What is Δm_{ee}^{2}?

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What is Δm^2_{ee} ?

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Abstract

The current short baseline reactor experiments, Daya Bay and RENO (Double Chooz) have measured (or are capable of measuring) an effective Δm^2 associated with the atmospheric oscillation scale of 0.5 km/MeV in electron anti-neutrino disappearance. In this paper, I compare and contrast the different definitions of such an effective Δm^2 and argue that the simple, L/E independent, definition given by $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$, i.e. "the ν_e weighted average of Δm^2_{31} and Δm^2_{32}," is superior to all other definitions and is useful for both short baseline experiments mentioned above and for the future medium baseline experiments JUNO and RENO 50.

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I. INTRODUCTION

The short baseline reactor experiments, Daya Bay [1], RENO [2], and Double Chooz [3] , have been very successful in determining the electron neutrino flavor content of the neutrino mass eigenstate with the smallest amount of ν_e, the state usually labelled ν_3. The parameter which controls the size of this flavor content is the mixing angle θ_{13}, in the standard PDG convention[1], and the current measurements indicate that $\sin^2 2\theta_{13} \approx 0.09$ with good precision ($\sim 5\%$).

The mass of the ν_3 eigenstate, has a mass squared splitting from the other two mass eigenstates, ν_1 and ν_2, of approximately $\pm 2.4 \times 10^{-3}$ eV^2 given by $\Delta m^2_{31} \equiv m^2_3 - m^2_1$ and $\Delta m^2_{32} \equiv m^2_3 - m^2_2$, the sign determines the atmospheric mass ordering. The mass squared difference between, ν_2 and ν_1, $\Delta m^2_{21} \equiv m^2_2 - m^2_1 \approx +7.5 \times 10^{-5}$ eV^2 is about 30 times smaller than both Δm^2_{31} and Δm^2_{32}, hence $\Delta m^2_{31} \approx \Delta m^2_{32}$. However, the difference between Δm^2_{31} and Δm^2_{32} is $\sim$3%.

Recently, two of these reactor experiments, Daya Bay, see [4] - [6] and RENO [7], have extended their analysis of their data, from just fitting $\sin^2 2\theta_{13}$, to a two parameter fit of both $\sin^2 2\theta_{13}$ and an effective Δm^2. The measurement uncertainty on this effective Δm^2 is approaching the difference between Δm^2_{31} and Δm^2_{32}. So it is now a pertinent question "What is the physical meaning of this effective Δm^2?" Clearly, the effective Δm^2 measured by these experiments is some combination of Δm^2_{31} and Δm^2_{32}. Answering the question "What is the combination of Δm^2_{31} and Δm^2_{32} is measured in such a short baseline reactor experiment?" is the primary purpose of this paper,

The outline of this paper is as follows: in Section II, I review the $\bar{\nu}_e$ survival probability as calculated in terms of an effective Δm^2 which naturally arises in this calculation, then this definition is applied to the short baseline reactor experiments, $L/E < 1$ km/GeV. In Section III, I compare and contrast other possible definitions of an effective Δm^2, including two new ones as well as the two definitions invented by the Daya Bay collaboration. The new effective Δm^2's are essentially equal to the effective Δm^2 of section II whereas the two invented by Daya Bay are L/E dependent and their original definition is discontinuous. This is followed by a conclusion and two appendices.

[1] A more informative notation for mixing angles $(\theta_{12},\ \theta_{13},\ \theta_{23})$ is $(\theta_{e2},\ \theta_{e3},\ \theta_{\mu 3})$, respectively, such that $U_{e2} = \cos\theta_{e3}\sin\theta_{e2}$, $U_{e3} = \sin\theta_{e3} e^{-i\delta}$ and $U_{\mu 3} = \cos\theta_{e3}\sin\theta_{\mu 3}$.

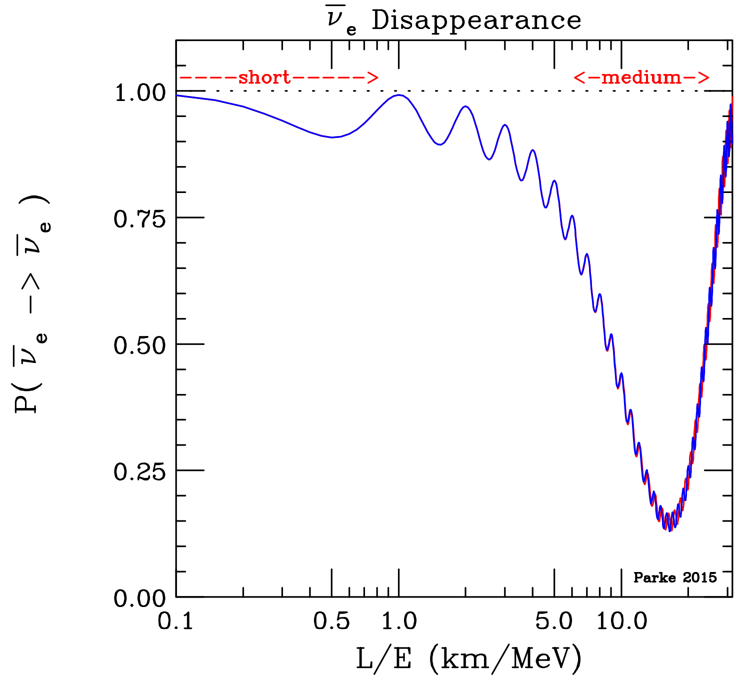

FIG. 1: The vacuum survival probability for $\bar{\nu}_e$ as a function of L/E. Blue is for the normal mass ordering (NO) and red is the inverted mass ordering (IO) with Δm^2_{31} and Δm^2_{32} chosen in such a fashion that the two survival probabilities are identical at small L/E, ie. $\Delta m^2_{31}(IO) = -\Delta m^2_{31}(NO) + 2\sin^2\theta_{12}\Delta m^2_{21}$. Near the solar oscillation minimum, $L/E \sim 15$ km/MeV, the phase of the θ_{13} oscillations advances (retards) for the normal (inverted) mass ordering and the two oscillation probabilities are distinguishable, in principle. Also near the solar minimum, the amplitude of the θ_{13} oscillations is significantly reduced compared to smaller values of L/E. The short baseline experiments, Daya Bay, RENO and Double Chooz, probe $L/E < 0.8$ km/MeV and the medium baseline, JUNO and RENO 50, probe $6 < L/E < 25$ km/MeV, as indicated.

