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# Gravity-dominated unequal-mass black hole collisions

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We continue our series of studies of high-energy collisions of black holes investigating unequal-mass, boosted head-on collisions in four dimensions. We show that the fraction of the center-of-mass energy radiated as gravitational waves becomes independent of mass ratio and approximately equal to 13% at large energies. We support this conclusion with calculations using black hole perturbation theory and Smarr’s zero-frequency limit approximation. These results lend strong support to the conjecture that the detailed structure of the colliding objects is irrelevant at high energies.

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**Introduction.** Numerical simulations of black hole (BH) collisions are an ideal framework to understand the behavior of gravity in the strong-field regime. These simulations allow us to answer fundamental questions and to verify (or disprove) some of our cherished beliefs about Einstein’s general relativity (GR). Are BH collisions subject to cosmic censorship, so that naked singularities are never the outcome of any such event? What is the upper limit of the fraction of kinetic energy of the system that can be radiated in gravitational waves (GWs) during these collisions? In the ultrarelativistic (UR) limit, what properties of the collision, if any, are dependent on the underlying structure of the colliding objects, here the spins of the BHs and their mass ratio?

Some years ago we started a long-term program to answer these questions. We first showed that the head-on collision of two equal-mass BHs at the speed of light will radiate no more than  $\sim 14 \pm 3\%$  of the energy of the system [1] (this result was recently confirmed independently by the RIT group [2], refining the limit to  $13 \pm 1\%$ ). This is less than half the upper limit of  $\sim 29\%$  predicted by Penrose in the seventies, but two orders of magnitude larger than the energy radiated when two BHs collide head-on from rest [3]. We found that collisions with finite impact parameter can be tuned to exhibit “zoom-whirl” behavior [4, 5] and that they can produce near-maximally spinning remnants [6]. We also used zero-frequency limit (ZFL) calculations pioneered by Smarr [7] and BH perturbation theory to clarify the structure of the radiation [8]. We studied grazing collisions with aligned spins, showing that in the UR limit the radiated energy

and scattering thresholds become spin-independent [9]. In principle, in the UR limit it may be possible to radiate all kinetic energy as GWs by fine-tuning the collision near threshold, but extrapolations of our numerical results suggest that there is an upper limit of  $\sim 50\%$  on the radiation that can be emitted (this number is consistent with perturbative calculations in the extreme-mass-ratio limit presented in [10]). By analyzing the evolution of the apparent horizon, we found that the other half of the kinetic energy is absorbed and converted into rest mass of the merger remnant, or of the scattering constituents in non-merging cases. Furthermore, we demonstrated that non-merging scattering events with spinning BHs can exhibit large center-of-mass recoil velocities due to GW emission, showing that the formation of a common horizon is not necessary to impart a kick to the system [11]. All of our calculations support Penrose’s cosmic censorship conjecture. Note however that the quoted results have been from studies in four spacetime dimensions ( $D = 4$ ); Ref. [12] presented evidence *suggesting* that high-speed collisions in  $D = 5$  can lead to naked singularities from a generic subset of initial conditions.

One of the main conclusions to be drawn from our simulations of equal-mass, spinning BH collisions is that spin does not matter in the high-energy limit. In GR and in  $D = 4$ , isolated BHs in vacuum are uniquely characterized by their masses and spins. Since classical GR has no intrinsic scale then, beyond spin the only way this “ultraviolet universality” may be violated is by varying the mass ratio. Here we investigate one aspect of this problem by asking the following question: does the binary mass ratio affect the maximum amount of energy that can be radiated in UR head-on collisions? This paper bridges the gap between our previous simulations of UR, equal-mass head-on collisions [1] and the simulations of [13], which considered nonrelativistic head-on collisions with mass ratios as small as  $q = 1/100$ . The main product of

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this work is another confirmation of the simplicity and elegance of UR collisions in GR: we find that the maximum fraction of the total energy radiated as GWs in this limit is  $\sim 0.13$ , irrespective of the binary mass ratio.

