leg. Insertion into the  $yp$  leg produces the term

$$t_L(z, p, y) = ig^2 C_F e^{i\bar{y}(pz)} \int_0^1 dt \int d^4 z_1 e^{iy(pz_1)} \times S^c(z_1) \not{z} D^c(z_1 - tz) \quad (\text{VII.72})$$

If the gluon is inserted into the  $\bar{y}p$  line, we start with

$$t_R(z, p, y) = ig^2 C_F \int_0^1 dt \int d^4 z_1 e^{i\bar{y}(pz_1)} \times \not{z} S^c(z - z_1) D^c(z_1 - tz). \quad (\text{VII.73})$$

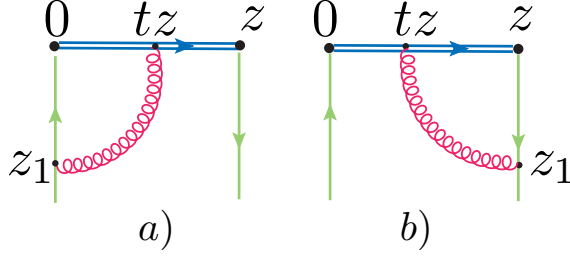


FIG. 14. Insertions of gluons coming out of the gauge link.

Shifting  $z_1 \rightarrow z_1 + z$ , changing  $z_1 \rightarrow -z_1$ ,  $t \rightarrow 1 - t$  and using that  $D^c(z)$  is an even function gives

$$t_R(z, p, y) = ig^2 C_F e^{i\bar{y}(pz)} \int_0^1 dt \int d^4 z_1 e^{-i\bar{y}(pz_1)} \times \not{z} S^c(z_1) D^c(z_1 - tz) . \quad (\text{VII.74})$$

Thus, these two cases are related by symmetry, and it is sufficient to analyze just the first of them, Eq. (VII.72). Integrating there over  $z_1$  gives

$$t_L(z, p, y) = i \frac{g^2}{16\pi^2} C_F e^{i\bar{y}(pz)} \int_0^1 dt \int_0^\infty \sigma_1 \frac{d\sigma_1 d\sigma_2}{(\sigma_1 + \sigma_2)^3} \times e^{i(yt(pz)\sigma_2 - \sigma_1 \sigma_2 t^2 z^2/4)/(\sigma_1 + \sigma_2)} \times \left( y \not{p} \not{z} + t \sigma_2 \frac{z^2}{2} \right) . \quad (\text{VII.75})$$

Switching to  $\alpha_i = 1/\sigma_i$ , introducing common  $\lambda = \alpha_1 + \alpha_2$ , and then relabelling  $\lambda = 1/\sigma$ , we obtain

$$t_L(z, p, y) = i \frac{g^2}{16\pi^2} C_F e^{i\bar{y}(pz)} \int_0^1 dt \int_0^\infty \frac{d\sigma}{\sigma} \times \int_0^1 d\beta e^{iyt\beta(pz)} e^{-i\sigma t^2 z^2/4} \left( y \bar{\beta} \not{p} \not{z} + t \sigma \frac{z^2}{2} \right) \quad (\text{VII.76})$$

Representing  $\not{p} \not{z} = (pz) + [\not{p}, \not{z}]/2$  and using this result in the expression for transition amplitude we end up with the operator  $\sim \epsilon_{\mu\alpha\beta\nu} p^\alpha z^\beta \bar{\psi}(0) \gamma^\nu \dots \psi(z)$  whose  $\langle p | \dots | 0 \rangle$  matrix element vanishes.

The second term in Eq. (VII.76) does not contribute to the hard tail since it lacks  $z^2$ -dependence after  $\sigma$ -integration. Thus, only the  $(pz)$ -part of the first term contributes to the hard tail, which in the coordinate representation is reflected by a logarithmic  $\ln z^2$  singularity resulting from the  $\sigma$  integration. As discussed earlier,  $\ln z^2$  reflects the DA evolution.

First, we are going to get rid of the integration over  $t$  specific for the vector gluons. Its calculation is complicated by the  $e^{-i\sigma t^2 z^2/4}$  factor. Let us represent

$$e^{-i\sigma t^2 z^2/4} = e^{-i\sigma z^2/4} + \left[ e^{-i\sigma t^2 z^2/4} - e^{-i\sigma z^2/4} \right] . \quad (\text{VII.77})$$

