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Atreya Chatterjee and David A. Lowe

Physics Department, Brown University, Providence, RI, 02912, USA

Abstract

The holographic dual of a gravitational theory around the de Sitter background is argued to be a Euclidean conformal gravity theory in one fewer dimensions. The measure for the holographic theory naturally includes a sum over topologies as well as conformal structures.

I. INTRODUCTION

There has been much success in describing gravity in Anti-de-Sitter spacetime using a holographic description on the boundary at infinity. In the holographic description, a conformal field theory (CFT) lives on the boundary. This marks a major step in quantizing gravity in asymptotically anti-de-Sitter spacetime. However according to experimental observation, our universe has positive cosmological constant. Thus it is interesting to consider a holographic description of gravity in the de-Sitter spacetime.

One way of approaching this problem is to analytically continue the AdS/CFT correspondence to dS/CFT correspondence [1]. There are many successes in this approach but there are many conceptual difficulties as well. In this paper we will try to clarify some of these difficulties. In the paper [2], we constructed a mapping between bulk field operators and boundary operators. As we saw in that paper, the boundary CFT has operators which violate cluster decomposition. Cluster decomposition is one of the basic assumptions of any interacting quantum field theory [3, 4].

To set the stage for understanding the problem in de Sitter spacetime, be begin by considering the well-understood problem in anti de Sitter spacetime. In that case, there is a positive energy theorem [5, 6] and the unitary representations of the conformal group $SO(d-1,2)$ that appear are lowest weight. Moreover the boundary conditions on conformal infinity $\mathscr{I}$ that preserve conformal flatness are compatible with the unitarity bound of [6]. In particular, with these boundary conditions, one obtains a complete set of modes for fluctuations around the anti-de Sitter background.

For de Sitter spacetime there is no global positive energy theorem [5] and the unitary representations of the conformal group $SO(d,1)$ corresponding to ordinary massive and massless fields are neither highest nor lowest weight, but are rather the principal series and the complementary series, which are unbounded. This leads to the problem of cluster decomposition violation in the boundary theory, noted in [2]. In the case of de Sitter, a complete set of modes (for the graviton) leads to configurations with a nontrivial conformal class at conformal infinity $\mathscr{I}$. Thus one cannot impose boundary conditions to maintain conformal flatness, without truncating the linearized spectrum of the theory [7, 8]. Therefore to describe a quantum theory with the full set of modes in a de Sitter background, the holographic description must accommodate a path integral over boundary metrics. The boundary theory will be invariant under the asymptotic symmetry group of the de Sitter spacetime that preserves this more general set of asymptotic boundary conditions. In this case, the asymptotic symmetry group is not just the conformal group, corresponding to isometries of de Sitter, but is rather the full group of diffeomorphisms of $\mathscr{I}$. This leads us to conjecture the holographic dual will be a theory of conformal gravity theory living at $\mathscr{I}$.

At first sight, this might seem a step backward, since theories of conformal gravity seem difficult to quantize [9]. Nevertheless, there are examples where progress has been made. For three dimensional pure conformal gravity, a Chern-Simons gauge formulation is available [10]. For conformal gravity arising in string theory, a twistor string formulation has been found [11]. So there is hope that the rather different conformal gravity theories considered here can be successfully quantized.

Having found a path integral over conformal classes of metric on $\mathscr{I}$ is needed to provide a holographic description of gravity in de Sitter, it is then natural to ask whether one must include a sum over topologies of $\mathscr{I}$ as well. In the case of anti-de Sitter, this question was addressed in [12]. There it was found that if $\mathscr{I}$ has positive curvature, it must be connected and cannot contain nontrivial topology, such as wormholes. This result is important for the basic consistency of AdS/CFT.

Some related questions have been considered in the context of dS/CFT in [13, 14]. However there it quickly becomes clear that ordinary matter will lead to nontrivial topology for $\mathscr{I}$ in four-dimensional de Sitter since a black hole already changes the topology from S^3 for empty de Sitter to $S^2 \times R$ for a black hole. Recall in AdS, the topology of $\mathscr{I}$ remains $S^2 \times R$ for empty AdS, or the AdS Schwarzschild black hole.

One can gain a more detailed understanding of this topology change in the case of three-dimensional de Sitter. As an example, we consider the solution for multi-black holes in three-dimensional de Sitter spacetime [15]. We show $\mathscr{I}$ can be mapped from a multi-sheeted sphere to a single cover with punctures. The resulting holographic dual is a theory of two-dimensional gravity, identical to a worldsheet string theory. At least in this example, there is a natural moduli space corresponding to a sum of worldsheet topologies. It remains an interesting open question whether such a sum over topologies can be defined in the higher dimensional case.