II. $\bar{\nu}_e$ SURVIVAL PROBABILITY IN VACUUM:

The exact $\bar{\nu}_e$ survival probability in vacuum, see Fig. 1, is given by[2]

$$\begin{aligned} P_x(\bar{\nu}_e \to \bar{\nu}_e) &= 1 - 4|U_{e2}|^2|U_{e1}|^2 \sin^2 \Delta_{21} \\ &\quad -4|U_{e3}|^2|U_{e1}|^2 \sin^2 \Delta_{31} - 4|U_{e3}|^2|U_{e2}|^2 \sin^2 \Delta_{32} \\ &= 1 - \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} \\ &\quad - \sin^2 2\theta_{13}\ (\cos^2 \theta_{12} \sin^2 \Delta_{31} + \sin^2 \theta_{12} \sin^2 \Delta_{32}), \end{aligned} \tag{1}$$

using $\Delta_{ij} \equiv \Delta m^2_{ij} L/4E$.

It was shown in [8], that to an excellent accuracy

$$\begin{aligned} \cos^2 \theta_{12} \sin^2 \Delta_{31} + \sin^2 \theta_{12} \sin^2 \Delta_{32} &\approx \sin^2 \Delta_{ee} \\ \text{where } \Delta m^2_{ee} &\equiv \cos^2 \theta_{12} \Delta m^2_{31} + \sin^2 \theta_{12} \Delta m^2_{32} \end{aligned}$$

for $L/E < 0.8$ km/MeV. A variant of this derivation is given in the Appendix V.

However, in this article we will use an exact formulation given in [9] which follows Helmholtz, [10], in combining the two oscillation frequencies, proportional to Δ_{31} and Δ_{32} into one frequency plus a phase. The exact survival probability is given by (see Appendix VI)

$$\begin{aligned} P_x(\bar{\nu}_e \to \bar{\nu}_e) &= 1 - \cos^4 \theta_{13} \sin^2 2\theta_{12} \sin^2 \Delta_{21} \\ &\quad -\frac{1}{2} \sin^2 2\theta_{13} \left(1 - \sqrt{1 - \sin^2 2\theta_{12} \sin^2 \Delta_{21}}\ \cos \Omega \right) \\ \text{with} \quad \Omega &= (\Delta_{31} + \Delta_{32}) + \arctan(\cos 2\theta_{12} \tan \Delta_{21}). \end{aligned} \tag{2}$$

Ω consists of two parts: one that is even under the interchange of Δ_{31} and Δ_{32} and is linear in L/E, $(\Delta_{31} + \Delta_{32})$, and the other which is odd under this interchange and contains both linear and higher (odd) powers in L/E, $\arctan(\cos 2\theta_{12} \tan \Delta_{21})$, remember $\Delta_{21} = \Delta_{31} - \Delta_{32}$.

The key point is the separation of the kinematic phase, Ω, into an effective 2Δ (linear in L/E) and a phase, ϕ. For short baseline experiments, it is natural to expand Ω in a power

[2] The standard PDG conventions with the kinematical phase given by $\Delta_{ij} \equiv \Delta m^2_{ij} L/4E$ or 1.267 $\Delta m^2_{ij} L/E$ depending on whether one is using natural or (eV^2, km, MeV) units. Also, matter effects shift the Δm^2 by $(1 + \mathcal{O}(E/10GeV))$, where $E < 10$ MeV, so are negligible for typical reactor neutrinos experiments.

series in L/E and identify the coefficient of the linear term in $L/2E$ as the effective Δm^2 and include all the higher order terms in the phase[3]. Then,

$$\Omega \;=\; 2\Delta_{ee} + \phi \tag{3}$$

$$\text{where} \quad \Delta m^2_{ee} \;\equiv\; \frac{\partial\, \Omega}{\partial(L/2E)}\Big|_{\frac{L}{E}\to 0} = \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32} \tag{4}$$

$$\text{and} \qquad \phi \;\equiv\; \Omega - 2\Delta_{ee} = \arctan(\cos 2\theta_{12}\tan\Delta_{21}) - \Delta_{21}\cos 2\theta_{12}. \tag{5}$$

With this separation, $2\Delta_{ee}$ varies at the atmospheric scale, 0.5 km/MeV, whereas ϕ varies at the solar oscillation scale, 15 km/MeV, and

$$\phi = 0, \quad \frac{\partial\, \phi}{\partial(L/2E)} = 0 \quad \text{and} \quad \frac{\partial^2\, \phi}{\partial(L/2E)^2} = 0 \quad \text{at} \;\frac{L}{E} = 0,$$

therefore, in a power series in L/E, ϕ starts at $(\Delta m^2_{21}L/E)^3$ (see eqn. 11).

Since Ω only appears as $\cos\Omega$, it is useful to redefine $\Omega = 2|\Delta_{ee}| \pm \phi$, so that the sign associated with the mass ordering appears only in front of ϕ. If and only if this sign is determined, is the mass ordering determined in ν_e disappearance experiments.

There are three things worth noting about writing the exact ν_e survival probability as in eqn. 2, with Ω given by eqn. 3:

- The effective atmospheric Δm^2 associated with θ_{13} oscillation is a simple combination of the fundamental parameters, see eqn. 4 above or in ref. [8] as they are identical,

$$\begin{aligned}\Delta m^2_{ee} \;&=\; \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32} \\ &=\; \Delta m^2_{31} - \sin^2\theta_{12}\Delta m^2_{21} = \Delta m^2_{32} + \cos^2\theta_{12}\Delta m^2_{21} \\ &=\; m^2_3 - (\cos^2\theta_{12}m^2_1 + \sin^2\theta_{12}m^2_2).\end{aligned}$$

  Thus Δm^2_{ee} is simple the "ν_e average of Δm^2_{31} and Δm^2_{32}," since the ν_e ratio of ν_1 to ν_2 is $\cos^2\theta_{12}$ to $\sin^2\theta_{12}$, and determines the L/E scale associated with the θ_{13} oscillations.

