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Phys. Rev. D **92**, 126006 — Published 14 December 2015

DOI: [10.1103/PhysRevD.92.126006](https://doi.org/10.1103/PhysRevD.92.126006)

Weak Gravity Conjecture in AdS/CFT

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We study implications of the weak gravity conjecture in the AdS/CFT correspondence. Unlike in Minkowski spacetime, AdS spacetime has a physical length scale, so that the conjecture must be generalized with an additional parameter. We discuss possible generalizations and translate them into the language of dual CFTs, which take the form of inequalities involving the dimension and charge of an operator as well as the current and energy-momentum tensor central charges. We then test these inequalities against various CFTs to see if they are universally obeyed by all the CFTs. We find that certain CFTs, such as supersymmetric QCDs, do not satisfy them even in the large N limit. This does not contradict the conjecture in AdS spacetime because the theories violating them are either unlikely or unclear to have weakly coupled gravitational descriptions, but it suggests that the CFT inequalities obtained here by naive translations do not apply beyond the regime in which weakly coupled gravitational descriptions are available.

PACS numbers: PACS numbers: 04.60.-m, 11.25.Hf

I. INTRODUCTION

There are many folk theorems that are believed to hold in quantum gravity. Some are qualitative such as the non-existence of continuous global symmetries, suggested by the physics of black holes as well as perturbative string theory. Others are more quantitative, which include the weak gravity conjecture [1]. These more quantitative theorems, however, generally have weaker foundations and their precise meanings are obscured beyond the semiclassical limit. For recent discussions on the weak gravity conjecture, see e.g. Refs. [2–5].

Since AdS/CFT duality [6, 7] provides a nonperturbative definition of quantum gravity, it is natural to explore how the folk theorems in quantum gravity may be realized in this framework. Ideally, a folk theorem can be translated into a universal statement in CFTs which may be tested, at least under some circumstances. Alternatively, one might find that such a universal statement is not possible, in which case one would learn that the theorem arises as a property that manifests itself only in a certain (weakly coupled gravitational) limit of the theory.

Motivated by these considerations, in this article we study the weak gravity conjecture in AdS/CFT. Since the original conjecture was formulated in asymptotically Minkowski spacetime, we first discuss possible generalizations in AdS spacetime (in Section III). Then, we translate the statements into the language of CFTs, all of which take the form that there must be an operator whose coupling to the energy-momentum tensor is smaller than that to the conserved current (in Section IV). Finally, we test these statements against known CFTs (in Section V). We find that the statements as formulated here do not apply universally to all the CFTs.

On the other hand, all the theories that do not satisfy them are those that are believed not to have weakly coupled gravitational descriptions or unclear to have such descriptions. It is, therefore, still consistent to postulate that the weak gravity bounds discussed here hold in asymptotically AdS spacetime. An alternative possibility is that there are some modified expressions that apply universally and reduce to the bounds discussed here when there are weakly coupled gravitational descriptions. This is discussed in Section VI.

II. WEAK GRAVITY CONJECTURE IN MINKOWSKI SPACETIME

Consider Einstein-Maxwell theory in D -dimensional (asymptotically-)flat Minkowski spacetime

$$S = \int d^D x \sqrt{-g} \left(\frac{R}{2\kappa_D^2} - \frac{1}{4e^2} F_{MN} F^{MN} + \text{matter} \right), \quad (1)$$

where $\kappa_D^2 = M_{\text{Pl}}^{2-D}$ is the D -dimensional Newton constant. The weak gravity conjecture states that a low energy effective theory of a consistent theory of quantum gravity must contain a particle with the mass m and charge q satisfying¹

$$\frac{m^2}{q^2} \leq C_D e^2 \kappa_D^{-2}. \quad (2)$$

Here, the coefficient C_D is determined such that the inequality is saturated by the extremal Reissner-Nordström

¹ There are two versions of the conjecture discussed in Ref. [1]. In this paper we focus on the weaker (more conservative) version.

(RN) black hole of mass m and charge q . (In the normalization of qe we will adopt later, $C_D = (D-2)/(D-3)$.)

An alternative, and essentially equivalent, formulation of the conjecture is given by the statement that extremal (non-BPS) RN black holes must be unstable, at least marginally. A connection between the two formulations is the following. Imagine that the weak gravity conjecture were violated. Then the particle with the smallest m/q has $\kappa_D^2 m^2 > C_D e^2 q^2$, so that gravitational attraction between two such particles is stronger than the gauge repulsion. This implies that we can form Kepler bound states composed of any number n of these particles, which are all absolutely stable and become extremal in the $n \rightarrow \infty$ limit. On the other hand, if there exists a particle with $m < \sqrt{C_D} qe/\kappa_D$, then extremal RN black holes can decay, except possibly for “quantum” ones with charges smaller than q , where we have assumed $q \gg 1$.