II. ASYMPTOTIC SYMMETRY GROUP

To specify the asymptotic structure of a spacetime we attempt to construct a set of boundary conditions that capture a wide-class of physically interesting solutions. The Penrose conformal compactification of the geometry provides an enormous simplification in treating these asymptotic boundary conditions, because solutions may more easily be studied on the compact unphysical spacetime (related by a Weyl transformation to the physical spacetime) where the group of diffeomorphisms is clearly defined [16].

If one considers linearized perturbations around de Sitter, the conformal group should have a well-defined action. In this limit, one can consider the perturbation on top of the fixed de Sitter background, which has as an isometry group $SO(d,1)$. These isometries induce a $SO(d,1)$ global conformal transformation on $\mathscr{I}$.

A. Four dimensions

At first sight, the situation for nonlinear solutions appears much less clear. We will restrict our discussion to four-dimensional de Sitter, and discuss the very special features of three dimensions later. As mentioned in the introduction, already black holes will tend to change the very topology of $\mathscr{I}$ and it is not clear if any precise asymptotic conditions can be formulated. Ashtekar et al. [7, 8] deal with this by focussing on isolated gravitating systems in de Sitter. Our approach will take a different viewpoint, and allow for arbitrary boundary metrics that respect the asymptotic de Sitter metric conditions locally

$$ds^2 = R_{dS}^2 \eta^{-2} \left(-d\eta^2 + (\delta_{ij} + h_{ij}) dx^i dx^j \right) \tag{1}$$

where we can perform a power series expansion of h_{ij} as

$$h_{ij}(\eta, x) = h_{(0)ij}(x) + \eta^2 h_{(2)ij}(x) + \eta^3 h_{(3)ij}(x) + \mathcal{O}(\eta^4) \tag{2}$$

following [17]. For now we will take $\mathscr{I}$ to have topology of the 3-sphere, thus we are considering globally asymptotically de Sitter spacetimes. We will consider more general topologies later in the paper. As we will see later, typical matter configurations only yield a single regular asymptotic region, so we take $\mathscr{I}$ to refer to either $\mathscr{I}^+$ or $\mathscr{I}^-$ but not a disconnected union of the two.

One can largely separate the issue of topology change by first restricting considerations to theories of gravity with conformally coupled matter. There powerful nonlinear stability theorems have been proven by Friedrich [18]. In particular, for an open set of initial data, it has been shown that a past asymptotic de Sitter spacetime can smoothly evolve to a future asymptotic de Sitter spacetime. The solutions obtained involve a metric at $\mathscr{I}$ in a nontrivial conformal class. These correspond to the usual long wavelength gravitons of the theory of inflation, which freeze out when stretched past the horizon scale. They induce a nontrivial Cotton tensor on $\mathscr{I}$.

As pointed out in [7, 8], demanding conformal flatness of the boundary projects out these graviton modes from de Sitter. Therefore if the holographic theory of de Sitter gravity was simply a conformal field theory, living on a background with a fixed conformal structure, the CFT would not be able to reproduce the full set of graviton modes. One may of course perturbatively correct for this by introducing sources on the boundary, however then one must specify a path integral measure for such sources in order to reproduce bulk observables, such as in-in correlators.

Let us try to establish the gauge symmetries of the boundary theory. If we consider general asymptotic boundary conditions of the form (1) the asymptotic symmetry group is much larger than the global conformal group. Instead, it consists of the full group of diffeomorphisms of $\mathscr{I}$. As we will see later, we can reconstruct part of the action of the holographic dual by considering the boundary action of the bulk theory, evaluated on solutions of the equation of motion. This boundary action then inherits the gauge symmetry of the bulk, associated with diffeomorphisms of $\mathscr{I}$.

The construction of the boundary theory is predicated on the Penrose compactification of the bulk spacetime. This is achieved by performing a general Weyl transformation of the bulk metric $g_{(unphys)\mu,\nu} = \Omega^2(\eta, x) g_{(phys)\mu,\nu}$ for some choice of smooth function Ω that vanishes on $\mathscr{I}$, but with non-vanishing normal derivative. Again, by reconstructing part of the action of the holographic dual involving the boundary metric, one sees the boundary theory must inherit this Weyl invariance as a gauge symmetry. We conclude then that the boundary theory must be a theory of Euclidean conformal gravity.

