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ABSTRACT

We extend an earlier investigation of the thermodynamics of static black holes in an Einstein-Horndeski theory of gravity coupled to a scalar field, by including now an electromagnetic field as well. By studying the two-parameter families of charged static black holes, we obtain much more powerful constraints on the thermodynamics since, unlike in the uncharged one-parameter case, now the right-hand side of the first law is not automatically integrable. In fact, this allows us to demonstrate that there must be an additional contribution in the first law, over and above the usual terms expected for charged black holes. The origin of the extra contribution can be attributed to the behaviour of the scalar field on the horizon of the black hole. We carry out the calculations in four dimensions and also in general dimensions. We also derive the ratio of viscosity to entropy for the dual boundary field theory, showing that the usual viscosity bound for isotropic solutions can be violated, with the ratio depending on the mass and charge.

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1 Introduction

Black holes are the most fundamental objects predicted by Einstein's theory of gravity. They have been extensively studied, and many important properties have been established. Subject to certain assumptions, the no-hair theorems establish that all properties of a black hole are completely characterized by a few conserved charges (see, for example, [1]). However, general arguments and numerical evidence suggests that black holes in Einstein-scalar theories can develop scalar "hair," both in asymptotically-AdS and asymptotically-flat spacetimes [2, 3]. Recently, many explicit examples of black holes carrying scalar hair have been found, in four and higher dimensions [4–13]. It was shown that the first law of thermodynamics of (charged) AdS black holes can be modified by the boundary conditions of the scalar field at asymptotic infinity [14, 15]. The first such explicit example, with a closed-form expression for the solution, was provided by the Kaluza-Klein AdS dyonic black hole [16]. It is important to emphasise that we take the first law, more or less by definition, to mean the relation between the infinitesimal variations of the parameters characterising the asymptotic form of the black hole solution near infinity and the variation of the parameters characterising the near-horizon expression for the black hole solution. In a simple example such as a Reissner-Nordström (RN) black hole of mass M and charge Q, this relation equates $\delta M - \Phi\,\delta Q$ at infinity (in the gauge where $\Phi = 0$ on the horizon) to $T\delta S$ on the horizon.

In a typical theory of Einstein gravity, matter fields couple to gravity minimally through the metric. However, theories such as Brans-Dicke theory [17], which involves a non-minimally coupled scalar, have also been widely studied. Recently, Galileon theories [18] have attracted attention, in which a scalar field has non-minimal derivative couplings to the curvature tensor. In fact, general classes of theories of this kind were constructed by Horndeski in the early 1970s [19]. These gravity-scalar theories are characterised by the property that both the gravity and the scalar field equations contain no more than second derivatives, which is analogous to the situation in Lovelock gravities [20].

The thermodynamics of black holes in Einstein-Horndeski gravity was studied in [21]. The theory studied there contains the usual Schwarzschild black hole, whose thermodynamics is the same as in Einstein gravity. A no-go theorem was established for asymptotically-flat spacetimes, showing that no further black hole solutions existed [22, 23]. When the theory contains a cosmological constant, there is in addition a branch of scalar-hairy black holes [24]. (See also [25–29].) Surprises emerged for these black hole solutions [21]. Firstly, the standard Wald entropy formula is no longer valid. The standard formula [30,31] is given by

$$S_W = -\frac{1}{8}\int_H d^{n-2}x\sqrt{h}\,\frac{\partial L}{\partial R^{abcd}}\epsilon^{ab}\epsilon^{cd}\,, \tag{1.1}$$

integrated over the horizon H, where L is defined by the action $I = \int d^n x\sqrt{-g}L$, $\sqrt{h}$ is the area density on the horizon and ϵ^{ab} is the normal bivector. As we shall see in section 4, we find that there is an additional contribution to the entropy, involving the scalar field, as a result of its non-minimal coupling to gravity via the Horndeski term.

Secondly, the quantum statistical relation $\Phi_{\rm thermo} = M - TS = I_E\,T$ [32] , where I_E is the regularised Euclidean action and $\Phi_{\rm thermo}$ is the thermodynamic potential (or free energy), appears to be no longer valid. This may be related to the theory's not admitting a Hamiltonian formulation. In particular the mass, as deduced from $\Phi_{\rm thermo} = I_E\,T$, disagrees with its calculation via the first law of thermodynamics if the entropy is determined simply using (1.1) [21].

To resolve the issue and derive a proper first law of black hole thermodynamics, the Wald procedure for computing a conserved Hamiltonian variation was applied to the black holes in [21], paying particular attention to details of the scalar contribution. It turns out that the first law for the black holes in this theory is modified as a consequence of the unusual behavior of the scalar field on the event horizon. It arises because the derivative of the scalar field χ diverges on the horizon, although there is no physical divergence since all invariants, including $g^{\mu\nu}\,\partial_\mu\chi\,\partial_\nu\chi$, remain finite.

The first law of black hole thermodynamics is a consequence of the equality between the variation of the Hamiltonian at asymptotic infinity and on the horizon, i.e.

$$\delta\mathcal{H}_\infty = \delta\mathcal{H}_+\,. \tag{1.2}$$

This identity was shown to be valid for the scalar-hairy black holes in Einstein-Horndeski gravity [21]. In order to be able to interpret (1.2) as a first law, one then needs to be able to express the terms appearing in the variations on each side of the equation in terms of thermodynamic variables characterising the solutions. For the black hole solutions in [21], one can then identify $\delta\mathcal{H}_\infty = \delta M$ and $\delta\mathcal{H}_+ = T\delta S$, and the first law $\delta M = T\delta S$ is thus ostensibly derived (with S effectively being *defined* by $\delta S = T^{-1}\,\delta\mathcal{H}_+$). The weak point in this argument is that the static scalar-hairy black holes studied in [21] depend on only one parameter, and hence there is no non-trivial integrability check in the above derivation. Furthermore, the spherically-symmetric solutions are asymptotically locally AdS, and hence it is difficult to find any independent method to calculate the mass. Thus in this first law, only T can be unambiguously calculated independently of the Wald procedure, and it follows that the validity of $dM = TdS$ for the scalar-hairy black hole in [21] remains questionable, as was discussed in that paper.

In this paper, we consider Einstein-Horndeski gravity minimally coupled to a Maxwell field. As in the case of the two-parameter RN black holes of pure Einstein-Maxwell theory, where the first law $dM = TdS + \Phi_e dQ_e$ implies that there is a non-trivial integrability condition $dT\wedge dS + d\Phi_e\wedge dQ_e = 0$, the Einstein-Horndeski-Maxwell (EHM) theory we shall consider admits two-parameter static black hole solutions, and so again we can obtain further non-trivial insights into the form of the first law by means of integrability considerations. The static electrically-charged black hole solutions in EHM gravity were obtained in [33]. (See also [34].) We shall use the Wald formalism to derive the first law. For a minimally-coupled Maxwell field, the electric charge Q_e and the electrostatic potential Φ_e (i.e the potential difference between the horizon and asymptotic infinity) can be independently and unambiguously defined. We find that for the charged black holes, the Wald identity (1.2) is indeed always satisfied, but reading off a first law from this is somewhat subtle. In a gauge choice where the gauge potential A vanishes on the horizon, we find that $\delta\mathcal{H}_\infty = \delta M - \Phi_e\delta Q_e$, as one would have expected. However, at the horizon we find $\delta\mathcal{H}_+ \neq T\delta S$, and in fact $T^{-1}\delta\mathcal{H}_+$ is not integrable. In order to complete the first law, we find it necessary to introduce the concept of a global scalar charge, which can give a non-trivial contribution to $\delta\mathcal{H}$ on the horizon.

The paper is organised as follows. In section 2 we introduce the charged Horndeski theory that we shall be considering, and present the equations of motion for a static black-hole ansatz. We then review three kinds of static black hole solutions in four dimensions in section 3. Next, we analyze the thermodynamics of these black holes in section 4, by using the Wald formalism. We find that the entropy has a standard contribution of one quarter the area of the event horizon (in accordance with the Wald entropy formula) but with an additional contribution from the scalar field. Furthermore, the first law also requires an additional contribution from the scalar field. In section 5, we generalize these four-dimensional results to arbitrary dimensions. For planar AdS black holes, there exists a global scaling symmetry. We study the corresponding Noether charge and obtain the generalized Smarr relation in section 6. In section 7, the ratio of viscosity to entropy density is calculated. The paper ends with conclusions in section 8.

2 The Theory and Equations of Motion

In this paper we consider an Einstein-Horndeski gravity, minimally coupled to a Maxwell field, with the action given by

$$I = \frac{1}{16\pi}\int d^n x\sqrt{-g}\,L\,,\quad L = \kappa(R - 2\Lambda - \tfrac{1}{4}F^2) - \tfrac{1}{2}(\alpha g^{\mu\nu} - \gamma G^{\mu\nu})\,\partial_\mu\chi\,\partial_\nu\chi\,, \tag{2.1}$$

where κ, α and γ are coupling constants, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$ is the Einstein tensor, and $F = dA$ is the electromagnetic field strength. The theory consists of the metric $g_{\mu\nu}$, the $U(1)$ gauge potential $A = A_\mu dx^\mu$ and the scalar χ. The scalar is axionic, appearing in the Lagrangian only through its derivative. Thus the Lagrangian is invariant under the constant shift transformation $\chi \to \chi + c$. One may gauge this symmetry and obtain Einstein-vector gravity with a non-minimally coupled vector field [35], but we shall not consider this possibility in the present paper.