A consequence of this research, along with previous studies of high-energy collisions of “stars” [14–16], is more solid evidence that the structure of the colliding objects is irrelevant at large energies. In other words, in this regime the outcome of colliding objects with a complex multipolar structure is equivalent to colliding Schwarzschild BHs. Any intricacies associated with matter interactions are hidden behind horizons and do not leave a distinguishable imprint on the GW signal.

**Setup:** Consider the collision of two nonspinning, electrically neutral BHs with rest masses  $m_{A,B}$ , total rest mass  $M_0 \equiv m_A + m_B$ , and mass ratio  $q \equiv m_A/m_B \leq 1$ . In the center of mass (CM) frame, where we measure all quantities, the velocity of BH  $A$  is  $v_A$  with corresponding Lorentz factor  $\gamma_A = (1 - v_A^2)^{-1/2}$ , and we can define its energy and momentum as  $E_A = \gamma_A m_A$  and  $P_A = \gamma_A m_A v_A$  respectively; likewise for BH  $B$ . In terms of these quantities, the CM frame is defined by

$$P \equiv P_A + P_B = m_B(q\gamma_A v_A + \gamma_B v_B) = 0. \quad (1)$$

The total energy of the spacetime is defined as  $M \equiv E_A + E_B$ , and we further introduce an effective Lorentz factor  $\gamma$  such that

$$\gamma M_0 \equiv M = E_A + E_B = m_B(q\gamma_A + \gamma_B). \quad (2)$$

In other words,  $1 - 1/\gamma$  measures the fraction of total energy that is initially in the form of kinetic energy.

In terms of parameters characterizing the initial data of each simulation, those most relevant to the collision problem are (i) the mass ratio  $q$ , which in this study takes the values  $q = 1, 1/2, 1/4, 1/10$ , and (ii) the effective Lorentz factor  $\gamma$ , or equivalently the velocities  $v_A$  and  $v_B$ . We use initial separations in the range  $d/M \approx 40$  to  $d/M \approx 100$ , observing that for such large values the radiation is essentially independent of  $d$ . The key diagnostic quantity is the amount of energy  $E_{\text{rad}}$  radiated in GWs, normalized to the total spacetime mass  $M$ , and excluding a contribution from an early burst of spurious (“junk”) radiation coming from the initial data.

Equal-mass collisions with  $q = 1$  were discussed in [1]. Here we perform additional simulations of BH collisions with unequal masses using the LEAN code [17] which is based on CACTUS [18, 19] and uses CARPET [20, 21] for mesh refinement, AHFINDERDIRECT [22, 23] and the spectral solver of Ref. [24] for initial data generation. For these new simulations, we fix the resolution by the scale  $m_A$  of the smaller hole to  $h = m_A/80$  near the BH singularities, and increase it by a factor 2 on each consecutive outer refinement level, for a total of 10 refinement levels when  $q = 1/2, 1/4$ , or 12 levels when  $q = 1/10$ . To measure gravitational radiation we compute the Newman-Penrose scalar  $\Psi_4$  at several radii, typically within a range  $[50 \dots 200]M$ . We then decompose

$\Psi_4$  into multipole modes  $\psi_{lm}$  of the spherical harmonics  ${}_{-2}Y_{lm}$  of spin-weight  $-2$ :

$$\Psi_4(t, r, \theta, \phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^l {}_{-2}Y_{lm}(\theta, \phi) \psi_{lm}(t, r). \quad (3)$$

Due to the symmetries of this problem, the only nonvanishing multipoles all have  $m = 0$ . The energy flux is then given by

$$\dot{E} = \sum_l \lim_{r \rightarrow \infty} \frac{r^2}{16\pi} \left| \int_{-\infty}^t \psi_{l0}(\tilde{t}) d\tilde{t} \right|^2 \equiv \sum_l \dot{E}_l, \quad (4)$$

where overdots ( $\dot{\phantom{x}}$ ) denote time derivatives.