In this paper we adopt the latter formulation, based on extremal RN black holes, and discuss how it may be generalized in AdS spacetime. We also see how the generalized conjecture may be interpreted in dual CFTs, using the AdS/CFT correspondence.

III. WEAK GRAVITY CONJECTURE IN ADS SPACETIME

How can we extend the weak gravity conjecture to asymptotically AdS spacetime? The answer is not obvious because of the following facts: (i) AdS spacetime can be regarded as a finite box, preventing Hawking radiation from escaping to “infinity”; (ii) Physical properties of black holes change when their size becomes larger than the AdS scale (making the $n \rightarrow \infty$ limit we took in the previous section less convincing); (iii) Unlike in Minkowski spacetime, there is no no-hair theorem in AdS spacetime, making it possible for a black hole to decay by a process that does not have a direct analogue in Minkowski spacetime.

Given these facts, in this paper we formulate our conjecture(s) in the following steps.² We first consider the requirement that small extremal AdS-RN black holes must be able to decay by a process that is also available in Minkowski spacetime. In particular, we require that there is a particle in the AdS theory to which small extremal AdS-RN black holes can decay. We call this condition the *simple kinematic conjecture*, and discuss its formulation in dual CFTs.

We next consider the condition that small extremal AdS-RN black holes decay by a dynamical process that is available (only) in AdS spacetime. In particular, we consider that the decay occurs through superradiant instability discussed in Refs. [10, 11]. We find that this gives a condition weaker than that of the simple kinematic

conjecture, and call it the *dynamical conjecture*. The difference between the simple kinematic and dynamical conjectures is purely AdS in nature—both these conjectures reduce to the Minkowski one in the appropriate large AdS radius limit.

We finally discuss possible additional constraints coming from large extremal AdS-RN black holes. We find that as long as either of the above conjectures is satisfied, a large extremal AdS-RN black hole can always have a microscopic “decay” process. Namely, a process in which a larger black hole is converted into a smaller one and the light quantum is always kinematically allowed. It is possible that this is indeed enough for the consistency of the theory.

On the other hand, in AdS spacetime the above process does not lead to a real decay of a large black hole because the finite-box nature of AdS makes a large black hole be in thermal equilibrium with the ambient space. To make the large black hole really unstable, we need to have a different process. In Ref. [12], it was advocated that this may in fact be the case—large extremal AdS-RN black holes have instabilities associated with the presence of a superconducting phase in strongly coupled dual CFTs. While we do not have a better argument for this conjecture than the authors of Ref. [12], we also discuss it for completeness.

IV. CFT FORMULATION

We now formulate our conjectures using the language of dual CFTs. Below, we focus on the case with $D = 5$, but the extension to other dimensions is straightforward.

In dual $d = 4$ CFTs, a conserved current J_μ and the energy-momentum tensor $T_{\mu\nu}$ have the two-point functions

$$\langle J_\mu(x) J_\nu(0) \rangle = \frac{C_V}{x^6} I_{\mu\nu}(x), \quad (3)$$

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = \frac{C_T}{x^8} I_{\mu\nu,\rho\sigma}(x), \quad (4)$$

where $I_{\mu\nu}(x) = \delta_{\mu\nu} - x_\mu x_\nu / x^2$ and $I_{\mu\nu,\rho\sigma}(x) = (I_{\mu\rho}(x) I_{\nu\sigma}(x) + I_{\mu\sigma}(x) I_{\nu\rho}(x)) / 2 - \delta_{\mu\nu} \delta_{\rho\sigma} / 4$. Crudely speaking, C_T counts the number of massless degrees of freedom in the CFT, while C_V counts the number of massless *charged* degrees of freedom in the CFT. (When the current J_μ is gauged, the leading-order beta function is proportional to C_V .) For our explicit normalization convention for these quantities, see Appendix A.

The AdS/CFT correspondence states that C_V and C_T are related to the kinetic terms of the bulk fields in AdS spacetime

$$S = \int d^5x \sqrt{-g} \left(\frac{1}{2\kappa_5^2} \left(R + \frac{12}{L^2} \right) - \frac{1}{4e^2} F_{MN} F^{MN} + \dots \right), \quad (5)$$

as

$$C_V = \frac{6L}{\pi^2} e^{-2}, \quad C_T = \frac{40L^3}{\pi^2} \kappa_5^{-2}, \quad (6)$$

² Related but different conjectures were discussed in Refs. [8, 9].

where L is the AdS radius. The existence of a bulk field of mass m implies that of a CFT operator of scaling dimension

$$\Delta = Lm + O(1), \quad (7)$$

where $O(1)$ corrections depend on the spin and detailed couplings, and we will discuss it only when necessary. It is natural to focus on Δ rather than m , since it corresponds to the conserved global energy in AdS spacetime. A (minimally-coupled) scalar field in AdS spacetime must satisfy the Breitenlohner-Freedman bound

$$m^2 L^2 \geq -4. \quad (8)$$

Note that a small negative mass-squared is allowed without causing an instability.