The equations of motion following from the variations of the metric $g_{\mu\nu}$, the scalar χ and the Maxwell potential $A = A_\mu dx^\mu$ are given by

$$\begin{aligned}
E_{\mu\nu} \equiv\ & \kappa(G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2}F^2_{\mu\nu} + \frac{1}{8}F^2 g_{\mu\nu}) \\
& -\tfrac{1}{2}\alpha\Big(\partial_\mu\chi\partial_\nu\chi - \tfrac{1}{2}g_{\mu\nu}(\partial\chi)^2\Big) - \tfrac{1}{2}\gamma\Big(\tfrac{1}{2}\partial_\mu\chi\partial_\nu\chi R - 2\partial_\rho\chi\,\partial_{(\mu}\chi\, R_{\nu)}{}^\rho \\
& -\partial_\rho\chi\partial_\sigma\chi\, R_\mu{}^\rho{}_\nu{}^\sigma - (\nabla_\mu\nabla^\rho\chi)(\nabla_\nu\nabla_\rho\chi) + (\nabla_\mu\nabla_\nu\chi)\Box\chi + \tfrac{1}{2}G_{\mu\nu}(\partial\chi)^2 \\
& -g_{\mu\nu}\big[-\tfrac{1}{2}(\nabla^\rho\nabla^\sigma\chi)(\nabla_\rho\nabla_\sigma\chi) + \tfrac{1}{2}(\Box\chi)^2 - \partial_\rho\chi\partial_\sigma\chi\, R^{\rho\sigma}\big]\Big) = 0\,,
\end{aligned}$$

$$E_\chi \equiv \nabla_\mu\big((\alpha g^{\mu\nu} - \gamma G^{\mu\nu})\nabla_\nu\chi\big) = 0\,, \qquad E_A^\mu \equiv \nabla_\nu F^{\nu\mu} = 0\,. \tag{2.2}$$

Although the theory involves a total of four derivatives, in the equations of motion no field carries more than two derivatives, and the total of four derivatives arise through non-linearities. This is one of the general characteristic properties of Horndeski theories, and in consequence the perturbations of any background are described by linear differential equations of only second order in derivatives, and hence they can be ghost free.

In this paper, we are concerned with the properties of static charged black hole solutions. The most general static ansatz, after coordinate gauge fixing, can be taken to be

$$ds^2 = -h(r)dr^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{n-2\,,\epsilon}\,, \quad \chi = \chi(r)\,, \quad A = \phi(r)\,dt\,, \tag{2.3}$$

where $d\Omega^2_{n-2\,,\epsilon}$, with $\epsilon = 1\,,0\,,-1$, is the metric for the unit S^{n-2}, the $(n-2)$-torus or the unit hyperbolic $(n-2)$-space. The electrostatic potential ϕ can be expressed in terms of the metric fields by using the Maxwell equation of motion $E_A^\mu = 0$, which implies

$$\phi' = \frac{q}{r^{n-2}}\sqrt{\frac{h}{f}}\,, \tag{2.4}$$

where hereafter, we use a prime to denote a derivative with respect to r. The scalar equation of motion $E_\chi = 0$ gives

$$\Big(r^{n-4}\sqrt{\frac{f}{h}}\Big(\gamma\big((n-2)rfh' + (n-2)(n-3)(f-\epsilon)h\big) - 2\alpha r^2 h\Big)\chi'\Big)' = 0\,. \tag{2.5}$$

There are two further independent equations that follow from $E_{\mu\nu} = 0$, namely

$$\begin{aligned}
&4\kappa\Big((n-2)rf' + (n-2)(n-3)(f-\epsilon) + 2\Lambda r^2\Big) + 2\kappa q^2 r^{6-2n} + 2\alpha r^2 f\chi'^2\\
&\qquad + \gamma(n-2)\Big(4rf\chi'' + \big(3rf' + (n-3)(f+\epsilon)\big)\chi'\Big)f\chi' = 0\,,\\
&4\kappa\Big((n-2)rfh' + (n-2)(n-3)h(f-\epsilon) + 2\Lambda r^2 h\Big) + 2\kappa q^2 r^{6-2n} h\\
&\qquad - 2\alpha r^2 fh\chi'^2 + \gamma(n-2)\Big(3rfh' + (n-3)(3f-\epsilon)h\Big)f\chi'^2 = 0\,.
\end{aligned} \tag{2.6}$$

3 Static Black Hole Solutions in Four Dimensions

In the previous section, we obtained the equations for the static and charged ansatz. The most general analytical solutions of these equations would be difficult to obtain explicitly. In [33], a special class of solutions was obtained, by solving the scalar equation (2.5) by taking

$$\gamma\big((n-2)rfh' + (n-2)(n-3)(f-\epsilon)h\big) - 2\alpha r^2 h = 0\,. \tag{3.1}$$

The resulting solutions turn out to describe black holes. In [21], it was shown that all the static neutral black hole solutions must satisfy (3.1). We expect the same to be true for the charged black holes. In this section, we review the black holes constructed in [33].

Asymptotically Minkowski solution:

When $\alpha = 0$ and $\Lambda = 0$, the theory admits an asymptotically Minkowski solution in four dimensions:

$$\chi' = \sqrt{-\frac{\kappa q^2}{2\gamma r^2 f}}\,, \qquad \phi = \phi_0 - \frac{q}{r} + \frac{q^3}{24 r^3}\,, \qquad f = \frac{64 r^4}{(8r^2 - q^2)^2}\, h\,,$$
$$h = 1 - \frac{\mu}{r} + \frac{q^2}{4r^2} - \frac{q^4}{192 r^4}\,. \tag{3.2}$$

The solution has three integration parameters, ϕ_0, q and μ, of which two correspond to non-trivial parameters. There are two curvature singularities; one is as usual at $r = 0$, whilst the other, where f diverges, is at $r = r_* = q/\sqrt{8}$. Here, we assume without loss of generality that $q \geq 0$. For the solution to describe a black hole, we must have an event horizon $r = r_0$, which is the largest root of $h = 0$. Furthermore, we must require that r_* be inside the event horizon. This implies that

$$h(r_*) = \frac{8}{3} - \frac{\sqrt{8}\mu}{q} < 0\,, \qquad \Longrightarrow \qquad \mu > \frac{2\sqrt{2}}{3} q\,. \tag{3.3}$$

With this condition satisfied, the functions h and f both run from 0 to 1 as r runs from the horizon at $r = r_0$ to $r = \infty$. The Hawking temperature can then be calculated in the standard way, leading to

$$T = \frac{8 r_0^2 - q^2}{32\pi r_0^3}\,. \tag{3.4}$$

Interestingly, the requirement that the singularity $r_* = q/\sqrt{8}$ be inside the event horizon implies that $T > 0$. The temperature T can approach 0, but can never reach 0. This behavior is different from that of the well-known RN black hole, and in fact is more in line with the behaviour of conventional systems with respect to the third law of thermodynamics.

It is worth remarking that these asymptotically-flat black holes provide counterexamples to the no-go theorem of [22, 23], in consequence of the presence of the Maxwell field.

Asymptotically AdS solution:

For generic $\alpha \neq 0$ and $\Lambda \neq 0$, there exist asymptotically (locally) AdS solutions. It is convenient to introduce two parameters $(g\,, \beta)$ in place of the original two parameters $(\alpha\,, \Lambda)$ in the Lagrangian, defined by

$$\alpha = \tfrac{1}{2}(n-1)(n-2) g^2 \gamma\,, \qquad \Lambda = -\tfrac{1}{2}(n-1)(n-2) g^2 \Big(1 + \frac{\beta\gamma}{2\kappa}\Big)\,. \tag{3.5}$$

First, we present the simpler case of planar black holes, which arise when $\epsilon = 0$. The four-dimensional solution is then given by

$$\phi = \phi_0 - \frac{q}{r} + \frac{\kappa q^3}{30 g^2 (4\kappa + \beta\gamma)\, r^5}\,,$$
$$\chi' = \sqrt{\beta - \frac{\kappa q^2}{6\gamma g^2 r^4}}\, \frac{1}{\sqrt{f}}\,, \qquad f = \frac{36(4\kappa + \beta\gamma)^2 g^4 r^8}{\left(\kappa q^2 - 6(4\kappa + \beta\gamma) g^2 r^4\right)^2}\, h\,,$$
$$h = g^2 r^2 - \frac{\mu}{r} + \frac{\kappa q^2}{(4\kappa + \beta\gamma) r^2} - \frac{\kappa^2 q^4}{60(4\kappa + \beta\gamma)^2 g^2 r^6}\,. \tag{3.6}$$

As in the previous asymptotically-Minkowski case, there is an additional curvature singularity, as well as the usual one at $r = 0$, located now at

$$r_* = \left(\frac{\kappa q^2}{6 g^2 (\beta\gamma + 4\kappa)}\right)^{\frac{1}{4}}\,, \tag{3.7}$$

where f diverges. The requirement that this singularity be inside the event horizon $r = r_0$ implies

$$\mu > \frac{16 \times 2^{\frac{1}{4}} g^{\frac{1}{2}} \kappa^{3/4} q^{3/2}}{5 \times 3^{3/4} (\beta\gamma + 4\kappa)^{3/4}}\,. \tag{3.8}$$

Provided this condition is satisfied, the solutions describe asymptotically-AdS planar black holes. The Hawking temperature is given by

$$T = -\frac{\kappa q^2 - 6 g^2 {r_0}^4 (\beta\gamma + 4\kappa)}{8\pi {r_0}^3 (\beta\gamma + 4\kappa)}\,. \tag{3.9}$$

As can be easily seen, the requirement that the singularity be inside the black hole event horizon $r_* < r_0$ again leads to the conclusion that $T > 0$, and that it can never reach 0.