In many ways, this is not a new statement. It has been advocated that the dS/CFT correspondence be viewed as a computation of a wavefunction via a CFT partition function

$$\Psi(h) = Z_{CFT}[h] \tag{3}$$

where h denotes the boundary metric. Our point is simply to compute bulk observables, one must make the further step of computing

$$\langle 0|O(x_1)O(x_2)|0\rangle = \int \mathcal{D}h \Psi^*(h) O_{CFT}(x_1) O_{CFT}(x_2) \Psi(h) = \int \mathcal{D}h \, Z^*_{CFT}[h] O_{CFT}(x_1) O_{CFT}(x_2) Z_{CFT}[h]$$

with some a priori unknown measure $\mathcal{D}h$, and some de Sitter spacetime operators O. Here the vacuum state $|0\rangle$ is to be understood as an interacting generalization of the Bunch-Davies vacuum. The operators O_{CFT} are the dual CFT operators. For matter fields in a fixed de Sitter background, these can be constructed [2]. To formulate a complete holographic description, one instead must build the integration measure into the theory. This gives rise to our conjecture that dS gravity is dual to a theory of conformal gravity on $\mathscr{I}$. In that case, the relevant correlator would be

$$\langle 0|O(x_1)O(x_2)|0\rangle = \langle O_{cgrav}(x_1) O_{cgrav}(x_2)\rangle \tag{4}$$

where the left-hand side is an in-in correlator in the bulk theory, and the right-hand side represents the map of these observables into the conformal gravity theory. The next goal is to try to specify as much as possible, this conformal gravity theory. If this can be established, it will then be necessary to revisit the boundary to bulk operator mapping after properly understanding the gauge invariant observables of the conformal gravity theory. In its current formulation [2], the mapping would only make sense for small perturbations around some classical background.

B. Quadratic action for holographic theory: 4d de Sitter

In general to build operators in the boundary theory from those in the bulk, one must use the integral transform method described in [2], and its generalizations. This can be viewed as an analog of the LSZ transform in constructing the S-matrix in asymptotically flat spacetime.

In anti-de Sitter spacetime, one has a much easier task, because the bulk to boundary mapping is much simpler, since the physical fields of interest have simple power law falloff, dependent on their masses. So while one must perform an integral transform to construct quasi-local bulk fields from boundary operators, the inverse operation reduces to taking a residue in the limit that the bulk operator approaches infinity.

Nevertheless, if we focus on the gravitational field, and massless minimally coupled scalars, for example, the results of AdS may be continued to de Sitter. This is the approach followed in [19, 20]. See also [21] for related discussion of these issues. Here let us generalize this to a massive scalar in de Sitter, with action

$$S_{mat} = \int d\eta d^3x \, \frac{1}{2}\sqrt{-\mathrm{det}g}\left(-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2 - \xi R\phi^2\right) . \tag{5}$$

In empty de Sitter, with metric (1) the solution of the equation of motion may be decomposed into the Bunch-Davies [22, 23] mode functions

$$u_k = \frac{1}{2^{5/2}\pi}\eta^{3/2}H_\mu^{(2)}(-k\eta)e^{ik\cdot x}$$

where

$$\mu^2 = \frac{9}{4} - 12\left(\frac{m^2}{R} + \xi\right) .$$

Let us for the moment take μ to be real, corresponding to the so-called complementary series representations of the conformal group. Note we will work with the future half of the slicing, so $-\infty < \eta < 0$. We wish to compute the on-shell action, which reduces to a boundary term as $\eta \to 0^-$. We take a solution with some fixed behavior on some late-time slice $\eta = \eta_c$

$$\phi(\eta, x) = \frac{\eta^{3/2}H_\mu^{(2)}(-k\eta)}{\eta_c^{3/2}H_\mu^{(2)}(-k\eta_c)}f_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}$$

and substitute into (5) to obtain

$$\begin{aligned} iS_{mat} &= iR_{dS}^2 \int d^3x \frac{1}{2\eta^2}\phi\partial_\eta\phi|_{\eta=\eta_c} \\ &= iR_{dS}^2 \int \frac{d^3k}{2\,(2\pi)^3} f_{\vec{k}}f_{-\vec{k}}\frac{1}{4\eta_c^3}\left(3 - 2\mu - \frac{2\eta_c k H_{\mu-1}^{(2)}(-k\eta_c)}{H_\mu^{(2)}(-k\eta_c)}\right) . \end{aligned} \tag{6}$$

