Merging compact binaries in hierarchical triple systems: Resonant excitation of binary eccentricity

Bin Liu, Dong Lai, and Ye-Fei Yuan

Merging Compact Binaries in Hierarchical Triple Systems: Resonant Excitation of Binary Eccentricity

Bin Liu[1,2,3], Dong Lai[3], and Ye-Fei Yuan[1,2]

[1] *Department of Astronomy, University of Science and Technology of China, Hefei, Anhui 230026, China*
[2] *CAS Key Laboratory for Research in Galaxies and Cosmology, Hefei, Anhui 230026, China and*
[3] *Cornell Center for Astrophysics and Planetary Science, Cornell University, Ithaca, NY 14853, USA*

We study the secular dynamics of compact binaries (consisting of white dwarfs, neutron stars or black holes) with tertiary companions in hierarchical triple systems. As the inner binary (with initially negligible eccentricity) undergoes orbital decay due to gravitational radiation, its eccentricity can be excited by gravitational forcing from the tertiary. This excitation occurs when the triple system passes through an "apsidal precession resonance", when the precession rate of the inner binary, driven by the gravitational perturbation of the external companion and General Relativity, matches the precession rate of the outer binary. The eccentricity excitation requires the outer companion to be on an eccentric orbit, with the mutual inclination between the inner and outer orbits less than $\sim 40°$. Gravitational wave (GW) signals from the inner binary can be significantly modified as the system evolves through the apsidal precession resonance. For some system parameters (e.g., a WD binary with a brown dwarf tertiary), the resonance can happen when the binary emits GWs in the $10^{-4} - 10^{-1}$ Hz range (the sensitivity band of LISA).

I. INTRODUCTION

Merging compact binaries containing white dwarfs (WDs), neutron star (NSs) and black holes (BHs) are of great importance and current interest in astronomy. First, depending on their total mass, WD-WD binaries may evolve into Type Ia supernova, NS formation through accretion-induced collapse, AM CVn binaries or R CrB stars (e.g., [1], [2]); merging NS-NS or NS-BH binaries have been conceived as the progenitors of short gamma-ray bursts (GRBs) (e.g., [3], [4], [5]). Second, merging compact binaries are promising sources of gravitational waves (GWs), detectable by the Laser Interferometer Space Antenna (LISA) (see [6], [7]) when the the corresponding GW frequencies are in the 10^{-4}-10^{-1} Hz band, or by the ground-based interferometers (LIGO/VIRGO) in the next decade (see [8]) when the GW frequencies lie in the 10-1000 Hz band. For detecting these GWs, accurate gravitational waveforms during the binary inspiral and merger phases are needed, and the latter requires 3D numerical simulations in full General Relativity (e.g., [9], [10], [11], [12]).

In this paper, we study the evolution of merging compact binaries influenced by an external third body. Three-body systems are ubiquitous in astrophysics, appearing in a wide range of configurations and scales. A significant fraction of stars are in triples [13]. It is possible that a tertiary companion orbits around the compact binary as part of the original stellar triple. A low-mass companion may also form in the envelope around the compact binary (during the common envelope phase of binary evolution). In such systems, long-term secular interaction between the inner and outer orbits may induce variations of the eccentricity and inclination of the inner binary. In particular, when the mutual inclination of two orbits lies between 40° and 140°, the eccentricity

and inclination may undergo oscillations – the so-called Lidov-Kozai oscillations (e.g., [14], [15]). The effects of Lidov-Kozai oscillations in the formation and evolution of various astrophysical systems have been extensively studied in recent years (e.g., [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30]). It has also been recognized that high-order (octupole) interactions between the inner and outer orbits can lead to rich dynamical behaviors in hierarchical triples (e.g., [31], [32], [33], [34], [35]).

In this paper we focus on systems with small mutual inclinations so that Lidov-Kozai oscillations do not occur. Several recent studies (e.g. [33], [36] and [37]) showed that when post-Newtonian effect of General Relativity (GR) is taken into account in the secular dynamics, significant eccentricity excitation can still occur under some conditions. In particular, periapse (apsidal) precession of the inner orbit due to GR plays an important role. However, these studies did not identify the physical origin and the condition of eccentricity excitation.