Our results are affected by three main sources of uncertainty: the finite extraction radius  $r_{\text{ex}}$ , the discretization error and the spurious initial radiation. We estimate the error arising from using a finite extraction radius by measuring the waveform components at several radii, and fitting the resultant flux to an expression of the form  $\dot{E}(r, t) = \dot{E}^{(0)}(t) + \dot{E}^{(1)}(t)/r$ . The estimated uncertainty is then given by the difference between the net radiated energy  $E_{\text{rad}}$  calculated using the extrapolated result  $\dot{E}^{(0)}$  and that calculated with  $\dot{E}(r_{\text{ex}})$  at the largest value of  $r_{\text{ex}}$ . We find that the fractional uncertainty in  $E_{\text{rad}}$  amounts to a maximum of 2% for low or vanishing boosts, 4% for moderate velocities  $v_A \approx 0.5$ , and 8% for the largest boosts  $v_A \approx 0.9$ .

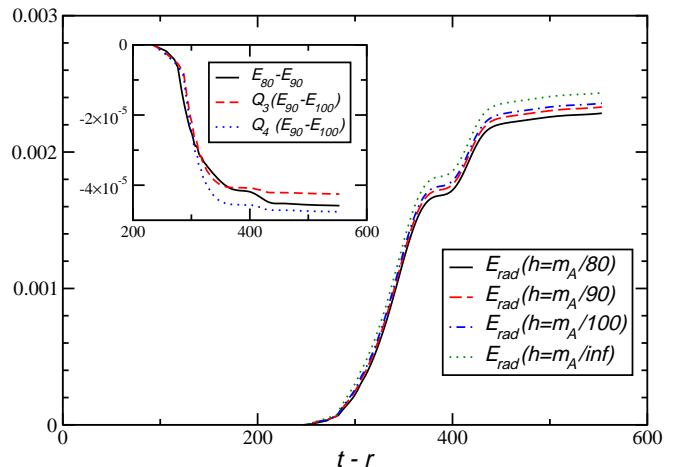


Figure 1. Convergence plot for a  $q = 0.10$ ,  $\gamma = 1.11$  binary where  $v_A = 0.87$ . The radiated energy is shown in units of the total mass  $M$ . The inset shows the deviations between coarse, medium and high resolution rescaled for third-order convergence ( $Q_3 = 1.56$ ) and fourth-order convergence ( $Q_4 = 1.75$ ). The dotted curve in the main panel shows the energy extrapolated to infinite resolution using the more conservative 3rd-order estimate.

To estimate discretization errors we have evolved one of the most challenging collisions, namely  $q = 1/10$  and  $v_A = 0.87$ , using additional resolutions  $h = m_A/90$ ,

$m_A/100$  (with all coarser refinement levels adjusted accordingly). Figure 1 shows that the quantity  $E_{\text{rad}}$  exhibits between third- and fourth-order convergence. A conservative estimate obtained assuming third-order convergence gives a fractional error of 6%, and we complement this with estimates of 2% obtained from the prior  $\gamma = 1$  study in [13].

Finally, the conformally flat puncture initial data contain spurious gravitational radiation, which increases strongly with boost  $\gamma$ . In order to extract physically meaningful information we must separate the spurious radiation from the radiation generated by the collision itself. This is done by “waiting” for the spurious radiation to pass the last extraction radius, and then discarding the earlier part of the GW signal. The exact choice of the time where to separate spurious initial radiation from that generated in the collision itself introduces an uncertainty, which we estimate by varying this choice guided by the quadrupole radiation, where spurious and physical radiation can be identified most clearly. For low boosts the resulting error is negligible, but it increases significantly to 6% for  $v_A \approx 0.6$  and 10% (12%) for  $q = 1/2$  ( $q = 1/4, 1/10$ ) at  $v \approx 0.9$ . In summary, the total error budget is about 4% when  $v_A = 0$ , 14% when  $v \approx 0.6$  and 24% (26%) when  $v_A \approx 0.9$  for  $q = 1/2$  ( $1/4, 1/10$ ).

**Results.** The waveforms and corresponding energy fluxes from a set of the most challenging runs are shown in Fig. 2. The waveforms have a structure familiar in BH dynamics [1]: a precursor, a main burst at the onset of the formation of a common apparent horizon, and a final ringdown tail. We find that, to a good approximation, the final BH rings down in the lowest QNM frequency as predicted by linear theory [25, 26].

