Renormalization-group evolution and nonperturbative behavior of chiral gauge theories with fermions in higher-dimensional representations

Yan-Liang Shi (施言亮) and Robert Shrock

Renormalization-Group Evolution and Nonperturbative Behavior of Chiral Gauge Theories with Fermions in Higher-Dimensional Representations

Yan-Liang Shi (石炎亮) and Robert Shrock

C. N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, N. Y. 11794

We study the ultraviolet to infrared evolution and nonperturbative behavior of a simple set of asymptotically free chiral gauge theories with an SU(N) gauge group and an anomaly-free set of n_{S_k} copies of chiral fermions transforming as the symmetric rank-k tensor representation, S_k, and $n_{\bar{A}_\ell}$ copies of fermions transforming according to the conjugate antisymmetric rank-ℓ tensor representation, $\bar{A}_\ell$, of this group with k, $\ell \geq 2$. As part of our study, we prove a general theorem guaranteeing that a low-energy effective theory resulting from the dynamical breaking of an anomaly-free chiral gauge theory is also anomaly-free. We analyze the theories with $k = \ell = 2$ in detail and show that there are only a finite number of these. Depending on the specific theory, the ultraviolet to infrared evolution may lead to a non-Abelian Coulomb phase, or may involve confinement with massless composite fermions, or fermion condensation with dynamical gauge and global symmetry breaking. We show that $S_k \bar{A}_k$ chiral gauge theories with $k \geq 3$ are not asymptotically free. We also analyze theories with fermions in S_k and $\bar{A}_\ell$ representations of SU(N) with $k \neq \ell$ and k, $\ell \geq 2$.

I. INTRODUCTION

The properties of chiral gauge theories are of fundamental field-theoretic interest. We recall the definition of such theories: with the fermions written in left-handed chiral form, they transform as complex representations of the gauge group. It is natural to restrict one's consideration to asymptotically free chiral gauge theories since this guarantees that in the deep ultraviolet (UV) at large Euclidean reference momentum scales μ, the gauge coupling is small and hence such theories are perturbatively calculable. In order to avoid a triangle anomaly in gauged currents that would spoil renormalizability, one also requires that the sum of the contributions from the fermions to this anomaly must vanish. Given that the theory is asymptotically free, its running gauge coupling $g(\mu)$ increases as the scale μ decreases toward the infrared (IR). A basic goal of quantum field theory is to understand this UV to IR evolution of the theory. A chiral gauge theory is said to be irreducibly chiral if it does not contain any vectorlike subsector. In this case, the chiral gauge symmetry precludes any fermion mass terms in the underlying Lagrangian. We shall focus on such theories here. If the fermion content of the theory satisfies the 't Hooft global anomaly-matching conditions, then, as the coupling becomes sufficiently strong in the infrared, the gauge interaction may confine and produce massless gauge-singlet composite spin-1/2 fermions [1]-[12]. Alternatively, at some scale $\mu = \Lambda$, the gauge interaction could become strong enough to produce one or more bilinear fermion condensates, thereby spontaneously breaking gauge and global chiral symmetries [9, 11, 12],[15]-[17]. Just as quarks gain constituent masses in quantum chromodynamics, the fermions involved in these condensates gain dynamical masses of order Λ. One then integrates them out to construct an effective theory with the remaining massless fields that is applicable as the reference scale μ decreases below Λ. In general, there can be several such stages of condensate formation and symme-try breaking, with an associated sequence of effective field theories that describe the physics at different intervals of μ. A third type of renormalization-group (RG) evolution that occurs in certain theories is that although the running gauge coupling grows as μ decreases, it reaches an infrared fixed point at sufficiently weak coupling that neither confinement nor fermion condensation occurs, and instead there is a (deconfined) non-Abelian Coulomb behavior in the infrared. Asymptotically free chiral gauge theories have been explored in the past in efforts to understand some of the questions that the Standard Model (SM) can accommodate but does not explain, such as the origin of fermion generations and the spectrum of quark and lepton masses. Some work along these lines includes [1]-[17].

In this paper we continue the recent investigations in [11, 12] of the UV to IR evolution of chiral gauge theories. In general, it is valuable to examine as many such theories as possible to gain insight into their behavior. Here we study a class of chiral gauge theories in which the underlying theory in the deep UV has only fermions transforming according to higher-dimensional representations of the gauge group, but no fermions in the fundamental or conjugate fundamental representation, and is irreducibly chiral, with no vectorlike subsector. Specifically, we construct and study asymptotically free chiral gauge theories with an SU(N) gauge group and an anomaly-free set of n_{S_k} copies of chiral fermions transforming as the symmetric rank-k tensor representation, denoted S_k, and $n_{\bar{A}_\ell}$ copies of fermions transforming according to the conjugate antisymmetric rank-ℓ tensor representation, denoted $\bar{A}_\ell$, of this group, with k, $\ell \geq 2$ [18]. We will equivalently refer to these copies as flavors. One basic property of these theories that differs with previously studied chiral gauge theories may be highlighted at the outset, namely the property that the combined requirements of asymptotic freedom and anomaly cancellation constrain the models so strongly that at most only a finite number of $S_k \bar{A}_\ell$ models satisfy these requirements. In con-

trast, previously studied chiral gauge theories typically form infinite families, such as the family of models with SU(N) gauge groups and chiral fermions transforming as $S_k + (N + 4)\bar{F}$ (defined for any $N \geq 3$), and the family with chiral fermions transforming as $A_k + (N - 4)\bar{F}$ (defined for any $N \geq 5$), and extensions thereof including vectorlike subsectors. As part of our study, we prove a general theorem guaranteeing that a low-energy effective theory resulting from the dynamical breaking of an anomaly-free chiral gauge theory is also anomaly-free. We carry out a detailed analysis of the simplest case, $k = \ell = 2$, i.e., $S_2\bar{A}_2$ theories. We give a complete enumeration of the finite set of such theories satisfying the constraints of anomaly cancellation and asymptotic freedom, and investigate the interesting variety of patterns of UV to IR evolution that these theories exhibit. We then proceed to analyze theories with higher values of k and/or ℓ. We restrict our consideration here to chiral gauge theories with only gauge and fermion fields, but without any scalar fields.

This paper is organized as follows. In Sect. II we briefly review our general theoretical framework and methods of analysis. In Sect. III we prove that dynamical breaking of an anomaly-free chiral gauge theory yields a low-energy effective theory that is also anomaly-free. In Sect. IV we present our new set of chiral gauge theories with gauge group SU(N) and chiral fermions transforming according to (an anomaly-free set of) the rank-2 symmetric and conjugate antisymmetric tensor representations of the gauge group. Sections V and VI are devoted to detailed analyses of the UV to IR evolution of models of this type with an SU(5) gauge group and with an SU(N) gauge group with $N \geq 6$, respectively. In Sect. VII we show that there are no asymptotically free $S_k\bar{A}_k$ chiral gauge theories with $k \geq 3$. In Sect. VIII we analyze $S_k\bar{A}_\ell$ chiral gauge theories with $k \neq \ell$ and k, $\ell \geq 2$. Sections IX and X are devoted to the two simplest of these, namely the $S_3\bar{A}_2$ and $S_2\bar{A}_3$ theories. We give our conclusions are in Sect. XI. Some auxiliary formulas are included in two appendices.

II. METHODS OF ANALYSIS

In this section we briefly review the methods of analysis that we use. As noted above, we consider asymptotically free chiral gauge theories with gauge group $G = $ SU(N) and denote the running gauge coupling measured at a Euclidean momentum scale as $g(\mu)$. It is also convenient to use the quantities $\alpha(\mu) = g(\mu)^2/(4\pi)$ and $a(\mu) \equiv g(\mu)^2/16\pi^2$. We will often suppress the argument μ in these running quantities. Without loss of generality, we write all fermion fields in terms of left-handed chiral components.

The ultraviolet to infrared evolution of the gauge coupling is described by the beta function, $\beta_g = dg/dt$, or equivalently, $\beta_\alpha = d\alpha/dt = [g/(2\pi)]\beta_g$, where $dt = d\ln\mu$.

This has the series expansion

$$\beta_\alpha = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ , \qquad (2.1)$$

where we have extracted an overall minus sign, b_ℓ is the ℓ-loop coefficient, and it will be useful to define the reduced ℓ-loop coefficient, $\bar{b}_\ell = b_\ell/(4\pi)^\ell$. The n-loop beta function, denoted $\beta_{\alpha,n\ell}$, is given by Eq. (2.1) with the upper limit on the ℓ-loop summation equal to n instead of ∞. The requirement of asymptotic freedom means that $\beta_\alpha < 0$ for small α, which holds if $b_1 > 0$. The one-loop and two-loop coefficients b_1 [19] and b_2 [20] are independent of the scheme used for regularization and renormalization, while the b_ℓ with $\ell \geq 3$ are scheme-dependent.

With $b_1 > 0$, if $b_2 < 0$, then the two-loop beta function, $\beta_{\alpha,2\ell}$, has an IR zero at $a_{IR,2\ell} = -b_1/b_2$, or equivalently, $\alpha_{IR,2\ell} = -\bar{b}_1/\bar{b}_2 = -4\pi b_1/b_2$. For small enough fermion content, b_2 is positive, but as one enlarges the fermion content in the theory, still retaining the property of asymptotic freedom, the sign of b_2 can become negative, thereby producing an infrared zero in $\beta_{\alpha,2\ell}$ at the above value. If this occurs, then, as the reference scale μ decreases from large values in the ultraviolet, $\alpha(\mu)$ increases toward this infrared zero. If the IR zero occurs at sufficiently weak coupling, then one expects that the theory evolves from the UV to the IR without confinement or spontaneous chiral symmetry breaking (SχSB), to a non-Abelian Coulomb phase. In this case, the infrared zero of the beta function is an exact IR fixed point (IRFP) of the renormalization group. This was discussed for vectorial gauge theories in [20, 21].

In a chiral gauge theory whose UV to IR evolution leads to a gauge coupling that becomes strong in the infrared, there are several tools that one may use in studying the possible resultant nonperturbative behavior. First, one may investigate whether the fermion content of the theory satisfies the 't Hooft anomaly-matching conditions. To do this, one determines the global flavor symmetry group under which the theory is invariant and then examines various candidate operator products for gauge-singlet composite spin 1/2 fermions to ascertain if these can match the anomalies in the global flavor symmetries. If this necessary condition is satisfied, then one possibility is that in the infrared the strong chiral gauge interaction may confine and produce massless composite spin 1/2 fermions (as well as massive gauge-singlet hadrons).

An alternative possibility in a strongly coupled chiral gauge theory is that the gauge interaction can produce bilinear fermion condensates. In an irreducibly chiral theory (without a vectorlike subsector), these condensates break the gauge symmetry, as well as global flavor symmetries. A method that has been widely used to predict which type of condensate is most likely to form in this case is the most-attractive-channel approach [15]. Thus, consider a bilinear fermion condensation channel in which fermions in the representations R_1 and R_2 of

a given gauge group form a condensate that transforms according to the representation $R_{cond.}$ of this group, denoted $R_1 \times R_2 \to R_{cond.}$. An approximate measure, based on one-gluon exchange, of the attractiveness of this condensation channel, is

$$\Delta C_2 \equiv C_2(R_1) + C_2(R_2) - C_2(R_{Ch}) , \qquad (2.2)$$

where $C_2(R)$ is the quadratic Casimir invariant for the representation R (see Appendix B) and $R_{Ch} \equiv R_{cond.}$. At this level of one-gluon exchange, if ΔC_2 is positive (negative), then the channel is attractive (repulsive). The most attractive channel is the one with the largest (positive) value of ΔC_2. According to the MAC approach, if several possible condensation channels might, a priori, occur, then the one that actually occurs is the channel that has the largest (positive) value of ΔC_2. The MAC approach is supported by the fact that in quantum chromodynamics, of the four a priori possible (Lorenz-invariant) bilinear quark condensation channels $3 \times \bar{3} \to 1 + 8$ and $3 \times 3 \to \bar{3}_a + 6_s$, only one occurs, and this is precisely the MAC, namely $3 \times \bar{3} \to 1$. The MAC method has been used in theoretical studies of abstract chiral gauge theories and in efforts to build reasonably UV-complete models with dynamical electroweak symmetry breaking [17].

An analysis of the Schwinger-Dyson equation for the propagator of a massless fermion transforming according to the representation R of a gauge group G shows that, in the ladder (i.e., iterated one-gluon exchange) approximation the minimum value of α for which fermion condensation occurs in a vectorial gauge theory is given by the condition that $3\alpha_{cr}C_2(R)/\pi \sim 1$, or equivalently, $3\alpha_{cr}\Delta C_2/(2\pi) = 1$, since $\Delta C_2 = 2C_2(R)$ in this case [22]. Therefore, a rough estimate for the minimal value of the running coupling which is sufficient to cause condensation in a given channel Ch is

$$\alpha_{cr,Ch} \sim \frac{2\pi}{3\Delta C_2(R)_{Ch}} , \qquad (2.3)$$

Because of the strong-coupling nature of the fermion condensation process, Eq. (2.3) is only a rough estimate. A measure of the likelihood that the coupling grows large enough in the infrared to produce fermion condensation in a given channel Ch is thus the ratio

$$\rho_{IR,Ch} \equiv \frac{\alpha_{IR,2\ell}}{\alpha_{cr,Ch}} . \qquad (2.4)$$

If this ratio is significantly larger (smaller) than unity, one may infer that condensation in the channel Ch is likely (unlikely). As with the use of the MAC, this ρ test is only a rough estimate.

In the case of evolution from the UV to an IR non-Abelian Coulomb phase, the perturbative field degrees of freedom remain the same. In the other types of UV to IR evolution, in general, they change. Given that one restricts to asymptotically free theories, it is always possible to enumerate these field degrees of freedom in the ultraviolet, and in many theories, one can also enumerate them in the infrared. Denoting the UV and IR measures as f_{UV} and f_{IR}, it was conjectured that $f_{UV} \geq f_{IR}$ for vectorial gauge theories in [23], and this conjecture was extended to chiral gauge theories in [9], and further studied in [10, 11] and in our previous work, [12], where several classes of chiral gauge theories were constructued and shown to yield further support for this conjectured inequality.

Large-N methods have also proved fruitful for the analysis of chiral gauge theories that form families extendable to infinite N [8]. However, as we will show, the families of $S_k \bar{A}_\ell$ chiral gauge theories that we construct and study here are only asymptotically free for a finite set of N values. Going beyond the various approaches discussed in this section, one would ideally hope to make use of fully nonperturbative methods that can be used for any N, such as a lattice formulation and numerical simulations. However, while the lattice formulation has been of great value for vectorial gauge theories such as quantum chromodynamics, it has been difficult to use lattice methods to study chiral gauge theories, owing to the presence of fermion doubling.