When μ is half-integer, this expression may be expanded near $\eta \to 0^-$ (i.e. $\mathscr{I}^+$) and interpreted as a series of counter-terms that must be subtracted to yield a finite boundary action. For example, the massless minimally coupled scalar corresponds to $m = 0, \xi = 0$ giving $\mu = 3/2$ and

$$iS_{mat} = R_{dS}^2 \int \frac{d^3k}{2\,(2\pi)^3} f_{\vec{k}}f_{-\vec{k}}\left(\frac{ik^2}{2\eta_c} - \frac{k^3}{2}\right)$$

as $\eta_c \to 0$. The imaginary divergent term might then be subtracted with a local $\int d^3x(\partial\phi)^2$ counterterm. The finite piece yields the expected boundary propagator of a quasi-primary field with conformal weight $\Delta = 3 = \frac{3}{2} + \mu$. The boundary action for the scalar field then has the following form, which is non-analytic in momenta

$$S_{boundary} = R_{dS}^2 \int d^3x \frac{1}{2} \phi \left(\Box\right)^{3/2} \phi \, .$$

Another simple example is the massless conformally coupled scalar, with $m = 0, \xi = 1/6$ which gives $\mu = 1/2$ and

$$iS_{mat} = R_{dS}^2 \int \frac{d^3k}{2\left(2\pi\right)^3} f_{\vec{k}} f_{-\vec{k}} \left(\frac{i}{2\eta_c^3} - \frac{k}{2\eta_c^2} \right)$$

which has a vanishing finite boundary action after subtracting the divergent counterterms. We will comment on this and the case of more general mass in a moment.

In a transverse traceless gauge, the action for metric fluctuations matches that of the massless minimally coupled scalar with a different normalization, giving the boundary action

$$S_{grav,boundary} = \frac{R_{dS}^2}{64\pi G} \int d^3x \, h_{ij}^{TT} \left(\Box\right)^{3/2} h_{TT}^{ij} \, .$$

As noted in [19] this gives a negative contribution to the 2-point function of the boundary stress energy tensor proportional to the central charge.

So far we have seen the boundary counter-term approach seems to work well for the metric and massless minimally coupled scalar matter. As noted in [2] this approach of extracting boundary operators for more general matter in de Sitter, by simply taking asymptotic limits of the fields, fails in general. If one were to evaluate (6) one would get oscillating cutoff (i.e. η_c) dependent expressions[24]. The correct approach is to apply an analog of the LSZ reduction formula of asymptotically flat spacetime, by performing an integral transform on the bulk fields to obtain a boundary expression that transforms covariantly under the conformal group [2]. For scalar fields, this gives

$$S_{matter,boundary} = R_{dS}^2 \int d^3x \frac{1}{2} \phi \left(\Box\right)^{\mu} \phi$$

for $\mu > 0$ real, corresponding to the complementary series of the conformal group and

$$S_{matter,boundary} = R_{dS}^2 \int d^3x \frac{1}{2} \left(\phi \left(\Box\right)^{\mu} \phi + \phi^* \left(\Box\right)^{-\mu} \phi^* \right)$$

for μ imaginary, corresponding to the principal series of the conformal group.

In the above, we have derived the quadratic terms that appear in the conformal gravity theory coupled to matter. Already we see the conformal gravity theory seems to be of a new kind, due to the non-polynomial nature of its derivatives appearing in the quadratic term. The theory appears to be free of ghosts, at least in the case when the matter is restricted so $\mu > 0$. At least at bulk tree-level, one should be able to recover the higher order terms in the holographic Lagrangian. The classical stability of de Sitter would seem to indicate this procedure should be completely well-defined.

An easy generalization of the above is to break parity in the bulk by adding an $\int R\hat{}R$ term, which corresponds to adding a Chern-Simons gravity term to the boundary. This yields the boundary Lagrangian for the well-studied case of topologically massive gravity [25], which is invariant under Weyl and diffeomorphism symmetries.

As has been emphasized in [2] the boundary theory violates cluster decomposition, which is one of the central axioms of Euclidean quantum field theory [3, 4]. It will be very interesting to construct interacting holographic duals. In the free limit, examples have been constructed in the context of higher spin gravity in de Sitter [26].