The bracketed term here formally vanishes for  $z^2 = 0$ , which means a suppression for small  $z^2$ . As a result, it does not contain  $\ln z^2$  terms and hence does not contribute to the hard tail. Integrating over  $t$  in the part corresponding to the first term, we get

$$t^{(1)}(z, p, y) = \gamma_\mu \alpha_g e^{i\bar{y}(pz)} \int_0^\infty \frac{d\sigma}{\sigma} e^{-i\sigma z^2/4} \times \int_0^1 d\bar{\beta} \frac{\bar{\beta}}{\beta} \left[ e^{i\bar{y}\beta(pz)} - 1 \right] , \quad (\text{VII.78})$$

where  $\alpha_g = C_F \alpha_s / (2\pi)$ . This contribution generates the evolution corresponding to the  $x \leq y$  part of the QCD ERL kernel:

$$t^{(1)}(z, p, y) = \gamma_\mu \alpha_g \ln(z^2) \int_0^1 dx e^{i\bar{x}(pz)} V_1(x, y) , \quad (\text{VII.79})$$

where

$$V_1(x, y) = \int_0^1 d\beta \delta(x - y + \beta y) \left( \frac{\bar{\beta}}{\beta} \right)_+ = \left( \frac{x}{y} \frac{\theta(x \leq y)}{y - x} \right)_+ . \quad (\text{VII.80})$$

Similarly, attaching the gluon to the  $\bar{y}p$  quark line, we get the  $x \geq y$  part

$$V_2(x, y) = \int_0^1 d\beta \delta(x - y - \beta \bar{y}) \left( \frac{\bar{\beta}}{\beta} \right)_+ = \left( \frac{\bar{x}}{\bar{y}} \frac{\theta(x \geq y)}{x - y} \right)_+ \quad (\text{VII.81})$$

of the ERL evolution kernel. Thus, in the convolution model we have an evolution contribution

$$\delta\varphi^{(1)}(x, z^2) = \alpha_g \ln(z^2) \int_0^1 dy V(x, y) \varphi_0(y) , \quad (\text{VII.82})$$

with the ERL kernel in its correct “plus prescription” form.

### 3. Noncollinear initial quarks

Let us now consider the case when we have an initial soft distribution described by a VDA  $\Phi_0(y, \sigma_0)$ . Again, it is sufficient to consider the diagram corresponding to the gluon insertion into the  $yp$  leg. The contribution of the  $\bar{y}p$  diagram can be added at the end using the symmetry considerations.

*a. Derivation of a general formula for VDA.* The result of integration over  $z_1$  has the structure similar to that obtained in the collinear quarks case (VII.75). Keeping again the  $\sim (pz)$  term only, and representing the  $z^2$ -dependent factor as a sum of its  $t = 1$  value and the rest (which formally vanishes for  $z^2 = 0$ , and thus does not

contribute to the hard tail), and integrating over  $t$  in this (“hard”) part, we get

$$t^{(h)}(z, p) = g^2 \int_0^1 dy e^{iy(pz)} \int_0^\infty d\sigma e^{-i\sigma z^2/4} \int_0^1 \frac{d\xi}{\xi} \times \int_0^1 \frac{d\beta}{\beta} \left( e^{iy\beta(pz)} - e^{iy\xi\beta(pz)} \right) \Phi_0(y, \xi\sigma/\bar{\beta}) . \quad (\text{VII.83})$$

In terms of VDA, we have

$$\Phi^{(h)}(x, \sigma) = \alpha_g \int_0^1 dy \int_0^1 \frac{d\xi}{\xi} \int_0^1 \frac{d\beta}{\beta} \Phi_0(y, \xi\sigma/\bar{\beta}) \times [\delta(x - y + y\beta) - \delta(x - y + y\xi\beta)] . \quad (\text{VII.84})$$

One can see that

$$\int_0^1 dx \Phi^{(h)}(x, \sigma) = 0 , \quad (\text{VII.85})$$

i.e. the hard addition  $\Phi^{(h)}(x, \sigma)$  does not change the  $x$ -integral of  $\Phi(x, \sigma)$ .

*b. Structure of hard TMDA.* The integrand in Eq. (VII.84) contains  $1/\beta$  and  $1/\bar{\xi}$  factors resulting in potential singularities for  $\beta = 0$  and  $\xi = 1$ . However, the two  $\delta$ -functions in Eq. (VII.84) coincide in these limits, and as a result there are no divergencies. Still, the most natural way to proceed with integrations in Eq. (VII.84) is to use the  $\delta$ -functions to eliminate the integral over  $\beta$ . Then the common  $1/\beta$  factor in both terms results in contributions proportional to the factor  $1/(y - x)$  singular for  $y = x$ . Of course, the singularity cancels because it is accompanied by a difference of two VDAs coinciding for  $y = x$ . However, each term produces a divergent contribution. To be able to keep these terms separately, we choose to regularize the original  $1/\beta$  singularity. To this end, we impose a cut-off for the lower limit of the  $\beta$ -integral at some finite  $\epsilon > 0$  value. Converting the result of  $\beta$ -integration into an expression for TMDA, we obtain