III. ANOMALY FREEDOM OF A LOW-ENERGY EFFECTIVE THEORY ARISING FROM DYNAMICAL BREAKING OF A CHIRAL GAUGE THEORY

Here we prove a general theorem that guarantees that a low-energy effective field theory that arises from an anomaly-free chiral gauge theory via dynamical gauge symmetry breaking is also anomaly-free. Let us consider a chiral gauge theory with a gauge group G and an anomaly-free set of chiral fermions transforming according to some set of representations $\{R_i\}$ of G. Without loss of generality, we take all of the fermions to be left-handed. Also without loss of generality, we assume that this theory is irreducibly chiral, i.e., does not contain any vectorlike subsector. This assumption does not entail any loss of generality because the fermions in a vectorlike subsector give zero contribution to a chiral anomaly. Because the theory is irreducibly chiral, the gauge symmetry precludes any fermion mass terms in the fundamental lagrangian. To begin with, we assume that G is a simple group and discuss later the straightforward generalization of our argument to the case where G is a direct-product group. Let us denote the contribution of a chiral fermion in the R_i representation to the triangle anomaly in gauged currents as $\mathcal{A}(R_i)$. The property that the initial theory is anomaly-free is the condition

$$\sum_i n_{R_i} \mathcal{A}(R_i) = 0 , \qquad (3.1)$$

where n_{R_i} denotes the number of copies of fermions in the representation R_i. This anomaly cancellation condition (3.1) also implies that if one restricts to a subgroup

$H \subset G$, which means decomposing each representation R_i into representations R_i' of H, then the sum of contributions is also zero. Now, assume that this theory is asymptotically free, so that as the Euclidean reference scale μ decreases from the UV to the IR, the running gauge coupling increases, and assume further that this gauge coupling becomes strong enough at a scale Λ to produce bilinear fermion condensates that break the original gauge symmetry G to a subgroup $H \subset G$. The fermions involved in the condensate gain dynamical masses of order Λ, and the gauge bosons in the coset G/H also gain masses of this order.

To construct the low-energy effective field theory that describes the physics as the reference scale μ decreases below Λ, one integrates out these massive states and enumerates the remaining H-nonsinglet massless fields. This enumeration involves decomposing each fermion representation R_i of G in terms of representations R_i' of H. The resultant anomaly cancellation condition in the low-energy effective theory that is the descendant of the original theory is

$$\sum_i n_{R_i'} \mathcal{A}(R_i') = 0 , \qquad (3.2)$$

where here $\mathcal{A}(R_i')$ refers to the contribution to the anomaly in the descendant theory from the fermions in the R_i' representation of the gauge group H. Now the fermions in the original theory that were involved in the condensate, and hence acquired dynamical masses and were integrated out, transform as singlets under H, and therefore, even if they were included in Eq. (3.2), they would make zero contribution to this sum. Combining this fact with Eq. (3.1), we deduce that the remaining H-nonsinglet fermions must also make zero net contribution in Eq. (3.2). This proves the theorem.

We make some further remarks on this result. In general, an asymptotically free chiral gauge theory that becomes strongly coupled and produces fermion condensates that dynamically break the gauge symmetry may undergo not just one, but several sequential stages of dynamical gauge symmetry breaking. Clearly, the theorem above applies not just to the first stage, but also to subsequent stages of symmetry breaking. As noted, it is straightforward to extend this theorem to the case where the gauge group of the theory is a direct-product group instead of a simple group. An example of this is given below in our analysis of the low-energy effective $SU(5) \otimes U(1)$ theory resulting as a descendant from an initial (anomaly-free) $SU(6)$ chiral gauge with fermions in the S_2 and $\bar{A}_2$ representations of $SU(6)$.

We next contrast our theorem with the situation concerning chiral gauge symmetry breaking and associated fermion mass generation by the vacuum expectation value (VEV) of a Higgs field. For example, consider the Standard Model, with gauge group $G_{SM} = SU(3) \otimes SU(2) \otimes U(1)_Y = SU(3) \otimes G_{EW}$ and Higgs field ϕ. In the fundamental Lagrangian, the chiral G_{EW} gauge symmetry forbids any mass terms for the SM quarks and (SM-nonsinglet) leptons [24]. Arranging the Higgs potential $V(\phi)$ to have a minimum with a nonzero vacuum expectation value of the Higgs field, $\langle\phi\rangle_0 = \binom{0}{v/\sqrt{2}}$, breaks G_{EW} to electromagnetic $U(1)_{em}$. The SM quarks and leptons gain masses through Yukawa interactions with the SM Higgs boson ϕ, via its nonzero vacuum expectation value. For example, consider the quarks. Denote the left-handed quark fields as

$$Q_{i,L} \equiv \binom{u_i}{d_i}_L , \quad i = 1, 2, 3 , \qquad (3.3)$$

and the right-handed quarks as $u_{i,R}$ and $d_{i,R}$, where here i is a generational index, and we suppress color indices. The Yukawa term in the SM that generates the mass matrix for the up-type $q = 2/3$ quarks is

$$\mathcal{L}_{Y,u} = - \sum_{i,j=1}^3 \bar{Q}_{i,L} Y_{ij}^{(u)} u_{j,R} \tilde{\phi} + h.c. , \qquad (3.4)$$

where $\tilde{\phi} \equiv i\sigma_1 \phi^*$ This yields the mass matrix for the charge $q = 2/3$ quarks,

$$M_{ij}^{(u)} = \frac{v}{\sqrt{2}} Y_{ij}^{(u)} . \qquad (3.5)$$

The diagonalization of this matrix yields the mass eigenstates for these quarks. For simplicity, assume that the Yukawa matrix is diagonal: $Y_{ij}^{(u)} = Y_{ii}^{(d)} \delta_{ij}$. One could, formally at least, envision taking one element of this matrix to be arbitrarily large, say $Y_{33}^{(u)} \gg 1$, so that the t quark would have a mass $m_t = M_{33}^{(u)} \gg v$. If one were to attempt to integrate out the top quark, the resulting theory would, at a perturbative level, appear to have gauge anomalies (as well as an incomplete third quark generation). The theorem that we have proved shows that this sort of complication never happens in a chiral gauge theory (without scalar fields) in which the gauge symmetry breaking and fermion mass generation is dynamical, due to the formation of bilinear fermion condensates. In the Higgs-Yukawa framework, if one were formally to take a Yukawa coupling to infinity, the problem of the apparently anomalous low-energy effective theory would be circumvented by the appearance of a nonperturbative topological Wess-Zumino term in the action [25]. However, there are complications with trying to take a Yukawa coupling to be arbitrarily large, because the beta function for the Yukawa coupling has an infrared zero at the value zero (the "triviality" property of Yukawa theories), as was shown by nonperturbative lattice measurements [26] and perturbative beta function calculations (e.g., [27] and references therein).

IV. $S\bar{A}$ THEORIES

A. Basic Construction

In this section we construct and analyze an interesting set of asymptotically free chiral gauge theories with

an SU(N) gauge group and chiral fermions transforming according to the rank-2 symmetric and conjugate antisymmetric tensor representations of this group. We denote these fermions generically as S_2, and $\bar{A}_2$ (suppressing possible flavor indices) and equivalently by the corresponding Young tableaux, $S_2 = \square\square$ and $\bar{A}_2 = \overline{\begin{array}{c}\square\\\square\end{array}}$. To keep the notation as simple as possible, we omit the subscripts where no confusion will result, setting

$$S_2 \equiv S, \quad \bar{A}_2 \equiv \bar{A} . \tag{4.1}$$

We denote the number of S and $\bar{A}$ fields as n_S and $n_{\bar{A}}$. An $S\bar{A}$ theory is irreducibly chiral, i.e., it does not contain any vectorial subset. The chiral gauge symmetry forbids any fermion mass term in the Lagrangian. The triangle anomaly $\mathcal{A}$ in gauged currents of our $S\bar{A}$ theory is

$$\begin{aligned} \mathcal{A} &= n_S \mathcal{A}(S) + n_{\bar{A}} \mathcal{A}(\bar{A}) \\ &= n_S \mathcal{A}(S) - n_{\bar{A}} \mathcal{A}(A) . \end{aligned} \tag{4.2}$$

Substituting $\mathcal{A}(S) = N + 4$ and $\mathcal{A}(A) = N - 4$ (see Appendix B), the condition that this $S\bar{A}$ theory should be free of a triangle anomaly in gauged currents is that

$$n_S(N + 4) - n_{\bar{A}}(N - 4) = 0 . \tag{4.3}$$

Thus, n_S and $n_{\bar{A}}$ take values in the ranges $n_S \geq 1$ and $n_{\bar{A}} \geq 1$, subject to the anomaly cancellation condition (4.3) and the requirement that the resultant $S\bar{A}$ theory must be asymptotically free. A member of this set of chiral gauge theories is thus defined as

$$S\bar{A}: \quad G = \text{SU}(N), \quad \text{fermions}: n_S S + n_{\bar{A}} \bar{A} , \tag{4.4}$$

where it is understood implicitly that N, n_S, and $n_{\bar{A}}$ satisfy the condition (4.3) and yield an asymptotically free theory. We denote such a theory, for short, as $(N; n_S, n_{\bar{A}})$.

The anomaly cancellation condition (4.3) is a linear diophantine equation. If $N = 4$, i.e., $G = \text{SU}(4)$, a nontrivial solution of this equation is not possible, because in this case the $\begin{array}{c}\square\\\square\end{array}$ representation is self-conjugate, and hence has zero anomaly, so there is no value of $n_{\bar{A}}$ which can cancel the contribution to the anomaly in gauged currents from the fermions in the $\square\square$ representation. Consequently, a nontrivial solution of the anomaly cancellation condition (4.3) requires that $N \geq 5$, and we restrict to this range. We define the ratio

$$\frac{n_{\bar{A}}}{n_S} = \frac{N + 4}{N - 4} \equiv p . \tag{4.5}$$

Since the right-hand side of Eq. (4.5) is greater than one, it follows that $n_S < n_{\bar{A}}$. Therefore, the theories of this type with minimal chiral fermion content have $n_S = 1$ and take the form

$$(N; n_S, n_{\bar{A}}) = (N; 1, p) , \tag{4.6}$$

with the understanding that $n_{\bar{A}}$ must be a (positive) integer. We find that there are precisely four solutions of Eq. (4.5) with $n_S = 1$ that satisfy this condition, namely (including the N characterizing the SU(N) gauge group)

$$(N; n_S, n_{\bar{A}}) = (5; 1, 9), \ (6; 1, 5), \ (8; 1, 3), \ (12; 1, 2) . \tag{4.7}$$

In the context of anomaly cancellation alone, before imposing the condition of asymptotic freedom, we observe a basic mathematical property. If $(N; n_S, n_{\bar{A}})$ is a solution of Eq. (4.3), then a theory with n_{cp} copies (abbreviated cp) of the fermion content also yields a solution of (4.3). That is,

$$(N; n_S, n_{\bar{A}}) \text{ is anom. free } \Longrightarrow$$

$$(N; n_{cp} n_S, n_{cp} n_{\bar{A}}) \text{ is anom. free for } n_{cp} \geq 2. \tag{4.8}$$

If one were not to require that the theory must be asymptotically free, then n_{cp} could be any positive integer, and hence the linear diophantine equation (4.3) would have an infinite number of solutions. However, we do require that our chiral gauge theories must be asymptotically free so that they are perturbatively calculable in the deep ultraviolet.

Given this requirement, the next step is to ascertain, for a given value of N, which values of n_{cp} are allowed by asymptotic freedom. To do this, we calculate the first two coefficients of the beta function. These coefficients are

$$b_1 = \frac{1}{3}\left[11N - \left\{n_S(N + 2) + n_{\bar{A}}(N - 2)\right\}\right] \tag{4.9}$$

and

$$b_2 = \frac{1}{3}\left[34N^2 - n_S\left\{5N + 3\frac{(N+2)(N-1)}{N}\right\}(N+2) \right.$$
$$\left. - n_{\bar{A}}\left\{5N + 3\frac{(N-2)(N+1)}{N}\right\}(N-2)\right] . \tag{4.10}$$

It will be useful to reexpress these coefficients in a convenient form for analysis of the minimal set of fermions, viz., $(n_S, n_{\bar{A}}) = (1, p)$, given explicitly in (4.7) and the sets involving n_{cp}-fold replication (copies, or flavors) of minimal sets,

$$(n_S, n_{\bar{A}}) = n_{cp}(1, p) = (n_{cp}, n_{cp} p) . \tag{4.11}$$

Thus, equivalently,

$$\begin{aligned} b_1 &= \frac{1}{3}\left[11N - n_{cp}\left\{(N+2) + p(N-2)\right\}\right] \\ &= \frac{1}{3}\left[11N - 2n_{cp}\left(\frac{N^2 - 8}{N - 4}\right)\right] \end{aligned} \tag{4.12}$$

and

$$b_2 = \frac{1}{3}\left[34N^2 - n_{cp}\left\{\left(5N + 3\frac{(N+2)(N-1)}{N}\right)(N+2)\right.\right.$$
$$\left.\left. + p\left(5N + 3\frac{(N-2)(N+1)}{N}\right)(N-2)\right\}\right]$$
$$= \frac{1}{3}\left[34N^2 - 8n_{cp}\left\{\frac{2N^4 - 19N^2 + 12}{N(N-4)}\right\}\right],$$
$$(4.13)$$

where here it is understood that, since n_S is taken to have its minimal value of 1, the value of N is restricted so that p is a (positive) integer. The requirement of asymptotic freedom, i.e., $b_1 > 0$, implies that n_{cp} is bounded above according to

$$n_{cp} < \frac{11N(N-4)}{2(N^2-8)} . \qquad (4.14)$$

As a rational number, this upper bound has the respective values (quoted to the indicated floating-point accuracy) 1.62, 2.36, 3.14, and 3.88 for $N = 5$, 6, 8, 12. Therefore, on the integers, we have the upper bounds

$$n_{cp} \leq \begin{cases} 1 & \text{for } N = 5 \\ 2 & \text{for } N = 6 \\ 3 & \text{for } N = 8, \ 12 \end{cases} \qquad (4.15)$$

Thus, the full set of (anomaly-free) asymptotically free $S\bar{A}$ chiral gauge theories of this type, $(N; n_S, n_{\bar{A}}) = (N; 1, p)$ and $(N; n_{cp}, n_{cp}p)$ with integer p, includes, in addition to the minimal set (4.7), also the additional theories with an n_{cp}-fold replication of the set (4.7), namely

$$(N; n_S, n_{\bar{A}}) = (6; 2, 10), \ (8; 2, 6), \ (8; 3, 9),$$

$$(12; 2, 4), \ (12; 3, 6) . \qquad (4.16)$$

There are also (asymptotically free) solutions of the anomaly cancellation condition (4.3) with nonminimal values $n_S > 1$ that are not of the form of simple replications of the minimal set (4.7), i.e., for which p is a (positive) rational, but not integer, number. We find that there are seven such solutions, namely

$$(N; n_S, n_{\bar{A}}) = (10; 3, 7), \ (16; 3, 5), \ (20; 2, 3), \ (20; 4, 6),$$

$$(28; 3, 4), \ (36; 4, 5), \ (44; 5, 6) . \qquad (4.17)$$

Thus, we find that there are sixteen $S\bar{A}$ anomaly-free asymptotically free chiral gauge theories; these consist of the four minimal ones of the form $(N; 1, p)$ in Eq. (4.7), the five theories of the form $(N; n_{cp}, n_{cp}p)$ in Eq. (4.16), and the seven additional ones in Eq. (4.17) with rational, but non-integral p. As noted in the introduction, a striking feature of this family of $S\bar{A}$ chiral gauge theories is that the combined requirements of anomaly cancellation and asymptotic freedom yields only a finite set of solutions, in contrast to generic families of chiral

gauge theories that have been studied in the past, such as $S + (N+4)\bar{F}$ and $A + (N-4)\bar{F}$, and extensions of these with vectorlike subsectors, which allow, respectively, the infinite ranges $N \geq 3$ and $N \geq 5$.