C. Three dimensions

The case of three-dimensional de Sitter is special, because then $\mathscr{I}$ is two-dimensional, and always locally conformally flat. In this case, the expansion of the metric (1) takes the form

$$h_{ij} = h_{(0)ij} + \eta^2 h_{(2)ij} + \mathcal{O}(\eta^3)\,.$$

We can nevertheless follow the strategy described above to compute the boundary term arising from the on-shell bulk action. Now we will find the boundary counter-term action

$$iS_{grav} = \frac{i}{16\pi G_3}\int d^2x 2\sqrt{-\det g} - \frac{1}{2}\log\left(-\eta_c\right)\sqrt{\det h_{(0)}}R_{(0)}\,.$$

The anomalous contribution proportional to $\log \eta_c$ must be cancelled for the theory to be conformally invariant. One way to approach the problem is to couple the boundary theory to a Liouville field theory with central charge adjusted so that a Weyl transformation, shifting $\eta_c \to \alpha\eta_c$ is compensated by the anomaly term coming from the Liouville theory. This renders the boundary theory diffeomorphism invariant and Weyl invariant.

In this way, the boundary theory takes the form of the Polyakov string. The central charge induced by the gravitational contribution to the conformal anomaly is

$$c = -\frac{3R_{dS}}{2G_3} \,. \tag{7}$$

For the theory to be Weyl invariant at the quantum level, this central charge must be cancelled by that of the Liouville field, leading to a boundary theory with vanishing conformal anomaly.

In the usual conformal gauge of string theory, for fixed boundary topology, the theory reduces to an ordinary conformal field theory (coupled to the Liouville field) and the details of conformal gravity may be forgotten. Moreover in string theory there is a well-defined path integral involving sums over nontrivial worldsheet topologies. Each topology is equipped with a well-defined moduli space. We expect this sum over topologies is important to properly understand the holographic theory describing quantum gravity in de Sitter, a question we turn to in the next section.

III. TOPOLOGY CHANGE

It is important for the consistency of AdS/CFT that there are strong restrictions on the topology of the bulk geometry M given the boundary. For example, Witten and Yau [12] showed that the boundary must be connected, and that the bulk Euclidean geometry satisfies $H_n(M, \mathbb{Z}) = 0$ if the boundary has positive scalar curvature.

Similar topological restrictions have been explored in the context of four-dimensional asymptotically de Sitter spacetimes in [13, 14]. For example, if $\mathscr{I}^+$ has infinite fundamental group, then one has the rather strong result if matter obeys the null energy condition, there is no regular $\mathscr{I}^-$. Similarly if $\mathscr{I}^+$ has positive first Betti number, then the bulk is past null geodesically incomplete. Nevertheless, there are many examples where at least $\mathscr{I}^+$ is well-defined. The case we will be most interested in is the case where $\mathscr{I}^+$ is a sphere with punctures. Isolated gravitating systems in de Sitter can reach $\mathscr{I}^+$ where they appear as punctures. In the work of [7, 8] the focus is on a single isolated gravitating system. Since here we are interested in building a holographic dual applicable to cosmology, we will be most interested in is the case where $\mathscr{I}^+$ is a sphere with multiple punctures.

If we wish to accommodate such isolated gravitating systems in the dual conformal gravity theory living on $\mathscr{I}^+$, we must therefore include a sum over topologies of the boundary. In the case of three-dimensional Euclidean geometries, it is not clear whether a path integral of conformal gravity over such a space can be defined. Though it nevertheless appears to be a simpler problem than the original proposals for four-dimensional Euclidean quantum gravity as a path integral over geometries.

In the case of three-dimensional asymptotically de Sitter geometries things are much simpler. Again, $\mathscr{I}^+$ is always conformally flat, but one nevertheless must deal with this sum over topologies. The sum over the moduli space of compact Riemann surfaces (including punctures), is well-understood in the context of string theory and leads to a complete proposal for the path integral of the conformal gravity theory. That is, if we are given a Lagrangian for a CFT with central charge (7), we can couple it to conformal gravity by performing a Weyl rescaling, and add in the Liouville sector to cancel the overall conformal anomaly. One can then fix conformal gauge, and treat the theory as one would with any worldsheet string theory.

In the remainder of this section, we consider an example of a multi-black hole solution in three-dimensional asymptotically de Sitter spacetime [15]. If the above proposal is correct, it should be possible to view $\mathscr{I}^+$ as a 2-sphere with punctures. However the original work [15] expressed the Cauchy slices as a multiple cover of a sphere with only two punctures at the north and south poles. In the following, we construct the covering space and show it is a single cover of a sphere with multiple non-degenerate punctures.