$$\begin{aligned} \psi^h(x, k_\perp^2) &= g^2 \int_0^1 \frac{dy}{y - x} \int_0^1 \frac{d\xi}{\xi} \left\{ \theta(y(1 - \epsilon) \geq x) \right. \\ &\times \left[ \psi_0\left(y, \frac{\xi k_\perp^2}{x/y}\right) - \psi_0\left(y, \frac{\xi^2 k_\perp^2}{x/y - \xi}\right) \theta\left(\bar{\xi} \leq \frac{x}{y}\right) \right] \\ &- \theta(y(1 - \epsilon) \leq x) \psi_0\left(y, \frac{\xi^2 k_\perp^2}{x/y - \xi}\right) \\ &\left. \times \theta\left(1 - \frac{x}{y} \leq \xi \leq (1 - \frac{x}{y})/\epsilon\right) \right\} . \quad (\text{VII.86}) \end{aligned}$$

**Consider first the  $y \leq x/(1 - \epsilon)$  part:**

$$\begin{aligned} \psi_{<}^h(x, k_\perp^2) &= -\alpha_g \int_x^{x/(1-\epsilon)} \frac{dy}{y - x} \\ &\times \int_0^1 \frac{d\xi}{\xi} \psi_0\left(y, \frac{\xi^2 k_\perp^2}{x/y - \xi}\right) \\ &\times \theta\left(1 - \frac{x}{y} \leq \xi \leq \frac{1}{\epsilon}(1 - \frac{x}{y})\right) . \quad (\text{VII.87}) \end{aligned}$$

Note that the interval of integration over  $y$  shrinks to zero when  $\epsilon \rightarrow 0$ . Nevertheless, the integral itself has a finite value in this limit.

Since the region of integration is specified by  $0 \leq 1 - x/y \leq \epsilon$ , we write  $1 - x/y = \epsilon z$  with a new variable  $z$  satisfying  $0 \leq z \leq 1$ . Furthermore, the limits on  $\xi$  correspond to  $\xi = (1 - x/y)\zeta$  with  $1 \leq \zeta \leq 1/\epsilon$ . Thus, we denote  $\xi = \nu z$  with  $\epsilon \leq \nu \leq 1$  to get

$$\begin{aligned} \psi_{<}^h(x, k_\perp^2) &= -\alpha_g \int_0^1 \frac{dz}{1 - \epsilon z} \int_\epsilon^1 \frac{d\nu}{1 - z\nu} \\ &\times \psi_0\left(\frac{x}{1 - \epsilon z}, \frac{\nu^2}{\nu - \epsilon} z k_\perp^2\right) . \quad (\text{VII.88}) \end{aligned}$$

Taking the  $\epsilon \rightarrow 0$  limit and introducing  $\tau = \nu z$ , we obtain

$$\psi_{<}^h(x, k_\perp^2) = \alpha_g \int_0^1 d\tau \frac{\ln \tau}{1 - \tau} \psi_0(x, \tau k_\perp^2) . \quad (\text{VII.89})$$

There is no change in the longitudinal momentum fraction  $x$  in this term, i.e. this contribution may be written as a  $\delta(y - x)$  term under the  $y$ -integral. Note also that this contribution is finite for  $k_\perp = 0$  if the initial TMDA  $\psi(x, k_\perp^2)$  is finite for  $k_\perp = 0$ . Namely,

$$\begin{aligned} \psi_{<}^h(x, k_\perp^2 = 0) &= \alpha_g \int_0^1 d\tau \frac{\ln \tau}{1 - \tau} \psi_0(x, k_\perp^2 = 0) \\ &= -\alpha_g \frac{\pi^2}{6} \psi_0(x, k_\perp^2 = 0) . \quad (\text{VII.90}) \end{aligned}$$

**Sudakov logarithm.** In the opposite limit of large  $k_\perp^2$ , the  $\tau$ -integral for  $\psi_{<}^h(x, k_\perp^2)$  is dominated by small values  $\tau \sim \Lambda^2/k_\perp^2$ , where  $\Lambda^2$  is the scale characterizing the decrease of TMDA with  $k_\perp^2$ . Approximating the  $\tau$ -integral as

$$\begin{aligned} &\int_0^{\Lambda^2/k_\perp^2} d\tau \ln \tau \psi_0(x, \tau k_\perp^2) \\ &= \frac{1}{k_\perp^2} \int_0^{\Lambda^2} d\kappa^2 \ln \frac{\kappa^2}{k_\perp^2} \psi_0(x, \kappa^2) \\ &= -\frac{1}{k_\perp^2} \ln\left(\frac{k_\perp^2}{\Lambda^2}\right) \int_0^\infty d\kappa^2 \psi_0(x, \kappa^2) + \dots \\ &= -\frac{1}{k_\perp^2} \varphi_0(x) \ln\left(\frac{k_\perp^2}{\Lambda^2}\right) + \dots . \quad (\text{VII.91}) \end{aligned}$$