- The modulation of the amplitude associated with the θ_{13} oscillation, is manifest in the square root multiplying the $\cos\Omega$ oscillating term, where

$$\sqrt{1-\sin^2 2\theta_{12}\sin^2\Delta_{21}} \;=\; \begin{cases} 1 & \text{at} \quad \Delta_{21} = n\pi \\ \cos 2\theta_{12} \approx 0.4 & \text{at} \quad \Delta_{21} = (2n+1)\pi/2 \end{cases} \tag{6}$$

[3] Appendix A of [9] contains a discussion of an effective Δm^2 as a function of L/E for arbitrary L/E. At $L/E = 0$ this definition is identical to Δm^2_{ee}.

for $n = 0, 1, 2, \cdots$. Thus, at solar oscillation minima, when $\Delta_{21} = 0, \pi, 2\pi, ...$, the oscillation amplitude is just $\sin^2 2\theta_{13}$, whereas at solar oscillation maxima, when $\Delta_{21} = \pi/2, 3\pi/2, ...$, the oscillation amplitude is $\cos 2\theta_{12} \sin^2 2\theta_{13}$ i.e. reduced by approximately 60%.

- The phase, ϕ, causes an advancement (retardation) of the θ_{13} oscillation for the normal (inverted) mass ordering of the neutrino mass eigenstates. ϕ is a "rounded" staircase function[4], which is zero and has zero first and second derivatives at $L/E = 0$ ($\Delta_{21} = 0$), but then between $L/E \sim 10 - 20$ km/MeV ($\Delta_{21} \sim \frac{\pi}{3} - \frac{2\pi}{3}$) rapidly jumps by $2\pi \sin^2\theta_{12}$, and this pattern is repeated for every increase of $L/E \sim 30$ km/MeV (Δ_{21} by π), i.e.

$$\phi(\Delta_{21} \pm \pi) = \phi(\Delta_{21}) \pm 2\pi \sin^2\theta_{12}, \tag{7}$$

see Fig. 2. Also shown on the same plot is $2|\Delta_{ee}|$ divided by 80. This number 80 was chosen so that $2|\Delta_{ee}|$ fits on the same plot and to demonstrate that $2|\Delta_{ee}| \geq 80\ \phi$ so that the shift in phase caused by ϕ is never bigger than a 1.25% effect. Also for $L/E < 5$ km/MeV, the shift in phase is much smaller than this, see next section.

A. Short Baseline Experiments ($0 < L/E < 1$ km/MeV)

For reactor experiments with baselines less than 2 km, the exact expression eqn. 2 contains elements which require measurement uncertainties on the oscillation probability to better than one part in 10^4. This is way beyond the capability of the current or envisaged experiments. This occurs because for experiments at these baselines, the following conditions on the kinematic phases are satisfied,

$$0 < |\Delta_{31}| \approx |\Delta_{32}| < \pi \quad \Rightarrow 0 < \Delta_{21} < 0.1, \tag{8}$$

and some elements of eqn. 2 dependent on higher powers of Δ_{21}. These elements are

- The modulation of the θ_{13} oscillation amplitude which when expanded in powers of

[4] In the limit, $\sin^2\theta_{12} \to \frac{1}{2}$, one recovers the well known result that this rounded staircase function becomes a true staircase or step function.

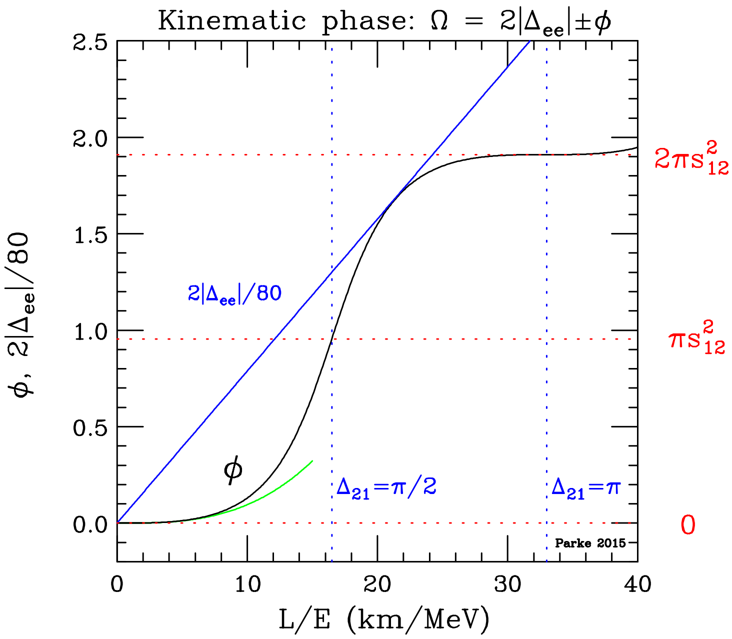

FIG. 2: The L/E dependence of the two components that make up the kinematic phase $\Omega = 2|\Delta_{ee}| \pm \phi$ associated with the θ_{13} oscillation, eqn. 2. ϕ is the black staircase function which increases by $2\pi \sin^2\theta_{12}$ for every increase in Δ_{21} by π, see eqn. 7. The blue straight line is $2|\Delta_{ee}|/80$, which is always greater than or equal to ϕ. The green curve is the Δ_{21}^3 approximation to ϕ given in eqn. 11, which is an excellent approximation for $L/E < 8$ km/MeV.