Here we show that eccentricity excitation arises from a resonance between the precessions of the inner and outer binaries due to the combined effects of mutual gravitational interaction and GR. In essence, the inner binary "gets" its eccentricity from the outer binary through this "Apsidal Precession Resonance". We also study how the eccentricity of the inner binary evolves as the system crosses the resonance driven by GW-induced orbital decay of the inner binary.

Our paper is organized as follows. In Section 2, we present the secular evolution equations for coplanar triple systems. In Section 3, a linear eigenmode analysis for the eccentricities of the inner and outer binaries is given; the result is used to identify the "apsidal precession resonance" and to calculate the maximum eccentricity that can be achieved in the resonance. In Section 4, we ex-

plore the parameter space for resonances in the various systems. In Section 5, we describe our numerical integrations for the evolution of the triple when the inner binary undergoes orbital decay due to GW emission. In Section 6, we extend our calculations to systems with arbitrary mutual inclinations. Our results and conclusions are summarized in Section 7.

II. SECULAR DYNAMICS OF STELLAR TRIPLES: EQUATIONS

A. Disturbing function

We consider the evolution of a hierarchical triple system, consisting of an inner binary with the masses of m_1, m_2 and a relatively distant third body of mass m_3. We introduce the orbital semi-major axes as $a_{1,2}$ and eccentricities as $e_{1,2}$, the subscript "1, 2" denote the inner and outer binaries, respectively. Each orbit is described by three angles: the inclination i, the longitude of the periapse ω and the longitude of the ascending node Ω, which are defined in a coordinate where z-axis is aligned with the total angular momentum. Thus, the relative inclination between the two orbits is $i_{\text{tot}} \equiv i_1 + i_2$, as shown in Fig. 1.

Based on the calculation in [33], [34] and [35], the interaction potential, averaged over both the inner and outer orbits and expanded up to the octupole order, is given by:

$$\langle\langle\Phi\rangle\rangle = -\frac{\mu_1\Phi_0}{8}\Big[2 + 3e_1^2 - (3 + 12e_1^2 - 15e_1^2\cos^2\omega_1)\sin^2 i_{\text{tot}}\Big]$$
$$+ \frac{15\mu_1\Phi_0\varepsilon_{\text{Oct}}}{64}\Bigg\{5e_1(1 - e_1^2)\sin i_{\text{tot}}\sin 2i_{\text{tot}}\sin\omega_1$$
$$\times \sin\omega_2 + \frac{e_1}{4}\Big[6 - 13e_1^2 + 5(2 + 5e_1^2)\cos 2i_{\text{tot}}$$
$$+ 70e_1^2\cos 2\omega_1\sin^2 i_{\text{tot}}\Big]$$
$$\times \Big(-\cos\omega_1\cos\omega_2 - \cos i_{\text{tot}}\sin\omega_1\sin\omega_2\Big)\Bigg\},$$
$$\tag{1}$$

where

$$\mu_1 = \frac{m_1 m_2}{m_1 + m_2},$$
$$\mu_2 = \frac{(m_1 + m_2)m_3}{m_1 + m_2 + m_3}$$
$$\tag{2}$$

are the reduced masses of the inner and outer orbits, respectively. In Eq.(1), the coefficients are defined as:

$$\Phi_0 \equiv \frac{Gm_3 a_1^2}{a_2^3(1 - e_2^2)^{3/2}},\tag{3}$$

where G is the gravitational constant and

$$\varepsilon_{\text{Oct}} \equiv \frac{m_1 - m_2}{m_1 + m_2}\frac{a_1}{a_2}\frac{e_2}{1 - e_2^2}\tag{4}$$
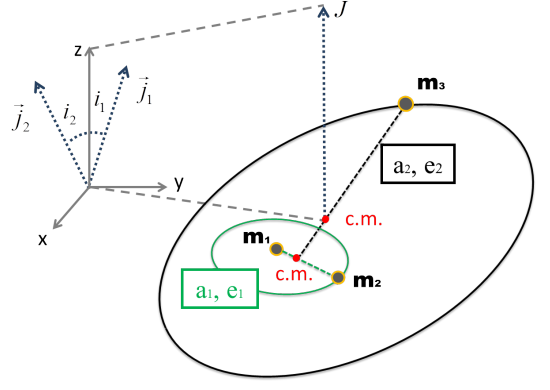

FIG. 1. Coordinate system used to describe the hierarchical triple system, where z-axis is aligned with the total angular momentum $\mathbf{J}$. Here, $\mathbf{j}_1$ and $\mathbf{j}_2$ denote the angular momenta of the inner and outer binaries, respectively, while "c.m." indicates the center of mass of each system.

quantifies the relative significance of the octupole term.