A. Simple kinematic conjecture

Let us first consider the simple kinematic bound coming from the requirement that there exists a particle that has a smaller ratio of the AdS energy Δ to the charge q than that of small extremal AdS-RN black holes (which have the horizon sizes smaller than the AdS radius L). As summarized in Appendix B, in AdS spacetime the mass-to-charge ratio, M/Q , of a small extremal black hole depends on the size of the black hole

$$\frac{M^2}{Q^2} = \frac{3e^2}{2\kappa^2} h(x), \quad (9)$$

where $h(x) = (3x^2/4)(\sqrt{1+x}-1)^{-2}(2\sqrt{1+x}+1)^{-1}$, and $x = 2M\kappa_5^2/L^2$ ($0 < x \lesssim 1$). Since $h(x)$ is a monotonically increasing function, however, requiring the bound for the smallest black hole, i.e. in the $x \rightarrow 0$ limit,³ ensures that all heavier black holes satisfy the corresponding bounds.

This leads to the condition that in 5D AdS spacetime there must be a particle whose AdS energy E and charge q satisfy

$$\frac{E^2}{q^2} \leq \frac{3}{2} e^2 \kappa_5^{-2}. \quad (10)$$

Using Eq. (6) and $\Delta = LE$, we can write this in terms

of the CFT data⁴

$$\frac{\Delta^2}{q^2} \leq \frac{9}{40} \frac{C_T}{C_V}. \quad (11)$$

This condition, by itself, does not tell us where the state exists, but it is natural to expect that it must be below the mass of the lightest 5D AdS-RN black hole.

The mass of the lightest 5D AdS-RN black hole depends on the size of the extra dimensions beyond 5D AdS we consider. It is not known how small the extra dimensions can be made in general, but it is possible that there is a lower bound on their size. For example, Ref. [1] argues that the volume of the extra dimensional space X must satisfy $(V_X/l_s^5) \gtrsim g_s(R/l_s)$, where l_s and g_s are the string length and coupling, respectively. Assuming that X has only one length scale, this implies that the state satisfying Eq. (11) must exist below $\Delta \sim C_T^{3/5}$.

Since black holes in the $x \rightarrow 0$ limit behave similarly to those in Minkowski spacetime, we expect that the condition discussed here is reduced to the original Minkowski bound when we take $L \rightarrow \infty$ (with the fixed Planck scale as well as any other scales). Indeed, using Eqs. (6, 7), we find that Eq. (11) yields Eq. (2) in the appropriate limit.

B. Dynamical conjecture

In general, the stability condition for a system in AdS spacetime is different from that in Minkowski spacetime. In particular, since there is no no-hair theorem in AdS spacetime, extremal AdS-RN black holes may have dynamical instabilities involving classical condensates, which are not available in Minkowski spacetime. Indeed, it is known that in the presence of a minimally coupled charged scalar field, extremal AdS-RN black holes may be unstable against scalar hair formation. For small extremal AdS-RN black holes, this instability can be interpreted as a superradiant instability.

According to Refs. [10, 11], the superradiant instability for small extremal black holes occurs when there is a minimally coupled charged scalar field in the bulk satisfying the condition (for the $r_+/L \rightarrow 0$ limit):

$$L^2 m^2 - \frac{3}{2} e^2 \kappa_5^{-2} L^2 q^2 \leq -4. \quad (12)$$

In terms of the CFT data, this leads to

$$\frac{(\Delta - 2)^2}{q^2} \leq \frac{9}{40} \frac{C_T}{C_V}, \quad (13)$$

where Δ is the dimension of the CFT operator corresponding to the charged scalar field in the bulk. Note

³ This limit must be taken such that the size of the black hole is still larger than the 5D Planck scale. In the CFT language, if we have a (5D) Planck-sized black hole, $\Delta \sim L\kappa_5^{-2/3} \sim C_T^{1/3}$. In comparison, we have $\Delta \gtrsim C_T$ for large black holes. Note that the 5D Planck scale is the largest conceivable cutoff for the 5D theory, but there can be lower scales such as the Kaluza-Klein or string scales. In fact, Ref. [1] argues that this must be the case, based on an analysis of 5D AdS spacetime cut off by a “UV brane.”