Δ_{21} is given by

$$\sqrt{1-\sin^2 2\theta_{12}\sin^2\Delta_{21}} = 1 - 2\sin^2\theta_{12}\cos^2\theta_{12}\Delta_{21}^2 + \mathcal{O}(\Delta_{21}^4) \quad (9)$$

$$= 1 + \mathcal{O}(< 10^{-3}) \quad (10)$$

Remember, this amplitude modulation factor is multiplied by $\frac{1}{2}\sin^2 2\theta_{13} \sim 0.05$. Reducing the effect of the amplitude modulation to less than one part in 10^4.

- The advancement or retardation of the kinematic phase, Ω, caused by ϕ whose sign depends on the mass ordering. For small values of Δ_{21} the advancing/retarding phase can be written as

$$\phi = \frac{1}{3}\cos 2\theta_{12}\sin^2 2\theta_{12}\Delta_{21}^3 + \mathcal{O}(\Delta_{21}^5) \quad (11)$$

then using this approximation in the kinematic phase Ω, we have

$$\begin{aligned}\cos(2|\Delta_{ee}| \pm \phi) &= \cos(2|\Delta_{ee}|)\cos\phi \mp \sin(2|\Delta_{ee}|)\sin\phi \\ &= \cos(2|\Delta_{ee}|) \mp \frac{1}{3}\cos 2\theta_{12}\sin^2 2\theta_{12}\Delta_{21}^3 \sin(2|\Delta_{ee}|) + \mathcal{O}(\Delta_{21}^5) \\ &= \cos(2|\Delta_{ee}|) + \mathcal{O}(< 10^{-4})\end{aligned} \tag{12}$$

Again remember, that we have a further reduction by $\frac{1}{2}\sin^2 2\theta_{13} \sim 0.05$. Making the phase advancement or retardation significantly smaller than even the amplitude modulation for these experiments.

Using this information in the ν_e survival probability, we can replace eqn. 2 by

$$P_{\text{short}}(\bar{\nu}_e \to \bar{\nu}_e) = 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \sin^2 2\theta_{13}\sin^2|\Delta_{ee}|. \tag{13}$$

which is accurate to better than one part in 10^{-4}. In Fig. 3 the fractional difference between eqn. 2 and 13 is shown for an experiment with a baseline of 1.6 km. Since the measurement uncertainty on the ν_e survival probability is much greater ($> 0.01\%$) than the difference between the exact, eqn. 2, and the approximate, eqn. 13, survival probabilities, use of either will result in the same measured values of the parameters $\sin^2 2\theta_{13}$ and $|\Delta m^2_{ee}|$ i.e. the measurement uncertainties will dominate.

If new, extremely precise, short baseline experiments ever need a more accurate survival probability, one could easily add the first correction of the amplitude modulation, giving

$$\begin{aligned}P_{\text{xshort}}(\bar{\nu}_e \to \bar{\nu}_e) &= 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} \\ &\quad - \sin^2 2\theta_{13}\,[\ \sin^2|\Delta_{ee}| + \sin^2\theta_{12}\cos^2\theta_{12}\Delta_{21}^2\cos(2|\Delta_{ee}|)\ ]\end{aligned} \tag{14}$$

and this would improve the accuracy of the approximation to better than one part in 10^5.

An alternative way to derive these approximate survival probability, eqn 13 & 14, is given in the Appendix V.

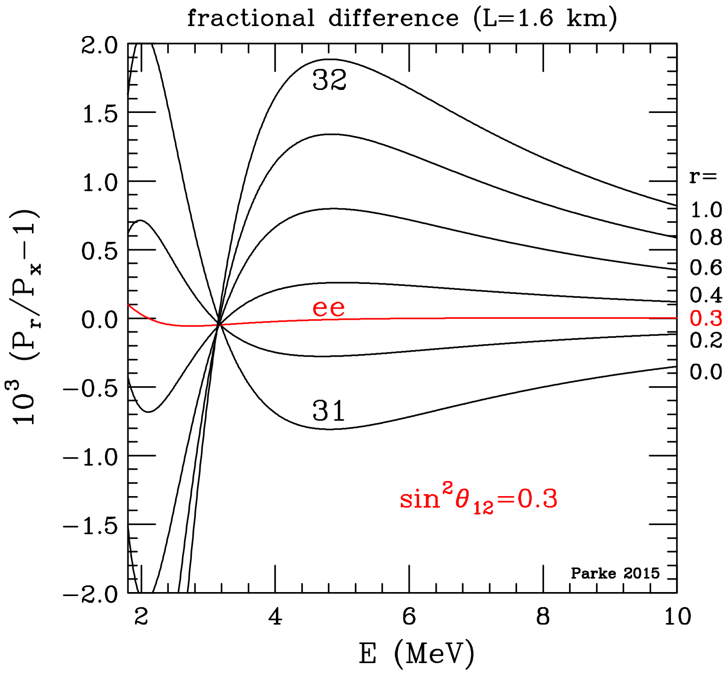

FIG. 3: The fractional difference between the exact survival probability, eqn. 1, and a sequence of approximate survival probabilities, where $\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32}$ is replaced with $\sin^2(\Delta m^2_{rr}L/4E)$ with $\Delta m^2_{rr} \equiv (1-r)\Delta m^2_{31} + r\Delta m^2_{32}$. Clearly, $r = \sin^2\theta_{12}$ minimizes the absolute value of the fractional difference between the exact and approximate survival probabilities. Thus, the approximation of replacing $\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32}$ with $\sin^2\Delta_{ee}$ gives an approximate survival probability that is better than one part in 10^4 over the L/E range of the Daya Bay, RENO and Double Chooz experiments.