Secondly, we consider $\epsilon = 1$, which yields static and spherically-symmetric black hole solutions, given by

$$\phi = \phi_0 - \frac{\kappa q \left(3 g^2 q^2 + 8\right)}{2r(\beta\gamma + 4\kappa)} + \frac{\kappa q^3}{6 r^3 (\beta\gamma + 4\kappa)} + \frac{\sqrt{3} g q \arctan\left(\frac{1}{\sqrt{3} g r}\right) \left(3 g^2 \kappa q^2 - 2\beta\gamma\right)}{2(\beta\gamma + 4\kappa)}\,,$$
$$\chi' = \sqrt{\frac{6\beta g^2 \gamma r^4 - \kappa q^2}{2\gamma r^2 (1 + 3 g^2 r^2)}}\, \frac{1}{\sqrt{f}}\,, \qquad f = \frac{4(4\kappa + \beta\gamma)^2 (1 + 3 g^2 r^2)^2 r^4}{\left(6(4\kappa + \beta\gamma) g^2 r^4 + \kappa(8 r^2 - q^2)\right)^2}\, h\,,$$
$$h = g^2 r^2 + \frac{4\kappa - \beta\gamma}{4\kappa + \beta\gamma} - \frac{\mu}{r} + \frac{\kappa^2 q^2 (3 g^2 q^2 + 16)}{4(4\kappa + \beta\gamma)^2 r^2} - \frac{\kappa^2 q^4}{12(4\kappa + \beta\gamma)^2 r^4}$$
$$- \frac{(2\beta\gamma - 3\kappa g^2 q^2)^2 \arctan(\frac{1}{\sqrt{3}\, g r})}{4\sqrt{3}(4\kappa + \beta\gamma)^2 g r} \tag{3.10}$$

Besides the usual curvature singularity at the origin, there is again a further curvature singularity at $r = r_*$, which is now given by

$$6 g^2 r_*^4 (\beta\gamma + 4\kappa) - \kappa \left(q^2 - 8 r_*^2\right) = 0\,. \tag{3.11}$$

For the solutions to describe black holes, we require that the largest root r^* be less than r_0, the radius of the event horizon. The Hawking temperature is

$$T = \frac{6g^2 r_0^4(4\kappa + \beta\gamma) + \kappa\left(8r_0^2 - q^2\right)}{8\pi r_0^3(4\kappa + \beta\gamma)}\,. \tag{3.12}$$

As in the previous case, the requirement that the singularity r_* should be inside the event horizon implies that $T > 0$. The requirement also sets a constraint between mass parameter μ and charge parameter q. Since the explicit expression is rather complicated, we shall not present it here.

4 Thermodynamics

Having described the charged black holes and their Hawking temperatures in the previous section, we now turn to a study of their thermodynamic properties, and in particular, we shall obtain the first law of black hole thermodynamics that they obey. The thermodynamics of the neutral black holes in Horndeski gravity was studied in [21]. It was shown there that the standard Wald entropy formula requires modification in Einstein-Horndeski gravity. The reason for this is that the axionic scalar χ has a branch-cut singularity at the horizon, with the associated consequence the radial vielbein component of its derivative $\partial_a\chi = E_a^\mu \partial_\mu \chi$ approaches a constant, rather than vanishing, on the horizon. This leads to an additional contribution in the expression for the entropy.

The neutral static black holes discussed in [21] depends on only one non-trivial parameter, and hence one can always find a mass functional M such that a first law of the form $dM = TdS$ ostensibly holds, since TdS is necessarily integrable. The situation is very different now for the case of charged static black holes, which depend on two non-trivial parameters, since integrability is no longer *a priori* guaranteed in the two-dimensional parameter space of solutions.

Although the Wald entropy formula acquires an additional scalar contribution, the Wald conserved Hamiltonian formalism still works for the neutral black holes, as analyzed in [21]. The subtlety lies in how to read off the first law of black hole thermodynamics from the Wald formalism. For this reason, we shall also employ the Wald formalism to study the charged black holes. We first give a brief review of the formalism, and then we apply it to the charged black holes discussed in the previous section.

4.1 Wald formalism

Wald has developed a procedure that can be used to derive the first law of thermodynamics, by calculating the variation of a Hamiltonian associated with a conserved Noether current [30, 31]. The Wald formalism has been applied to the study of the first law of thermodynamics for asymptotically-AdS black holes in a variety of theories, including Einstein-scalar [14, 15], Einstein-Proca [36], Einstein-Yang-Mills [37], in gravities extended with quadratic-curvature invariants [38], and also for Lifshitz black holes [39]. Starting from a Lagrangian $\mathcal{L}$, its variation under a general variation of the fields can be written as

$$\delta\mathcal{L} = \text{e.o.m.} + \sqrt{-g}\,\nabla_\mu J^\mu\,, \tag{4.1}$$

where e.o.m. denotes terms proportional to the equations of motion for the fields. From this one can define a 1-form $J_{(1)} = J_\mu dx^\mu$ and its Hodge dual $\Theta_{(n-1)} = (-1)^{n+1} * J_{(1)}$. The next step is to specialise to a variation that is induced by an infinitesimal diffeomorphism $\delta x^\mu = \xi^\mu$. One can show that

$$J_{(n-1)} \equiv \Theta_{(n-1)} - i_\xi *\mathcal{L} = \text{e.o.m} - d*J_{(2)}\,, \tag{4.2}$$

where i_ξ denotes a contraction of ξ^μ on the first index of the n-form $*\mathcal{L}$. One can thus define an $(n-2)$-form $Q_{(n-2)} \equiv *J_{(2)}$, such that $J_{(n-1)} = dQ_{(n-2)}$. Note that we use the subscript notation $_{(p)}$ to denote a p-form. To make contact with the first law of black hole thermodynamics, we take ξ^μ to be the Killing vector that becomes null on the horizon. Wald shows that the variation of the Hamiltonian with respect to the integration constants of a specific solution is given by

$$\delta\mathcal{H} = \frac{1}{16\pi}\delta\int_c J_{(n-1)} - \int_c d\big(i_\xi\Theta_{(n-1)}\big) = \frac{1}{16\pi}\int_{\Sigma_{(n-2)}}\big(\delta Q - i_\xi\Theta_{(n-1)}\big)\,, \tag{4.3}$$

where c denotes an $(n-1)$-dimensional Cauchy surface and $\Sigma_{(n-2)}$ is its boundary, which has two components, one at infinity and one on the horizon. Thus according to the Wald formalism, the first law of black hole thermodynamics is a consequence of the identity (1.2), which follows from the fact that $\delta Q - i_\xi\Theta_{(n-1)}$ is an exact form. It is worth commenting that the identity is valid as long as the spacetime is well behaved in the region between the two surfaces. It applies also for solitons, which have no horizon. In this case it therefore implies $\delta\mathcal{H}_\infty = 0$. As we shall see in the next subsection, the Wald identity is satisfied by all the solutions presented in section 3, which is consistent with the fact that these are all well-defined black holes. As we shall see, there are subtleties involved in identifying appropriate thermodynamical quantities that are compatible with interpreting the Wald identity as a first law.

4.2 Results for charged static black holes

The Wald formalism for the general EHM theory (2.1) is rather involved, and we shall present here only the key results for the static ansatz (2.3). (A detailed discussion of the Wald formalism for Einstein-Horndeski gravity can be found in [21].) We find

$$
\begin{aligned}
(\delta Q - i_\xi \Theta)_{\rm min} &= -r^{n-2}\sqrt{\frac{h}{f}}\Big[\kappa\Big(\frac{n-2}{r}\delta f + \frac{f}{h}\,\phi\delta\phi' + \frac{\phi\phi'}{2}(\frac{\delta f}{h} - \frac{f\delta h}{h^2})\Big) + \alpha f\chi'\delta\chi\Big]\,\Omega_{(n-2)}\,,\\
(\delta Q - i_\xi \Theta)_{\rm non} &= \tfrac12(n-2)\gamma\, r^{n-3}\sqrt{\frac{h}{f}}f^2\Big[-\tfrac32\chi'^2\frac{\delta f}{f} - \delta(\chi'^2)\\
&\qquad + \Big(\frac{n-3}{r}(1-\frac{\epsilon}{f}) + \frac{h'}{h}\Big)\chi'\delta\chi\Big]\,\Omega_{(n-2)}\,,
\end{aligned}
\tag{4.4}
$$

where the subscript "min" or "non" indicates terms that are related to the minimally-coupled part or the non-minimally coupled part of the action (2.1) respectively. We find that the coefficient of the term $\chi'\delta\chi$ is proportional to the constraint equation (3.1), and so for the solutions we are considering this contribution gives zero. This is easily understood, since this term comes from the variation of χ in $(\alpha g_{\mu\nu} - \gamma G_{\mu\nu})\partial^\mu\chi\partial^\nu\chi$. The contributions that survive are then given by

$$
\begin{aligned}
(\delta Q - i_\xi \Theta)_{\rm total} &= -r^{n-2}\frac{n-2}{r}\sqrt{\frac{h}{f}}\Big(\kappa\delta f + \frac{3\gamma}{4}f\chi'^2\delta f + \frac{\gamma}{2}f^2\delta(\chi'^2)\Big)\,\Omega_{(n-2)}\\
&\quad -r^{n-2}\sqrt{\frac{h}{f}}\kappa\Big(\frac{f}{h}\,\phi\delta\phi' + \frac{\phi\phi'}{2}(\frac{\delta f}{h} - \frac{f\delta h}{h^2})\Big)\,\Omega_{(n-2)}\,.
\end{aligned}
\tag{4.5}
$$

For our solution, we can write $\delta(\chi'^2)$ in terms of δf, and so

$$
\begin{aligned}
(\delta Q - i_\xi \Theta)_{\rm total} &= -r^{n-2}\frac{n-2}{r}\sqrt{\frac{h}{f}}\Big(\kappa + \frac{\gamma}{4}f\chi'^2\Big)\delta f\,\Omega_{(n-2)}\\
&\quad -r^{n-2}\sqrt{\frac{h}{f}}\kappa\Big(\frac{f}{h}\,\phi\delta\phi' + \frac{\phi\phi'}{2}(\frac{\delta f}{h} - \frac{f\delta h}{h^2})\Big)\,\Omega_{(n-2)}\,.
\end{aligned}
\tag{4.6}
$$

It is now straightforward to verify that the Wald identity (1.2) is indeed satisfied for all the charged black holes in section 3.