We label the fermion fields in a given $S\bar{A}$ theory as

$$S_i : \ \psi^{ab}_{i,L} = \psi^{ba}_{i,L} , \quad 1 \leq i \leq n_S \qquad (4.18)$$

and

$$\bar{A}_j : \ \chi_{ab,j,L} = -\chi_{ba,j,L}, \quad 1 \leq j \leq n_{\bar{A}} , \qquad (4.19)$$

where a, b are SU(N) gauge indices and i, j are flavor indices.

B. Global Flavor Symmetry

The classical global flavor (fl) symmetry group of the $(N; n_S, n_{\bar{A}})$ $S\bar{A}$ theory is

$$G_{fl,cl} = \mathrm{U}(n_s) \otimes \mathrm{U}(n_{\bar{A}})$$

$$= \begin{cases} \mathrm{SU}(n_{\bar{A}}) \otimes \mathrm{U}(1)_S \otimes \mathrm{U}(1)_{\bar{A}} & \text{if } n_s = 1 \\ \mathrm{SU}(n_S) \otimes \mathrm{SU}(n_{\bar{A}}) \otimes \mathrm{U}(1)_S \otimes \mathrm{U}(1)_{\bar{A}} & \text{if } n_s \geq 2 \end{cases}$$
$$(4.20)$$

The operation of the elements of these global groups on the fermion fields is as follows. For fixed SU(N) group indices a, b, the theory is invariant under the action of an element $\mathcal{U}_S \in \mathrm{U}(n_S)$ on the n_S-dimensional vector $(\psi^{ab}_{1,L}, \psi^{ab}_{2,L}, ... \psi^{ab}_{n_S,L})$

$$\psi^{ab}_{i,L} \rightarrow \sum_{j=1}^{n_S} (\mathcal{U}_S)_{ij} \psi^{ab}_{j,L} \qquad (4.21)$$

and separately under the action of an element of an element $\mathcal{U}_{\bar{A}} \in \mathrm{U}(n_{\bar{A}})$ on the $n_{\bar{A}}$-dimensional vector $(\chi_{ab,1,L}, \chi_{ab,2,L}, ... \chi_{ab,n_{\bar{A}},L})$

$$\chi_{ab,i,L} \rightarrow \sum_{j=1}^{n_{\bar{A}}} (\mathcal{U}_{\bar{A}})_{ij} \chi_{ab,j,L} . \qquad (4.22)$$

The $\mathrm{U}(1)_S$ and $\mathrm{U}(1)_{\bar{A}}$ global symmetries are both broken by SU(N) instantons [28]. As before in our analysis of different chiral gauge theories [11], we define a vector whose components are comprised of the instanton-generated contributions to the breaking of these symmetries. In the basis $(S, \bar{A})$, this vector is

$$\vec{v} = \left(n_S T(S), \ n_{\bar{A}} T(\bar{A})\right) = n_{np}\left(\frac{N+2}{2}, \ p\left(\frac{N-2}{2}\right)\right)$$

$$= \frac{n_{np}}{2}\left(N+2, \ \frac{(N+4)(N-2)}{N-4}\right) . \qquad (4.23)$$

We can construct one linear combination of the two original currents that is conserved in the presence of SU(N) instantons. We denote the corresponding global U(1)

flavor symmetry as U(1)′ and the fermion charges under this U(1)′ as

$$\vec{Q}' = \left(Q'_S, \ Q'_{\bar{A}} \right) . \tag{4.24}$$

The U(1)′ current is conserved if and only if

$$\sum_f n_f T(R_f) Q'_f = \vec{v} \cdot \vec{Q}' = 0 . \tag{4.25}$$

Clearly, this condition determine the vector $\vec{Q}'$ only up to an overall multiplicative constant. A solution is

$$\vec{Q}' = \left((N-2)(N+4), \ -(N+2)(N-4) \right) . \tag{4.26}$$

The actual global chiral flavor symmetry group (preserved in the presence of instantons) is then

$$G_{fl} = \begin{cases} \mathrm{SU}(n_{\bar{A}}) \otimes \mathrm{U}(1)' & \text{if } n_S = 1 \\ \mathrm{SU}(n_S) \otimes \mathrm{SU}(n_{\bar{A}}) \otimes \mathrm{U}(1)' & \text{if } n_S \geq 2 \end{cases} \tag{4.27}$$

So far, our discussion of global flavor symmetries applies generally to all of the $S\bar{A}$ chiral gauge theories. We next determine the most attractive channel for fermion condensation, which differs for $N = 5$ and $N \geq 6$, and then proceed with analyses of specific $S\bar{A}$ theories.

C. Fermion Condensation Channels

For $N \neq 5$, the most attractive channel for the formation of a bilinear fermion condensate in a $(N; n_S, n_{\bar{A}})$ $S\bar{A}$ chiral gauge theory is

$$S \times \bar{A} \to adj , \tag{4.28}$$

where adj denotes the adjoint representation of SU(N). This has

$$\Delta C_2 = \frac{(N+2)(N-2)}{N} \quad \text{for } S \times \bar{A} \to adj . \tag{4.29}$$

Substituting this expression for ΔC_2 into Eq. (2.3) for the estimate of the minimum critical coupling for condensation in this channel, we obtain

$$\alpha_{cr} \simeq \frac{2\pi N}{3(N+2)(N-2)} \quad \text{for } S \times \bar{A} \to adj . \tag{4.30}$$

In general, there are several stages of fermion condensation, as will be evident in the analyses of specific theories below. In the $S\bar{A}$ theory with $G = \mathrm{SU}(5)$, the most attractive channel is $\bar{A} \times \bar{A} \to F$ instead of (4.28) and will be discussed in the section devoted to this theory.

V. $S\bar{A}$ THEORY WITH $G = \mathrm{SU}(5)$

A. General

Our SU(5) $S\bar{A}$ theory has $(n_S, n_{\bar{A}}) = (1,9)$. Since $n_S = 1$ for this theory, we use a simplified notation without the flavor index on the S field, namely $\psi^{ab}_{i=1,L} \equiv \psi^{ab}_L$.

We recall that the $S = \square\square$ and $\bar{A} = \overline{\begin{array}{c}\square\\\square\end{array}}$ representations of SU(5) have dimensionalities 15 and 10, respectively, and we shall equivalently refer to them in this section by these dimensionalities. From Eq. (4.27), this theory has a (nonanomalous) global flavor symmetry

$$G_{fl} = \mathrm{SU}(9)_{\bar{A}} \otimes \mathrm{U}(1)' . \tag{5.1}$$

We have not found SU(5)-gauge-singlet composite-fermion operators that satisfy the 't Hooft anomaly matching conditions for this theory. Indeed, the minimal fermionic operator products, such as $S^{ab}\bar{A}_{bc}S^{cd}$, $\epsilon^{abcde}\bar{A}_{ab}\bar{A}_{cd}\bar{A}_{ef}$, are not SU(5) gauge singlets.

We list the values of the first two coefficients of the beta function for this theory in Table I. This beta function has an IR zero which occurs, at the two-loop level, at a value $\alpha_{IR,2\ell} = 0.645$. As will be discussed further below, we find that this value is close to an estimate of the minimal critical value, α_{cr} for the formation of a bilinear fermion condensate in the most attractive channel, with associated spontaneous chiral symmetry breaking. Consequently, we shall analyze both possibilities (i.e., retaining or breaking chiral symmetry) for the UV to IR evolution of this SU(5) $S\bar{A}$ theory.

The most attractive channel for condensation is

$$\text{MAC for } (5;1,9): \quad \overline{\begin{array}{c}\square\\\square\end{array}} \times \overline{\begin{array}{c}\square\\\square\end{array}} \to \square, \ i.e., \ \bar{A} \times \bar{A} \to F , \tag{5.2}$$

where $F = \square$ is the fundamental representation. Equivalently, in terms of dimensionalities, this is the channel $\overline{10} \times \overline{10} \to 5$. Since $C_2(A_2) = 18/5$ and $C_2(F) = 12/5$ for SU(5), the measure of attractiveness for this channel is

$$\Delta C_2 = \frac{24}{5} \quad \text{for } \overline{10} \times \overline{10} \to 5 . \tag{5.3}$$

(The next-most attractive channel is $S \times \bar{A} \to$ adj, with $\Delta C_2 = 21/5$.) The rough Schwinger-Dyson estimate of the critical coupling for condensate formation in the $\overline{10} \times \overline{10} \to 5$ channel is $\alpha_{cr} \sim 5\pi/36 = 0.44$ To compare $\alpha_{IR,2\ell}$ with α_{cr}, we use the ratio ρ defined in Eq. (2.4). We calculate $\rho = 1.5$ for the channel (5.2). This value of ρ is close enough to unity that we cannot make a definite conclusion concerning the presence or absence of spontaneous chiral symmetry breaking. There are thus two possibilities for the first step in the UV to IR evolution of this SU(5) $S\bar{A}$ theory, and we investigate both of these.

B. Evolution of SU(5) $S\bar{A}$ Theory to a Non-Abelian Coulomb Phase in the IR

First, the SU(5) (5;1,9) $S\bar{A}$ theory might evolve downward in μ without any spontaneous chiral symmetry breaking, yielding a (deconfined) non-Abelian Coulomb phase (NACP) in the infrared. We denote this possibility as

$$(5;1,9): \quad \mathrm{UV} \to \mathrm{IR} \ \ \mathrm{NACP} . \tag{5.4}$$

In this case, the global flavor symmetry in the IR is the same as in the UV, namely (5.1).

C. Dynamical Breaking of SU(5) to SU(4) Gauge Symmetry

Second, the gauge coupling of the (5;1,9) SU(5) $S\bar{A}$ theory might become sufficiently strong to lead to non-perturbative behavior. Since we have not found gauge-singlet operator products that satisfy 't Hooft anomaly matching conditions in this SU(5) theory, we infer that this nonperturbative behavior would lead to the formation of a bilinear fermion condensate, breaking the SU(5) gauge symmetry. We denote this possibility as

$$(5;1,9): \quad \text{UV} \to \text{IR}: \quad S\chi SB \implies \text{SU(4)} . \quad (5.5)$$

We proceed to analyze this possibility. Thus, we assume that as the reference Euclidean momentum scale μ decreases below a value that we denote Λ_5, the gauge coupling becomes large enough to form a bilinear condensate in the most attractive channel, $\bar{A} \times \bar{A} \to F$, Eq. (5.2), dynamically breaking the SU(5) gauge symmetry to SU(4) (and also breaking the global flavor symmetry (5.1)). The associated fermion condensate is of the form $\langle \epsilon^{abdef} \chi^T_{bd,L} C \chi_{ef,j,L} \rangle$, where C is the charge-conjugation Dirac matrix. With no loss of generality, we may choose the uncontracted index a to be $a = 5$. By a vacuum alignment argument similar to that used in [12], we infer that the actual condensates are of the form

$$\langle \epsilon^{5bdef} \chi^T_{bd,j,L} C \chi_{ef,j,L} \rangle \propto \Big[\langle \chi^T_{12,j,L} C \chi_{34,j,L} \rangle$$
$$- \langle \chi^T_{13,j,L} C \chi_{24,j,L} \rangle + \langle \chi^T_{14,j,L} C \chi_{23,j,L} \rangle \Big]$$
$$(5.6)$$

for $1 \le j \le 9$. Since the gauge interaction is independent of the flavor index, it follows that these condensates have a common value independent of the flavor index, j. The fermions involved in these condensates thus gain a common dynamical mass of order Λ_5 and are integrated out of the low-energy effective field theory applicable at energy scales $\mu < \Lambda_5$. When SU(N) breaks to SU($N-1$) there are $2N-1$ gauge bosons in the coset SU(N)/SU($N-1$), corresponding to the broken generators. Here, with $N = 5$, there are nine gauge bosons in the coset SU(5)/SU(4), and these also gain masses of order Λ_5.