III. OTHER POSSIBLE DEFINITIONS OF AN EFFECTIVE Δm^2

A. A New Definition of the Effective Δm^2

Another possible way to define an effective Δm^2, here I will use the symbol Δm^2_{XX}, is as follows

$$\Delta m^2_{XX} \equiv \sqrt{\cos^2\theta_{12}\ (\Delta m^2_{31})^2 + \sin^2\theta_{12}\ (\Delta m^2_{32})^2}\ . \tag{15}$$

Clearly this definition is independent of L/E and it guarantees that, in the limit $L/E \to 0$, that

$$\sin^2\Delta_{XX} = \cos^2\theta_{12}\ \sin^2\Delta_{31} + \sin^2\theta_{12}\ \sin^2\Delta_{32}. \tag{16}$$

One can then show, that

$$|\Delta m^2_{XX}| \; = \; |\Delta m^2_{ee}| \; (1 + \mathcal{O}\Big[\; \Big(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} \Big)^2 \; \Big]) \tag{17}$$

So $|\Delta m^2_{XX}|$ is essentially equal to $|\Delta m^2_{ee}|$ up to correction on the order of 10^4, including the effects of the solar mixing angle[5].

A variant of this definition of an effective Δm^2 (here I will used the subscripts "xx"), is defined in terms of the position of the first extremum of $(\cos^2\theta_{12} \; \sin^2\Delta_{31} + \sin^2\theta_{12} \; \sin^2\Delta_{32})$ in L/E. If this extremum occurs at $(L/E)|_1$, then define

$$\Delta m^2_{xx} \; \equiv \; \frac{2\pi}{(L/E)|_1}, \tag{18}$$

so that, at this extremum, $\frac{\Delta m^2_{xx} L}{4E} = \frac{\pi}{2}$. With this definition it is again easy to show that,

$$|\Delta m^2_{xx}| \; = \; |\Delta m^2_{ee}| \; (1 + \mathcal{O}\Big[\; \Big(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}} \Big)^2 \; \Big]). \tag{19}$$

Again, essentially equal to Δm^2_{ee}.

In both $|\Delta m^2_{XX}|$ and $|\Delta m^2_{xx}|$, the corrections of order $\left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right)^2$, come from the amplitude modulation of the θ_{13} oscillation and the coefficients are $\frac{1}{2}\sin^2\theta_{12}\cos^2\theta_{12}$ and $\sin^2\theta_{12}\cos^2\theta_{12}$ respectively. Note, these corrections are mass ordering independent.

B. Daya Bay's Original Definition of the Effective Δm^2

In ref. [4] & [5], the Daya Bay experiment used the following definition for an effective Δm^2, here I will use the symbol Δm^2_{YY},

$$\sin^2\Delta_{YY} \; \equiv \; \cos^2\theta_{12} \; \sin^2\Delta_{31} + \sin^2\theta_{12} \; \sin^2\Delta_{32}. \tag{20}$$

which implies that

$$\Delta m^2_{YY} \; \equiv \; \left(\frac{4E}{L}\right) \arcsin\left[\sqrt{(\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32})}\right]. \tag{21}$$

For $L/E < 0.3$ km/MeV, so that $\sin^2\Delta_{3i} = \Delta^2_{3i}$ is a good approximation, Δm^2_{YY} is approximately independent of L/E. However, for larger values of L/E, Δm^2_{YY} is L/E dependent,

[5] The following, useful identity is easy to prove by writing $\Delta m^2_{21} = \Delta m^2_{31} - \Delta m^2_{32}$:
$(\cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32})^2 = [\cos^2\theta_{12}(\Delta m^2_{31})^2 + \sin^2\theta_{12}(\Delta m^2_{32})^2] - \cos^2\theta_{12}\sin^2\theta_{12}(\Delta m^2_{21})^2$.

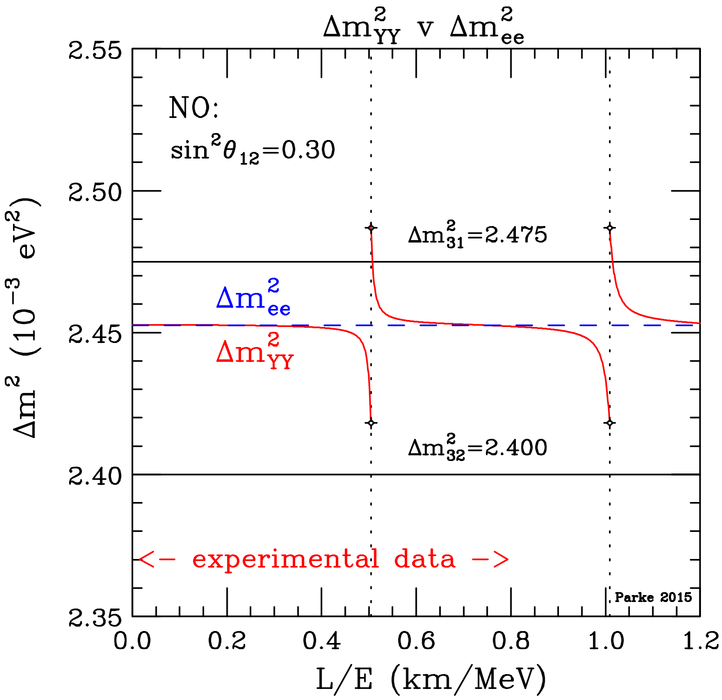

FIG. 4: Daya Bay's original definition, see [4] and [5], for an effective Δm^2, Δm^2_{YY}, is given by the solid red line. Notice the sizeable L/E dependence near oscillation minimum and maximum (vertical black dotted lines). At all oscillation extrema, this definition is discontinuous and the size of the discontinuity is $\sin 2\theta_{12}\Delta m^2_{21} \sim 3\%$. The first discontinuity occurs in the middle of the experimental data of the Daya Bay, RENO and Double Chooz experiments. The L/E independent lines: $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$ is the blue dashed, Δm^2_{31} and Δm^2_{32} are the labelled black lines. This figure is for normal mass ordering with $\sin^2\theta_{12} = 0.30$ and $\Delta m^2_{ee} = 2.453 \times 10^{-3}$ eV^2.

exactly in the L/E region, $0.3 < L/E < 0.7$ km/MeV, where the bulk of the experimental data from the far detectors of the Daya Bay experiment is obtained. In the center of this L/E region, $L/E \approx 0.5$ km/MeV, is the position of the oscillation minimum.