We now turn to the question of interpreting (1.2) as the first law of thermodynamics for these black hole solutions. Let us first consider $\delta\mathcal{H}_\infty$. We shall choose a gauge for the Maxwell field such that it vanishes on the horizon. We then find, as we shall see later, that

$$
\delta\mathcal{H}_\infty = \delta M - \Phi\delta Q\,. \tag{4.7}
$$

where Φ is the electric potential at infinity. More precisely, Φ is the potential difference between the horizon and asymptotic infinity, since A vanishes on the horizon in our gauge choice. The quantities M and Q can be interpreted as the mass and electric charge of the solution. This result is the same as in the case of RN (AdS) black holes.

The situation is rather different from that for RN black holes when we consider $\delta\mathcal{H}_+$ on the horizon. This can be studied in a rather general way, as follows. We suppose that the black hole event horizon is located at $r = r_0$, and that the metric functions h and f near the horizon have Taylor expansions of the form

$$f = f_1(r - r_0) + f_2(r - r_0)^2 + \dots\,, \qquad h = h_1(r - r_0) + h_2(r - r_0)^2 + \dots\,. \tag{4.8}$$

(It is easy to check that such expansions indeed arise for all the solutions we have presented.) For the class of black hole solutions we have considered, the scalar field χ has the near-horizon behaviour $\chi' \sim \frac{1}{\sqrt{f}}$, and so in the near-horizon region the function χ has an expansion of form

$$\chi = \tilde{\chi}_0 + \tilde{\chi}_1(r - r_0)^{\frac{1}{2}} + \tilde{\chi}_2(r - r_0)^{\frac{3}{2}} + \cdots\,. \tag{4.9}$$

The electrostatic potential ϕ, on the other hand, vanishes on the horizon in our gauge choice.

Taking the above expansions into account, we find that $\delta\mathcal{H}$ on the horizon is given by

$$\delta\mathcal{H}_+ = \big(\kappa + \tfrac{1}{4}\gamma f\chi'^2\big|_{r_0}\big)\, T\,\delta\big(\frac{\mathcal{A}}{4}\big)\,, \tag{4.10}$$

where $\mathcal{A} = r_0^2\,\omega_{2,\epsilon}$ is the area of the horizon, and $T = \frac{\sqrt{f_1 h_1}}{4\pi}$ is the Hawking temperature. This result was first obtained in [21] for neutral black holes in Einstein-Horndeski gravity. The formula remains unchanged for the charged black holes we are considering here, provided we make our gauge choice in which $\phi = 0$ on the horizon.

In the case of the neutral black holes considered in [21] , the solutions are characterised by a single parameter, which may be taken to be the horizon radius r_0, and so the expression

$$\big(\kappa + \tfrac{1}{4}\gamma f\chi'^2\big|_{r_0}\big)\delta\big(\frac{\mathcal{A}}{4}\big) = \delta S\,, \tag{4.11}$$

can always be integrated, so that $\delta\mathcal{H}_+ = T\,\delta S$. This was done in [21], although questions about the validity of interpreting the resulting function S as the entropy were also raised there, in view of the fact that there was no non-trivial integrability check and furthermore that there was no independent calculation of the entropy.

For the charged black holes, we find that the differential

$$\big(\kappa + \tfrac{1}{4}\gamma f\chi'^2\big|_{r_0}\big)\delta\big(\frac{\mathcal{A}}{4}\big) \tag{4.12}$$

cannot be interpreted as δS for any function S, since it is now a variation in a two-dimensional parameter space, and this variation, thought of as a 1-form, is not an exact one. One may take the two parameters to be the horizon radius r_0 and the electric charge Q_e of the black hole. The non-integrability is a consequence of the fact that $f\chi'^2|_{r_0}$ is a function of both r_0 and Q_e, whilst the area of the horizon depends only on r_0.

To resolve this issue, we introduce the concept of a "scalar charge" defined by $Q_\chi = \int \sqrt{(\partial\chi)^2}$, where the integration is over the level surfaces $d\Omega_{2,\epsilon}$. This charge can be evaluated both on the horizon and at infinity:

$$\begin{aligned} Q_\chi^+ &= \int_{r=r_0} d\Omega_{2,\epsilon} \sqrt{(\partial\chi)^2_+} = \omega_{2,\epsilon}\sqrt{f}\,\chi'\Big|_{r=r_0}\,, \\ Q_\chi^\infty &= \int_{r\to\infty} d\Omega_{2,\epsilon} \sqrt{(\partial\chi)^2_\infty} = \omega_{2,\epsilon}\sqrt{f}\,\chi'\Big|_{r=\infty}\,. \end{aligned} \tag{4.13}$$

An analogous kind of definition was used previously in the literature in the case of a (non-conserved) Yang-Mills "charge" [40, 41], i.e.

$$P^{\rm YM} = \int_{r\to\infty} \sqrt{F^a_{\theta\phi}F^a_{\theta\phi}}\, d\theta d\varphi\,. \tag{4.14}$$

Note that this function of the Yang-Mills field vanishes when evaluated on the horizon. It was shown [37] that this definition of a global Yang-Mills charge is indeed consistent with the first law of thermodynamics, by using the Wald-formalism approach. The first law of AdS-Yang-Mills black holes were shown to be [37]

$$dM = TdS + \Phi_{\rm YM}\, dP^{\rm YM}\,. \tag{4.15}$$

In fact the electric charge can be defined in analogous way also, namely

$$\frac{1}{\omega_{2,\epsilon}}\int \sqrt{|F|^2} d\Sigma = \frac{1}{\omega_{2,\epsilon}}\int \sqrt{2}\, q d\Omega_{2,\epsilon} = \sqrt{2}\, q \propto Q_e\,. \tag{4.16}$$

With this new scalar "charge" Q_χ^+ on the horizon, we can write $\delta\mathcal{H}_+$ in our case as

$$\delta\mathcal{H}_+ = (\kappa + \frac{\gamma}{4} f\chi'^2|_{r_0}) T\delta(\frac{\mathcal{A}}{4}) = T\delta S + \Phi_\chi^+ \delta Q_\chi^+\,, \tag{4.17}$$

where the entropy and the conjugate potential Φ_χ for the scalar charge are given by

$$S = \Big(\kappa + \frac{\gamma}{4}(f\chi'^2)|_{r_0}\Big)\frac{\mathcal{A}}{4}\,, \qquad \Phi_\chi^+ = -\frac{\gamma\,\mathcal{A}\,T}{8\omega_{2,\epsilon}}\sqrt{f\chi'^2|_{r_0}}\,. \tag{4.18}$$

The entropy has the standard contribution of one quarter of the area of the horizon, plus a modification involving the scalar field χ. Having interpreted both $\delta\mathcal{H}_\infty$ and $\delta\mathcal{H}_+$, the Wald identity (1.2) then yields the first law in the form

$$dM = TdS + \Phi_e dQ_e + \Phi_\chi^+ dQ_\chi^+\,. \tag{4.19}$$

It is worth commenting that an analogous contribution from the scalar in the last term of the above first law were also derived for general scalar hairy AdS black holes [14,16,36]. An important difference is that in the previous examples, this extra term arose in $\delta\mathcal{H}_\infty$, rather than $\delta\mathcal{H}_+$ as it does here.

To recapitulate, before we analyse the explicit examples, the Wald identity (1.2) is satisfied for the all black holes discussed in section 3. To interpret it as a first law, we find that the last term in (4.19) is unavoidable, since for the gauge choice where the vector potential dA vanishes on the horizon, the quantity $T^{-1}\,\delta\mathcal{H}_+$ is not integrable. In the first law (4.19), three quantities are unambiguous; they are the Hawking temperature T, the electrostatic potential Φ_e and the electric charge Q_e. The other quantities are less clear cut, and hence our interpretation is not unique, but in any case the introduction of a contribution from a global "charge" for the scalar χ is unavoidable. We have also examined other possibilities, and the first law as written in (4.19) seems to give the most reasonable interpretation of the Wald identity. As we shall see presently, for all the charged black holes the entropy defined as above has the property that $S\sim T$ in the "extremal" limit where $T\to 0$. Some further justifications will be given in sections 6 and 7.

We now turn to the explicit solutions.

Asymptotically-Minkowski black holes

For the asymptotically-Minkowski solutions (3.2), we find that $\delta\mathcal{H}$ evaluated at infinity and on the horizon are given by

$$\delta\mathcal{H}_\infty = \tfrac12\kappa\delta\mu + \tfrac14\phi_0\,\kappa\,\delta q\,,\qquad \delta\mathcal{H}_+ = \frac{\kappa\left(q^2-8{r_0}^2\right)^2\delta r_0}{128 r_0^4}\,. \tag{4.20}$$

It is easy to verify that the Wald identity (1.2) is indeed satisfied. Following the earlier discussion, the thermodynamical quantities are given by

$$\begin{aligned} M &= \tfrac12\kappa\mu\,,\qquad \Phi_e=\frac{q(24r_0^2-q^2)}{24r_0^3}\,,\qquad Q_e=\tfrac14\kappa q\,,\\ T &= \frac{8r_0^2-q^2}{32\pi r_0^3}\,,\qquad S=\tfrac18\pi\kappa\left(8r_0^2-q^2\right)=4\kappa\,\pi^2 r_0^3\,T\,,\\ Q_\chi^+ &= 4\pi\sqrt{-\tfrac{\kappa q^2}{2\gamma r_0^2}}\,,\qquad \Phi_\chi^+=-\tfrac18\gamma r_0^2 T\,\sqrt{-\tfrac{\kappa q^2}{2\gamma r_0^2}}\,. \end{aligned} \tag{4.21}$$

It easy to verify that the first law (4.19) is satisfied. Note that for the solution in this case to be real, we must require that $\gamma<0$. This implies that Φ_χ^+ is positive.