we see that the term  $\psi_{<}^h(x, k_\perp^2)$  behaves like  $-(\ln k_\perp^2)/k_\perp^2$ , i.e., has an extra  $\ln k_\perp^2$  on top of expected  $1/k_\perp^2$  hard tail factor. After integration over  $k_\perp^2$  till  $xQ^2$  in the transition amplitude, such a term produces a negative doubly-logarithmic contribution  $\sim -\ln^2(xQ^2/\Lambda^2)$ , i.e. a Sudakov logarithm. However, it is known that there should be no such logarithms in the total result for transition form factor.



**So, let us consider the  $y(1-\epsilon) \geq x$  part:**

$$\begin{aligned} \psi_{>}^h(x, k_{\perp}^2) = & \alpha_g \int_{x/(1-\epsilon)}^1 \frac{dy}{y-x} \int_0^1 \frac{d\xi}{\xi} \left[ \psi_0 \left( y, \frac{\xi k_{\perp}^2}{x/y} \right) \right. \\ & \left. - \psi_0 \left( y, \frac{\xi^2 k_{\perp}^2}{x/y - \xi} \right) \theta \left( \bar{\xi} \leq \frac{x}{y} \right) \right] . \end{aligned} \quad (\text{VII.92})$$

Since the two terms in the square brackets coincide when  $y = x$ , the integral over  $y$  converges even if we take  $\epsilon = 0$ . Another potential divergence is due to the  $1/\bar{\xi}$  factor in the  $\xi$ -integral. Again, the two terms in the square brackets coincide when  $\xi = 1$ , thus we can safely put  $\epsilon = 0$ .

In particular, for small  $k_{\perp}^2$ , assuming that  $\psi(y, k_{\perp}^2 = 0)$  is finite, we get

$$\psi_{>}^h(x, k_{\perp}^2 = 0) = \alpha_g \int_x^1 \frac{dy}{y-x} \ln \left( \frac{y}{x} \right) \psi_0(y, k_{\perp}^2 = 0) . \quad (\text{VII.93})$$

However, we also need to see that the  $y(1-\epsilon) \geq x$  part contains a term that cancels the Sudakov contribution that we obtained for the  $y(1-\epsilon) \geq x$  part.

For large  $k_{\perp}^2$ , the leading  $\sim 1/k_{\perp}^2$  behavior is obtained from integration over the regions where the second argument of  $\psi_0$  vanishes, i.e.  $\xi \rightarrow 0$  for both terms in Eq. (VII.93). In this limit, the singularity of the integrand for  $\xi = 1$  is irrelevant, so to avoid formal complications related to this singularity, we write  $1/\bar{\xi} = 1 + \xi/\bar{\xi}$  and observe that the  $\xi/\bar{\xi}$  factor produces a suppression in the  $\xi \rightarrow 0$  region resulting in extra powers of  $1/k_{\perp}^2$  for large  $k_{\perp}^2$ , i.e. the terms accompanied by it do not contribute to the leading behavior. Thus we have

$$\begin{aligned} \psi_{>}^h(x, k_{\perp}^2) = & \alpha_g \int_{x/(1-\epsilon)}^1 \frac{dy}{y-x} \int_0^1 d\xi \left[ \psi_0 \left( y, \frac{\xi k_{\perp}^2}{x/y} \right) \right. \\ & \left. - \psi_0 \left( y, \frac{\xi^2 k_{\perp}^2}{x/y - \xi} \right) \theta \left( \xi \geq 1 - \frac{x}{y} \right) \right] + \dots . \end{aligned} \quad (\text{VII.94})$$

We keep finite  $\epsilon$  here because we want again to treat separately the two terms of this expression. The first term looks similar to what we had in the nongauge case, and can be written as

$$\begin{aligned} \psi_{>}^{h,1}(x, k_{\perp}^2) = & \frac{\alpha_g}{k_{\perp}^2} \int_{x/(1-\epsilon)}^1 \frac{dy}{y-x} V_0(x, y) \\ & \times \int_0^{k_{\perp}^2/V_0(x,y)} d\kappa^2 \psi_0(y, \kappa^2) . \end{aligned} \quad (\text{VII.95})$$