Furthermore, the definition given by Eqn. 20, is discontinuous at oscillation minimum (OM). This occurs because as you increase L/E, the L.H.S. eqn. 20 can go to 1, whereas the R.H.S. never reaches 1. So to satisfy Eqn. 20, as you increase L/E, your effective Δm^2 must be discontinuous at OM and the size of this discontinuity is given by[6]

$$\delta\ \Delta m^2_{EE}|_{OM} \ = \ \sin 2\theta_{12}\Delta m^2_{21} \tag{22}$$

[6] The following identity is useful to understand this point, $\sin^2(\frac{\pi}{2} \pm \epsilon) \approx 1 - \epsilon^2$ where here $\epsilon = s_{12}c_{12}\Delta_{21}$. Similarly at oscillation maximum, $\sin^2(\pi \pm \epsilon) \approx \epsilon^2$.

which is of order of 3%. In Fig. 4, the various Δm^2's are plotted as a function of L/E.

The relationship between Daya Bay's Δm^2_{YY} and that of the previous section is as follows

$$\Delta m^2_{YY}|_{L/E\to 0} = \Delta m^2_{ee}\sqrt{\left(1+\sin^2\theta_{12}\cos^2\theta_{12}\left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right)^2\right)}\,. \tag{23}$$

Therefore they are identical up to corrections of $\mathcal{O}(10^{-4})$ as L/E $\to 0$.

Given that Δm^2_{YY} is L/E dependent one should take the average of Δm^2_{YY} over the L/E range of the experiment

$$\langle\Delta m^2_{YY}\rangle \;=\; \frac{\int_{(L/E)_{min}}^{(L/E)_{max}} d(L/E)\;\Delta m^2_{YY}}{[(L/E)_{max}-(L/E)_{min}]}\,. \tag{24}$$

For the current experiments this range is from [0,0.8] km/MeV and then from Fig, 4 it is clear that

$$\langle\Delta m^2_{YY}\rangle \;\approx\; \Delta m^2_{ee}, \tag{25}$$

if the discontinuity at OM is averaged over in a symmetric way. In practice, of course, one needs to weight the average over the L/E range by the experimental L/E sensitivity. This is something that can only be performed by the experiment. This was not performed in ref. [4] or [5].

C. Daya Bay's New Definition of the Effective Δm^2

After the issue with Δm^2_{YY} was pointed out to the Daya Bay collaboration [11], the Daya Bay collaboration defined a new effective Δm^2 in the supplemental material of ref. [6]. Here I will use the symbol Δm^2_{ZZ} for this new definition which is defined in terms of the kinematic phase, Ω, given eqn. 3, as

$$\begin{aligned}\Delta m^2_{ZZ} \;&\equiv\; \frac{2E}{L}\,\Omega, \\ &=\; |\Delta m^2_{ee}| \pm \frac{2E}{L}\,\phi.\end{aligned} \tag{26}$$

Unfortunately, since ϕ is not a linear function in L/E, Δm^2_{ZZ} is also L/E dependent. In contrast remember, from eqn. 4, $\Delta m^2_{ee} \equiv \frac{\partial\;\Omega}{\partial(L/2E)}\Big|_{\frac{L}{E}\to 0}$.

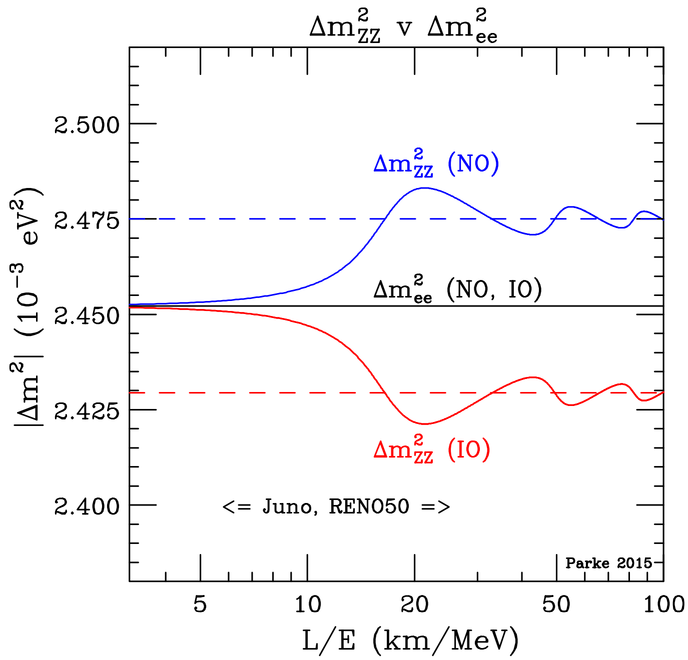

FIG. 5: Daya Bay's new definition, see [6], of an effective Δm^2, Δm^2_{ZZ}, for $\bar{\nu}_e$ disappearance compared $\Delta m^2_{ee} \equiv \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}$. The L/E range appropriate for JUNO and RENO-50 is 6 to 25 km/MeV, exactly the range in which Δm^2_{ZZ} changes by $\pm 1\%$. Yet, the expected accuracy of these two experiments is better than 0.5%. The sign of the variation of Δm^2_{ZZ} is mass ordering dependent. The blue and red dashed lines are Δm^2_{31} for NO and IO respectively.

For short baseline experiments, such as Daya Bay, RENO and Double Chooz, this dependence is small, and can be calculated analytically from eqn. (11),

$$\begin{aligned}\Delta m^2_{ZZ} &= |\Delta m^2_{ee}| \left[1 \pm \frac{1}{6}\cos 2\theta_{12}\sin^2 2\theta_{12}\left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right)\Delta^2_{21} + \mathcal{O}\left(\left(\frac{\Delta m^2_{21}}{\Delta m^2_{ee}}\right)\Delta^4_{21}\right)\right] \\ &\approx |\Delta m^2_{ee}| \left[1 \pm 6\times 10^{-6}\left(\frac{L/E}{0.5\ \mathrm{km/MeV}}\right)^2\right]. \end{aligned} \tag{27}$$

Given the current and expected future accuracy of the current short baseline experiments, the L/E dependence in Δm^2_{ZZ} can be ignored.