D. Analysis of SU(4) Descendant Theory

Since the low-energy effective field theory resulting as a descendant from the breaking of the SU(5) $S\bar{A}$ gauge symmetry is invariant under an SU(4) gauge symmetry, in order to analyze it, we decompose the remaining massless fermions into SU(4) representations. For this purpose, we make use of the following general results for an SU(N) group:

$$\Box\Box_{\,\text{SU}(N)} = [\, \Box\Box + \Box + 1 \,]_{\,\text{SU}(N-1)} , \quad (5.7)$$

where 1 denotes a singlet, and, for $N \ge 4$, and

$$\overline{\overline{\Box}}_{\,\text{SU}(N)} = [\, \overline{\Box} + \overline{\Box} \,]_{\,\text{SU}(N-1)} . \quad (5.8)$$

Here, in terms of dimensionalities, these decompositions read

$$15_{\text{SU}(5)} = 10_{\text{SU}(4)} + 4_{\text{SU}(4)} + 1 \quad (5.9)$$

and

$$\overline{10}_{\text{SU}(5)} = 6_{\text{SU}(4)} + \bar{4}_{\text{SU}(4)} . \quad (5.10)$$

Note that $6_{\text{SU}(4)}$ is self-conjugate, i.e., $6_{\text{SU}(4)} \approx \bar{6}_{\text{SU}(4)}$. The massless SU(4)-nonsinglet fermions in the SU(4) theory thus consist of ψ^{ab}_L with $1 \le a$, $b \le 4$, ψ^{a5}_L with $1 \le a \le 4$, and $\chi_{a5,j,L}$ with $1 \le a \le 4$ and $1 \le j \le 9$. In terms of Young tableaux, these are $\Box\Box + \Box + 9\overline{\Box}$ under SU(4), or in equivalent notation, the theory is

$$\text{SU(4), fermions}: \; S + F + 9\bar{F} = S + 8\bar{F} + 1\{F + \bar{F}\} \quad (5.11)$$

Thus the SU(4)-nonsinglet fermion content of the theory is precisely the $N = 4$, $p = 1$ special case of the Sp model presented in [4] and further analyzed in [5, 10, 11], so we can apply the results from these previous studies here. This SU(4) theory also contains the massless SU(4)-singlet chiral fermion ψ^{55}_L inherited from the SU(5) theory, but this does not affect the SU(4) dynamics. We recall that the Sp model is defined by the gauge group and chiral fermion content

$$Sp: \; G = \text{SU}(N), \text{ fermions}: \; S + (N+4)\bar{F} + p\{F + \bar{F}\} \quad (5.12)$$

where the first part, $S + (N+4)\bar{F}$ is irreducibly chiral and the second part is a vectorlike subsector consisting of p copies of $\{F + \bar{F}\}$. As dictated by our theorem proved above, this SU(4) descendant theory is anomaly-free. This is evident from the count

$$\mathcal{A}(\Box\Box_{\,\text{SU}(4)}) \; + \; \mathcal{A}(\Box_{\,\text{SU}(4)}) - 9\,\mathcal{A}(\Box_{\,\text{SU}(4)})$$
$$= 8 + 1 - 9 = 0 . \quad (5.13)$$

The first two coefficients of the beta function of this SU(4) low-energy effective field theory have the same sign (explicitly, $\bar{b}_1 = 0.7427$ and $\bar{b}_2 = 0.1831$), so this beta function has no IR zero at the maximal scheme-independent, two-loop order. Thus, as the Euclidean momentum scale μ decreases below Λ_5, the SU(4) gauge coupling inherited from the original SU(5) theory continues to increase until it exceeds the region where it can be described by the perturbative beta function. There are then several possibilities for the next stage of RG evolution to lower scales. We discuss these next.

1. Confinement in SU(4) Theory with Massless Composite Fermions

The first of these possibilities for the SU(4) theory is present because of the fact that (for general values of N

and p where there is confinement) the Sp model satisfies the 't Hooft anomaly-matching conditions [4, 10, 11]. Owing to this, as the gauge coupling continues to increase in the infrared, the gauge interaction could confine the (massless) SU(4)-nonsinglet fermions, producing massless spin 1/2 composite fermions as well as massive SU(4)-singlet hadrons (mesons, glueballs, and mass eigenstates that are linear combinations of mesons and glueballs). The massless fermion spectrum would also contain the SU(4)-singlet chiral fermion ψ_L^{55} from the original SU(5) $S\bar{A}$ theory.

2. Formation of Fermion Condensates Breaking SU(4) Gauge Symmetry

The second of these possibilities for the UV to IR evolution of the SU(4) low-energy effective field theory resulting from the breaking of the SU(5) $S\bar{A}$ theory is further fermion condensation in the most attractive channel in this SU(4) theory. The MAC is the channel $\square\!\square \times \bar{\square} \to \square$, i.e.,

$$S \times \bar{F} \to F \ . \tag{5.14}$$

The next-most attractive channel is $F \times \bar{F} \to 1$, with

$$\Delta C_2 = 2C_2(F) = \frac{N^2-1}{N} \ . \tag{5.15}$$

The fact that the $S \times \bar{F} \to F$ channel is the MAC is evident from the property that it has a larger ΔC_2 value than the $F \times \bar{F} \to 1$ channel:

$$\frac{(N+2)(N-1)}{N} - \frac{N^2-1}{N} = \frac{N-1}{N} > 0 \ . \tag{5.16}$$

For generality, we discuss the physics of the $S \times \bar{F} \to F$ channel for general N, although our specific application will be to $N = 4$. The attractiveness measure for this channel is

$$\Delta C_2 = C_2(S) = \frac{(N+2)(N-1)}{N} \quad \text{for } S \times \bar{F} \to F \ . \tag{5.17}$$

Substituting this into Eq. (2.3) for the estimate of the minimum critical coupling for condensation in this channel, we obtain

$$\alpha_{cr} \simeq \frac{2\pi N}{3(N+2)(N-1)} \quad \text{for } S \times \bar{F} \to F \ . \tag{5.18}$$

For present case of $N = 4$, this yields the estimate $\Delta C_2 = 4.5$. We denote the Euclidean scale μ at which the running coupling $\alpha(\mu)$ exceeds the critical value for condensation in this MAC as Λ_4. The condensation (5.14) breaks SU(4) to SU(3). The associated condensate has the general form $\langle \sum_{b=1}^{4} \psi_L^{ab\ T} C \chi_{5b,j,L} \rangle$. Without loss of generality, we can denote the breaking axis as $a = 4$ and

label the copy (flavor) index of the $\bar{F}$ fermion $\chi_{5b,j,L}$ involved in this condensate as $j = 9$, so that the condensate is

$$\langle \sum_{b=1}^{4} \psi_L^{4b\ T} C \chi_{b5,9,L} \rangle \ . \tag{5.19}$$

The fermions ψ_L^{b4} and $\chi_{b5,9,L}$ with $1 \le b \le 4$ involved in this condensate thus get common dynamical masses of order Λ_4. The seven gauge bosons in the coset SU(4)/SU(3) also get masses of order Λ_4. These fermions and bosons are integrated out of the low-energy effective field theory that is operative for scales $\mu < \Lambda_4$.

This low-energy effective field theory is invariant under an (anomaly-free) SU(3) gauge symmetry and contains the massless SU(3)-nonsinglet chiral fermions ψ_L^{ab} with $1 \le a, b \le 3$, transforming as $S = \square\!\square$ of SU(3), ψ_L^{a5}, with $1 \le a \le 3$, transforming as $F = \square$ of SU(3), and the $\chi_{a5,j,L}$ with $1 \le a \le 3$ and $1 \le j \le 8$; that is,

$$G = \text{SU(3), fermions}: \quad S+F+8\,\bar{F} = S+7\bar{F}+1\{F+\bar{F}\} \tag{5.20}$$

The SU(3)-nonsinglet fermion content of this theory is the $N = 3$, $p = 1$ special case of the Sp model (5.12). This SU(3) theory also contains a number of massless SU(3)-singlet chiral fermions. In addition to the ψ_L^{55} SU(4)-singlet fermion remaining from the SU(5) $\to$ SU(4) breaking at the higher scale Λ_5, there are also nine massless SU(3)-singlet fermions remaining from the SU(4) $\to$ SU(3) breaking at Λ_4, namely ψ_L^{45} and the $\chi_{45,j,L}$ with $1 \le j \le 8$.

As discussed in [4, 10, 11], the further evolution into the infrared of this SU(3) Sp model might lead to confinement with resultant massless composite fermions or to further condensation in the most attractive channel, which is $S \times \bar{F} \to F$, breaking SU(3) to SU(2) and then breaking SU(2) completely. In the latter case, the full sequence of gauge symmetry breaking of (5;1,9) theory would be as follows: $\bar{A} \times \bar{A} \to F$, breaking SU(5) to SU(4), followed in the resultant SU(4) descendant theory by the condensation $S \times \bar{F} \to F$, breaking SU(3) to SU(2), followed again by condensation in the respective $S \times \bar{F} \to F$ channel, breaking SU(2) completely.

VI. $S\bar{A}$ MODELS WITH $N \ge 6$

A. General Analysis

We next proceed to analyze the $S\bar{A}$ models $(N; n_S, n_{\bar{A}})$ with $N \ge 6$. In contrast to the SU(5) $S\bar{A}$ theory $(5; 1, 9)$, if $N \ge 6$, the most attractive channel for bilinear fermion condensation is $S \times \bar{A} \to adj$, as given in Eq. (4.28): This condensation produces, as the first stage of dynamical gauge symmetry breaking, the pattern

$$\text{SU}(N) \to \text{SU}(N-1) \otimes \text{U}(1) \ . \tag{6.1}$$

The values of ΔC_2 and α_{cr} for this channel were given in Eqs. (4.29) and (4.30). The resultant estimates for α_{cr}

for condensation in this channel in specific $(N; n_S, n_{\bar{A}})$ models with $N \geq 6$ are listed in Tables I and II. In these tables we also list the (reduced) beta function coefficients $\bar{b}_1$ and $\bar{b}_2$, the resultant IR zero in the two-loop beta function, if it exists, and the ratio $\rho = \alpha_{IR,2\ell}/\alpha_{cr}$ from Eq. (2.4). In cases where the beta function has no IR zero, the coupling increases with decreasing reference scale μ until it exceeds the perturbatively calculable regime. For a generic $S\bar{A}$ $(N; n_S, n_{\bar{A}})$ theory, we do not find solutions for 't Hooft anomaly matching, although, as will be discussed later, in certain cases, resultant low-energy effective descendant field theories with different fermion content (e.g., the SU(4) Sp model below), do satisfy these matching conditions. Thus, as regards the initial UV to IR evolution of the $(N; n_S, n_{\bar{A}})$ theory, if the gauge coupling becomes sufficiently strong, one expects fermion condensation. The resulting expectations for whether or not fermion condensation and associated spontaneous chiral symmetry breaking occur are listed for the sixteen $S\bar{A}$ theories in Tables I and II.

1. Flow to Chirally Symmetric Non-Abelian Coulomb Phase in IR

Referring to these Tables I and II, in the six cases where the value of ρ is substantially less than unity, we infer that the theory is likely to evolve smoothly from the UV to a (deconfined) chirally symmetric non-Abelian Coulomb phase in the IR. Explicitly, we infer that this IR behavior occurs for the (6;2,10), (8;3,9), (10;3,7), (20;4,6), (36;4,5), and (44;5,6) $S\bar{A}$ theories.

2. Flow to IR with Spontaneous Chiral Symmetry Breaking

We next discuss the situation in which, as the reference scale μ decreases from the UV to the IR, the coupling becomes large enough so that nonperturbative behavior occurs. As noted, in the absence of sets of fermionic operator products that yield solutions to 't Hooft anomaly matching conditions, one infers that this nonperturbative behavior entails fermion condensation and associated spontaneous breaking of the SU(N) gauge symmetry (although after some stage(s) of such symmetry breaking, a low-energy descendant theory may satisfy these matching conditions). For technical simplicity, we restrict our discussion to the minimal theories $(N; 1, p)$; corresponding analyses can be given for the other $(N; n_S, n_{\bar{A}})$ models. As is evident from Table I, all three of the $(N; 1, p)$ theories with $N \geq 6$, namely (6;1,5), (8;1,3), and (12;1,2) have the property that the gauge coupling becomes sufficiently strong to produce further bilinear fermion condensation. As before, we denote the scale where this occurs as Λ_N. A vacuum alignment argument implies that the symmetry breaking is such as to leave the largest residual symmetry. This implies that the condensate breaks the original SU(N) gauge symmetry to SU$(N-1) \otimes$ U(1). Without loss of generality, we take the breaking direction in SU(N) to be $a = N$. To show how this occurs, we recall the decompositions of $\square_{\text{SU}(N)}$ and $\boxbar_{\text{SU}(N)}$ under SU$(N-1)$ given, respectively, in Eqs. (5.7) and (5.8) above. Using these decompositions, we have

$$\square_{\text{SU}(N)} \times \overline{\boxbar}_{\text{SU}(N)} = \left(\square_{\text{SU}(N-1)} + \square_{\text{SU}(N-1)} + 1 \right) \times \left(\overline{\boxbar}_{\text{SU}(N-1)} + \overline{\square}_{\text{SU}(N-1)} \right) . \tag{6.2}$$

Among the various products, we see that $\square_{\text{SU}(N-1)} \times \overline{\square}_{\text{SU}(N-1)}$ yields a singlet of SU$(N-1)$ and hence is favored by the vacuum alignment argument. The associated condensate thus has the form (with no sum on a)

$$\langle T_a^a \rangle = \begin{cases} \kappa & \text{for } 1 \leq a \leq N-1 \\ -\kappa(N-1) & \text{for } a = N \end{cases} \tag{6.3}$$

where κ is a constant. Thus, in terms of the fermion fields, the $\langle T_a^a \rangle$ condensate is of the form $\langle T_a^a \rangle = \langle \psi_L^{ad\ T} C \chi_{da, n_{\bar{A}}, L} \rangle$ (with no sum on a or d). The fermions involved in this condensate gain dynamical masses of order Λ_N. The $2N$ gauge bosons in the coset SU(N)/[SU$(N-1) \otimes$ U(1)] also gain masses of order Λ_N. In the low-energy effective field theory that is applicable at scales $\mu < \Lambda_N$, one thus integrates out these fields with masses $\sim \Lambda_N$.

The massless nonsinglet chiral fermion content of the resultant low-energy effective field theory with SU$(N-1) \otimes$ U(1) gauge invariance thus consists of the $\square_{\text{SU}(N-1)}$, p copies of the $\boxbar_{\text{SU}(N-1)}$, and $p-1$ copies of the $\overline{\square}_{\text{SU}(N-1)}$, with corresponding U(1) charges. We summarize this SU$(N-1)$-nonsinglet content as

$$\text{SU}(N-1): \quad \text{fermions}: \ S + p\bar{A} + (p-1)\bar{F} \tag{6.4}$$

where here S, $\bar{A}$, and $\bar{F}$ refer to the SU$(N-1)$ gauge symmetry. This SU$(N-1)$ effective theory also contains the massless SU$(N-1)$-singlet fermion ψ_L^{NN}. As guaranteed by the theorem proved above, this descendant theory is free of anomalies in gauged currents. This is evident for the SU$(N-1)^3$ triangle anomaly, for example, since this is given (with $M \equiv N-1$) by:

$$\text{SU}(N-1)^3 \ \mathcal{A} = \mathcal{A}(S) + p\mathcal{A}(\bar{A}) + (p-1)\mathcal{A}(\bar{F})$$

$$= (M+4) - p(M-4) - (p-1) = 0 , \tag{6.5}$$

where the last line follows upon substitution of p from Eq. (4.5). Similar cancellations hold for the $SU(N-1)^2\,U(1)$ and $U(1)^3$ anomalies.