Note that the requirement that the curvature singularity at $r_*\equiv q/\sqrt{8}$ lies inside the horizon (i.e. $r_*<r_0$) not only guarantees that the temperature T is positive, but also that

the entropy S is positive. The requirement also implies that

$$M > M_* \equiv \tfrac{4\sqrt{2}}{3} Q_e \,. \tag{4.22}$$

Unlike the RN black hole, the mass/charge bound cannot be saturated. As M approaches M_*, both the temperature and entropy approach zero, and Q_χ^+ becomes a numerical constant. The first law then reduces to

$$dM \to \Phi_e dQ_e \to dM_* \,, \qquad \text{with} \qquad \Phi_e \to \tfrac{4\sqrt{2}}{3} \,. \tag{4.23}$$

<u>**AdS planar black holes ($\epsilon = 0$)**</u>

Substituting the asymptotically-AdS planar black hole solutions (3.6) into the Wald formula gives

$$\delta\mathcal{H}_\infty = \tfrac{1}{32\pi}(4\kappa + \beta\gamma)\,\delta\mu + \tfrac{1}{16\pi}\phi_0\,\kappa\,\delta q \,, \quad \delta\mathcal{H}_+ = \frac{\left(\kappa q^2 - 6g^2 {r_0}^4(\beta\gamma + 4\kappa)\right)^2 \delta r_0}{384\pi g^2 {r_0}^6(\beta\gamma + 4\kappa)} \,. \tag{4.24}$$

Note that we have chosen $\omega_{2,0} = 1$ (a unit area 2-torus), so the extensive quantities here may in general be interpreted as densities. It is easy to verify that the Wald identity (1.2) is satisfied, from which we can read off the thermodynamical quantities

$$\begin{aligned}
M &= \tfrac{1}{32\pi}(4\kappa + \beta\gamma)\mu \,, \qquad \Phi_e = \phi_0 \,, \qquad Q_e = \tfrac{1}{16\pi}\kappa q \,,\\
T &= \frac{3g^2 r_0}{4\pi} - \frac{\kappa\, q^2}{8(4\kappa + \beta\gamma)\pi r_0^3} \,, \qquad S = \frac{(4\kappa + \beta\gamma)\pi r_0}{12 g^2} T \,,\\
Q_\chi^+ &= \sqrt{\beta - \frac{\kappa q^2}{6\gamma g^2 r_0^4}} \,, \qquad \Phi_\chi^+ = -\tfrac{1}{8}\gamma\, r_0^2\, T\, Q_\chi^+ \,.
\end{aligned} \tag{4.25}$$

Here too, the requirement that the additional curvature singularity at $r = r_*$ should lie inside the horizon ensures that both the temperature T and entropy S are positive. This condition also implies that

$$M > M_* \equiv \frac{32 \times 2^{\frac{1}{4}} \sqrt{\pi g}(4\kappa + \beta\gamma)^{\frac{1}{4}}}{5 \times (3\kappa)^{\frac{3}{4}}} \, Q^{\frac{3}{2}} \,. \tag{4.26}$$

The equality cannot be saturated. As M approaches M_*, both the temperature and entropy vanish, and furthermore Q_χ^+ becomes a pure numerical constant. The first law then reduces to

$$dM \to \Phi_e dQ_e \sim dM_* \,. \tag{4.27}$$

<u>**(Lacally) AdS black holes ($\epsilon = 1$)**</u>

For spherically-symmetric solutions, we have $\omega_{2,1} = 4\pi$. We find that the Wald formalism gives

$$\begin{aligned} \delta\mathcal{H}_\infty &= \tfrac{1}{8}(4\kappa + \beta\gamma)\,\delta\mu - \tfrac{1}{4}\phi_0\,\kappa\,\delta q\,, \\ \delta\mathcal{H}_+ &= \frac{\left(6g^2 r_0^4 (4\kappa+\beta\gamma) + \kappa\left(8r_0^2 - q^2\right)\right)^2 \delta r_0}{32 r_0^4 \left(3g^2 r_0^2 + 1\right)(4\kappa+\beta\gamma)}\,, \end{aligned} \tag{4.28}$$

and from these we can read off the thermodynamical quantities, namely

$$\begin{aligned} M &= \tfrac{1}{8}(4\kappa+\beta\gamma)\mu\,, \qquad \Phi = \phi_0\,, \qquad Q = \tfrac{1}{4}\kappa\,q\,, \\ T &= \frac{3g^2 r_0}{4\pi} + \frac{\kappa(8r_0^2 - q^2)}{8(4\kappa+\beta\gamma)\pi r_0^3}\,, \quad S = \frac{(4\kappa+\beta\gamma)\pi^2 r_0^3}{1+3g^2 r_0^2} T\,, \\ Q_\chi^+ &= 4\pi\sqrt{-\tfrac{\kappa q^2 - 6\beta\gamma g^2 r_0^4}{2\gamma r_0^2(1+3g^2 r_0^2)}}\,, \qquad \Phi_\chi^+ = -\frac{\gamma r_0^2}{32\pi} T\,Q_\chi^+\,. \end{aligned} \tag{4.29}$$

There is a smooth limit under which $g \to 0$, leading back to the asymptotically flat case. Again here, the requirement that the singularity at $r = r_*$ lie inside the horizon ensures that both the temperature and the entropy are positive. It implies that $M > M_*$ where M_* is a complicated function of the electric charge Q_e. As $M \to M_*$, we have $S \sim T \to 0$, so Q_χ^+ becomes a purely numerical constant and the first law reduces to (4.27).

5 Generalization to General Dimensions

In the previous sections, we studied three classes of static black hole solutions in the four-dimensional charged Horndeski theory, and obtained the first law of thermodynamics for these black holes. We shall now generalize these results to an arbitrary spacetime dimension n.

Asymptotically-Minkowski black holes

The solution in a general dimension n is given by

$$\begin{aligned} \chi'^2 &= -\frac{\kappa q^2}{\gamma(n-2)(n-3)r^{2(n-3)}\,f}\,, \qquad f = \frac{16(n-2)^2(n-3)^2 r^{4n}}{(q^2 r^6 - 4(n-2)(n-3)r^{2n})^2}\,h\,, \\ h &= 1 - \frac{\mu}{r^{n-3}} + \frac{q^2}{2(n-2)(n-3))r^{2(n-3)}} - \frac{q^4}{48(n-2)^2(n-3)^2 r^{4(n-3)}}\,, \\ \phi &= \phi_0 - \frac{q}{(n-3)r^{n-3}} - \frac{q^3}{12(n-2)(n-3)^2 r^{3(n-3)}}\,. \end{aligned} \tag{5.1}$$

The solutions contain two non-trivial parameters, which we make take to be μ and q. As we shall see presently, these parameterise the mass and the electric charge respectively. The further integration constant ϕ_0 is a pure gauge parameter, and as before we shall adopt the gauge choice where the potential A vanishes on the horizon. The solution contains two curvature singularities: the usual one at the origin $r = 0$, and the other located at $r = r_*$, given by

$$4(n-2)(n-3)r_*^{2n-6} - q^2 = 0\,. \tag{5.2}$$

The event horizon $r = r_0$ is at the largest root of $h = 0$. Furthermore, we must have $r_0 > r_*$, which implies

$$\frac{\mu}{q} > \frac{4}{3\sqrt{(n-2)(n-3)}}\,. \tag{5.3}$$

Applying the Wald formalism, we find

$$\begin{aligned}\delta\mathcal{H}_\infty &= \frac{\kappa}{16\pi}(n-2)\,\delta\mu - \frac{1}{16\pi}\phi_0\,\kappa\,\delta q\,,\\ \delta\mathcal{H}_+ &= \frac{\kappa\left(q^2 - 4(n-2)(n-3){r_0}^{2n-6}\right)^2\delta r_0}{16(n-2)(n-3){r_0}^{3n-8}}\,.\end{aligned} \tag{5.4}$$

It is then easy to establish that the Wald identity (1.2) is satisfied. This leads to the first law (4.19) with

$$\begin{aligned} M &= \frac{\kappa}{16\pi}(n-2)\omega\,\mu\,, \quad \Phi_e = \frac{q\big(12(n-2)(n-3)r_0^{2(n-3)} - q^2\big)}{12(n-2)(n-3)^2\,r_0^{3(n-3)}}\,, \quad Q_e = \frac{\kappa}{16\pi}\omega\,q\,,\\ T &= \frac{4(n-2)(n-3)r_0^{2n-6} - q^2}{16\pi(n-2)r_0^{2n-5}}\,, \quad S = \frac{\kappa\pi\omega}{n-3}\,r_0^{n-1}T\,,\\ Q_\chi^+ &= \sqrt{-\frac{\kappa q^2}{\gamma(n-2)(n-3){r_0}^{2n-6}}}\,, \qquad \Phi_\chi^+ = -\frac{\gamma}{8\omega}r_0^{n-2}\,T\,Q_\chi^+\,,\end{aligned} \tag{5.5}$$

where ω is the volume of the unit S^{n-2}.