However for future experiments such as JUNO, [12], and RENO-50, [13], the L/E dependence of Δm^2_{ZZ} is significant, see Fig. 5. These experiments explore an L/E range from 6 to 25 km/MeV. In this range, Δm^2_{ZZ} changes by $\sim 1\%$ whereas the expected accuracy of the measurement is better than 0.5%, see [12]. So this definition of Δm^2_{ZZ} is not appropriate for these experiments unless the experiments want to do the L/E averaging as discussed in the previous section.

IV. CONCLUSIONS

Having a single, L/E independent effective Δm^2 which can be used for reactor experiments of any L/E is highly desirable. Δm^2_{ee}, defined in eqn. 4, is the best effective Δm^2 for ν_e disappearance in the literature for the following reasons:

- Is independent of L/E for all values of L/E.
- Is a simple combination of fundamental parameters:

$$\begin{aligned}\Delta m^2_{ee} &\equiv \cos^2\theta_{12}\ \Delta m^2_{31} + \sin^2\theta_{12}\ \Delta m^2_{32} \\ &= \Delta m^2_{31} - \sin^2\theta_{12}\ \Delta m^2_{21} = \Delta m^2_{32} + \cos^2\theta_{12}\ \Delta m^2_{21}. \\ &= m^2_3 - (\cos^2\theta_{12} m^2_1 + \sin^2\theta_{12} m^2_2).\end{aligned}$$

- Has a direct, simple, physical interpretation:
  Δm^2_{ee} is "the ν_e weighted average of Δm^2_{31} and Δm^2_{32}," since the ratio of the ν_e fraction in $\nu_1 : \nu_2$ is $\cos^2\theta_{12} : \sin^2\theta_{12}$.
- Can be used in the future medium baseline reactor experiments, $L/E > 6$ and < 25 km/MeV, using the exact oscillation probability,

$$\begin{aligned}P(\bar\nu_e \to \bar\nu_e) &= 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} \\ &\quad -\frac{1}{2}\sin^2 2\theta_{13}\left(1 - \sqrt{1-\sin^2 2\theta_{12}\sin^2\Delta_{21}}\ \cos(2|\Delta_{ee}| \pm \phi)\right),\end{aligned}$$

  where $\phi \equiv \arctan(\cos 2\theta_{12}\tan\Delta_{21}) - \Delta_{21}\cos 2\theta_{12}$. This probability can be used to determine solar parameters $\sin^2\theta_{12}$ and Δm^2_{21} as well as $|\Delta m^2_{ee}|$ with unprecedented precision and may be able to determine the atmospheric mass ordering, if the sign in front of ϕ can be determined at high enough confidence level.
- Can be used in the current short baseline reactor experiments, $L/E < 1$ km/MeV, using the approximate oscillation probability,

$$P(\bar\nu_e \to \bar\nu_e) \approx 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} - \sin^2 2\theta_{13}\sin^2\Delta_{ee}.$$

This is trivially obtained from the exact expression, eqn. 2, by setting both the amplitude modulation to one and the phase advancement or retardation to zero,

$$\sqrt{1-\sin^2 2\theta_{12}\sin^2\Delta_{21}} \rightarrow 1 \quad \text{and} \quad \phi \rightarrow 0$$

as these are higher order effects. This approximates the exact oscillation probability to better than 1 part in 10^4 and can be improved in a systematic way, see Eqn. 29. This probability, using the current experimental data, allows for an accurate determination of mixing angle θ_{13} and the atmospheric mass splitting $|\Delta m^2_{ee}|$, independent of the atmospheric mass ordering, and only very weakly dependent on our current knowledge of the solar parameters, through the solar term. From a measured value of $|\Delta m^2_{ee}|$, using short baseline reactor experiments, it is simple to calculate Δm^2_{31} for both mass orderings. However, the uncertainties on Δm^2_{31} will be more dependent on solar parameters, measured by other experiments, than $|\Delta m^2_{ee}|$.

Furthermore, Δm^2_{ee}, defined by eqn. 4, naturally appears as the renormalized atmospheric Δm^2 in neutrino propagation in matter, see [14], as using this renormalized Δm^2 significantly reduces the complexity of the oscillation probabilities.

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V. APPENDIX A

In this Appendix, an alternative derivation of why $\sin^2\Delta_{ee}$ is the most accurate approximation for $\cos^2\theta_{12}\sin^2\Delta_{31}+\sin^2\theta_{12}\sin^2\Delta_{32}$ is given. Starting with the following linear combination of Δm^2_{31} and Δm^2_{32}, given by

$$\Delta_{rr} \equiv (1-r)\Delta_{31} + r\Delta_{32} \quad \text{then} \quad \Delta_{31} \;=\; \Delta_{rr} + r\Delta_{21}, \quad \Delta_{32} = \Delta_{rr} - (1-r)\Delta_{21},$$

since $\Delta_{21} = \Delta_{31} - \Delta_{32}$ and r is a number between [0,1]. The relevant range of kinematic phases is $0 \leq |\Delta_{31}| \sim |\Delta_{32}| < \pi$ and $0 \leq \Delta_{21} < \pi/30 \approx 0.1$. So it is a simple exercise to perform a Taylor series expansion about Δ_{rr} using expansion parameter Δ_{21}, and obtain (using $c^2_{12} \equiv \cos^2\theta_{12}$ and $s^2_{12} \equiv \sin^2\theta_{12}$)

$$\begin{aligned} c^2_{12}\sin^2\Delta_{31} + s^2_{12}\sin^2\Delta_{32} \;=\; & \sin^2\Delta_{rr} \\ & +[c^2_{12}r - s^2_{12}(1-r)]\ \Delta_{21}\sin(2\Delta_{rr}) \\ & +[c^2_{12}r^2 + s^2_{12}(1-r)^2]\ \Delta^2_{21}\cos(2\Delta_{rr}) \\ & -\frac{2}{3}[c^2_{12}r^3 - s^2_{21}(1-r)^3]\ \Delta^3_{21}\sin(2\Delta_{rr}) \\ & -\frac{1}{3}[c^2_{12}r^4 + s^2_{12}(1-r)^4]\ \Delta^4_{21}\cos(2\Delta_{rr}) \\ & +\mathcal{O}(\Delta^5_{21}). \end{aligned} \tag{28}$$