B. SU(6) Theory with $n_S = 1$, $n_{\bar A} = 5$

The renormalization-group evolution of a $(N; n_S, n_{\bar A})$ theory into the infrared depends on the specific theory. For definiteness, we shall focus on the (6;1,5) theory for our further discussion. We list the values of the first two coefficients of the beta function for this theory in Table I. As before, since $n_S = 1$ for this theory, we use a simplified notation without the flavor index on the S field, namely $\psi^{ab}_{i=1,L} \equiv \psi^{ab}_L$. Our discussion for general $N \geq 6$ applies, in particular, to this theory.

1. Initial Condensation and Breaking of SU(6) to SU(5) ⊗ U(1)

Since the ratio ρ is substantially larger than unity (see Table I) and since we have not found composite fermion operators that satisfy the 't Hooft anomaly matching conditions, we infer that as the reference scale μ decreases from the UV to the IR, the gauge interaction produces a bilinear fermion condensate in the $S \times \bar A \to adj$ channel. Using the notation introduced above, this occurs at a scale denoted Λ_6. By convention, we take the breaking direction as $a = 6$ and the copy (flavor) label of the $\bar A_2$ fermion involved to be $j = p = 5$. In the notation of Eq. (6.3), the condensate can then be written as

$$\langle T^a_a \rangle = \langle \psi^{a6\,T}_L \chi_{6a,5,L} \rangle \tag{6.6}$$

where $1 \leq a \leq 5$, and there is no sum on a. The fermions involved in this condensate gain dynamical masses of order Λ_6.

2. Analysis of Descendant SU(5) ⊗ U(1) Theory

We next consider the descendant $SU(5) \otimes U(1)$ theory that emerges from the self-breaking of the $SU(6)$ theory at Λ_6. The relevant decomposition of the $SU(6)$ S and $\bar A$ representations under $SU(5) \otimes U(1)$, are indicated as follows, in terms of Young tableaux and $SU(5)$ dimensionalities, with $U(1)$ charges given as subscripts (normalized according to the conventions of [29]):

$$\square\square_{SU(6)} = [\,\square\square + \square + 1\,]_{SU(5)}$$
$$= 15_2 + 5_{-4} + 1_{-10} \tag{6.7}$$

and

$$\overline{\square}_{SU(6)} = [\,\overline{\square} + \overline{\square}\,]_{SU(5)}$$
$$= \overline{10}_{-2} + \bar 5_4 \tag{6.8}$$

We will again use the shorthand notation $S \equiv S_2$, $\bar A \equiv \bar A_2$, and $\bar F$ for the $\square\square$, $\overline{\square}$ and $\bar\square$, where these now refer to $SU(5)$. We will indicate the $U(1)$ charge of the S field as η_S and so forth for the other fermion fields. The massless $SU(5)$-nonsinglet fermion content in this effective theory is thus

$$SU(5): \quad \text{fermions}: \ S + 5\,\bar A + 4\,\bar F . \tag{6.9}$$

Explicitly, these fermions (with dimensions of the $SU(5)$ representations indicated in parentheses) are

$$S(15): \quad \psi^{ab}_L \text{ with } 1 \leq a,\ b \leq 5,$$

$$5\,\bar A(10): \quad \chi_{ab,j,L} \text{ with } 1 \leq a,\ b \leq 5 \text{ and } 1 \leq j \leq 5,$$

$$4\,\bar F(5): \quad \chi_{6b,j,L} \text{ with } 1 \leq b \leq 5 \text{ and } 1 \leq j \leq 4 . \tag{6.10}$$

This theory also contains the massless $SU(5)$-singlet fermion ψ^{66}_L from the original $SU(6)$ theory. From Eq. (6.7), it follows that this fermion has $U(1)$ charge $\eta_1 = -10$.

As an illustration of our theorem, it is instructive to see explicitly the cancellation of contributions to the anomalies in various gauge currents in this $SU(5) \otimes U(1)$ descendant gauge theory. There are three triangle anomalies that are relevant, namely the $SU(5)^3$, $SU(5)^2\,U(1)$, and $U(1)^3$ anomalies. We have

$$SU(5)^3\ \mathcal{A} = \mathcal{A}(S) + 5\,\mathcal{A}(\bar A) + 4\,\mathcal{A}(\bar F)$$
$$= 9 + 5(-1) + 4(-1) = 0 \tag{6.11}$$

and

$$SU(5)^2\,U(1)\ \mathcal{A} = T(S)\eta_S + 5\,T(\bar A)\eta_{\bar A} + 4\,T(\bar F)\eta_{\bar F}$$
$$= \frac{7}{2}2 + 5\frac{3}{2}(-2) + 4\frac{1}{2}(4) = 0 . \tag{6.12}$$

For the $U(1)^3$ anomaly cancellation, we must also include the contribution of the $SU(5)$-singlet fermion ψ^{66}_L since it carries a nonzero $U(1)$ charge:

$$U(1)^3\ \mathcal{A} = \dim(S)\eta_S^3 + 5\dim(\bar A)\eta_{\bar A}^3$$
$$+ 4\dim(\bar F)\eta_{\bar F}^3 + \eta_1^3$$
$$= 15(2^3) + 5(10)(-2)^3 + 4(5)(4^3) + (-10)^3 = 0 . \tag{6.13}$$

We also observe that the mixed gauge-gravitational anomaly vanishes:

$$(\text{grav})^2\, U(1)\ \mathcal{A} = \dim(S)\eta_S + 5\dim(\bar{A})\eta_{\bar{A}}$$

$$+\ 4\dim(\bar{F})\eta_{\bar{F}} + \eta_1$$

$$=\ 15(2) + 5(10)(-2) + 4(5)(4) + (-10) = 0 \ .$$

$$(6.14)$$

The first two (reduced) coefficients of the SU(5) beta function are $\bar{b}_1 = 0.76925$ and $\bar{b}_2 = -0.22882$, so that this two-loop SU(5) beta function has an IR zero at $\alpha_{IR,2\ell} = -\bar{b}_1/\bar{b}_2 = 3.36$. The U(1) beta function is not asymptotically free, so that as the reference scale μ decreases, the running U(1) gauge coupling inherited from the original SU(6) theory decreases. As regards the SU(5) dynamics, the most attractive channel for fermion condensation is $S \times \bar{F} \to F$, with $\Delta C_2 = 28/5$. (The next-most attractive channel is $\bar{A} \times \bar{A} \to F$, with $\Delta C_2 = 24/5$.) Fermion condensation in this most attractive channel causes the gauge symmetry breaking

$$\mathrm{SU}(5) \otimes \mathrm{U}(1) \to \mathrm{SU}(4) \ . \qquad (6.15)$$

The fact that the fermion condensate breaks the U(1) gauge symmetry is evident, since $\eta_S + \eta_{\bar{F}} = 6 \neq 0$. The estimated minimum critical coupling for condensation in this MAC to occur is $\alpha_{cr} \simeq 0.38$. Since the ratio $\rho = \alpha_{IR,2\ell}/\alpha_{cr} = 9.0$, and since we have not found $\mathrm{SU}(5) \otimes \mathrm{U}(1)$-invariant fermionic operator products that satisfy 't Hooft anomaly matching, we anticipate that fermion condensation occurs in this most attractive channel. We denote the scale at which this occurs as Λ_5. The $S\bar{F}$ condensate is of the form $\langle \sum_{b=1}^{5} \psi_L^{ab\ T} C\chi_{b6,j,L}\rangle$. By convention, we denote the breaking direction as $a = 5$ and choose the copy index on the $\chi_{b6,j,L}$ field to be $j = 4$, so that the condensate is

$$\langle \sum_{b=1}^{5} \psi_L^{5b\ T} C\chi_{b6,4,L}\rangle \ . \qquad (6.16)$$

The fermions ψ_L^{5b} and $\chi_{b6,4,L}$ with $1 \leq b \leq 5$ that are involved in this condensate gain dynamical masses of order Λ_5. The ten gauge bosons in the coset $[\mathrm{SU}(5) \otimes \mathrm{U}(1)]/\mathrm{SU}(4)$ also gain masses of order Λ_5. These fields are integrated out in the construction of the SU(4)-invariant low-energy effective field theory applicable at scales $\mu < \Lambda_5$.

3. Analysis of Descendant SU(4) Theory

The massless SU(4)-nonsinglet chiral fermion content of this effective low-energy theory consists of ☐☐, 5 copies of ⊟, and eight copies of ☐̄, i.e.

$$\mathrm{SU}(4): \quad \text{fermions}: \ S + 5\,\bar{A} + 8\,\bar{F} \qquad (6.17)$$

Explicitly, these fermions (with dimensions of the SU(5) representations indicated in parentheses) are

$$S(10): \quad \psi_L^{ab} \ \text{with} \ 1 \leq a, \ b \leq 4,$$

$$5\ \bar{A}(6): \quad \chi_{ab,j,L} \ \text{with} \ 1 \leq a, \ b \leq 4 \ \text{and} \ 1 \leq j \leq 5,$$

$$5\ \bar{F}(4): \quad \chi_{5b,j,L} \ \text{with} \ 1 \leq b \leq 4 \ \text{and} \ 1 \leq j \leq 5$$

$$3\ \bar{F}(4): \quad \chi_{6b,j,L} \ \text{with} \ 1 \leq b \leq 4 \ \text{and} \ 1 \leq j \leq 3$$

$$(6.18)$$

This theory also contains the massless SU(4)-singlet fermions ψ_L^{66} and $\chi_{65,j,L}$ with $1 \leq j \leq 3$.

In accordance with our theorem, we show explicitly that this SU(4) descendant theory is anomaly-free:

$$\mathrm{SU}(4)^3\ \mathcal{A} = \mathcal{A}(S) + 5\,\mathcal{A}(\bar{A}) + 8\,\mathcal{A}(\bar{F})$$

$$= 8 + 0 + 8(-1) = 0 \ . \qquad (6.19)$$

where we have used the fact that the ⊟ = $\bar{A}$ representation of SU(4) is self-conjugate.

The first two (reduced) coefficients of the beta function of this SU(4) descendant theory are $\bar{b}_1 = 0.5305$ and $\bar{b}_2 = 0.1224$, with the same sign, so at the maximal scheme-independent, two-loop level, the beta function has no IR zero. Hence, as the reference scale decreases below Λ_5, the SU(4) gauge coupling inherited from the SU(5) theory continues to increase, eventually exceeding the range where it is perturbatively calculable. The most attractive channel for the formation of a bilinear fermion condensate is

$$\bar{A} \times \bar{A} \to 1 \ , \qquad (6.20)$$

with $\Delta C_2 = 2C_2(A) = 5$. Clearly, this fermion condensation preserves the SU(4) gauge symmetry. The estimated minimal critical coupling for condensation in this channel is $\alpha_{cr} = 2\pi/15 = 0.42$. The associated condensates are of the form $\langle \epsilon^{abde}\chi_{ab,j,L}^T C\chi_{de,k,L}\rangle$, where the copy indices take on values in the interval $1 \leq j, k \leq 5$. Applying a vacuum alignment argument, we may take $j = k$, so that these condensates are

$$\langle \epsilon^{abde}\chi_{ab,j,L}^T C\chi_{de,j,L}\rangle \quad \text{for } 1 \leq j \leq 5 \qquad (6.21)$$

(where there is no sum on j). These condensates are equal and hence preserve an O(5) isospin symmetry. We denote the scale at which this condensation takes place as $\Lambda_{\bar{A}\bar{A}}$. Owing to this condensation, all of the $\chi_{ab,j,L}$ fields gain masses of order $\Lambda_{\bar{A}\bar{A}}$.

This leaves a descendant (anomaly-free) chiral gauge theory with massless SU(4)-nonsinglet fermion content $S + 8\bar{F}$, given by Eq. (6.18) with the $\bar{A}$ fields removed. This theory has been studied before [4, 10, 11, 30], and we can combine the known results with the new ingredients here for our analysis. The first two (reduced) coefficients

in the beta function are $\bar{b}_1 = 0.7958$ and $\bar{b}_2 = 0.2913$, with the same sign, so that this beta function has no IR zero. Hence, as the scale μ decreases below $\Lambda_{\bar{A}\bar{A}}$, the SU(4) gauge coupling continues to increase from its value at $\Lambda_{\bar{A}\bar{A}}$.

Since it is known that this $S + 8\bar{F}$ theory satisfies the 't Hooft anomaly matching conditions [4, 10, 11], one possibility is that it confines without any spontaneous chiral symmetry breaking, producing massless composite fermions and massive hadrons. An alternate type of IR behavior is fermion condensation in the most attractive channel, which is $S \times \bar{F} \to F$, breaking SU(4) to SU(3), followed by further fermion in the respective MAC $S \times \bar{F} \to F$ channels in the descendant SU(3) and SU(2) theories, finally breaking the gauge symmetry completely.

4. Discussion

It is of interest to contrast our present SU(N) $S\bar{A}$ theories with $N \geq 6$, and hence a most attractive channel of the form $S \times \bar{A} \to adj$, with the theories analyzed in Ref. [16]. One of the purposes of Ref. [16] was to investigate how a fermion condensate transforming as the adjoint representation of a simple SU(N) gauge theory would dynamically break the gauge symmetry, and to contrast this with the types of gauge symmetry breaking patterns that one obtains if one uses a fundamental Higgs field transforming according to the adjoint representation of the SU(N) group. The type of theory considered in [16] had a direct-product gauge group of the form

$$G_{UV} = G \otimes G_b , \qquad (6.22)$$

where G is a chiral gauge symmetry and G_b is a vectorial gauge symmetry. As constructed, the G_b gauge interaction becomes strong in the infrared and leads to a condensate involving a fermion field transforming as a nonsinglet under both G and G_b, and specifically as the adjoint representation of G, thereby breaking G to a subgroup, H. At the stage where this breaking occurs, the G gauge interaction is still weak. With $G = \mathrm{SU}(N)$ regarded as a hypothetical grand unification group, Ref. [16] addressed the question of what the pattern of induced dynamical breaking of a grand unified theory would be and how it would differ from the pattern obtained with a nonzero vacuum expectation value of a fundamental Higgs field in the adjoint representation. This exploration of possible dynamical symmetry breaking of a grand unified theory is reminiscent of, although different from, the old idea of dynamical breaking of electroweak gauge symmetry by means of a vectorial, strongly coupled, confining gauge theory which would produce bilinear fermion condensates involving fermion(s) that transform under both the electroweak gauge group and the strongly coupled gauge group [13, 31]. In the latter case, the breaking of the electroweak gauge symmetry G_{EW} is caused by a bilinear fermion condensate transforming as the fundamen-

tal, rather than adjoint, representation of weak $\mathrm{SU}(2)_L$ with weak hypercharge $Y = 1$.