⁴ Whether this inequality is satisfied or not is related to a certain convexity of the CFT operator spectrum in the large spin limit [13]. We thank João Penedones for bringing this to our attention.

that the bound on Δ is shifted by two units compared with that in Eq. (11). This is because the condensation effect can make the AdS energy per charge lower than that of the collection of quanta. In fact, in the range allowed by unitarity, $\Delta \geq 1$, the bound in Eq. (13) is weaker than that in Eq. (11).

In the appropriate Minkowski limit (sending $L \rightarrow \infty$ while keeping m), Eq. (13) is also reduced to the Minkowski bound in Eq. (2). This implies that the difference between the two bounds in Eqs. (11) and (13) is purely AdS in nature—it is important only for low Δ .

We note that, unlike the corresponding objects in Minkowski spacetime, extremal AdS-RN black holes do not saturate the BPS bound (i.e. they cannot be supersymmetric), except in the limit $r_+/L \rightarrow 0$. (See Appendix B.) The decay processes described above, therefore, may occur non-marginally even in theories with supersymmetry.

C. Large black holes

In AdS spacetime, we have large extremal AdS-RN black holes ($r_+ > L$), which do not possess a simple flat spacetime limit. While the weak gravity bound in Minkowski spacetime does not directly lead to the conclusion that these black holes must be unstable, it is interesting to see what bounds on CFTs can be obtained by requiring that they are indeed unstable. In fact, the idea that the planar extremal AdS-RN black branes (which are equivalent to AdS-RN black holes in the $r_+/L \rightarrow \infty$ limit) should be unstable was advocated in Ref. [12], in relation to the presence of a superconducting phase in strongly coupled CFTs.

The instability condition on an AdS-RN black hole with respect to the formation of (minimally coupled) scalar hair condensation depends on the size of the black hole r_+ . While the general condition can be found in Ref. [11], here we quote only two representative cases. We expect that the true bound is obtained by the union of the conditions for all values of $r_+ \gtrsim L$.

In the limit of a planar extremal AdS-RN black brane (i.e. $r_+/L \rightarrow \infty$), the horizon topology becomes $\text{AdS}_2 \times R^3$, and the instability appears when the effective mass of a charged field near the horizon becomes below the AdS_2 Breitenlohner-Freedman bound. In our normalization, we find ([12] for $D = 4$ and [14] for $D = 5$)

$$\frac{3(\Delta - 1)(\Delta - 3)}{2q^2} \leq \frac{9}{40} \frac{C_T}{C_V}. \quad (14)$$

On the other hand, for an “intermediate” extremal AdS-RN black hole (i.e. $r_+ \sim L$ or $\Delta_{\text{BH}} \sim C_T$), the condition that it must be unstable gives (for $r_+ = L$)

$$\frac{4(\Delta - 2)^2}{3q^2} \leq \frac{9}{40} \frac{C_T}{C_V}. \quad (15)$$

The shift in Δ is the same as in Eq. (13), but we have an additional factor of $4/3$ in the left-hand side.

The conditions in Eqs. (14, 15) give stronger bounds than the original weak gravity bound, Eq. (2), in the naive flat-space limit $\Delta \gg 1$. This, however, does not mean the existence of a stronger bound than Eq. (2) in Minkowski spacetime. In the true Minkowski limit, large black holes considered here disappear from the spectrum, and so do the corresponding bounds.

V. TESTING WITH EXAMPLES

In this section, we study if the bounds discussed in the previous section are indeed satisfied in various known CFTs. Since our conjectures are about “generic” CFTs that have weakly coupled gravitational descriptions, and these theories are not well understood, we need to “test” them against theories in our hands, which are not necessarily in a class to which the conjectures must apply. Nevertheless, we find some interesting lesson—all the theories that we find do not satisfy the bounds are those that are believed not to have weakly coupled gravitational descriptions (or unclear to have such descriptions). In particular, we find that supersymmetric theories that have weakly coupled gravitational descriptions (although in 10D) do satisfy the bounds.

A. Known AdS/CFT with supersymmetry

We first study if our conjectures are satisfied in known examples of the AdS/CFT correspondence. Since these theories have weakly coupled gravitational descriptions in 10D, our analysis in the previous section need not a priori apply. Moreover, their 10D bulk descriptions possess high supersymmetries that relate gravity with $U(1)$ gauge forces, reducing the significance of the conjectures in some cases. Nevertheless, we find it nontrivial that all these theories satisfy the bounds applied naively, especially given that not all the CFTs satisfy them as we will see in later subsections.