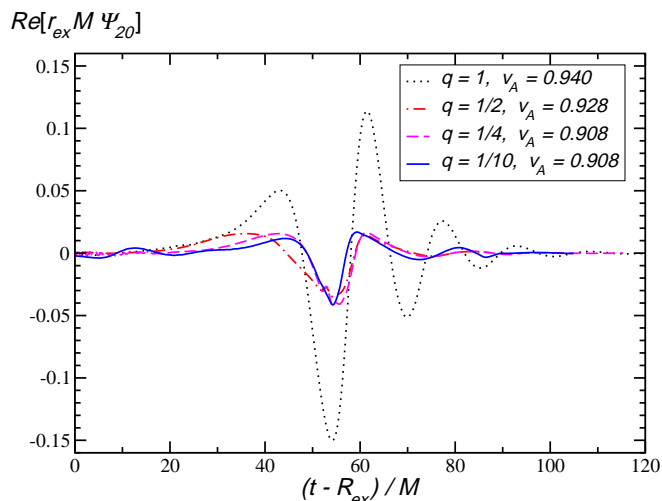


Figure 2. The dominant multipole  $\psi_{20}$  of the Newman-Penrose scalar extracted from the most relativistic collisions considered for each mass ratio. The imaginary part of  $\psi_{20}$  vanishes for all cases due to symmetry.

The total integrated energy  $E_{\text{rad}}$  radiated in GWs (normalized by the total center-of-mass energy  $M$ ) is

shown in Fig. 3 for all the simulations we studied. This

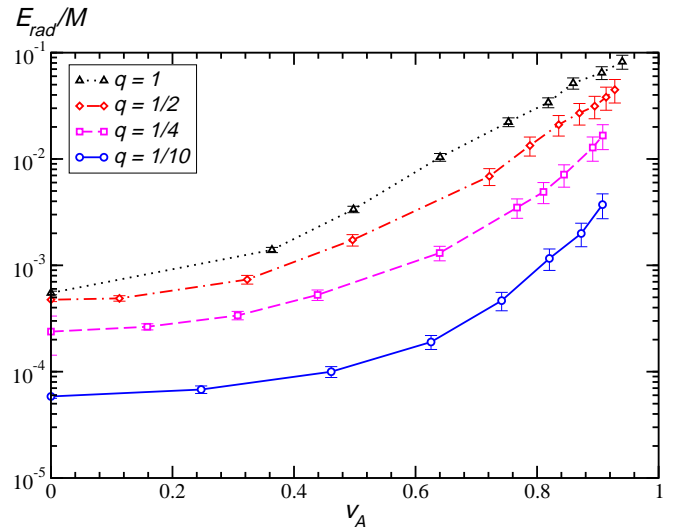


Figure 3. Total energy radiated  $E_{\text{rad}}/M$  as a function of  $v_A$ .

quantity rapidly increases for large boosts. To understand the limiting behavior we resort to two perturbative calculations: the ZFL and point-particle approximations.

The ZFL [8] has been very successful at describing the functional dependence of the nonlinear results for equal-mass collisions [1, 2]. We therefore use the ZFL result for generic *unequal-mass* collisions [7], in particular the spectrum per unit solid angle

$$f(\theta, q, v_A) \equiv \frac{1}{\gamma_A^2 m_A^2} \frac{d^2 E_{\text{rad}}}{d\omega d\Omega} \quad (5)$$

$$= \frac{v_A^2 \sin^4 \theta}{4\pi^2} \left[ \frac{v_A + v_B}{(1 - v_A \cos \theta)(1 + v_B \cos \theta)} \right]^2.$$