At large  $k_{\perp}^2$ , this term gives

$$\psi_{>}^{h,1}(x, k_{\perp}^2) = \frac{\alpha_g}{k_{\perp}^2} \int_{x/(1-\epsilon)}^1 \frac{dy}{y-x} V_0(x, y) \varphi_0(y) + \dots , \quad (\text{VII.96})$$

involving the expected part of the ERBL evolution kernel. The  $y$ -integral is singular at the lower limit produc-

ing  $\ln(1/\epsilon)$  term, namely

$$\begin{aligned} \psi_{>}^{h,1}(x, k_{\perp}^2) = & \frac{\alpha_g}{k_{\perp}^2} \ln \frac{1}{\epsilon} \int_0^{k_{\perp}^2} d\kappa^2 \psi_0(y, \kappa^2) \\ & + \text{regular part} . \end{aligned} \quad (\text{VII.97})$$

This term should be compensated by the second term in Eq. (VII.94). Introducing  $t \equiv (1-x/y)$ , the latter may be written as

$$\begin{aligned} \psi_{>}^{h,2}(x, k_{\perp}^2) = & -\alpha_g \int_{\epsilon}^{1-x} \frac{dt}{t(1-t)} \\ & \times \int_t^1 d\xi \psi_0 \left( \frac{x}{1-t}, \frac{\xi k_{\perp}^2}{1-t/\xi} \right) . \end{aligned} \quad (\text{VII.98})$$

Its singular part comes from the  $t \rightarrow 0$  region of integration of the  $t$ -integral. Getting it, we can take  $t = 0$  in the  $\xi$ -integral. The resulting singular part is evidently opposite to that in Eq. (VII.97). To get a more accurate estimate, let us neglect  $t$  only when it stays in  $1-t$  combination, but keep it in other places. Namely, consider

$$\begin{aligned} \psi_{>}^{h,2,\text{approx}}(x, k_{\perp}^2) = & -\alpha_g \int_{\epsilon}^{1-x} \frac{dt}{t} \\ & \times \int_t^1 d\xi \psi_0 \left( x, \frac{\xi k_{\perp}^2}{1-t/\xi} \right) . \end{aligned} \quad (\text{VII.99})$$

Representing  $\xi = t(1+u)$  we can write

$$\begin{aligned} \psi_{>}^{h,2,\text{approx}}(x, k_{\perp}^2) = & -\alpha_g \int_{\epsilon}^{1-x} dt \int_0^{1/t-1} du \\ & \times \psi_0 \left( x, t k_{\perp}^2 \frac{(1+u)^2}{u} \right) . \end{aligned} \quad (\text{VII.100})$$

Note that  $(1+u)^2/u$  is always larger than 4, which means the leading large  $k_{\perp}^2$  power behavior comes from integration over small  $t \sim \Lambda^2/k_{\perp}^2$ , where  $\Lambda^2$  is a scale characterizing the fall-off of TMDA. Thus we can substitute the upper limit of integration over  $u$  by infinity without changing the leading large- $k_{\perp}^2$  power behavior of the integral and write

$$\begin{aligned} \psi_{>}^{h,2,\text{approx}}(x, k_{\perp}^2) = & -\alpha_g \int_{\epsilon}^{\Lambda^2/k_{\perp}^2} dt \int_0^{\infty} du \\ & \times \psi_0 \left( x, t k_{\perp}^2 \frac{(1+u)^2}{u} \right) + \dots . \end{aligned} \quad (\text{VII.101})$$

For large  $u$ , we can approximate  $(1+u)^2/u$  by  $u$  to get

$$\begin{aligned} \psi_{>}^{h,2,\text{approx}}(x, k_{\perp}^2) = & -\frac{\alpha_g}{k_{\perp}^2} \int_{\epsilon}^{\Lambda^2/k_{\perp}^2} \frac{dt}{t} \\ & \times \int_0^{\infty} d\kappa^2 \psi_0(x, \kappa^2) + \dots \\ = & -\frac{\alpha_g}{k_{\perp}^2} \varphi_0(x) \left[ \ln \frac{1}{\epsilon} - \ln \left( \frac{k_{\perp}^2}{\Lambda^2} \right) \right] + \dots . \end{aligned} \quad (\text{VII.102})$$



Thus, we conclude that the  $\psi_{>}^{h,2}(x, k_{\perp}^2)$  part contains the  $\ln 1/\epsilon$  term canceling the singularity of the  $\psi_{>}^{h,1}(x, k_{\perp}^2)$  part. In other words, it supplies the subtraction term

$$\psi_{>}^{h,\text{subtr}}(x, k_{\perp}^2) = -\frac{\alpha_g}{k_{\perp}^2} \varphi_0(x) \left[ \ln \frac{1}{\epsilon} \right] \quad (\text{VII.103})$$

generating the plus prescription for the part of the ERL kernel displayed in Eq. (VII.96).