The difference with respect to our present work is that here we study a chiral gauge theory with a single gauge group rather than a direct product, and the chiral gauge interaction may produce condensates that self-break the strongly coupled chiral gauge symmetry instead of having a weakly coupled chiral gauge symmetry broken by a condensate of fermions that are nonsinglets under both G and G_b. The common feature shared by the dynamical gauge symmetry breaking in studied in [16] and the $S\bar{A}$ theories with $N \geq 6$ here is that the bilinear fermion condensate transforms as an adjoint of the SU(N) gauge symmetry and breaks it at the highest stage according to the pattern (6.1). To see how this differs with the situation with a Higgs field Φ in the adjoint representation, we recall the Higgs potential (with a $\Phi \to -\Phi$ symmetry imposed for technical simplicity),

$$V = \frac{\mu^2}{2}\mathrm{Tr}(\Phi^2) + \frac{\lambda_1}{4}[\mathrm{Tr}(\Phi^2)]^2 + \frac{\lambda_2}{4}\mathrm{Tr}(\Phi^4) , \quad (6.23)$$

where μ^2, λ_1, and λ_2 are real for hermiticity. One chooses $\mu^2 < 0$ to produce the symmetry breaking. Assuming $N \geq 4$, for the comparison here, it follows that the two quartic terms in (6.23) are independent, and the requirement that V be bounded below implies that $\lambda_1 > 0$. This boundedness condition allows λ_2 to take on a restricted range of negative values depending on λ_1 and N, namely [16]

$$-\left(\frac{N(N-1)}{N^2 - 3N + 3}\right)\lambda_1 < \lambda_2 < 0 . \qquad (6.24)$$

With λ_2 in this interval, V is minimized with a Higgs VEV such that the SU(N) gauge symmetry is broken according to (6.1). However, if $\lambda_2 > 0$, then V is minimized for a Higgs VEV that yields a different symmetry breaking: if N is even, then the breaking pattern is

$$\mathrm{SU}(N) \to \mathrm{SU}(N/2) \otimes \mathrm{SU}(N/2) \otimes \mathrm{U}(1) , \qquad (6.25)$$

while if N is odd, then the breaking is

$$\mathrm{SU}(N) \to \mathrm{SU}((N+1)/2) \otimes \mathrm{SU}((N-1)/2) \otimes \mathrm{U}(1) . \ (6.26)$$

This comparison elucidates the difference between the breaking of a gauge symmetry by the VEV of a fundamental Higgs field and the dynamical symmetry breaking by a fermion condensate produced by a strongly coupled gauge interaction.

VII. INVESTIGATION OF $S_k \bar{A}_k$ CHIRAL GAUGE THEORIES WITH $k \geq 3$

It is natural to ask whether the type of asymptotically free (anomaly-free) chiral gauge theories that we have constructed and studied here with chiral fermions transforming according to the rank-2 symmetric and conjugate antisymmetric representations of SU(N) can be extended to corresponding chiral gauge theories with chiral

fermions in the rank-k symmetric and rank-ℓ conjugate antisymmetric representations of SU(N) with k, $\ell \geq 3$. We show here that this cannot be done for the diagonal case $k = \ell$ because such theories are not asymptotically free. Thus, our $S\bar{A}$ theories are the unique realization of asymptotically free S_k $\bar{A}_\ell$ chiral gauge theories with diagonal $k = \ell \geq 2$.

Let us then consider a chiral gauge theory with chiral fermions transforming according to the rank-k symmetric and conjugate antisymmetric representations of SU(N), denoted as the S_k and $\bar{A}_k$. We denote the number of these fermions as n_{S_k} and $n_{\bar{A}_k}$, respectively, and the theory itself as

$$(N; k; n_{S_k}, n_{\bar{A}_k}) \ . \tag{7.1}$$

The correspondence of this notation with the shorthand notation in the previous part of the text, which studied the $k = 2$ case, is

$$(N; 2; n_{S_2}, n_{\bar{A}_2}) \equiv (N; n_S, n_{\bar{A}}) \ . \tag{7.2}$$

The condition that the theory must be free of any triangle anomaly in gauged currents is

$$n_{S_k}\mathcal{A}(S_k) + n_{\bar{A}_k}\mathcal{A}(\bar{A}_k)$$
$$= n_{S_k}\mathcal{A}(S_k) - n_{\bar{A}_k}\mathcal{A}(A_k) = 0 \ , \tag{7.3}$$

where $\mathcal{A}(S_k)$ and $\mathcal{A}(A_k)$ are given in Eqs. (B13) and (B14) of Appendix B. A solution of this equation has the ratio of copies of fermions in the S_k and $\bar{A}_k$ representations given by

$$\frac{n_{\bar{A}_k}}{n_{S_k}} = \frac{\mathcal{A}(S_k)}{\mathcal{A}(A_k)} \equiv p_k$$
$$= \frac{(N + 2k)(N + k)!(N - k - 1)!}{(N - 2k)(N + 2)!(N - 3)!} \ . \tag{7.4}$$

In the case $k = 2$ discussed in detail above, $p_k = p$ given in Eq. (4.5). For $k \geq 3$, p_k can also be expressed as

$$p_k = \frac{(N + 2k)\left[\prod_{j=3}^{k}(N + j)\right]}{(N - 2k)\left[\prod_{j=3}^{k}(N - j)\right]} \quad \text{for } k \geq 3 \ . \tag{7.5}$$

For example,

$$p_3 = \frac{(N + 6)(N + 3)}{(N - 6)(N - 3)} \tag{7.6}$$

and

$$p_4 = \frac{(N + 8)(N + 3)(N + 4)}{(N - 8)(N - 3)(N - 4)} \ . \tag{7.7}$$

For a physical solution, this ratio (7.4) must be positive, which requires that

$$N \geq 2k + 1 \ , \tag{7.8}$$

and we restrict N to this range. From Eq. (7.4) it follows that if $k \geq 2$, then $n_{\bar{A}_k} > n_{S_k}$. Therefore the theories of this type with minimal chiral fermion content have the form

$$(N; k; n_S, n_{\bar{A}}) = (N; k; 1, p_k) \ , \tag{7.9}$$

with the understanding that p_k must be a (positive) integer. If $k = 3$, there are only two solutions of Eq. (7.4) with $n_S = 1$ that satisfy this condition that p_k be an integer, namely

$$(N; 3; 1, p_3) = \{(9; 3; 1, 10) \, , (12; 3; 1, 5)\} \ . \tag{7.10}$$

The number of solutions decreases as k increases. Thus, if $k = 4$, then there is only one such theory, viz.,

$$(N; 4; 1, p_4) = \{(10; 4; 1, 39)\} \ , \tag{7.11}$$

and similarly, if $k = 5$, there is only one solution,

$$(N; 5; 1, p_5) = \{(11; 5; 1, 210)\} \ , \tag{7.12}$$

while we have not found solutions with integer p_k for $k \geq 6$. Theories with n_{cp} copies of this minimal fermion content also satisfy the anomaly cancellation condition (7.3), e.g., if $k = 3$, then the theories $(9; 3; n_{cp}, 10n_{cp})$ and $(12; 3; n_{cp}, 5n_{cp})$ for $n_{cp} \geq 2$ also satisfy the anomaly cancellation condition.

To test whether any of these solutions yield theories that are asymptotically free, we begin by calculating the first coefficient of the beta function for cases with minimal fermion content, with $n_{cp} = 1$, which is

$$b_1 = \frac{1}{3}\left[11N - 2(T_{S_k} + p_k T_{\bar{A}_k})\right] \ . \tag{7.13}$$

If $k = 3$, this is

$$k = 3 \ \rightarrow \ b_1 = \frac{1}{3}\left[11N - \frac{(N + 3)(N^2 - 12)}{N - 6}\right]$$
$$= \frac{-N^3 + 8N^2 - 54N + 36}{3(N - 6)} \ . \tag{7.14}$$

Evaluating this for the two solutions (7.10), we obtain $\bar{b}_1 = -4.695$ for $(N; k; 1, p_3) = (9; 3; 1, 10)$ and $\bar{b}_1 = -5.252$ for $(N; k; 1, p_3) = (12; 3; 1, 5)$. These are both negative, so neither of these theories is asymptotically free.

In a similar manner, we find that the anomaly-free $S_k \bar{A}_k$ theories with higher k are also not asymptotically free. Substituting the value $k = 4$ into Eq. (7.13) yields

$$k = 4 \ \rightarrow \ b_1 = \frac{1}{3}\left[11N - \frac{(N + 3)(N + 4)^2(N - 4)}{3(N - 8)}\right]$$
$$= \frac{-N^4 - 7N^3 + 37N^2 - 152N + 192}{9(N - 8)} \ .$$
$$\tag{7.15}$$

Evaluating this for the solution $(N; 4; 1, p_4) = (10; 4; 1, 39)$, we obtain $\bar{b}_1 = -64.67$. Finally, for $k = 5$,

$$k = 5 \rightarrow$$

$$
\begin{aligned}
b_1 &= \frac{1}{3}\left[11N - \frac{(N+3)(N+4)(N+5)(N^2-20)}{12(N-10)} \right] \\
&= \frac{-N^5 - 12N^4 - 27N^3 + 312N^2 - 380N + 1200}{36(N-10)} .
\end{aligned}
$$
(7.16)

Evaluating this for the solution $(N; 5; 1, p_5) = (11; 5; 1, 210)$, we get $\bar{b}_1 = -746.94$. For each of these theories, letting n_{cp} be larger than 1 makes b_1 more negative, so the respective theories with $n_{cp} \geq 2$ are also not asymptotically free.

Thus, we find that there are no anomaly-free chiral gauge theories with fermions in the k-fold symmetric and conjugate antisymmetric representations of SU(N) for $k \geq 3$.

VIII. INVESTIGATION OF $S_k\bar{A}_\ell$ CHIRAL GAUGE THEORIES WITH $k \neq \ell$ AND $k, \ell \geq 2$

One can also consider generalizations of our $S\bar{A} = S_2\bar{A}_2$ chiral gauge theories to theories with chiral fermions transforming as the rank-k symmetric representation and the conjugate rank-ℓ antisymmetric representation of SU(N), where $k \neq \ell$ and $k, \ell \geq 2$. We consider theories of this type here. We denote the number of fermions transforming as the rank-k symmetric representation of SU(N) as n_{S_k} and the number of fermions transforming as the rank-ℓ conjugate antisymmetric representation as $n_{\bar{A}_\ell}$, respectively, and the theory itself as

$$(N; k; \ell; n_{S_k}, n_{\bar{A}_\ell}) .$$
(8.1)

Here the condition that there theory should have no anomaly in gauged currents reads

$$n_{S_k}\mathcal{A}(S_k) + n_{\bar{A}_\ell}\mathcal{A}(\bar{A}_\ell)$$

$$= n_{S_k}\mathcal{A}(S_k) - n_{\bar{A}_\ell}\mathcal{A}(A_\ell) = 0 .$$
(8.2)

This anomaly cancellation condition is satisfied if and only if

$$
\begin{aligned}
\frac{n_{\bar{A}_\ell}}{n_{S_k}} &= \frac{\mathcal{A}(S_k)}{\mathcal{A}(A_\ell)} \equiv p_{\bar{A}_\ell/S_k} \\
&= \frac{(N+k)!(N+2k)(N-\ell-1)!(\ell-1)!}{(N-3)!(N-2\ell)(N+2)!(k-1)!}
\end{aligned}
$$
(8.3)

Although $k \neq \ell$ here, we note that if one took $k = \ell$ as in the previous part of this paper, then the correspondence in notation with Eqs. (4.5) and (7.4) is $p_{\bar{A}_k/S_k} \equiv p_k$ and

$p_{\bar{A}_2/S_2} \equiv p_2 \equiv p$. For the ratio (8.3) to be a physical, positive number, it is necessary that

$$N \geq 2\ell + 1 ,$$
(8.4)

and we shall restrict N to this range. As explicit examples, we discuss the $S_3\bar{A}_2$ and $S_2\bar{A}_3$ theories.

IX. $S_3\bar{A}_2$ THEORY

Here, in accordance with (8.4), we restrict N to the range $N \geq 5$. For this theory, the ratio $n_{\bar{A}_2}/n_{S_3}$ is

$$\frac{n_{\bar{A}_2}}{n_{S_3}} \equiv p_{\bar{A}_2/S_3} = \frac{(N+3)(N+6)}{2(N-4)} .$$
(9.1)

Unlike p_k for the case of diagonal $S_k\bar{A}_k$ theories, this ratio $p_{\bar{A}_2/S_3}$ is not a monotonic function of N. It decreases from the value 44 at $N = 5$ to a formal minimum at the real value $N = 4 + \sqrt{70} = 12.367$, where it is equal to $(17 + 2\sqrt{70})/2 = 16.667$, and then increases without bound as N increases further. Since $p_{\bar{A}_2/S_3}$ is always larger than unity, it is natural to consider models of this type with n_{S_3} equal to its smallest value, namely $n_{S_3} = 1$. We have found many of these, but none of them is asymptotically free. As allowed by the non-monotonicity of $p_{\bar{A}_2/S_3}$, there are two values of N that yield a minimal value of $p_{\bar{A}_2/S_3}$, namely $N = 11$ and $N = 14$, both of which give $p_{\bar{A}_2/S_3} = 17$. To minimize the fermion content with this value of $p_{\bar{A}_2/S_3}$, we choose $n_{S_3} = 1$ and $n_{\bar{A}_2} = 17$. For $N = 11$, we have $\mathcal{A}(S_3) = 119$ and $\mathcal{A}(\bar{A}_2) = -7$, while for $N = 14$, we have $\mathcal{A}(S_3) = 170$ and $\mathcal{A}(\bar{A}_2) = -10$. To test whether any of these solutions of the anomaly cancellation condition yields an asymptotically free theory, we calculate the one-loop coefficient of the beta function. In general for this type of theory,

$$b_1 = \frac{1}{3}\left[11N - 2n_{S_3}\{T_{S_3} + p_{\bar{A}_2/S_3}T_{\bar{A}_2}\} \right] .$$
(9.2)

With the minimal choice $n_{S_3} = 1$, this is

$$
\begin{aligned}
b_1 &= \frac{1}{3}\left[11N - \frac{(N+3)(N^2+N-10)}{N-4} \right] \\
&= \frac{-N^3 + 7N^2 - 37N + 30}{3(N-4)} .
\end{aligned}
$$
(9.3)

For the case with $N = 11$, $\bar{b}_1 = -3.263$, while for $N = 14$, $\bar{b}_1 = -4.934$. These are both negative, i.e., these theories are not asymptotically free. Solutions of the anomaly conditions with larger values of n_{S_3} and $n_{\bar{A}_2}$ yield values of b_1 that are even more negative. Thus, we do not find any anomaly-free, asymptotically free theories of this $S_3\bar{A}_2$ type.