B. Equations of motion with GR effects

Various short-range effects, such as General Relativity (GR), tidal and rotational distortions, can affect the secular dynamics or the stellar triples (e.g., [17], [35]). Since the tidal and rotational effects on the compact objects are small compared to the GR effect, here we only consider the leading order GR effect on the inner binary in the absence of energy dissipation due to gravitational waves (see Section III). Thus, the total potential $\langle\langle\Phi_{\text{tot}}\rangle\rangle = \langle\langle\Phi\rangle\rangle + \langle\langle\Phi_{\text{GR}}\rangle\rangle$ is conserved, where

$$\langle\langle\Phi_{\text{GR}}\rangle\rangle = -\frac{3G^2 m_1 m_2(m_1 + m_2)}{a_1^2 c^2}\frac{1}{(1 - e_1^2)^{1/2}},\tag{5}$$

is the post-Newtonian potential associated with the GR-induced pericenter advance (e.g., [16]).

Our numerical calculations (see Section VI) will consider general inclinations between the inner and outer binaries. Here we focus on coplanar configurations, with $i_{\text{tot}} = 0$. We define

$$\varpi_1 \equiv \omega_1 + \Omega_1, \qquad \varpi_2 \equiv \omega_2 + \Omega_2.\tag{6}$$

Substituting Eq.(6) into Eq.(1) and Eq.(5), the total potential can only be expressed in terms of $e_{1,2}$ and $\varpi_{1,2}$. The orbital elements are governed by the following Lagrange's equations [38]:

$$\mu_1\dot{e}_1 = \frac{\sqrt{1 - e_1^2}}{n_1 a_1^2 e_1}\frac{\partial\langle\langle\Phi_{\text{tot}}\rangle\rangle}{\partial\varpi_1},\tag{7}$$

$$\mu_2\dot{e}_2 = \frac{\sqrt{1 - e_2^2}}{n_2 a_2^2 e_2}\frac{\partial\langle\langle\Phi_{\text{tot}}\rangle\rangle}{\partial\varpi_2},\tag{8}$$

$$\mu_1\dot{\varpi}_1 = -\frac{\sqrt{1-e_1^2}}{n_1 a_1^2 e_1}\frac{\partial\langle\langle\Phi_{\rm tot}\rangle\rangle}{\partial e_1} - \frac{\tan(i_{\rm tot}/2)}{n_1 a_1^2\sqrt{1-e_1^2}}\frac{\partial\langle\langle\Phi_{\rm tot}\rangle\rangle}{\partial i_{\rm tot}},\tag{9}$$

$$\mu_2\dot{\varpi}_2 = -\frac{\sqrt{1-e_2^2}}{n_2 a_2^2 e_2}\frac{\partial\langle\langle\Phi_{\rm tot}\rangle\rangle}{\partial e_2} - \frac{\tan(i_{\rm tot}/2)}{n_2 a_2^2\sqrt{1-e_2^2}}\frac{\partial\langle\langle\Phi_{\rm tot}\rangle\rangle}{\partial i_{\rm tot}},\tag{10}$$

where $n_1 = G^{1/2}(m_1+m_2)^{1/2}/a_1^{3/2}$ and $n_2 = G^{1/2}(m_1+m_2+m_3)^{1/2}/a_2^{3/2}$ are the mean motions of the inner and outer binaries. Thus, the evolution equations for the eccentricities (to the octupole order) are

$$\dot{e}_1 = -\frac{15}{64}n_1 e_2\frac{\sqrt{1-e_1^2}(4+3e_1^2)}{(1-e_2^2)^{5/2}}\frac{m_3(m_1-m_2)}{(m_1+m_2)^2}$$
$$\times\left(\frac{a_1}{a_2}\right)^4\sin(\varpi_1-\varpi_2),\tag{11}$$

and

$$\dot{e}_2 = \frac{15}{64}n_2 e_1\frac{(4+3e_1^2)}{(1-e_2^2)^{5/2}}\frac{m_1 m_2(m_1-m_2)}{(m_1+m_2)^3}$$
$$\times\left(\frac{a_1}{a_2}\right)^3\sin(\varpi_1-\varpi_2).\tag{12}$$