The choice of $r = s^2_{12}$, making $\Delta_{rr} = \Delta_{ee}$, does two great things for this Taylor series expansion:

1. the coefficient of Δ_{21} vanishes, since $[c^2_{12}r - s^2_{12}(1-r)] = 0$,
2. and, the coefficient of Δ^2_{21} is a minimized, since

$$\frac{\partial}{\partial r}[c^2_{12}r^2 + s^2_{12}(1-r)^2]\bigg|_{r=s^2_{12}} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial^2 r}[c^2_{12}r^2 + s^2_{12}(1-r)^2] > 0.$$

No other value of r satisfies either of these requirements. Thus, using $r = s^2_{12}$ makes $\sin^2\Delta_{ee}$ the best possible approximation to $c^2_{12}\sin^2\Delta_{31} + s^2_{12}\sin^2\Delta_{32}$ for a constant Δm^2 and the corrections are tiny, of $\mathcal{O}(10^{-3})$ for L/E < 1 km/MeV.

Using this expansion the ν_e survival probability can be written as

$$
\begin{aligned}
P_{\rm xshort}(\bar{\nu}_e \to \bar{\nu}_e) \;=\; & 1 - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} \\
& - \sin^2 2\theta_{13} \Bigg[\; \sin^2|\Delta_{ee}| \\
& \quad + \sin^2\theta_{12}\cos^2\theta_{12}\Delta_{21}^2\cos(2|\Delta_{ee}|) \;] \\
& \quad \mp \frac{1}{6}\cos 2\theta_{12}\sin^2 2\theta_{12}\;\Delta_{21}^3\sin(2|\Delta_{ee}|) \\
& \quad - \frac{1}{48}\sin^2 2\theta_{12}\;[4 - 3\sin^2 2\theta_{12}]\;\Delta_{21}^4\cos(2|\Delta_{ee}|) \\
& \quad + \mathcal{O}(\Delta_{21}^5)\Bigg]. \qquad (29)
\end{aligned}
$$

VI. APPENDIX B

The simplist way to show that

$$\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32} \;=\; \frac{1}{2}\left(1 - \sqrt{1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}}\;\cos\Omega\right) \qquad (30)$$

with

$$
\begin{aligned}
\Omega \;&=\; 2\Delta_{ee} + \phi \qquad (31) \\
\text{where} \quad \Delta m^2_{ee} \;&\equiv\; \frac{\partial\,\Omega}{\partial(L/2E)}\Big|_{\frac{L}{E}\to 0} = \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32} \qquad (32) \\
\text{and} \qquad \phi \;&\equiv\; \Omega - 2\Delta_{ee} = \arctan(\cos 2\theta_{12}\tan\Delta_{21}) - \Delta_{21}\cos 2\theta_{12}, \qquad (33)
\end{aligned}
$$

is to write

$$c^2_{12}\sin^2\Delta_{31} + s^2_{12}\sin^2\Delta_{32} \;=\; \frac{1}{2}\Big(1 - (c^2_{12}\cos 2\Delta_{31} + s^2_{12}\cos 2\Delta_{32})\Big), \qquad (34)$$

using $c^2_{12} \equiv \cos^2\theta_{12}$ and $s^2_{12} \equiv \sin^2\theta_{12}$.

Then, if we rewrite $2\Delta_{31}$ and $2\Delta_{32}$ in terms of $(\Delta_{31} + \Delta_{32})$ and Δ_{21}, we have

$$
\begin{aligned}
c^2_{12}\cos 2\Delta_{31} + s^2_{12}\cos 2\Delta_{32} \;&=\; c^2_{12}\cos(\Delta_{31} + \Delta_{32} + \Delta_{21}) + s^2_{12}\cos(\Delta_{31} + \Delta_{32} - \Delta_{21}) \\
&=\; \cos(\Delta_{31} + \Delta_{32})\cos\Delta_{21} - \sin(\Delta_{31} + \Delta_{32})\cos 2\theta_{12}\sin\Delta_{21}.
\end{aligned}
$$

Since

$$\cos^2\Delta_{21} + \cos^2 2\theta_{12}\sin^2\Delta_{21} = 1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}$$

we can then write

$$c_{12}^2 \cos 2\Delta_{31} + s_{12}^2 \cos 2\Delta_{32} \;=\; \sqrt{1 - \sin^2 2\theta_{12} \sin^2 \Delta_{21}}\; \cos\Omega, \tag{35}$$

where

$$\Omega \;=\; \Delta_{31} + \Delta_{32} + \arctan(\cos 2\theta_{12} \tan \Delta_{21}).$$

Applying the prescription given in Sec. II to sepearate Ω into an effective 2Δ and a phase, ϕ, we find

$$\frac{\partial\,\Omega}{\partial L/2E}\Big|_{\frac{L}{E}\to 0} \;=\; \cos^2\theta_{12}\Delta m_{31}^2 + \sin^2\theta_{12}\Delta m_{32}^2 = \Delta m_{ee}^2$$

$$\text{and} \quad \phi \;=\; \Omega - 2\Delta_{ee} = \arctan(\cos 2\theta_{12} \tan \Delta_{21}) - \Delta_{21}\cos 2\theta_{12}$$

thus

$$\Omega \;=\; 2\Delta_{ee} + (\arctan(\cos 2\theta_{12} \tan \Delta_{21}) - \Delta_{21}\cos 2\theta_{12}), \tag{36}$$

qed.

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