A. Example: multi-black hole solution in $\mathbf{dS}_3$

Deser and Jackiw have found the metric of 2+1 dimensional gravity asymptotically de-Sitter spacetime [15] in the presence of N stationary massive particles. It is given by

$$\begin{aligned}
ds^2 &= M^2(r)dt^2 + f(r)dzdz^* \\
f(z) &= \frac{\epsilon}{\lambda V(z)V^*(z^*)\cosh^2\left(\sqrt{\epsilon}\,(\zeta-\zeta_0)\right)} \\
M(z) &= \epsilon\tanh\left(\sqrt{\epsilon}\,(\zeta-\zeta_0)\right) \\
V(z) &= c^{-1}\prod_{n=1}^{N}(z-z_n) \\
\zeta(z) &= \frac{1}{2}\left(\int\frac{dz}{V(z)}+\int\frac{dz^*}{V^*(z^*)}\right) = \ln\left(\prod_n |z-z_n|^{c_n}\right) \\
c_n &= \prod_{n'\neq n}\frac{c}{z_n-z_{n'}} \\
\sum_{n=1}^{N} c_n &= 0\,.
\end{aligned} \tag{8}$$

Here $\lambda > 0$ is the cosmological constant. The first equation gives the metric in complex plane in terms of $f(z), M(z)$. $V(z)$ is the master function in terms of which the solution is given. z_n are the punctures in the complex plane where particles are inserted and c is a free parameter. We demand that c_n be real for single valuedness of the solution. The coordinate transformation

$$\begin{aligned}
\sin\omega &= \frac{1}{\cosh\left(\sqrt{\epsilon}(\zeta-\zeta_0)\right)} \\
\phi &= \frac{\epsilon}{2i}\left(\int\frac{dz}{V(z)}-\int\frac{dz^*}{V^*(z^*)}\right)
\end{aligned} \tag{9}$$

takes us to the familiar static coordinates

$$ds^2 = -\cos^2\omega dt^2 + \lambda^{-1}\left(d\omega^2+\sin^2\omega d\phi^2\right)\,. \tag{10}$$

Note that all the particles are located at $\sin\omega = 0$ so that $\omega = 0, \pi$. The further coordinate change $\sqrt{\lambda}R = \sin\omega$ takes us to the static Schwarzschild-de-Sitter coordinates

$$ds^2 = -(1-\lambda R^2)dt^2 + (1-\lambda R^2)^{-1}dR^2 + R^2 d\phi^2$$

which covers the full space, but the range of ϕ goes from $[0, 2\pi\alpha_n)$ at the location of n^{th} particle where $\alpha_n = \sqrt{\epsilon}c_n = 1 - 4Gm_n$. This is the familiar conical deficit of 3-dimensional gravity. Locally the metric is same as pure de-Sitter and has constant curvature.

Now we will investigate the geometry. First we will consider 3-particle case before generalizing to the N particle case. Uniqueness of the solution requires that the 3 particles are all in a line and c_n sum to zero. Let us take $c = 1, z_1 = -3, z_2 = 1, z_3 = 2$. Then let us choose

$$\begin{aligned} c_1 &= \frac{1}{(z_1 - z_2)(z_1 - z_3)} = \frac{1}{20} \\ c_2 &= -\frac{1}{4} \\ c_3 &= \frac{1}{5} \\ c_1 + c_2 + c_2 &= 0\,. \end{aligned}$$

Thus this configuration satisfies all the constraints. Now let us look at the functions that determine the geometry

$$\begin{aligned} V(z) &= (z+3)(z-1)(z-2) \\ \frac{1}{V(z)} &= \frac{1}{20(z+3)} - \frac{1}{4(z-1)} + \frac{1}{5(z-2)} \\ \zeta &= \frac{1}{2}\left(\int \frac{dz}{V(z)} + \int \frac{dz^*}{V^*(z^*)}\right) = \frac{1}{20}\ln\left(\frac{|z+3||z-2|^4}{|z-1|^5}\right)\,. \end{aligned}$$

At $z = -3, 2,\ \zeta = -\infty$ and at $z = 1,\ \zeta = \infty$. In z coordinates, we have punctures at 3 points. We now want to understand the picture in the ω, ϕ coordinates using (9)..

The points $z = -3, 1, 2$ correspond to $\sin\omega = 0 \implies \omega = 0, \pi$. Thus two of the particles are at south pole and one at north pole. But then it is not immediately clear whether the particles at the south pole are overlapping or they are multiple disconnected sheets or they are sphere connected at some points etc. To understand the topology, we first note that we can have a path between any two particles without crossing the other particle. This implies that the sheets are connected. Secondly, the distance between any two particles is non-zero.