X. $S_2 \bar{A}_3$ THEORIES

A. General Analysis

Here we consider an SU(N) theory with n_{S_2} chiral fermions in the S_2 representation and $n_{\bar{A}_3}$ chiral fermions in the $\bar{A}_3$ representation. In accord with (8.4), we restrict N to the range $N \geq 7$. For this theory, the ratio $n_{\bar{A}_3}/n_{S_2}$ is

$$\frac{n_{\bar{A}_3}}{n_{S_2}} \equiv p_{\bar{A}_3/S_2} = \frac{2(N+4)}{(N-3)(N-6)} . \qquad (10.1)$$

This ratio decreases monotonically as a function of N from the value value $11/2$ at $N = 7$ and approaches zero as $N \to \infty$. The ratio (10.1) takes on an integer value for only one value of N, namely $N = 10$, where it is equal to 1. This reflects the equality $\mathcal{A}_{S_2} = 14 = \mathcal{A}_{A_3}$ for SU(10). A theory with $N = 10$ and $n_{cp} \geq 2$ copies of the S_2 and $\bar{A}_3$ representations is also anomaly-free.

For $N = 10$ and $n_{S_2} = n_{\bar{A}_3} = 1$, we calculate the reduced one-loop coefficient in the beta function to be $\bar{b}_1 = 1.8569$, so this theory satisfies the requirement of being asymptotically free. We compute the reduced two-loop coefficient to be $\bar{b}_2 = 0.086545$, so at the maximal scheme-independent level, i.e., the two-loop level, this theory has no IR zero in the beta function. Hence, as the reference scale μ decreases from the UV to the IR, the SU(10) gauge coupling continues to increase.

The condition of the cancellation of anomalies in gauged currents is also satisfied in a theory in which the chiral fermion content is replicated n_{cp} times. However, we find that only one of these nonminimal theories is asymptotically free, namely the one with $n_{cp} = 2$. For this theory with $N = 10$ and $n_{S_2} = n_{\bar{A}_3} = n_{cp} = 2$, we calculate $\bar{b}_1 = 0.795775$ and $\bar{b}_2 = -7.00383$. Thus, the two-loop beta function of this second theory has an IR zero at $\alpha_{IR,2\ell} = 0.1136$.

B. SU(10) Theory with $n_{S_2} = n_{\bar{A}_3} = 1$

1. Initial Breaking of SU(10) to SU(6)

We will focus here on the simplest SU(10) theory of this type, with $n_{cp} = 1$ and thus $n_{S_2} = n_{\bar{A}_3} = 1$. This theory has a classical global symmetry $G_{fl,cl} = \mathrm{U}(1)_{S_2} \otimes \mathrm{U}(1)_{\bar{A}_3}$. Both of these U(1) symmetries are broken by SU(10) instantons, but one can construct a linear combination U(1)$'$ that is invariant in the presence of these instantons. Since (in a notation analogous to Eq. (4.23)) $\vec{v} = (T_{S_2}, T_{\bar{A}_3}) = (6, 14)$, U(1)$'$ has the charge assignments

$$(Q_{S_2}, Q_{\bar{A}_3}) \propto (7, -3) . \qquad (10.2)$$

We have not found a set of gauge-singlet composite fermion operators satisfying the 't Hooft anomaly matching conditions for this U(1)$'$ symmetry. Therefore, we

infer that as the SU(10) gauge coupling increases sufficiently, fermion condensation will occur. We find that the most attractive channel is

$$MAC: \quad \bar{A}_3 \times \bar{A}_3 \to A_4 \qquad (10.3)$$

with

$$\Delta C_2 = 9.90 \quad \text{for } \bar{A}_3 \times \bar{A}_3 \to A_4 . \qquad (10.4)$$

We denote the S_2 and $\bar{A}_3$ fermion fields as ψ_L^{ab} and $\chi_{abcd,L}$. The condensate for the channel $\bar{A}_3 \times \bar{A}_3 \to A_4$ is of the form

$$\langle \epsilon^{7\,8\,9\,10\,\{a_1...a_6\}} \chi_{a_1 a_2 a_3, L}^T C \chi_{a_4 a_5 a_6, L} \rangle , \qquad (10.5)$$

where, by convention, we take the four uncontracted indices to be 7, 8, 9, and 10, and the summed indices to be $a_1, ..., a_6 \in \{1, ..., 6\}$. We denote the scale at which this condensate forms as Λ_{10}. This condensate breaks the SU(10) gauge symmetry to SU(6) and also breaks the global U(1)$'$ symmetry. The 64 gauge bosons in the coset SU(10)/SU(6) also gain masses of this order. In order to construct the low-energy effective SU(6) gauge theory that is operative at reference scales $\mu < \Lambda_{10}$, we first enumerate the chiral fermions that are involved in the condensate (10.5) and that consequently gain dynamical masses of order Λ_{10} and are integrated out to form this low-energy effective theory. The representation $\bar{A}_3$ has dimension $\binom{10}{3} = 120$ in SU(10). One can choose the three antisymmetrized group indices $a_1, a_2, a_3 \in \{1, ..., 6\}$ in the first fermion in (10.5) in any of $\binom{6}{3} = 20$ ways, and the remaining group indices a_4, a_5, a_6 in any of $\binom{3}{3} = 1$ ways, so of the initial 120 components in the $\bar{A}_3$ fermion, the 20 components with gauge indices in the set $\{1, ..., 6\}$ gain masses and are integrated out of the SU(6) theory.

We next must determine how the remaining massless fermions transform under SU(6). For this purpose, let us use group indices $a, b, .. \in \{1, ..., 6\}$ to refer to indices of the residual SU(6) gauge symmetry and $\alpha, \beta, .. \in \{7, 8, 9, 10\}$ to refer to the indices along the broken directions of SU(10). The remaining 100 massless components of the $\bar{A}_3$ fermion can be classified and enumerated as follows. First, there are the $\binom{4}{3} = 4$ components $\chi_{\alpha\beta\gamma,L}$ for which $\alpha, \beta, \gamma \in \{7, 8, 9, 10\}$, which are singlets under SU(6). Second, there are the $6 \times \binom{4}{2} = 36$ components $\chi_{a\alpha\beta,L}$ with $1 \leq a \leq 6$ and $7 \leq \alpha, \beta \leq 10$, which form six $\bar{F}$s of SU(6). Third, there are $\binom{6}{2} \times 4 = 60$ components $\chi_{ab\alpha,L}$ with $1 \leq a, b \leq 6$ and $7 \leq \alpha \leq 10$, which comprise four copies of $\bar{A}_2$ in SU(6). For the symmetric rank-2 tensor representation, we have

$$(S_2)_{\mathrm{SU(10)}} = (S_2)_{\mathrm{SU(6)}} + 4\,F_{\mathrm{SU(6)}} + 10\,(1)_{\mathrm{SU(6)}} . \quad (10.6)$$

where $(1)_{\mathrm{SU(6)}}$ is the singlet. Recall that $\dim(S_k) = (1/k!) \prod_{j=0}^{k-1}(N+j)$. Thus, the 55-dimensional $(S_2)_{\mathrm{SU(10)}}$ representation of SU(10) decomposes into the sum of the the $(S_2)_{\mathrm{SU(6)}}$ representation of SU(6) with its 21 component fields ψ_L^{ab} with $1 \leq a, b \leq 6$, plus four copies of

the fundamental representation of SU(6) with fields $\psi_L^{a\alpha}$, $1 \le a \le 6$ and $7 \le \alpha \le 10$, and ten SU(6)-singlet fields $\psi_L^{\alpha\beta}$ with $7 \le \alpha, \beta \le 10$. We summarize the massless SU(6)-nonsinglet chiral fermion content of the low-energy SU(6) theory:

$$\text{SU}(6): \quad \text{fermions}: \ S_2 + 4\bar{A}_2 + 4\,F + 6\,\bar{F} \ . \quad (10.7)$$

The explicit fermion fields (with dimensionalities in parentheses) are

$$S_2(21): \quad \psi_L^{ab} \text{ with } 1 \le a, \ b \le 6,$$

$$4\,\bar{A}_2(15): \quad \chi_{ab\alpha,L} \text{ with } 1 \le a, \ b \le 6 \text{ and } 7 \le \alpha \le 10,$$

$$4\,F(6): \quad \psi^{a\alpha,L} \text{ with } 1 \le a \le 6 \text{ and } 7 \le \alpha \le 10,$$

$$6\,\bar{F}(6): \quad \chi_{a\alpha\beta,L} \text{ with } 1 \le a \le 6 \text{ and } 7 \le \alpha, \ \beta \le 10 \ .$$

$$(10.8)$$

As guaranteed by our theorem above, this low-energy effective SU(6) theory is anomaly-free; the contributions to the anomaly are

$$\begin{aligned} \mathcal{A} &= \mathcal{A}(S_2) - 4\mathcal{A}(A_2) + 4\mathcal{A}(F) + 6\mathcal{A}(\bar{F}) \\ &= 10 - (4 \times 2) + 4 - 6 = 0 \ . \quad (10.9) \end{aligned}$$

2. Breaking of SU(6) to SU(5)

We calculate the reduced one-loop and two-loop coefficients of the beta function of this SU(6) theory (10.7) to be $\bar{b}_1 = 0.84883$ and $\bar{b}_2 = -0.56465$, so the two-loop beta function has an IR zero at $\alpha_{IR,2\ell} = 1.503$. The most attractive channel for fermion condensation is $S_2 \times \bar{F} \to F$, with $\Delta C_2 = 20/3$ and the resultant estimate $\alpha_{cr} \simeq 0.31$. (The next-most attractive channel is $F \times \bar{F} \to 1$ with $\Delta C_2 = 35/6$.) The ratio $\rho = \alpha_{IR,2\ell}/\alpha_{cr} = 4.8$, which is considerably larger than unity. We will explore evolution toward the infrared that involves further fermion condensation, breaking the SU(6) gauge symmetry to SU(5). We denote the scale at which such condensation occurs as Λ'_6 (where the prime is included to avoid confusion with the scale Λ_6 introduced in our discussion above of the SU(6) $S_2\bar{A}_2$ theory). By convention, we label the breaking direction as $a = 6$ and the α, β indices of the $\bar{F}$ fermion as $\alpha = 9$, $\beta = 10$. The associated $S_2\bar{F}$ fermion condensate is then

$$\langle \sum_{b=1}^{6} \psi_L^{6b\,T} C \chi_{b\alpha\beta,L} \rangle \quad \text{with } (\alpha,\beta) = (9,10) \ . \quad (10.10)$$

The fermions involved in this condensate gain dynamical masses of order Λ_6, as do the 11 gauge bosons in the coset SU(6)/SU(5).

3. Breaking of SU(5) to SU(4)

To analyze the subsequent evolution into the infrared, we enumerate the massless SU(5)-nonsinglet chiral fermion content of the resultant low-energy effective SU(5) theory. By the same methods as before, we find that this content is

$$\text{SU}(5): \text{fermions}: \quad S_2 + 4\bar{A}_2 + 4F + 9\bar{F} \ . \quad (10.11)$$

The explicit fermion fields (with dimensionalities in parentheses) are listed below. For this purpose, we relabel the group indices such that $a, b \in \{1, ..., 5\}$ are SU(5) indices and $\alpha, \beta \in \{6, ..., 10\}$. We have

$$S_2(15): \quad \psi_L^{ab} \text{ with } 1 \le a, \ b \le 5,$$

$$4\,\bar{A}_2(10): \quad \chi_{ab\alpha,L} \text{ with } 1 \le a, \ b \le 5 \text{ and } 7 \le \alpha \le 10,$$

$$4\,F(5): \quad \psi_L^{a\alpha} \text{ with } 1 \le a \le 5 \text{ and } 7 \le \alpha \le 10,$$

$$9\,\bar{F}(5): \quad \chi_{a\alpha\beta,L} \text{ with } 1 \le a \le 5$$

$$\text{and } 6 \le \alpha, \ \beta \le 10 \text{ except } (\alpha,\beta) = (9,10)$$

$$(10.12)$$

We calculate the reduced one-loop and two-loop coefficients of the beta function of this SU(5) theory to be $\bar{b}_1 = 0.61009$ and $\bar{b}_2 = -0.61384$, so the two-loop beta function has an IR zero at $\alpha_{IR,2\ell} = 0.994$. The most attractive channel for fermion condensation is $\bar{A}_2 \times \bar{A}_2 \to F$, with $\Delta C_2 = 24/5$ and the resultant estimate $\alpha_{cr} \simeq 0.44$. The resultant ratio $\rho = \alpha_{IR,2\ell}/\alpha_{cr} = 2.3$, suggesting that this condensation could plausibly occur. With condensation in the $\bar{A}_2 \times \bar{A}_2 \to F$ channel, and with the breaking direction taken to be $a = 5$, the condensates are

$$\langle \epsilon^{abcd5} \chi_{ab\alpha,L}^T C \chi_{cd\beta,L} \rangle \propto \Big[\langle \chi_{12\alpha,L}^T C \chi_{34\beta,L} \rangle$$
$$- \langle \chi_{13\alpha,L}^T C \chi_{24\beta,L} \rangle + \langle \chi_{14\alpha,L}^T C \chi_{23\beta,L} \rangle \Big] \ , \quad (10.13)$$

where $6 \le \alpha, \ \beta \le 10$ as specified above. We denote the scale at which these condensates form as Λ'_5. The fermions involved in these condensates, as well as the nine gauge bosons in the coset SU(5)/SU(4), gain masses of order Λ'_5.

4. IR Evolution of the Descendant SU(4) Theory

We determine the massless SU(4)-nonsinglet chiral fermion content of the resultant SU(4) descendant theory to be

$$\begin{aligned} \text{SU}(4): \quad \text{fermions}: \ & S_2 + 5F + 13\bar{F} \\ &= S_2 + 8\bar{F} + 5\{F + \bar{F}\} \ . \quad (10.14) \end{aligned}$$

We see that this is precisely the $N = 4$, $p = 5$ special case of the Sp model of Eq. (5.12) studied in [4, 10, 11]. For this theory we calculate the beta function coefficients $\bar{b}_1 = 0.5303$ and $\bar{b}_2 = -0.2496$, so the two-loop beta function has an IR zero at $\alpha_{IR,2\ell} = 2.125$. The fermion content of this theory satisfies the 't Hooft anomaly matching conditions [4, 10, 11], so one possibility is that as the gauge interaction becomes strong, the theory confines and produces massless composite SU(4)-singlet spin 1/2 fermions. Another possibility is that the gauge interaction produces fermion condensation. The most attractive channel is $S_2 \times \bar{F} \to F$ with $\Delta C_2 = 9/2$, so the rough estimate of α_{cr} is $\alpha_{cr} \simeq 0.42$. The resultant ratio $\rho = 5.1$ is well above unity, which renders it likely that either the gauge interaction confines and produces the above-mentioned massless composite fermions or it produces fermion condensation in this $S_2 \times \bar{F} \to F$ channel. These possibilities and the further evolution into the IR were discussed in detail in [11].