The argument of periapse for the inner and outer binaries evolve according to

$$\dot{\varpi}_1 = \frac{3}{4}\left(\frac{a_1}{a_2}\right)^3 n_1\frac{m_3}{m_1+m_2}\frac{\sqrt{1-e_1^2}}{(1-e_2^2)^{3/2}}\Bigg[1-$$
$$\frac{5a_1 e_2(4+9e_1^2)(m_1-m_2)\cos(\varpi_1-\varpi_2)}{16 a_2 e_1(1-e_2^2)(m_1+m_2)}\Bigg]+\dot{\omega}_{\rm GR},\tag{13}$$

and

$$\dot{\varpi}_2 = \frac{3}{8}\left(\frac{a_1}{a_2}\right)^2 n_2\frac{m_1 m_2}{(m_1+m_2)^2}\frac{2+3e_1^2}{(1-e_2^2)^2}\Bigg[1-\frac{5a_1 e_1}{8a_2 e_2}$$
$$\times\frac{(4+3e_1)(1+4e_2^2)(m_1-m_2)\cos(\varpi_1-\varpi_2)}{(2+3e_1^2)(1-e_2^2)(m_1+m_2)}\Bigg].\tag{14}$$

Note that, the first term of Eqs.(13)-(14) corresponds to the quadrupole potential and the second term is induced by the octupole effects. Our results agree with [36] in the limit of small eccentricity and small m_2 of the inner binary.

The only effect of GR potential (Eq.5) is to provide an additional precession rate of the inner binary (e.g., [17]):

$$\dot{\omega}_{\rm GR} = \frac{3n_1}{1-e_1^2}\frac{G(m_1+m_2)}{a_1 c^2}.\tag{15}$$

Therefore, Eq.(13) describes the pericenter precession of the inner binary due to the combined effects of GR and the external body m_3.

III. LINEAR ANALYSIS: APSIDAL PRECESSION RESONANCE

A. Analytical equations

For $e_1, e_2 \ll 1$, Eqs.(11)-(14) can be reduced to

$$\dot{e}_1 = -\frac{15}{16}n_1 e_2\frac{m_3(m_1-m_2)}{(m_1+m_2)^2}\left(\frac{a_1}{a_2}\right)^4\sin(\varpi_1-\varpi_2),\tag{16}$$

$$\dot{e}_2 = \frac{15}{16}n_2 e_1\frac{m_1 m_2(m_1-m_2)}{(m_1+m_2)^3}\left(\frac{a_1}{a_2}\right)^3\sin(\varpi_1-\varpi_2),\tag{17}$$

$$\dot{\varpi}_1 = \frac{3}{4}\left(\frac{a_1}{a_2}\right)^3 n_1\frac{m_3}{m_1+m_2}$$
$$\times\left[1-\frac{5}{4}\frac{a_1}{a_2}\frac{e_2}{e_1}\frac{m_1-m_2}{m_1+m_2}\cos(\varpi_1-\varpi_2)\right]+\dot{\omega}_{\rm GR},\tag{18}$$

$$\dot{\varpi}_2 = \frac{3}{4}\left(\frac{a_1}{a_2}\right)^2 n_2\frac{m_1 m_2}{(m_1+m_2)^2}$$
$$\times\left[1-\frac{5}{4}\frac{a_1}{a_2}\frac{e_1}{e_2}\frac{m_1-m_2}{m_1+m_2}\cos(\varpi_1-\varpi_2)\right].\tag{19}$$