The  $\psi_{>}^{h,2}(x, k_{\perp}^2)$  part also contains the  $(\ln k_{\perp}^2/\Lambda^2)/k_{\perp}^2$  contribution that cancels a similar logarithm contained in the  $\psi_{>}^h(x, k_{\perp}^2)$  term, thus guaranteeing that in  $\alpha_g$  order there will be no  $\ln^2 Q^2$  Sudakov double logarithms in the transition form factor.

**Summarizing,** after properly taking the  $\beta$ -integral in the original hard contribution (VII.84), the resulting hard addition to the TM DA is given by a sum of the  $\psi_{>}^h(x, k_{\perp}^2)$  term (VII.89) and the  $\psi_{>}^h(x, k_{\perp}^2)$  term (VII.93) in which one can safely take  $\epsilon = 0$ . Furthermore, the “unwanted”  $(\ln k_{\perp}^2/\Lambda^2)/k_{\perp}^2$  Sudakov logarithms present in both terms cancel, while the remaining  $\sim 1/k_{\perp}^2$  terms generate evolution governed by the relevant part of the ERL kernel with a correct plus-prescription.

**Full result for the link contribution.** Finally, we should add the contribution of the diagram with the insertion into the  $\bar{y}p$  leg (which is obtained by changing  $y \rightarrow \bar{y}$  and  $x \rightarrow \bar{x}$ ) to get hard contribution due to gluons coming from the gauge link

$$\begin{aligned} \psi_{\text{link}}^{h,\text{total}}(x, k_{\perp}^2) &= \alpha_g \int_x^1 \frac{dy}{y-x} \int_0^1 \frac{d\xi}{\xi} \\ &\times \left[ \psi_0 \left( y, \frac{\xi k_{\perp}^2}{x/y} \right) - \psi_0 \left( y, \frac{\xi^2 k_{\perp}^2}{x/y - \xi} \right) \theta \left( \bar{\xi} \leq \frac{x}{y} \right) \right] \\ &+ \alpha_g \int_0^1 d\tau \frac{\ln \tau}{1-\tau} \psi_0(x, \tau k_{\perp}^2) \\ &+ \left\{ y \rightarrow \bar{y}, x \rightarrow \bar{x} \right\}. \end{aligned} \quad (\text{VII.104})$$

We can also transform this result into the impact parameter space

$$\begin{aligned} \varphi_{\text{link}}^{h,\text{total}}(x, z_{\perp}^2) &= \alpha_g \int_1^{\infty} \frac{d\nu}{\nu-1} \left\{ -\varphi_0(x, \nu z_{\perp}^2) \ln \nu \right. \\ &+ \int_x^1 \frac{dy}{y-x} \frac{x}{y} \left[ \varphi_0 \left( y, \nu z_{\perp}^2 \frac{x}{y} \right) - \theta \left( y \leq x \frac{\nu}{\nu-1} \right) \right. \\ &\times \varphi_0 \left( y, (\nu x/y - \nu + 1) \nu z_{\perp}^2 \right) \left. \right\} \\ &+ \alpha_g \int_1^{\infty} d\nu \int_x^1 \frac{dy}{y} \theta \left( y \leq x \frac{\nu}{\nu-1} \right) \\ &\times \varphi_0 \left( y, (\nu x/y - \nu + 1) \nu z_{\perp}^2 \right) + \left\{ y \rightarrow \bar{y}, x \rightarrow \bar{x} \right\}. \end{aligned} \quad (\text{VII.105})$$

**Large  $k_{\perp}^2$ .** As we have seen, for large  $k_{\perp}^2$  this expression contains all the relevant terms of the  $\sim 1/k_{\perp}^2$  hard tail. It also contains contributions that have a large- $k_{\perp}^2$  behavior similar to that of the soft TM DA  $\psi_0(x, k_{\perp}^2)$ .

For a Gaussian Ansatz with a flat profile,  $\psi_0^F(y, k_{\perp}^2) = e^{-k_{\perp}^2/\Lambda^2}/\Lambda^2$ , Fig. 15 illustrates the  $k_{\perp}$ -dependence of the hard addition  $\psi_{\text{link}}^{h,\text{total}}(x, k_{\perp}^2)$  for  $x = 0.5$  and  $x = 0.1$ . One can see that the product  $k_{\perp}^2 \psi^h$  flattens for  $k_{\perp}^2 \gtrsim 4\Lambda^2$ ,

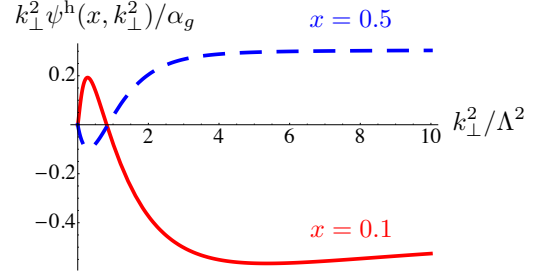


FIG. 15.  $k_{\perp}$ -dependence of the hard tail for a Gaussian model with flat DA for  $x = 0.1$  (solid line, red online) and  $x = 0.5$  (dashed line, blue online).

clearly demonstrating the presence of a  $\sim 1/k_{\perp}^2$  contribution in this case.