To be specific, we focus on type IIB string theory compactified on $\text{AdS}_5 \times Y_{p,q}$ with the coprime numbers $p > q$, which have weakly coupled supergravity descriptions with the second order bulk actions. In these theories, the Kaluza-Klein reduction is consistent and most 5D asymptotic AdS solutions (including black holes) can be uplifted to 10D solutions [15] despite the intrinsic 10D nature of these theories. The compact spaces $Y_{p,q}$ are nontrivial examples of Sasaki-Einstein five folds, whose explicit construction can be found in Ref. [16].

The resulting dual CFTs preserve $\mathcal{N} = 1$ superconformal symmetry in 4D with $U(1)_R$ symmetry. The central charges for the energy-momentum tensor and the R current can be computed both from the gravitational and

field theory points of view, giving [17]

$$C_T = \frac{10N^2}{\pi V_{p,q}}, \quad (16)$$

$$C_R = \frac{N^2}{\pi V_{p,q}}, \quad (17)$$

where N and $V_{p,q}$ are the number of branes and the volume of $Y_{p,q}$, respectively (which we will not use). These theories also have $U(1)_F \times U(1)_B \times SU(2)$ global symmetries. As long as we have a scalar chiral operator, the superconformal R charge always saturates the simple kinematic bound in Section IV A (and thus satisfies the weaker bound in Section IV B). We are therefore more interested in other global symmetries.

Let us first consider the $U(1)_F$ symmetry. In the theories under consideration, there are three types of chiral primary operators with their $U(1)_F$ charges given in terms of the R charges as

$$q_F(O_1) = y_1 R(O_1), \quad (18)$$

$$q_F(O_2) = -y_2 R(O_2), \quad (19)$$

$$q_F(O_3) = -\frac{1}{2}(y_1 + y_2)R(O_3), \quad (20)$$

where

$$y_1 = \frac{1}{4p}(2p - 3q - \sqrt{4p^2 - 3q^2}), \quad (21)$$

$$y_2 = \frac{1}{4p}(2p + 3q - \sqrt{4p^2 - 3q^2}). \quad (22)$$

Since these are scalar chiral primary operators, they satisfy $\Delta(O_i) = (3/2)R(O_i)$. The AdS/CFT as well as direct field theory computations give the central charge for $U(1)_F$ as

$$C_F = \frac{N^2}{8\pi V_{p,q}} \frac{\sqrt{4p^2 - 3q^2}}{p^2} (2p - \sqrt{4p^2 - 3q^2}). \quad (23)$$

Using these formulae, we can calculate the ratios Δ^2/q_F^2 for $O_{1,2,3}$. We find that the simple kinematic bound

$$\frac{\Delta^2}{q_F^2} \leq \frac{9}{40} \frac{C_T}{C_F}, \quad (24)$$

is always satisfied by O_1 and O_2 (but not necessarily O_3). Note that in order to be consistent with the weak gravity conjecture, we only need one operator (e.g. O_1 here) that satisfies the bound. The most stringent case is the $p \gg q$ limit, but we still have a factor of 3 margin there.

As far as we have checked, in all known examples of the AdS/CFT correspondence with weakly coupled gravity descriptions, the simple kinetic bound in Section IV A (and thus also the dynamical bound in Section IV B) is satisfied for the R symmetries and Abelian flavor symmetries. The further such examples include $\text{AdS}_5 \times L_{p,q,r}$ compactification of type IIB string theory [18]. We find this nontrivial.

As for the baryonic symmetry, the situation is less clear. In the examples considered, the lightest object charged under the baryonic symmetry has $\Delta \sim N \sim C_T^{1/2}$, so that it is heavier than the 5D Planck scale, $\Delta \sim L\kappa_5^{-2/3} \sim C_T^{1/3}$. This, however, may not mean a violation of the bound if the size of extra dimensions is necessarily larger than the (effective) 5D Planck scale; see discussions after Eq. (11).

Let us now turn to the bound coming from large black holes, discussed in Section IV C. Recall that this bound is related, in the limit $r_+/L \rightarrow \infty$, to the (in)stability of planar extremal AdS-RN black branes, since in this limit the horizon can be approximated by a plane with \mathbf{R}^3 topology. In fact, there had been some interests in the stability of these objects in string compactification [12, 14, 19]. The motivation there was mainly applications to condensed matter physics, in which the instability of these objects corresponds to the instability of zero temperature CFTs under the introduction of chemical potentials. In Section IV C, we discussed a possible instability due to a scalar hair formation. In the dual CFT language, this corresponds to an instability of the system due to a scalar condensate, leading to a superfluidity or superconductivity phase transition.