Following [38], we introduce the complex variable $I_\alpha \equiv e_\alpha e^{i\varpi_\alpha}$ (with $\alpha = 1, 2$). Then Eqs.(16)-(19) can be combined to give:

$$\dot{I}_1 = iA_{11}I_1 + iA_{12}I_2, \qquad \dot{I}_2 = iA_{21}I_2 + iA_{22}I_2.\tag{20}$$

where the coefficients A_{ij} are given by

$$A_{11} = \tfrac{3}{4}n_1\tfrac{m_3}{m_1+m_2}\left(\tfrac{a_1}{a_2}\right)^3 + \dot{\omega}_{\rm GR}|_{e_1\to 0},\tag{21}$$

$$A_{12} = -\tfrac{15}{16}n_1\left(\tfrac{a_1}{a_2}\right)^4\tfrac{m_3(m_1-m_2)}{(m_1+m_2)^2},\tag{22}$$

$$A_{21} = -\tfrac{15}{16}n_2\left(\tfrac{a_1}{a_2}\right)^3\tfrac{m_1 m_2(m_1-m_2)}{(m_1+m_2)^3},\tag{23}$$

$$A_{22} = \tfrac{3}{4}n_2\tfrac{m_1 m_2}{(m_1+m_2)^2}\left(\tfrac{a_1}{a_2}\right)^2.\tag{24}$$

Clearly, A_{11} is the total apsidal precession rate of the inner binary induced by the outer companion (to the quadrupole order) and the GR corrections, while A_{22} is the precession rate of the outer binary driven by the quadrupole potential of the inner binary.

The standard Eq.(20) can also be found in [38] in the case of $m_2, m_3 \ll m_1$ but without requiring $a_1/a_2 \ll 1$. Setting $I_\alpha \sim e^{ig_\star t}$, we obtain the two eigenvalues:

$$g_{\star 1,2} = \frac{A_{11}+A_{22}\mp\sqrt{(A_{11}-A_{22})^2+4A_{12}A_{21}}}{2},\tag{25}$$

where $g_{\star 1}$ refers to the upper sign, and $g_{\star 2}$ refers to the lower sign. The corresponding linear eigenvectors are:

$$\xi_1 = \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} = \begin{pmatrix} \frac{g_{\star 1} - A_{22}}{A_{21}} \\ 1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \xi_{21} \\ \xi_{22} \end{pmatrix} = \begin{pmatrix} \frac{g_{\star 2} - A_{22}}{A_{21}} \\ 1 \end{pmatrix}. \tag{26}$$

B. Excitation of eccentricity due to resonance

The general solution of Eq.(20) can be written in the form

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = A \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} e^{ig_{\star 1}t} + B \begin{pmatrix} \xi_{21} \\ \xi_{22} \end{pmatrix} e^{ig_{\star 2}t}. \tag{27}$$

The resulting time evolution of the eccentricity of the inner binary is then

$$e_1 = \left| (A\xi_{11})^2 + (B\xi_{21})^2 + 2AB\xi_{11}\xi_{21}\cos[(g_{\star 1} - g_{\star 2})t] \right|^{1/2}, \tag{28}$$

and for the outer binary

$$e_2 = \left| (A\xi_{12})^2 + (B\xi_{22})^2 + 2AB\xi_{12}\xi_{22}\cos[(g_{\star 1} - g_{\star 2})t] \right|^{1/2}, \tag{29}$$

where A and B are determined by the initial eccentricities:

$$e_{1,0} = A\xi_{11} + B\xi_{21}, \quad e_{2,0} = A + B. \tag{30}$$

Here the subscript "0" denotes $t = 0$.

Considering the case where $e_{1,0} = 0$ and $e_{2,0} \neq 0$. Then $e_1(t)$ will oscillate between 0 and $e_{1,\max}$, which is given by

$$e_{1,\max} = |A\xi_{11} - B\xi_{21}| = 2e_{2,0} \left| \frac{A_{12}}{g_{\star 2} - g_{\star 1}} \right|. \tag{31}$$