XI. CONCLUSIONS

In summary, in this paper we have constructed and studied asymptotically free chiral gauge theories with an SU(N) gauge group and n_{S_k} copies of massless chiral fermions transforming according to the symmetric rank-k representation and $n_{\bar{A}_\ell}$ copies of fermions transforming according to the conjugate antisymmetric rank-ℓ representation of this group, with k, $\ell \geq 2$. As part of our work, we have proved a general theorem guaranteeing that a low-energy effective theory resulting from the dynamical breaking of an anomaly-free chiral gauge theory is also anomaly-free. We have explored the restrictions due to the constraints of asymptotic freedom and anomaly cancellation and have shown that for a given N, k, and ℓ, these lead to, at most, a finite set of theories satisfying these restrictions. For the case $k = \ell = 2$, i.e., $S_2\bar{A}_2$ chiral gauge theories, we have given a detailed analysis of the UV to IR evolution of some simple theories, including an SU(5) theory with $n_{S_2} = 1$ and $n_{\bar{A}} = 9$ and an SU(6) model with $n_{S_2} = 1$ and $n_{\bar{A}} = 5$. We have shown that $S_2\bar{A}_2$ theories exhibit a considerable variety of types of UV to IR evolution, ranging from an infrared non-Abelian Coulomb phase to sequential chiral symmetry breaking of both gauge and global chiral symmetry groups and possible confinement with massless gauge-singlet composite fermions. We have also shown that there are no asymptotically free SU(N) $S_k\bar{A}_k$ chiral gauge theories with $k \geq 3$. Finally, we have also studied chiral gauge theories with chiral fermions in S_k and $\bar{A}_\ell$ representations of SU(N) with $k \neq \ell$ and k, $\ell \geq 2$. We believe that the results obtained here give useful new insights concerning the properties of chiral gauge theories.

Appendix A: Beta Function Coefficients and Relevant Group Invariants

For reference, we list the one-loop and two-loop coefficients [19, 20] in the beta function (2.1) for a non-Abelian chiral gauge theory with gauge group G and a set of chiral fermions comprised of N_i fermions transforming according to the representations R_i:

$$b_1 = \frac{1}{3}\left[11C_2(G) - 2\sum_{R_i} N_i T(R_i)\right] \qquad (A1)$$

and

$$b_2 = \frac{1}{3}\left[34C_2(G)^2 - 2\sum_{R_i} N_i\{5C_2(G) + 3C_2(R_i)\}T(R_i)\right] . \qquad (A2)$$

Appendix B: Relevant Group Invariants

We list below the group invariants that we use for the relevant case $G = $ SU(N). The symmetric and antisymmetric rank-k representations of SU(N) are denoted S_k and $A_k \equiv [k]_N$. In terms of Young tableaux, $S_1 = A_1 = \square$, $S_2 = \square\square$, $A_2 = \begin{array}{c}\square\\\square\end{array}$, etc. (In the text, where no confusion would result, we denote $S_2 \equiv S$ and $A_2 \equiv A$.) For a representation R, the Casimir invariants $C_2(R)$ and $T(R)$ are defined as

$$\sum_{i,j=1}^{\dim(R)} \mathcal{D}_R(T_a)_{ij}\mathcal{D}_R(T_b)_{ji} = T(R)\delta_{ab} \qquad (B1)$$

and

$$\sum_{a=1}^{o(G)}\sum_{j=1}^{\dim(R)} \mathcal{D}_R(T_a)_{ij}\mathcal{D}_R(T_a)_{jk} = C_2(R)\delta_{ik} , \qquad (B2)$$

where T_a are the generators of G, and $\mathcal{D}_R$ is the matrix representation (*Darstellung*) of R. These satisfy

$$T(R)\,o(G) = C_2(R)\dim(R) , \qquad (B3)$$

where $o(G) = N^2 - 1$ for SU(N) and $\dim(R)$ is the dimension of the representation R.

For the adjoint representation, $C_2(adj) \equiv C_2(G) = T(Adj) = N$. For the rank-$k$ symmetric and antisymmetric representations S_k and A_k,

$$T(S_k) = \frac{\prod_{j=2}^{k}(N+j)}{2(k-1)!} \qquad (B4)$$

$$T(A_k) = \frac{1}{2}\binom{N-2}{k-1} = \frac{\prod_{j=2}^{k}(N-j)}{2(k-1)!} \qquad \text{(B5)}$$

$$C_2(S_k) = \frac{k(N+k)(N-1)}{2N} \qquad \text{(B6)}$$

and

$$C_2(A_k) = \frac{k(N-k)(N+1)}{2N} \ . \qquad \text{(B7)}$$

Hence, in particular, with T_2 standing for the rank-2 tensor representation S_2 (+ sign) or A_2 (− sign) here and below, one has

$$T(T_2) = \frac{N \pm 2}{2} \qquad \text{(B8)}$$

$$C_2(T_2) = \frac{(N \pm 2)(N \mp 1)}{N} \qquad \text{(B9)}$$

$$T(T_3) = \frac{(N \pm 2)(N \pm 3)}{4} \qquad \text{(B10)}$$

and

$$C(T_3) = \frac{3(N \pm 3)(N \mp 1)}{2N} \ . \qquad \text{(B11)}$$

The anomaly produced by chiral fermions transforming according to the representation R of a group G is defined as

$$\text{Tr}_R(T_a, \{T_b, T_c\}) = \mathcal{A}(R)d_{abc} \qquad \text{(B12)}$$

where the d_{abc} are the totally symmetric structure constants of the corresponding Lie algebra. Thus, $\mathcal{A}(\square) = 1$ for SU(N). For S_k and A_k [32]

$$\mathcal{A}(S_k) = \frac{(N+k)!(N+2k)}{(N+2)!(k-1)!} \qquad \text{(B13)}$$

and

$$\mathcal{A}(A_k) = \frac{(N-3)!(N-2k)}{(N-k-1)!(k-1)!} \ . \qquad \text{(B14)}$$

Hence, in particular,

$$\mathcal{A}(T_2) = N \pm 4 \qquad \text{(B15)}$$

and

$$\mathcal{A}(T_3) = \frac{(N \pm 3)(N \pm 6)}{2} \ . \qquad \text{(B16)}$$

[1] G. 't Hooft, in *Recent Developments in Gauge Theories, 1979 Cargèse Summer Institute* (Plenum, New York, 1980), p. 135.
[2] S. Dimopoulos, S. Raby, and L. Susskind, Nucl. Phys. B **173**, 208 (1980).
[3] S. Weinberg and E. Witten, Phys. Lett. B **96**, 59 (1980).
[4] I. Bars and S. Yankielowicz, Phys. Lett. B **101**, 159 (1981).
[5] I. Bars, Nucl. Phys. B **208**, 77 (1982).
[6] J. L. Goity, R. D. Peccei, and D. Zeppenfeld, Nucl. Phys. B **262**, 95 (1985).
[7] H. Georgi, Nucl. Phys. B **266**, 274 (1986).
[8] E. Eichten, R. D. Peccei, J. Preskill, and D. Zeppenfeld, Nucl. Phys. B **268**, 161 (1986).
[9] T. Appelquist, A. Cohen, M. Schmaltz, and R. Shrock, Phys. Lett. B **459**, 235 (1999).
[10] T. Appelquist, Z. Duan, and F. Sannino, Phys. Rev. D **61**, 125009 (2000).
[11] T. Appelquist and R. Shrock, Phys. Rev. D **88**, 105012 (2013).
[12] Y. Shi and R. Shrock, Phys. Rev. D **91**, 045004 (2015) [arXiv:1411.2042].
[13] S. Weinberg, Phys. Rev. D **13**, 974 (1976).
[14] S. Dimopoulos and L. Susskind, Nucl. Phys. **B155**, 237 (1979); E. Eichten and K. Lane, Phys. Lett. **B90**, 125 (1980).
[15] S. Raby, S. Dimopoulos, and L. Susskind, Nucl. Phys. B **169**, 373 (1980).
[16] N. Chen, T. A. Ryttov, and R. Shrock, Phys. Rev. D **82**, 116006 (2010).
[17] See, e.g., T. Appelquist and J. Terning, Phys. Rev. D **50**, 2116 (1994); T. Appelquist and R. Shrock, Phys. Lett. B **548**, 204 (2002); Phys. Rev. Lett. **90**, 201801 (2003); T. Appelquist, M. Piai, and R. Shrock, Phys. Rev. D **69** (2004); N. C. Christensen and R. Shrock, Phys. Rev. Lett. **94**, 241801 (2005); Phys. Rev. D **72**, 035013 (2005); T. A. Ryttov and R. Shrock, Phys. Rev. D **81**, 115013 (2010).
[18] Clearly, our study of $S_k \bar{A}_\ell$ chiral gauge theories is equivalent to the study of the set with fermions in the conjugate representations $\bar{S}_k$ and A_ℓ.
[19] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. **30**, 1346 (1973); G. 't Hooft, unpublished.
[20] W. E. Caswell, Phys. Rev. Lett. **33**, 244 (1974); D. R. T. Jones, Nucl. Phys. B **75**, 531 (1974).
[21] T. Banks and A. Zaks, Nucl. Phys. B **196**, 189 (1982).
[22] K. Yamawaki, M. Bando, and K. Matumoto, Phys. Rev. Lett. **56**, 1335 (1986); T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. **57**, 957 (1986); V. A. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific, Singapore, 1993).
[23] T. Appelquist, A. Cohen, and M. Schmaltz, Phys. Rev. D **60**, 045003 (1999).
[24] In the extended Standard Model that includes SM-singlet right-handed neutrinos, there can be bare Majorana mass

terms for these.

[25] E. D'Hoker and E. Farhi, Nucl. Phys. B **248**, 59, 77 (1984).

[26] I-H. Lee, R. E. Shrock, and J. Shigemitsu, Nucl. Phys. B **330** 225, (1990); Nucl. Phys. B **334**, 265 (1990).

[27] O. Antipin, S. Di Chiara, M. Mojaza, E. Mølgaard, and F. Sannino, Phys. Rev. D **86**, 085009 (2012); E. Mølgaard and R. Shrock, Phys. Rev. D **89**, 105007 (2014).

[28] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1978).

[29] R. Slansky, Phys. Repts. **79**, 1 (1981).

[30] M. Kurachi, R. Shrock, and K. Yamawaki, Phys. Rev. D **91**, 055032 (2015).

[31] S. Weinberg, Phys. Rev. D **19**, 1277 (1979); L. Susskind, Phys. Rev. D **20**, 2619 (1979).

[32] J. Banks and H. Georgi, Phys. Rev. D **14**, 1159 (1976); S. Okubo, Phys. Rev. D **16**, 3528 (1977).

TABLE I: Properties of SU(N) $S\bar{A}$ chiral gauge theories with (i) minimal fermion content $n_S = 1$ and $n_{\bar{A}} = p = (N+4)/(N-4)$ and (ii) n_{cp}-fold replicated fermion content $n_s = n_{cp}$ and $n_{\bar{A}} = n_{cp}p$. The quantities listed are $(N; n_S, n_{\bar{A}})$, p, n_{cp}, $\bar{b}_1$, $\bar{b}_2$, and, for negative $\bar{b}_2$, $\alpha_{IR,2\ell} = -\bar{b}_1/\bar{b}_2$, α_{cr} for the relevant first condensation channel, and the ratio ρ given by Eq. (2.4). The dash notation $-$ means that the two-loop beta function has no IR zero. The likely IR behavior is indicated in the last column, where SχSB indicates spontaneously broken chiral symmetry, χS indicates a chirally symmetric behavior, and ESR stands for "either symmetry realization", χS or SχSB. See text for further discussion of descendant theories.

$(N; n_S, n_{\bar{A}})$	p	n_{cp}	$\bar{b}_1$	$\bar{b}_2$	$\alpha_{IR,2\ell}$	$\alpha_{cr,ch}$	ρ_{IR}	comment
(5;1,9)	9	1	0.5570	-0.8638	0.645	0.44	1.5	ESR
(6;1,5)	5	1	1.008	-0.1182	8.53	0.39	22	SχSB
(8;1,3)	3	1	1.592	0.9056	$-$	0.28	$-$	SχSB
(12;1,2)	2	1	2.600	3.519	$-$	0.18	$-$	SχSB
(6;2,10)	5	2	0.2653	-2.820	0.0941	0.39	0.24	χS
(8;2,6)	3	2	0.8488	-2.782	0.3051	0.28	1.1	ESR
(12;2,4)	2	2	1.698	-3.297	0.5149	0.18	2.9	SχSB
(8;3,9)	3	3	0.1061	-6.470	0.0164	0.28	0.059	χS
(12;3,6)	2	3	0.7958	-10.113	0.07869	0.18	0.44	ESR

TABLE II: Properties of SU(N) $S\bar{A}$ chiral gauge theories with other $(N; n_S, n_{\bar{A}})$ than those in Table I. The quantities listed are $(N; n_S, n_{\bar{A}})$, $\bar{b}_1$, $\bar{b}_2$, and, for negative $\bar{b}_2$, $\alpha_{IR,2\ell} = -\bar{b}_1/\bar{b}_2$, α_{cr} for the relevant first condensation channel, and the ratio ρ given by Eq. (2.4). The dash notation $-$ means that the two-loop beta function has no IR zero. The likely IR behavior is indicated in the last column, where SχSB indicates spontaneously broken chiral symmetry, χS indicates a chirally symmetric behavior, and ESR stands for "either symmetry realization", χS or SχSB. See text for further discussion of descendant theories.

$(N; n_S, n_{\bar{A}})$	$\bar{b}_1$	$\bar{b}_2$	$\alpha_{IR,2\ell}$	$\alpha_{cr,ch}$	ρ_{IR}	comment
(10;3,7)	0.4775	-8.116	0.0588	0.22	0.27	χS
(16;3,5)	1.379	-14.931	0.0924	0.13	0.69	ESR
(20;2,3)	3.236	-4.265	0.759	0.11	7.2	SχSB
(20;4,6)	0.6366	-37.238	0.0171	0.11	0.16	χS
(28;3,4)	3.024	-35.286	0.0857	0.075	1.1	SχSB
(36;4,5)	1.963	-102.512	0.0191	0.058	0.33	χS
(44;5,6)	0.05305	-218.913	0.242e-3	0.048	0.0051	χS