For sufficiently large  $k_{\perp}^2$ , the hard correction is positive for the middle point  $x = 0.5$  and negative for  $x = 0.1$ . In Fig. 16 we show the  $x$ -dependence of  $\psi_{\text{link}}^{h,\text{total}}(x, k_{\perp}^2)$  for a particular value  $k_{\perp}^2 = 5\Lambda^2$  and initially flat  $x$ -distribution. Clearly, the hard correction tends to make the combined distribution narrower.

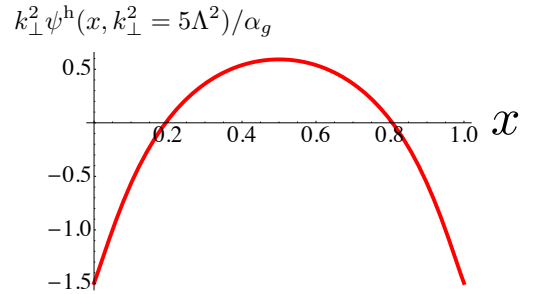


FIG. 16.  $x$ -dependence of the hard tail for a Gaussian model with flat DA for  $k_{\perp}^2 = 5\Lambda^2$ .

The situation changes when one takes an initially asymptotic profile, e.g.  $\psi_0^{\text{as}}(y, k_{\perp}^2) = 6y\bar{y}e^{-k_{\perp}^2/\Lambda^2}/\Lambda^2$ . As one can see from Fig.17, instead of flattening to a nonzero value, the product  $k_{\perp}^2 \psi^h$  continues to decrease to zero for large  $k_{\perp}^2$  thus demonstrating the absence of the  $\sim 1/k_{\perp}^2$  hard tail in this case, because the convolution of the ERL kernel with the asymptotic profile vanishes.

## VIII. SUMMARY AND OUTLOOK

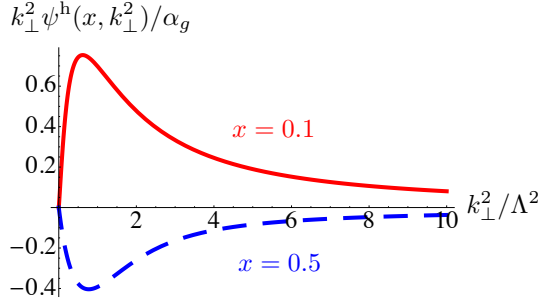


FIG. 17.  $k_{\perp}$ -dependence of the hard tail for a Gaussian model with asymptotic DA for  $x = 0.1$  (solid line, red online) and  $x = 0.5$  (dashed line, blue online).

**Small  $k_{\perp}$ .** For  $k_{\perp}^2 \rightarrow 0$ , Eq. (VII.104) has a finite limit, namely

$$\begin{aligned} \psi_{\text{link}}^{\text{h,total}}(x, k_{\perp}^2 = 0) = \alpha_g \int_0^1 dy \left\{ \left[ \frac{\theta(y > x)}{y - x} \ln \left( \frac{y}{x} \right) \right]_+ \right. \\ \left. + \left[ \frac{\theta(y < x)}{x - y} \ln \left( \frac{1 - y}{1 - x} \right) \right]_+ \right\} \psi_0(y, k_{\perp}^2 = 0). \end{aligned} \quad (\text{VII.106})$$

For a Gaussian Ansatz with a flat profile,  $\psi_0^{\text{F}}(y, k_{\perp}^2 = 0) = 1/\Lambda^2$ , this gives

$$\psi_{\text{link}}^{\text{h,total,F}}(x, k_{\perp}^2 = 0) = \frac{\alpha_g}{2\Lambda^2} \left[ \ln^2 \frac{\bar{x}}{x} - \frac{\pi^2}{3} \right]. \quad (\text{VII.107})$$

Similarly, for a Gaussian Ansatz with asymptotic profile,  $\psi_0^{\text{as}}(y, k_{\perp}^2 = 0) = 6x(1 - x)/\Lambda^2$ , we have

$$\begin{aligned} \psi_{\text{link}}^{\text{h,total,as}}(x, k_{\perp}^2 = 0) = \frac{3\alpha_g}{\Lambda^2} \left[ -\frac{3}{2} + 5x\bar{x} + (\bar{x} - x) \ln \frac{\bar{x}}{x} \right. \\ \left. + x\bar{x} \left[ \ln^2 \frac{\bar{x}}{x} - \frac{\pi^2}{3} \right] \right]. \end{aligned} \quad (\text{VII.108})$$