To see this we note that constant ω corresponds to constant $\frac{|z+3||z-2|^4}{|z-1|^5}$ curves in the complex plane.

1. $\frac{|z+3||z-2|^4}{|z-1|^5} = \pm\infty$ would correspond to north and south pole $\omega = 0, \pi$.

2. $\frac{|z+3||z-2|^4}{|z-1|^5} = 1$ corresponds to the equator $\omega = \pi/2$.

First we note that at $z = -3, 2$, $\frac{|z+3||z-2|^4}{|z-1|^5} = 0 < 1$ and at $z = 1$, $\frac{|z+3||z-2|^4}{|z-1|^5} = \infty > 1$. Thus we are sure that $\frac{|z+3||z-2|^4}{|z-1|^5} = 1$ contour will pass between (-3 and 1) and also between (1 and 2). We can verify it by plotting the contours as shown in Figure (1).

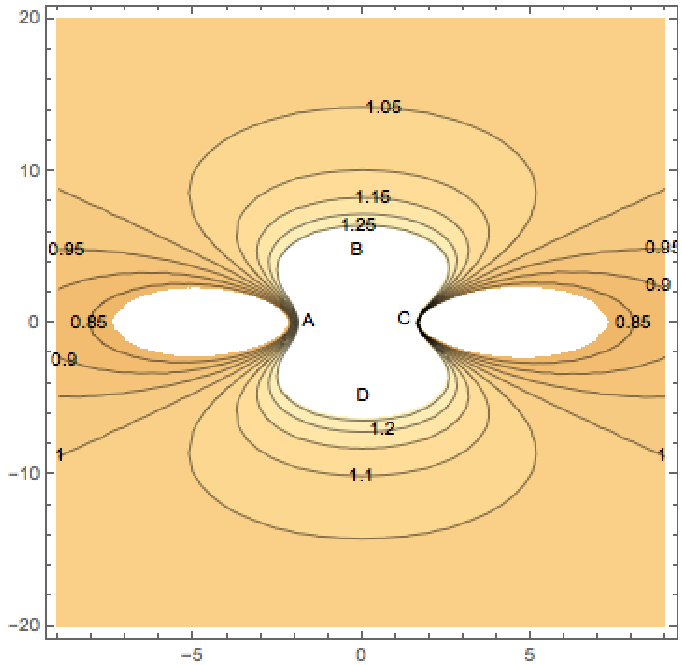

Figure 1. Contour plot of $e^{20\zeta} = \frac{|z+3||z-2|^4}{|z-1|^5}$ in the complex z-plane.

The plot clearly shows that from $z = -3$ to $z = 2$ we have to cross $\frac{|z+3||z-2|^4}{|z-1|^5} = 1$ contour at least twice. That is we have to cross equator at least twice. To go from $z = -3, 2$ to $z = 1$ we have to cross $\frac{|z+3||z-2|^4}{|z-1|^5} = 1$ contour or the equator at least once. Thus we are getting a picture where we have two spheres. The south pole of one sphere corresponds to $z = -3$ and the south pole of other sphere correspond to $z = 2$. The north pole of both the spheres correspond to $z = 1$. This means the two spheres have common northern hemispheres ($\zeta > 0$) and separate southern hemispheres ($\zeta < 0$).

How does this all look in the ω, ϕ coordinate? First let us look at the contours for $\zeta > 0 \implies \omega < \pi/2$. These curves are connected and close around $z = 1$. As we move along each contour ϕ ranges from $[0, \alpha_1 2\pi)$ where $1 - \alpha_1$ is the conical deficit of the particle at $z = 1$. For convenience let us mark the point A as $\phi = 0$. Then as we move along the curve we reach $B(\phi = \alpha_1\pi/2), C(\phi = \alpha_1\pi), D(\phi = 3\alpha_1\pi/2)$ and when we come back to A, ϕ changes by $\alpha_1 2\pi$. These points are shown in figure (1). This is true for all the contours $\omega < \pi/2$. For $\omega = \pi/2$, contour splits at $C, D \to \infty$. Topologically one then has a sphere with 3 punctures, also know as the pants diagram. To see that explicitly, we do the transformation $z \to \frac{1}{z'} + 1$. This sends $z = 1, \infty$ to $z' = \infty, 0$ respectively. The new function that determines the geometry is shown in Figure (2).