Asymptotically-AdS planar solutions ($\epsilon = 0$)

The solutions are given by

$$\begin{aligned}\phi &= \phi_0 - \frac{q}{(n-3)r^{n-3}} + \frac{\kappa q^3}{g^2(n-1)(n-2)(3n-7)(4\kappa+\beta\gamma)r^{3n-7}}\,,\\ \chi' &= \sqrt{\beta - \frac{\kappa q^2}{\gamma g^2(n-1)(n-2)r^{2(n-2)}}}\,\frac{1}{\sqrt{f}}\,,\\ f &= \frac{g^4(n-1)^2(n-2)^2r^{4n}(4\kappa+\beta\gamma)^2}{\left(\kappa q^2r^4 - g^2(n-1)(n-2)r^{2n}(4\kappa+\beta\gamma)\right)^2}\,h\,,\\ h &= g^2r^2 - \frac{\mu}{r^{n-3}} + \frac{2\kappa q^2}{(n-2)(n-3)(4\kappa+\beta\gamma)r^{2(n-3)}}\\ &\quad - \frac{\kappa^2q^4}{g^2(n-1)(3n-7)(n-2)^2(4\kappa+\beta\gamma)^2r^{4n-10}}\,.\end{aligned} \tag{5.6}$$

Besides the usual curvature singularity $r = 0$, there is also an additional curvature singularity r_*, which is given by

$$(n-1)(n-2)g^2r_*^{2n-4}(\beta\gamma+4\kappa) - \kappa q^2 = 0\,. \tag{5.7}$$

For the solution to be free from naked curvature singularities, namely the event horizon must be outside the second singularity, $r = r_0 > r_*$, we must have

$$\mu > \frac{2^{\frac{2n-5}{n-2}}(n-2)^{\frac{3n-7}{2(n-2)}} g^{\frac{n-3}{n-2}}}{(n-3)(3n-7)(n-1)^{\frac{n-1}{2(n-2)}}}\Big(1 + \frac{\beta\gamma}{4\kappa}\Big)^{-\frac{n-1}{2(n-2)}} q^{\frac{n-1}{n-2}}\,. \tag{5.8}$$

Evaluating $\delta\mathcal{H}$ both at infinity and on the horizon gives

$$\begin{aligned} \delta\mathcal{H}_\infty &= \frac{1}{64\pi}(n-2)(\beta\gamma + 4\kappa)\,\delta\mu - \frac{1}{16\pi}\phi_0\,\kappa\,\delta q\,, \\ \delta\mathcal{H}_+ &= \frac{\big(\kappa q^2 - (n-1)(n-2)g^2 {r_0}^{2n-4}(\beta\gamma + 4\kappa)\big)^2\,\delta r_0}{64\pi g^2 (n-1)(n-2)(\beta\gamma + 4\kappa){r_0}^{3n-6}}\,. \end{aligned} \tag{5.9}$$

We find that indeed the Wald identity (1.2) is satisfied. The first law of thermodynamics is then given by (4.19), with

$$\begin{aligned} M &= \frac{1}{64\pi}(n-2)(\beta\gamma + 4\kappa)\mu\,, \qquad Q_e = \frac{\kappa}{16\pi}q\,, \\ \Phi_e &= \frac{q r_0^{3-n}}{n-3} - \frac{\kappa q^3 r_0^{7-3n}}{(n-1)(n-2)(3n-7)g^2(4\kappa + \beta\gamma)}\,, \\ T &= \frac{(n-1)g^2 r_0}{4\pi} - \frac{\kappa q^2 r_0^{5-2n}}{4(n-2)\pi(4\kappa + \beta\gamma)}\,, \qquad S = \frac{(4\kappa + \beta\gamma)\pi}{4(n-1)g^2}\, r_0^{n-3}\, T\,, \\ Q_\chi^+ &= \sqrt{\beta - \frac{\kappa q^2 r_0^{2(2-n)}}{(n-1)(n-2)g^2\gamma}}\,, \qquad \Phi_\chi^+ = -\frac{\gamma}{8} r_0^{n-2} T Q_\chi^+\,. \end{aligned} \tag{5.10}$$

Note that we have taken the $(n-2)$-torus sections to have unit area, and so in general the extensive quantities (M, S, Q_e, Q_χ^+) are to be interpreted as densities.

Asymptotically-AdS spherical black holes ($\epsilon = 1$)

With the spherically-symmetric ansatz, we may follow the same procedure to solve the equations of motion and obtain the solution in a general dimension n. The solutions now are more complicated and for a general dimensions n they are naturally expressed in terms of hypergeometric functions. The f and χ functions are related to h as usual, and are given by

$$\begin{aligned} \frac{f}{h} &= \frac{(n-2)^2(4\kappa + \beta\gamma)^2\big((n-1)g^2r^2 + n - 3\big)^2}{\Big((n-1)(n-2)(4\kappa + \beta\gamma)g^2r^2 + \kappa\big(4(n-2)(n-3) - q^2 r^{2(3-n)}\big)\Big)^2}\,, \\ \chi' &= \sqrt{\frac{\beta\gamma g^2(n-1)(n-2)r^{2n-4} - \kappa q^2}{\gamma(n-2)\big((n-1)g^2r^2 + n - 3\big)r^{2(n-3)}}}\sqrt{\frac{1}{f}}\,. \end{aligned} \tag{5.11}$$

Once f/h is known, the function ϕ can be easily integrated out, and is given by

$$\phi = \phi_0 - \frac{q}{(n-3)r^{n-3}} + \frac{\beta\gamma(n-3)q}{g^2(n-1)^2(4\kappa + \beta\gamma)r^{n-1}}$$

$$
-\frac{\beta\gamma(n-3)^2 q\, {}_2F_1\left(1,\frac{n+1}{2};\frac{n+3}{2};\frac{3-n}{(n-1)g^2r^2}\right)}{g^4(n-1)^2(n+1)(4\kappa+\beta\gamma)r^{n+1}}
+\frac{\kappa q^3\, {}_2F_1\left(1,\frac{1}{2}(3n-7);\frac{1}{2}(3n-5);\frac{3-n}{(n-1)g^2r^2}\right)}{g^2(n-1)(n-2)(3n-7)(4\kappa+\beta\gamma)r^{3n-7}}\,. \tag{5.12}
$$

The integration constant ϕ_0 is pure gauge and we shall as usual choose it so that ϕ vanishes on the horizon. The function h is given by $h=\bar{h}+h_q$ with

$$
\begin{aligned}
h_q =&\ \frac{2\kappa q^2}{(n-2)(n-3)(4\kappa+\beta\gamma)r^{2n-6}} - \frac{2\beta\gamma\kappa(n-3)q^2}{g^2(n-1)^2(n-2)(4\kappa+\beta\gamma)^2r^{2n-4}}\\
&+\frac{2\beta\gamma\kappa(n-3)^2q^2\, {}_2F_1\left(1,\frac{n+1}{2};\frac{n+3}{2};\frac{3-n}{(n-1)g^2r^2}\right)}{g^4(n+1)(n-1)^2(n-2)(4\kappa+\beta\gamma)^2r^{2n-2}}\\
&-\frac{\kappa^2q^4\, {}_2F_1\left(1,\frac{1}{2}(3n-7);\frac{1}{2}(3n-5);\frac{3-n}{(n-1)g^2r^2}\right)}{g^2(n-1)(n-2)^2(3n-7)(4\kappa+\beta\gamma)^2r^{2(2n-5)}}\,.
\end{aligned} \tag{5.13}
$$

The hypergeometric functions in (5.13) are well defined for all integers $n\ge 4$. For $\bar{h}$, it is more convenient to present separate expressions in even and in odd dimensions. When n is even, we find

$$
\begin{aligned}
\bar{h} =&\ \frac{8g^2\kappa(2\kappa+\beta\gamma)}{(4\kappa+\beta\gamma)^2}r^2+\frac{16\kappa^2}{(4\kappa+\beta\gamma)^2}-\frac{\mu}{r^{n-3}}\\
&+\frac{\beta^2\gamma^2g^2r^2}{(4\kappa+\beta\gamma)^2}\, {}_2F_1[1,-\tfrac{1}{2}(n-1);-\tfrac{1}{2}(n-3);\tfrac{3-n}{(n-1)g^2r^2}]\,.
\end{aligned} \tag{5.14}
$$

This expression becomes divergent if n is an odd integer. The divergence occurs because the third argument in each of these two functions becomes a negative integer. This can be handled by noting that the hypergeometric functions are of the special form ${}_2F_1(1,b;b+1;-x)$, and making use of the identity

$$
{}_2F_1[1,b;b+1;-x]=\frac{b}{(b-1)x}\, {}_2F_1[1,1-b;2-b;-\frac{1}{x}]+\pi b\csc\pi b\, x^{-b}\,, \tag{5.15}
$$

in the term where the problem arises. It is then evident that the divergence comes purely from the $\pi b\csc\pi b\, x^{-b}$ term, and furthermore, that these lead to terms only at order $1/r^{n-3}$ in the expression for h, with diverging coefficients. The infinities can therefore be absorbed into an infinite additive renormalisation of the mass coefficient μ in the $-\mu/r^{n-3}$ term in h. By using the identity (5.15) and redefining the integration constants, we get

$$
\begin{aligned}
\bar{h} =&\ \frac{8g^2\kappa(2\kappa+\beta\gamma)}{(4\kappa+\beta\gamma)^2}r^2+\frac{16\kappa^2}{(4\kappa+\beta\gamma)^2}-\frac{\mu}{r^{n-3}}\\
&+\frac{(n-1)^2\beta^2\gamma^2g^4r^4}{(n-1)(n-3)(4\kappa+\beta\gamma)^2}\, {}_2F_1[1,\tfrac{1}{2}(n+1);\tfrac{1}{2}(n+3);\tfrac{(n-1)g^2r^2}{3-n}]\,,
\end{aligned} \tag{5.16}
$$

for odd $n\ge 5$.