In both cases, the correction decreases TMDA in the middle of the  $x$ -interval, and enhances it at the ends. One can check that the integral of  $\psi^{\text{h,total}}(x, k_{\perp}^2 = 0)$  over  $x$  in these two cases gives zero. In fact, as follows from Eq. (VII.85), integrating  $\psi_{\text{link}}^{\text{h,total}}(x, k_{\perp}^2)$  over  $x$  gives zero for any initial  $\psi_0(x, k_{\perp}^2)$  and for all  $k_{\perp}$ .

One may ask if it makes sense to consider a perturbatively obtained hard term in the small  $k_{\perp}$  region. To this end, we remind that we already use a model soft TMDA in this region, to which our hard term gives just a small correction. So, using the “soft+hard” combination is simply another model for the small- $k_{\perp}$  region. We think that an approach in which the hard term is naturally finite for small  $k_{\perp}$  has advantages compared to a usual practice when the explosion of the  $1/k_{\perp}^2$  hard term for small  $k_{\perp}$  is stopped by an arbitrarily chosen cut-off.

In the present paper, we outlined a new approach to transverse momentum dependence in hard processes. Its starting point, just like in the OPE formalism, is the use of coordinate representation. At handbag level, the structure of a hadron with momentum  $p$  is described by a matrix element of the bilocal operator  $\mathcal{O}(0, z)$ , treated as a function of  $(pz)$  and  $z^2$ . It is parametrized through a *virtuality distribution*  $\Phi(x, \sigma)$ , in which the variable  $x$  is Fourier-conjugate to  $(pz)$ , and has the usual meaning of a parton momentum fraction. Another parameter,  $\sigma$ , is conjugate to  $z^2$  through an analog of Laplace transform.

Projecting  $\mathcal{O}(0, z)$  onto a spacelike interval with  $z_+ = 0$ , we introduce *transverse momentum distributions*  $\Psi(x, k_{\perp})$  and show that they can be written in terms of virtuality distributions  $\Phi(x, \sigma)$ . This fact opens the possibility to convert the results of covariant calculations, written in terms of  $\Phi(x, \sigma)$ , into expressions involving  $\Psi(x, k_{\perp})$ . This procedure being a crucial feature of our approach, is illustrated in the present paper by its application to hard exclusive transition process  $\gamma^* \gamma \rightarrow \pi^0$  at the handbag level (which is analogous to the 2-body Fock state approximation). Starting with scalar toy models, we then extend the analysis onto the case of spin-1/2 quarks and vector gluons.

We studied a few simple models for soft VDAs/TMDAs, and then used them for comparison of VDA results with experimental (BaBar and BELLE) data on the pion transition form factor.

A natural next step is going beyond the handbag approximation. In QCD, an important feature is that quark-gluon interactions generate a hard  $\sim 1/k_{\perp}^2$  tail for TMDAs. To demonstrate the capabilities of the VDA approach in this direction, we describe the basic elements of generating hard tails from soft primordial TMDAs.

Another direction is to include the contribution due to a 3-body quark-gluon TMDA. In the OPE it starts with a  $\bar{q}Gq$  operator related to  $\bar{q}D_{\perp}^2 q$  operator that appears in our treatment of the handbag contribution as an explicit  $k_{\perp}^2/Q^2$  correction to the leading term.

Among other possible directions for future studies is the use of the VDA approach for a systematic study of quark virtuality corrections to the one-gluon contribution for the pion electromagnetic form factor  $F_{\pi}(Q^2)$ . As we have seen, for the transition form factor such corrections strongly reduce its magnitude in the region of moderately large  $Q^2$ . In case of  $F_{\pi}(Q^2)$ , one should expect even more drastic reduction, since two TMDAs are involved.

One more direction, also suggested by the pion form factor analysis, is a study of TMDAs corresponding to pseudoscalar  $\bar{\psi}\gamma_5\psi$  and tensor  $\bar{\psi}\sigma^{\mu\nu}\psi$  bilocal operators. It was argued [41, 42] that such chiral-odd projections may play an important role in understanding JLab data on the deeply virtual electroproduction of pions [43]. It should be emphasized that “purely collinear” pQCD formulas in these cases are known to produce diverging integrals like  $\varphi(x)/x^2$ . Within the TMDA formalism,

transverse momentum effects are expected to regulate such singularities, which necessitates the extension of the TMDA approach onto distributions related to chiral-odd bilocal operators.

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