With the physically reasonable assumption that there is a cutoff frequency at  $\omega_c \sim X(q)/M$ , we get

$$\frac{E_{\text{rad}}}{M} = X(q) \frac{m_A^2 \gamma_A^2}{M^2} F(v_A, q), \quad (6)$$

where  $F(v_A, q) = \int d\Omega f(\theta, q, v_A)$  can be computed analytically. In other words, the ZFL gives an analytical prediction with only one unknown parameter  $X(q)$ , and for very large CM energies  $E_{\text{rad}}/M \rightarrow X(q)/\pi$ . By fitting the last three points in Fig. 3 for each value of  $q$  to Eq. (6) we can get the percentage of energy radiated for each mass ratio in the UR limit,  $\epsilon(q) \equiv 100X(q)/\pi$ :

$$\begin{aligned} \epsilon(1) &= 12.7 \pm 1.5, & \epsilon(1/2) &= 11.2 \pm 2.7, \\ \epsilon(1/4) &= 11.6 \pm 3.0, & \epsilon(1/10) &= 12.0 \pm 3.0. \end{aligned} \quad (7)$$

Our results for  $\epsilon(q)$  consistently lie in the 11–13% interval for all mass ratios. This strongly supports the conjecture that the structure of the colliding objects becomes irrelevant at large center-of-mass energies.

There is another limit which is amenable to a semi-analytic treatment, and that is when a small BH  $A$  with

energy  $E_A$  collides with a large BH with mass  $M$  such that  $E_A \ll M$ . This is the point-particle limit [8, 27–30]. Nonlinear head-on collision results agree extremely well with point-particle predictions even when the mass ratios in the simulations approach unity, at least for low-energy encounters [13]. To test whether they also agree for high-energy collisions we consider, as a representative example, our smallest mass-ratio runs with  $q = 1/10$  (which are marginally within the regime of validity of perturbation theory). By computing the radiation for a point-like particle with velocity  $v_A = 0.91$  and energy  $E_A = 2.38m_0$  (where  $m_0$  is the particle’s rest mass) falling into a massive BH of mass  $M$  through a numerical integration of the Zerilli equation for multipoles up to  $l = 6$  with two independent codes we find

$$\frac{ME_{\text{rad}}}{E_A^2} = 0.090. \quad (8)$$

To make contact with our results, we take  $M$  to be the CM energy  $M = E_A + E_B$  (we could equally well take  $M = E_B$ , as the two are equivalent in the point-particle limit; this intrinsic ambiguity would not affect the agreement between the point-particle results and the numerics shown below). Our full nonlinear, numerical simulations for  $q = 1/10$  yield  $(E_B + E_A)E_{\text{rad}}/E_A^2 = 0.104$ . This surprisingly good agreement ( $\sim 10\%$ ) provides further support to our results.

**Conclusions.** The main conclusion of the present study is to confirm the expectation, borne out of our previous work [1, 6, 9, 15], that the structure of the colliding objects does not matter in gravity-dominated collisions. We have previously demonstrated that the effect of spin on the radiated energy and scattering threshold becomes negligible for grazing collisions in the UR limit [9]. In this work we show that the effect of the binary mass ratio on the radiated energy becomes negligible in the UR limit for head-on collisions. It will be interesting to verify whether the effect of mass ratio is likewise irrelevant in the UR limit for *grazing* collisions.

Another interesting extension will be to consider space-time dimensions  $D > 4$ . The modeling of  $D$ -dimensional spacetimes with  $SO(D-3)$  symmetry can be performed in the framework of standard “3+1” numerical relativity by adding extra fields [31–34] and has been applied to study the stability of Myers-Perry BHs [35, 36] and

BH collisions [12, 37, 38]. Progress on UR collisions in higher dimensions has been slower than expected, however, due to technical complications in achieving code stability. A new formulation of the higher-dimensional Einstein equations now allows our LEAN code to compute the radiation from head-on collisions up to  $D = 10$ , so that the number of extra dimensions considered possible in higher-dimensional gravity scenarios [39] falls within the range achievable by the code. The study of unequal-mass UR collisions in higher dimensions is of particular interest because perturbative calculations suggest that the percentage of kinetic energy radiated in GWs may reach a minimum as a function of  $D$ , and then increase again [30]. The perturbative calculations do not hold for  $D \geq 13$ , since they predict a total radiation output which breaks the assumptions behind the formalism [30], and therefore our understanding of radiation in large- $D$  spacetimes is still lacking. We plan to investigate this problem in the near future.

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