In all the examples studied in Refs. [12, 14, 19], the extremal AdS-RN black branes are indeed (marginally) unstable due to such scalar condensates. References [12, 14] studied (mainly) R -charged extremal AdS-RN black branes in which R -charged scalar fields, typically dual to chiral primary operators in the CFTs, trigger the instability. In Ref. [19], a more intricate situation with baryon charges was studied and the system was still marginally (un)stable.⁵ These authors interpreted this observation as a manifestation of the weak gravity conjecture applied to extremal AdS-RN branes. In our viewpoint, these examples suggest that extremal AdS-RN black holes are unstable in the large black hole limit. Correspondingly, in these dual CFTs, there exists an operator that (marginally) satisfies the bound such as Eqs. (14, 15).

B. Free theories

We now study if our bounds, as formulated in Eqs. (11), (13), (14), and (15), can be universally valid for all the CFTs regardless of the existence of a weakly coupled gravitational picture.

For this purpose, let us consider free field theories. We find that the naive bound in Eq. (11) cannot be universal. Take a free complex scalar with a $U(1)$ global symmetry.

⁵ Strictly speaking, what they obtained by their tree-level computations is that the potential vanishes. It is, however, conjectured that higher order corrections make the system unstable. We thank Igor Klebanov for discussions.

This theory has an operator (free complex scalar itself) with $\Delta = 1$. Normalizing the charge of this scalar to be unity, $q = 1$, we find that $C_T/C_V = 8/3$. The bound in Eq. (11) then leads to $1 \leq 3/5$, which is clearly not satisfied. The existence of other operators does not help, since they all have $|\Delta/q| \geq 1$. A similar conclusion is also obtained for a free fermion.

The situation is different for the dynamical conjecture in Eq. (13), which gives a weaker bound. This bound is satisfied by a free scalar and a free fermion due to the shift in the left-hand side. It is trivially satisfied for a charged free scalar ϕ because of the existence of the ϕ^2 operator, which has $(\Delta, q) = (2, 2)$. For a free fermion ψ , we have a $(\Delta, q) = (3, 2)$ scalar operator (i.e. ψ^2), and since the theory has $C_T/C_V = 2$, the bound is satisfied. Therefore, at this point, the bound in Eq. (13) still has a chance to be universal.

Finally, we discuss the bounds in Eqs. (14, 15), arising from considerations of large black holes. These bounds are also satisfied by free scalars and fermions. The meaning of this fact, however, is not clear. In a weakly coupled gravitational description, we might as well formulate the conjecture in a form more physical from the CFT point of view: the zero temperature CFTs must be unstable under the introduction of a chemical potential. (This issue was studied in Ref. [20].) In this form, however, we know that a free fermion system does not satisfy the conjecture, since it is stable under the introduction of large chemical potentials. This casts some doubt on adopting Eqs. (14, 15) as the conditions applying universally beyond the weakly coupled gravity limit.⁶

C. Supersymmetric QCDs in the large N limit

Consider supersymmetric QCDs with $SU(N_c)$ gauge groups and N_f flavors of quarks (i.e. N_f Q 's and N_f \bar{Q} 's in the fundamental and anti-fundamental representations of $SU(N_c)$, respectively) in the conformal window $\frac{3}{2}N_c \leq N_f \leq 3N_c$.⁷ This theory possesses a $U(1)_B$ symmetry, $Q(+1)$ and $\bar{Q}(-1)$, in addition to the $SU(N_f) \times SU(N_f)$ flavor symmetry and the R -symmetry. Since the theory has scalar chiral superconformal primary operators, the R -symmetry automatically satisfies the bounds in Eqs. (11, 13). We thus focus on the $U(1)_B$ symmetry below.

While the theory is strongly coupled away from the perturbative regime $N_f \sim 3N_c$, one can compute the exact value of the $U(1)_B$ -current central charge C_B from the supersymmetric formula

$$C_B = -\frac{9}{4\pi^4} \text{Tr}[RBB] = \frac{9}{4\pi^4} (2N_f N_c) \frac{N_c}{N_f}. \quad (25)$$

Similarly, the exact value of the energy-momentum tensor central charge is [21]

$$C_T = \frac{5}{2\pi^4} (7N_c^2 - \frac{9N_c^4}{N_f^2}), \quad (26)$$

leading to

$$\frac{C_T}{C_B} = \frac{5}{9} (7 - \frac{9N_c^2}{N_f^2}). \quad (27)$$

Because of the gauge invariance, the lightest baryonic charged operator is $\epsilon QQQQ\dots$ (with N_c Q 's), which has

$$q_B = N_c, \quad \Delta = \frac{3}{2} N_c (1 - \frac{N_c}{N_f}). \quad (28)$$

We thus find that when

$$\frac{N_f}{N_c} > \frac{3}{11} (6 - \sqrt{3}) \simeq 2.1, \quad (29)$$

all the bounds are violated—there is no light (protected, chiral) state that satisfies any of the bounds. Note that the shift of Δ in Eq. (13) does not help because the dimensions of relevant operators are of $O(N_c) \gg 2$. While it is logically possible that some unprotected operator satisfies a bound, we find it unlikely. Furthermore, even this loophole is closed when the theory is close to free, $N_f/N_c \simeq 3$.