Applying the Wald formalism, we find

$$
\begin{aligned}
\delta\mathcal{H}_\infty &= \frac{1}{64\pi}(n-2)(\beta\gamma+4\kappa)\,\delta\mu - \frac{1}{16\pi}\phi_0\,\kappa\,\delta q\,,\\
\delta\mathcal{H}_+ &= \frac{\left[(n-1)(n-2)g^2{r_0}^{2n+2}(\beta\gamma+4\kappa)+\kappa\left(4(n-2)(n-3){r_0}^{2n}-q^2{r_0}^6\right)\right]^2\delta r_0}{64\pi(n-2)(\beta\gamma+4\kappa)\left[(n-1)g^2{r_0}^2+n-3\right]{r_0}^{3n+4}}\,. \qquad (5.17)
\end{aligned}
$$

It is then straightforward to verify that the Wald identity (1.2) is indeed satisfied by the general spherically-symmetric black holes. With this identity we can establish the first law (4.19), with

$$
\begin{aligned}
M &= \frac{\omega}{64\pi}(n-2)(\beta\gamma+4\kappa)\mu\,, \quad \Phi=\phi_0\,, \quad Q=\frac{\kappa\omega}{16\pi}q\,,\\
T &= \frac{(n-1)g^2 r_0}{4\pi}+\frac{\kappa\big(4(n-2)(n-3)r_0^{2(n-3)}-q^2\big)}{4\pi(n-2)(4\kappa+\beta\gamma)r_0^{2n-5}}\,,\\
S &= \frac{\pi(4\kappa+\beta\gamma)\omega}{4\big((n-1)g^2r_0^2+n-3\big)}r_0^{n-1}T\,,\\
Q_\chi^+ &= \omega\sqrt{-\tfrac{\kappa q^2-(n-1)(n-2)\beta\gamma g^2 r_0^{2(n-2)}}{(n-2)\gamma((n-1)g^2r_0^2+n-3)r_0^{2(n-3)}}}\,, \qquad \Phi_\chi^+=-\frac{\gamma}{8\omega}r_0^{n-2}TQ_\chi^+\,. \qquad (5.18)
\end{aligned}
$$

Here ω is the volume of the unit S^{n-2}. It is worth noting that the general forms of the thermodynamical quantities are the same for both even and odd dimensions.

The following properties apply to all the solutions discussed above. In addition to $r=0$, a spacetime curvature singularity can also arise at r_*, which is the root of h/f. The requirement that $r_*<r_0$ implies that $M>M_*$ where M_* is some function of the electric charge Q_e. As $M\to M_*$, we have $S\sim T\to 0$. The scalar charge Q_χ^+ can be expressed as

$$
(Q_\chi^+)^2=-\frac{4\kappa}{\gamma}+\frac{16}{\gamma\mathcal{A}}S\,, \qquad (5.19)
$$

which becomes purely numerical as $S\to 0$. Thus as $M\to M_*$, the first law reduces to (4.27).

6 Noether Charge and Scaling Symmetry

Having obtained the first law of thermodynamics (4.19) from the Wald identity (1.2), we shall probe the results further in this and the next section. As we have seen, the right-hand side of the first law involves six thermodynamical quantities, of which three can be unambiguously defined independently of the Wald formalism. These are the Hawking temperature T, and the electrostatic potential and charge (Φ_e,Q_e). Our interpretation for the definitions of the other three quantities; the entropy S and (Φ_χ^+,Q_χ^+), and also the mass M, would be more satisfactory if we could find a different method to calculate them. In this section, we shall examine these quantities from a different angle.

As discussed in [42] and also in [21], the existence of a scaling symmetry for AdS planar black holes has the important consequence that there exists a generalized Smarr formula, which relates the thermodynamic mass M to the other thermodynamical quantities. To see the scaling property more explicitly, we rewrite the ansatz for the AdS planar black-hole solutions in the form

$$ds^2 = d\rho^2 - a(\rho)^2 dt^2 + b(\rho)^2 (dx^i dx^i)\,, \qquad \chi = \chi(\rho)\,, \qquad \phi = \phi(\rho)\,, \tag{6.1}$$

where the new radial coordinate ρ is related to r by $f^{-1/2}\, dr = d\rho$. The equations of motion can be derived from the effective one-dimensional Lagrangian

$$\begin{aligned} L &= \frac{1}{16\pi} a b^{n-2} \Big[\kappa(R + \frac{\dot\phi^2}{2a^2} - 2\Lambda) - \tfrac12 \alpha \dot\chi^2 + \tfrac12 \gamma G_{11} \dot\chi^2\Big]\,,\\ R &= -\frac{2\ddot a}{a} - \frac{2(n-2)\ddot b}{b} - \frac{2(n-2)\dot a \dot b}{ab} - \frac{(n-2)(n-3)\dot b^2}{b^2}\,,\\ G_{11} &= \frac{(n-2)\dot a\dot b}{ab} + \frac{(n-2)(n-3)\dot b^2}{2b^2}\,, \end{aligned} \tag{6.2}$$

where a dot denotes a derivative with respect to ρ. The Lagrangian is invariant under the global scaling

$$a \to \lambda^{2-n}\, a\,, \qquad b \to \lambda\, b\,, \qquad \phi \to \lambda^{2-n}\, \phi\,. \tag{6.3}$$

This global symmetry implies that there is an associated conserved Noether charge, given by

$$\mathcal{Q}_N = \frac{1}{32\pi} (n-2) a b^{n-2} \Big[(4\kappa + \gamma\dot\chi^2)(\frac{\dot a}{a} - \frac{\dot b}{b}) - \frac{2\kappa\phi\dot\phi}{a^2}\Big]\,. \tag{6.4}$$

In terms of the coordinates of the original ansatz (2.3) with $\epsilon = 0$, we have

$$\mathcal{Q}_N = \frac{1}{16\pi}(n-2) r^{n-2} \Big(\sqrt{hf}(\kappa + \frac{\gamma}{4} f \chi'^2)(\frac{h'}{h} - \frac{2}{r}) - \kappa\sqrt{\frac{f}{h}}\phi\phi'\Big)\,. \tag{6.5}$$

Substituting the AdS planar black hole solution into this Noether charge formula and evaluating at asymptotic infinity, we find

$$\mathcal{Q}_N\Big|_\infty = (n-1)M - (n-2)\Phi_e Q_e\,. \tag{6.6}$$

(Here we have, as usual, chosen the gauge where A vanishes on the horizon.) In this equation, Φ_e and Q_e are defined unambiguously, using the standard properties of the Maxwell field. The mass M read off from (6.6) turns out to be identical to the one we obtained previously from the Wald formalism. This result is the same as that for the RN-AdS planar black hole, and it can be viewed as an independent derivation of the mass. (See also [43] for another discussion of the calculation of the mass of black holes in Horndeski gravity based on Wald formalism.)

If we evaluate the Noether charge on the horizon, we find

$$\mathcal{Q}_N\Big|_+ = (n-2)\,\frac{\sqrt{h'(r_0)f'(r_0)}}{4\pi}\,\Big(\kappa+\tfrac14\gamma(f\chi'^2)|_{r_0}\Big)\frac{\mathcal{A}}{4} = (n-2)TS\,, \tag{6.7}$$

where S is precisely the entropy defined according to (4.18), thus strongly supporting the naturalness of this interpretation. We therefore have the generalized Smarr formula

$$M = \frac{n-2}{n-1}\Big(\Phi Q+TS\Big)\,, \tag{6.8}$$

which takes the same form as for the RN-AdS planar black holes of Einstein-Maxwell theory. Note that the newly introduced thermodynamical quantities Φ_χ^+ and Q_χ^+ do not enter the generalized Smarr formula. This is completely consistent with their specific scaling behaviours. Interestingly, it was shown that the scalar contributions via $\delta\mathcal{H}_\infty$ to the first law of the scalar hairy AdS-planar black holes do not enter the generalized Smarr relation either [42].

It is worth pointing out that in the neutral AdS planar black hole in Einstein-Horndeski gravity, the quantity $f\chi'^2 = \beta$ is a non-thermodynamical constant, and hence the term $\Phi_\chi^+ dQ_\chi^+$ does not enter the first law. The necessity of introducing a scalar charge Q_χ^+ does not arise in that case. Correspondingly, the first law can be simply stated as $dM = TdS$ with the generalized Smarr relation $M = \frac{n-2}{n-1}TS$ [21]. However, when the electric charge is turned on, the quantity $f\chi'^2$ ceases to be constant, and hence the first law acquires the extra contribution, as in (4.19).

7 Viscosity/Entropy Ratio

Based on the AdS/CFT correspondence, the AdS planar black holes can be viewed as dual to some ideal fluid in the boundary theory. One can study the linear response of the graviton modes and determine the shear viscosity η of the boundary fluid [44, 45]. One can then calculate the viscosity/entropy ratio. In typical situations, the evaluation of the entropy is straightforward, and it can be obtained simply by using the standard Wald entropy formula (1.1). However, as we have seen, in the EHM theory the determination of the entropy is far from trivial. In this section, we shall first evaluate the viscosity using a standard method in the literature, and then compute the η/S ratio. The elegance of the result may be viewed as providing further support for the naturalness of our entropy formula (4.18).

To calculate the shear viscosity of the boundary field theory, we consider a transverse and traceless perturbation of the AdS planar black hole, namely

$$ds^2 = -hdt^2+\frac{dr^2}{f}+r^2\big(dx_idx_i+2\Psi(r,t)\,dx_1dx_2\big)\,, \tag{7.1}$$

where the background solution is given by (5.6). We find that the mode $\Psi(r,t)$ satisfies the linearised equation

$$\begin{aligned}
&2rhf\left(\kappa q^2-(n-1)(n-2)(4\kappa+\beta\gamma)g^2r^{2(n-2)}\right)\Psi''\\
&+2r\left((n-1)(n-2)(4\kappa-\beta\gamma)g^2r^{2(n-2)}+\kappa q^2\right)\ddot{\Psi}\\
&-h\Big[\ \Big((3n-7)\kappa q^2+(n-1)^2(n-2)(4\kappa+\beta\gamma)g^2r^{2(n-2)}\Big)f\\
&\qquad+r\Big((n-1)(n-2)(4\kappa+\beta\gamma)g^2r^{2(n-2)}-\kappa q^2\Big)\big((n-1)g^2r+f'\big)\Big]\,\Psi'=0\,, \qquad (7.2)
\end{aligned}$$

where we are using a prime to denote a derivative with respect to r, and in this section a dot denotes a derivative with respect to t. For an infalling wave which is purely ingoing at the horizon, the solution for a wave with low frequency ω is given by

$$\begin{aligned}
\Psi(r,t)&=e^{-\mathrm{i}\omega t}\,\psi(r)\,,\qquad \psi(r)=\exp\Big[-\mathrm{i}\omega K\log\frac{h(r)}{g^2r^2}\Big]\left(1-i\omega U(r)+\mathcal{O}(\omega^2)\right),\\
U(r)&=1-\int dr\,\frac{4\kappa qK\phi}{(4\kappa+\beta\gamma)r^{n-2}h}\,,\\
K&=\frac{1}{4\pi T}\sqrt{\frac{(n-1)(n-2)(4\kappa-\beta\gamma)g^2r_0^{2(n-2)}+\kappa q^2}{(n-1)(n-2)(4\kappa+\beta\gamma)g^2r_0^{2(n-2)}-\kappa q^2}}\,. \qquad (7.3)
\end{aligned}$$

Note that the constant parameter K is determined by the horizon boundary condition. The overall integration constant is fixed so that Ψ is unimodular asymptotically, as $r\to\infty$.