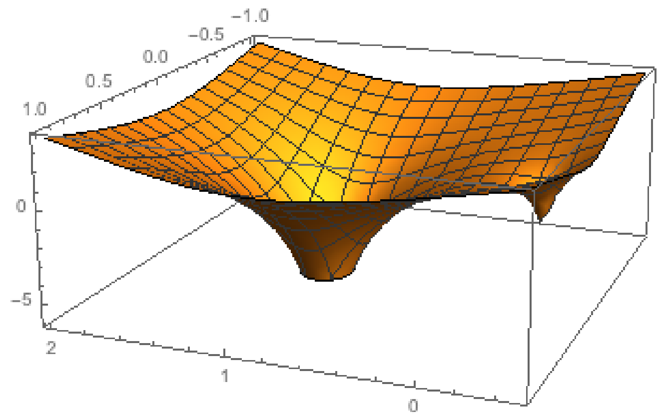

Figure 2. 3D plot of $e^{20\zeta}$ after the transformation $z \to \frac{1}{z'} + 1$ in z' plane. This shows that the topology of boundary is a pair of pants.

Generalizing to N particles this will be N punctures on the Riemann sphere. Single valuedness of the solution requires that all the punctures (position of the particles) lie on a line. Without loss of generality we can take this line to be real axis. Mass defects are given by absolute value of the c_n, which should sum to 0. We can choose $c = 1$ in the solution (8). and take the positions z_n such that they satisfy $\sum c_n = 0$. Then the solution is given by (8). Let us label the positions such that $z_1 > z_2 > ... > z_{N-1} > z_N$. Then

$$\begin{aligned} c_1 &= \frac{1}{(z_1 - z_2)(z_1 - z_3)...(z_1 - z_N)} > 0 \\ c_2 &= \frac{1}{(z_2 - z_1)(z_2 - z_3)...(z_2 - z_N)} < 0 \\ c_3 &> 0 \\ ... \end{aligned}$$

$c_{2m+1} > 0, c_{2m} < 0$. Thus we see from equation (8) that $\zeta(z_{2m+1}) = -\infty, \zeta(z_{2m}) = \infty$. This solution in z coordinate is transformed to the de-Sitter like metric (10) using (9). We see that at $z = z_n \implies \omega = 0, \pi$. That is particles are either at the south or north pole, corresponding to a multi-sheeted sphere with 2 punctures.

To see the geometry more clearly we look at the equator that is $\omega = \pi/2 \implies \sin\omega = 1 \implies \zeta(z) = \zeta_0$ contours. Let us check these contours generate the Riemann sphere with N punctures.

Since this is a compact manifold, all the contours must be closed. The punctures live at $|\zeta(z_n)| = \infty$. So none of the contours with finite value of ζ_0, end at the punctures. The second observation is that the $\zeta(z) = 0$ contour continuously extends to $z = \infty$. Thus all the contours with $\zeta(z) = 0$ are connected at $z = \infty$. A third observation is that $\zeta(z_{2m+1}) = -\infty, \zeta(z_{2m}) = \infty$. That is $\zeta(z)$ at successive punctures are of opposite sign. Thus, $\zeta(z) = 0$ contour separates any two successive punctures. Thus there are $N - 1$ $\zeta(z) = 0$ contours joined at $z = \infty$. These contours divide the Riemann sphere into N segments. Each segment contains exactly one puncture, and we have mapped the geometry to a single-cover of the N-punctured sphere.

IV. CONCLUSION

We have conjectured the holographic dual of an asymptotically de Sitter spacetime in $d+1$-dimensions is a d-dimensional theory of Euclidean conformal gravity living on $\mathscr{I}$. Various quadratic terms in the action of the conformal gravity have been constructed, which indicate the boundary metric becomes a dynamical variable. This then forces one to consider whether the path integral over the boundary metric includes a sum over topologies.

This is a sharp departure from the simplicity of the conformal field theory/anti-de Sitter correspondence, where we have many examples of suitable large N conformal field theories and the boundary metric is not dynamical. In the case of de Sitter, we instead get holographic theories that violate the usual axioms of Euclidean field theory [2] and examples are hard to come by. The massless higher spin theories have provided some examples where these issues can be explored in detail [26, 27]. Optimistically one might hope that the new feature of coupling to conformal gravity solves some of these problems. More pessimistically it suggests that the natural UV completion of de Sitter gravity may not be some lower dimensional holographic theory, but is rather to be understood as an unstable background in some larger complete theory [28, 29].

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