A reasonable conclusion is that none of the inequalities in Eqs. (11, 13, 14, 15) applies universally, at least as are written. This does not contradict the existence of the corresponding weak gravity bounds in the limit that theories admit weakly coupled gravity dual descriptions. Even in the limit of large N_c , the supersymmetric QCDs considered here are expected not to have weakly coupled gravitational duals, as suggested e.g. by the presence of higher spin protected operators and a violation of the holographic central charge equality $c = a$. The analysis here simply says that the bounds as are written cannot be true universally for all the CFTs.

The weak gravity bounds we present, therefore, must be corrected when we deviate from weakly coupled Einstein gravitational descriptions. In the example considered here, we are taking the large N_c limit. Therefore, these corrections must be understood as higher derivative corrections (such as higher curvature terms). It was claimed that higher derivative terms must contrive such that the original weak gravity conjecture holds without modification [22]. (See also Ref. [23] for related AdS discussions.) Our example suggests that this might not be the case in general. Note, however, that the relation between the two analyses is not strict. For example, unlike in the Minkowski case, in AdS spacetime one cannot take a simple large black hole limit to make higher derivative terms be small perturbations to the system.

⁶ We thank Sean Hartnoll for discussions on this point.

⁷ We take the Veneziano limit $N_c, N_f \rightarrow \infty$ for the purpose of simplifying the formula.

D. CFT dual of extremal AdS/RN branes

Suppose the weak gravity bound from large black holes holds. Then, any attempt to construct a dual field theory model for extremal AdS/RN branes must exhibit some instability, at least if we can take the weakly coupled limit in the gravity side.

There is, in fact, some attempt to construct field theories that model extremal AdS/RN branes in the large N limit with long range interactions [24]. The claim is that it is possible to reproduce states with a large degeneracy matching with the Bekenstein-Hawking entropy of RN black branes. An important thing for us is that these theories do not seem to show an instability suggested by the conjecture.

Similarly, in another recent paper [25], a universal behavior of scaling dimensions, $\Delta \sim Q^{(D-1)/(D-2)}$, in a large charge sector of certain (non-large N) $(D-1)$ -dimensional CFTs was discussed. The observation relevant to us is that the scaling behavior of Δ as a function of Q is precisely that of large AdS-RN black holes in D dimensions. Again, as long as their effective field theory building on large charge expansion is valid, there does not seem any instability.

These analyses, however, do not immediately imply that the weak gravity bound from large black holes is invalid, since it is not clear if the theories analyzed have weakly coupled gravitational descriptions. It would be interesting to study if these constructions can be applied in the regime in which the weakly coupled gravity limit can surely be taken. If such a limit can indeed be taken, the weak gravity conjecture for large black holes would imply that the effective field theory description discussed in Ref. [25] must possess an additional instability mode.

VI. DISCUSSION

In this paper, we have discussed possible generalizations of the weak gravity conjecture to AdS spacetime. We have considered the conditions arising from both small and large AdS-RN black holes, and translated them into the language of dual CFTs. While these conditions need to be satisfied a priori only in the regime in which weakly coupled gravitational descriptions are available, we have tested them against a wider range of CFTs. We have found that the bounds as formulated in this paper are not universally satisfied by all the CFTs, and yet all the examples that we found do not satisfy them are theories that are expected not to have, or unclear to have, weakly coupled gravitational descriptions.

Although the bounds as written here do not apply universally to all the CFTs, it is possible that a similar, modified bound exists that is universally valid. If such a bound exists, it must arise purely from consistency conditions applicable to all the CFTs with a $U(1)$ symmetry. One candidate for such consistency conditions is the conformal bootstrap condition for correlation functions,

and indeed there have been some studies on bounds of current central charges C_V using this method (with or without fixing the dimensions of operators or the energy-momentum tensor central charge C_T) [26].

While these bounds given by the conformal bootstrap are rigorous within numerical precision, they mostly give lower bounds on C_V , yielding *lower* bounds on the strength of gravity. Obtaining an upper bound is difficult because of the possibility that another (non-conserved) spin one operator mimics the current operator in question in a single bootstrap equation. This makes it hard to isolate the relevant contribution. In this respect, a promising case is a theory with $\mathcal{N} = 2$ supersymmetry, in which a conserved current multiplet has an isolated contribution to the bootstrap equation so that the above problem can be avoided. This allows us to obtain an upper bound on C_V [27], although current theoretical technology still seems unable to extract a useful bound in this way for large values of C_T , which we are interested in.