In order to study the boundary field theory using the AdS/CFT correspondence, we substitute the ansatz with the linearised perturbation into the action. The quadratic terms in the Lagrangian, after removing the second-derivative contributions by including a Gibbons-Hawking term, can be written as

$$\mathcal{L}_2=P_1\,\Psi'^2+P_2\,\Psi\,\Psi'+P_3\,\Psi^2+P_4\,\dot{\Psi}^2\,, \qquad (7.4)$$

with

$$\begin{aligned}
P_1&=-\frac{1}{8}r^{n-2}\sqrt{hf}\left(4\kappa+\gamma f\chi'^2\right),\quad P_2=\frac{1}{4}r^{n-3}\sqrt{hf}\left[-8\kappa+\gamma\left((n-1)g^2r^2+(n-3)f\right)\chi'^2\right],\\
P_3&=\frac{1}{4r^{n-2}}\sqrt{\frac{h}{f}}\Big[\ (n-1)(n-3)\kappa fr^{2(n-3)}+(n-1)\kappa g^2r^{2n-3}\Big(\frac{(n-1)g^2r}{f}-\frac{f'}{f}\Big)\\
&\qquad-\Big(\kappa q^2+(n-1)g^2r^{2(n-2)}\left(2(n-4)\kappa+(n-2)\beta\gamma\right)-(n-1)\kappa r^{2n-5}f'\Big)\Big]\,,\\
P_4&=\frac{3r^{n-2}}{8\sqrt{hf}}(-4\kappa+\gamma f\chi'^2) \qquad (7.5)
\end{aligned}$$

Note that $P_3=\frac{1}{2}P_2'$. We then find that the terms quadratic in Ψ in the Lagrangian are given by

$$\mathcal{L}_2=\frac{d}{dr}\left(P_1\Psi\,\Psi'+\tfrac{1}{2}P_2\Psi^2\right)+\frac{d}{dt}(P_4\,\Psi\,\dot{\Psi})-\Psi\left[P_1\,\Psi''+P_1'\,\Psi'+P_4\,\ddot{\Psi}\right]. \qquad (7.6)$$

The last term , enclosed in square brackets, vanishes by virtue of the linearised perturbation equation (7.2), and so the quadratic Lagrangian is a total derivative. The viscosity is determined from the $P_1\Psi\Psi'$ term, following the procedure described in [46, 47]. Using this, we find that the viscosity is given by

$$\eta = \frac{K}{16\pi}\Big[\frac{n-1}{4}(4\kappa+\beta\gamma)\mu - \kappa\phi_0 q\Big] = K\left[\frac{n-1}{n-2}M - \Phi Q\right]. \tag{7.7}$$

According to the generalized Smarr formula (6.8), this implies

$$\eta = KTS\,. \tag{7.8}$$

We therefore find that the viscosity/entropy ratio is given by

$$\frac{\eta}{S} = KT = \frac{1}{4\pi}\sqrt{\frac{(n-1)(n-2)(4\kappa-\beta\gamma)g^2 r_0^{2(n-2)} + \kappa q^2}{(n-1)(n-2)(4\kappa+\beta\gamma)g^2 r_0^{2(n-2)} - \kappa q^2}} \tag{7.9}$$

for the charged AdS planar Horndeski black holes. Note that when $q=0$, the result reduces to that for the neutral black hole, which was obtained in [21]. The viscosity/entropy ratio can also be expressed as

$$\frac{\eta}{S} = \frac{1}{4\pi}\sqrt{\frac{\kappa r_0^{n-2}}{2S} - 1} = \frac{1}{4\pi}\sqrt{\frac{2(n-1)\kappa g^2 r_0}{(4\kappa+\beta\gamma)\pi T} - 1}\,. \tag{7.10}$$

Although the final result (7.9) looks quite complicated, it still has a rather elegant form, especially when it is expressed in terms of the entropy or temperature, as in (7.10).

The viscosity (7.7) is given in terms of quantities that are defined asymptotically at infinity, whose interpretations are unambiguous. The only thing that was uncertain previously was the definition of the entropy S. Our formula (4.18) provides a simpler and more elegant form for the ratio (7.9) than alternative possibilities. Nevertheless, the viscosity/entropy ratio (7.9) is still rather complicated. It depends not only on the parameters in the theory, such as (κ, β, γ), but also on the integration constants in the solution, namely the mass and charge. The ratio is dependent on r_0 and T, which is qualitatively analogous to what was found for charged black holes in Einstein-Gauss-Bonnet-Maxwell theory. (See e.g. [48, 49].) However, the detailed dependence is quite different. For fixed charge Q_e, in the "extremal" limit as $T \to 0$, the viscosity and the entropy both vanish, as

$$\eta\Big|_{T\to 0} \to \sqrt{\tfrac{\kappa(4\kappa+\beta\gamma)r_0^{2n-5}}{128(n-1)\pi g^2}\,T}\,, \qquad S\Big|_{T\to 0} \to \tfrac{(4\kappa+\beta\gamma)\pi r_0^{n-3}}{4(n-1)g^2}\,T\,. \tag{7.11}$$

It follows that the viscosity/entropy ratio diverges in the low temperature $T \to 0$ limit. On the other hand, in the large temperature limit $T \to \infty$, the q terms in (7.9) become negligible, and η/S approaches a constant that is independent of the temperature, and in fact is equal to the ratio obtained for the neutral black hole.

8 Conclusions

In this paper, we considered the Einstein-Horndeski-Maxwell (EHM) theory where the derivative of the axionic scalar χ couples non-minimally to the Einstein tensor, as in (2.1). We studied the thermodynamics of the static black hole solutions in this theory by using the Wald formalism, and we found that the scalar field has a direct contribution to the first law. The contribution of the scalar can be attributed to a branch-cut singularity in the behavior of the scalar on the horizon. However, the scalar is axionic and enters the theory only through its derivative, and in an orthonormal frame $\partial_a \chi$ is regular everywhere, both on and outside the horizon, and all invariants involving the scalar field are finite everywhere.

In the Wald formalism, the first law of black hole thermodynamics can be read off from the Wald identity (1.2), provided one can attach an appropriate thermodynamic meaning to the various quantities appearing on the two sides of the equation. We found that the Wald identity is valid for all the charged solutions, supporting their interpretation as regular black holes. However, it is a non-trivial matter to translate the Wald identity into a first law, since to do this one needs to identify the quantities appearing in the two sides of (1.2) in terms of thermodynamic variables. In a previous paper [21] where neutral solutions were studied, the static black holes had only one non-trivial parameter, and so the integrability of $\delta\mathcal{H}_+/T$ was always guaranteed, allowing one to define an entropy and hence obtain the first law $dM = TdS$ for these black holes. For the charged solutions considered in this paper, on the other hand, there are two non-trivial parameters, namely the mass and the charge, and we found that $\delta\mathcal{H}_+/T$ was not an exact differential. It therefore appears to be unavoidable that one must introduce an additional pair of thermodynamical quantities. Inspired by the definition of a global "charge" in Yang-Mills theory, we defined an analogous scalar "charge" Q_χ^+ evaluated on the horizon, and also its thermodynamical conjugate potential Φ_χ^+. We were then able to obtain an extended first law of thermodynamics (4.19).

We explicitly analysed three kinds of static black-hole solutions in four dimensions, applied the Wald formalism, and obtained the thermodynamic variables and derived the first law. The black hole entropy is the usual one quarter of the horizon area (which is the result that follows form the standard Wald entropy formula) plus a modification from the scalar field. Apart from the usual curvature singularity at $r = 0$, a second curvature singularity at some radius $r = r_*$ emerges in these black hole solutions. For suitable choices of the parameter ranges the second singularity lies inside the event horizon. This also ensures that the Hawking temperature and the entropy are positive. We also obtained generalisations of all these results to arbitrary spacetime dimensions.

We then tested the validity of our interpretation of the entropy by considering independent methods for calculating the thermodynamical quantities. One such method, applicable to the case of planar black holes, involved deriving the Noether charge associated with the scaling symmetry of the planar black hole solutions. We also calculated the viscosity/entropy ratio in the dual boundary theory for the planar black hole examples, and the generalized Smarr relation. These results provide some support for the validity of our calculations of the thermodynamical quantities for the charged black holes.

There are several important differences between the charged black holes in EHM theory and the RN black holes of Einstein-Maxwell theory. Although both have "extremal" limits where the temperature approaches zero, the temperature of the charged black holes in the EHM theory can never reach absolute zero. Furthermore, as $T \to 0$, the black hole entropy in EHM theory also approaches zero, rather than some non-vanishing fixed value as in the case of RN black holes. Thus the charged black holes in EHM theory appear to have properties more in line with conventional systems as far as the third law of thermodynamics is concerned. Such characteristics of the black hole solutions in the EHM theory have not previously been seen in the black holes of more conventional theories, and hence it is of great interest to investigate them further.

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