Another direction would be to study the consistency of the CFT spectrum on nontrivial geometries such as $S_1 \times S_{d-1}$. There is a constraint on the spectrum from the modular invariance in $d = 2$ cases. There, the information of the central charges is also encoded in the torus partition function, and the promising results have been reported in Ref. [29]. In higher dimensions, however, due to the lack of manifest modular properties, it is an open question if we can derive an interesting bound.

Acknowledgments

We would like to thank Sean Hartnoll, Simeon Hellerman, Igor Klebanov, and Tomoki Ohtsuki for useful discussions. Y.N.1 thanks Weizmann Institute of Science and Y.N.2 thanks Kavli Institute for the Physics and Mathematics of the Universe, University of Tokyo for hospitality during their visits in which a part of this work was carried out. The work of Y.N.1 was supported in part by a Sherman Fairchild Senior Research Fellowship at California Institute of Technology and Department of Energy (DOE) grant number de-sc0011632 as well as the World Premier International Research Center Initiative (WPI Initiative), MEXT. The work of Y.N.2 was supported in part by the Director, Office of Science, Office of High Energy and Nuclear Physics, of the U.S. DOE under Contract DE-AC02-05CH11231, and in part by the National Science Foundation under grants PHY-1214644 and PHY-1521446.

Appendix A: Normalization Convention

Here we present our normalization convention for C_V and C_T in Eqs. (3, 4). In the dual gravitational description, our normalization for C_V corresponds to taking that of q_e so that $C_D = (D-2)/(D-3)$ in Eq. (2).

We focus on $D = 5$, i.e. 4-dimensional CFTs. For a single complex scalar with a unit charge, we take its contribution to C_V and C_T as (see, e.g., Ref. [28]):

$$C_V = \frac{1}{S_4^2}, \quad C_T = \frac{8}{3} \frac{1}{S_4^2}, \quad (\text{A1})$$

where S_4 is the volume of the unit four-sphere, $S_4 = 2\pi^2$. For a single Weyl fermion with a unit (chiral) charge, we then have

$$C_V = 2 \frac{1}{S_4^2}, \quad C_T = 4 \frac{1}{S_4^2}. \quad (\text{A2})$$

The contribution from a free massless vector field is $C_T = 16/S_4^2$.

The coefficient in the weak gravity bound, e.g. in the right-hand side of Eq. (11) can be worked out by noticing that the bound in the Minkowski limit (i.e. $1 \ll \Delta \ll C_T$) becomes identical to the BPS bound for the superconformal R -current in superconformal field theories, which is saturated by chiral primaries having $\Delta/q_R = 3/2$ and $C_T/C_R = 10$.

Appendix B: Extremal AdS-RN Black Holes

In our normalization, the metric of the 5D AdS-RN black hole is given by

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_3^2, \quad (\text{B1})$$

where

$$f(r) = 1 - \frac{2\kappa_5^2}{3r^2} M + \frac{\kappa_5^2 e^2}{6r^4} Q^2 + \frac{r^2}{L^2}, \quad (\text{B2})$$

and M and Q are the mass and charge of the black hole, respectively. The gauge potential is given by

$$A_t = \text{const.} - \frac{e^2 Q}{r^2}. \quad (\text{B3})$$

The outer horizon is located at $r = r_+$, where

$$\frac{2\kappa_5^2 M}{3} = r_+^2 + \frac{\kappa_5^2 e^2 Q^2}{6r_+^2} + \frac{r_+^4}{L^2}. \quad (\text{B4})$$

The extremal limit is defined by

$$Q^2 = \frac{6r_+^4}{e^2 \kappa_5^2} \left(1 + 2 \frac{r_+^2}{L^2} \right), \quad (\text{B5})$$

so that $f(r)$ has a double zero with zero temperature. In this limit, the mass of the black hole is given by

$$M = \frac{3r_+^2}{\kappa_5^2} \left(1 + \frac{3}{2} \frac{r_+^2}{L^2} \right). \quad (\text{B6})$$

Note that the BPS condition $M^2/Q^2 = C_D e^2 \kappa_D^{-2}|_{D=5} = (3/2)e^2 \kappa_5^2$ is not satisfied except for $r_+/L \rightarrow 0$, even though the black hole has zero temperature.

The transition between small and large black holes occurs at $r_+ \sim L$. At that point, $M \sim L^2 \kappa_5^{-2}$, and the corresponding conformal dimension is $\Delta \sim L^3 \kappa_5^{-2} \sim C_T$.

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