Clearly, $e_{1,\max}$ achieves the largest value, $e_{1,\mathrm{peak}}$, when $|g_{\star 1} - g_{\star 2}|$ is smallest, or equivalently,

$$A_{11} = A_{22}. \tag{32}$$

Physically, the condition (32) corresponds to the $\dot{\varpi}_1 = \dot{\varpi}_2$ at the quadrupole order. The peak value in $e_{1,\max}$ at this *"Apsidal Precession Resonance"* is then

$$e_{1,\mathrm{peak}} = e_{2,0} \left(\frac{A_{12}}{A_{21}} \right)^{1/2} = e_{2,0} \left(\frac{n_1}{n_2} \frac{a_1}{a_2} \frac{m_3}{\mu_1} \right)^{1/2}$$
$$= e_{2,0} \left[\frac{(m_1 + m_2)^3}{m_1 + m_2 + m_3} \right]^{1/4} \left(\frac{m_3}{m_1 m_2} \right)^{1/2} \left(\frac{a_2}{a_1} \right)^{1/4}. \tag{33}$$

Therefore, $e_{1,\mathrm{peak}}$ approaches $e_{2,0}$ only in the special case of $A_{12} = A_{21}$, where the "Apsidal Precession Resonance" is satisfied up to the octupole order.

IV. PARAMETER REGIME FOR RESONANCES IN VARIOUS SYSTEMS

A. Timescales

We are interested in inner binaries undergoing orbital decay due to gravitational radiation. The timescale of orbital decay is given by [41]

$$T_{\mathrm{GW}} \simeq 3.6 \times 10^{12} yr \left(\frac{m_1}{M_\odot} \right)^{-1} \left(\frac{m_2}{M_\odot} \right)^{-1}$$
$$\times \left(\frac{m_1 + m_2}{M_\odot} \right)^{-1} \left(\frac{a_1}{0.05\mathrm{AU}} \right)^4. \tag{34}$$

We require T_{GW} to be less than the Hubble time. This puts an upper limit on a_1, given by

$$T_{\mathrm{GW}}(a_{1,\mathrm{Hubble}}) = T_{\mathrm{Hubble}} \simeq 1.4 \times 10^{10}\mathrm{yr}. \tag{35}$$

For $a_1 > a_{1,\mathrm{Hubble}}$, the only way to have a shorter merging time ($T_{\mathrm{GW}} < T_{\mathrm{Hubble}}$) is to rely on a tertiary companion to induce Lidov-Kozai oscillations of the inner binary (e.g., [22]).

If resonant excitation of e_1 occurs during the shrinkage of the inner binary, the gravitational wave (GW) will be affected. The peak GW frequency can be approximated by [27]:

$$f_{\mathrm{GW}} = \frac{(1 + e_1)^{1.1954}}{\pi} \sqrt{\frac{G(m_1 + m_2)}{a_1^3(1 - e_1^2)^3}}. \tag{36}$$

For $e_1 = 0$, this becomes

$$f_{\mathrm{GW}} \sim 6.34 \times 10^{-8}\mathrm{Hz} \left(\frac{a_1}{\mathrm{AU}} \right)^{-3/2} \left(\frac{m_1 + m_2}{M_\odot} \right)^{1/2}. \tag{37}$$

In the following sections, we focus on the merging compact binaries and choose the sufficiently small initial values of semimajor axis of the inner orbit $a_{1,0}$, so that the corresponding GWs frequency at the apsidal precession resonance lies in the LIGO or LISA bands.

B. Perturbers in resonance

To explore the dependence of the apsidal precession resonance on the third body, we rewrite the resonance condition (32) as (using Eqs. 21 and 24):

$$m_3 = \left[\sqrt{\frac{m_1 + m_2 + m_3}{m_1 + m_2}} \sqrt{\frac{a_1}{a_2}} \frac{m_1 m_2}{(m_1 + m_2)^2} - 4\left(\frac{a_1}{a_2} \right)^3 \right.$$
$$\left. \times \frac{G(m_1 + m_2)}{a_1 c^2} \right] (m_1 + m_2). \tag{38}$$

A wide range of parameters can satisfy Eq.(38). Since the resonance is determined by Eq. (32) for any mass sets

TABLE I. Parameters for the canonical compact binaries. Here $a_{1,\text{Hubble}}$ is the semi-major axis at which the timescale of gravitational radiation of the binary equals the Hubble time (see Eq.35), $a_{1,\text{LISA}}$ (10^{-4}Hz) is the semi-major axis of the binary corresponding to GW frequency of 10^{-4} Hz (see Eq.36) and similarly for $a_{1,\text{LISA}}$ (10^{-1}Hz) and $a_{1,\text{LIGO}}$ (10Hz).

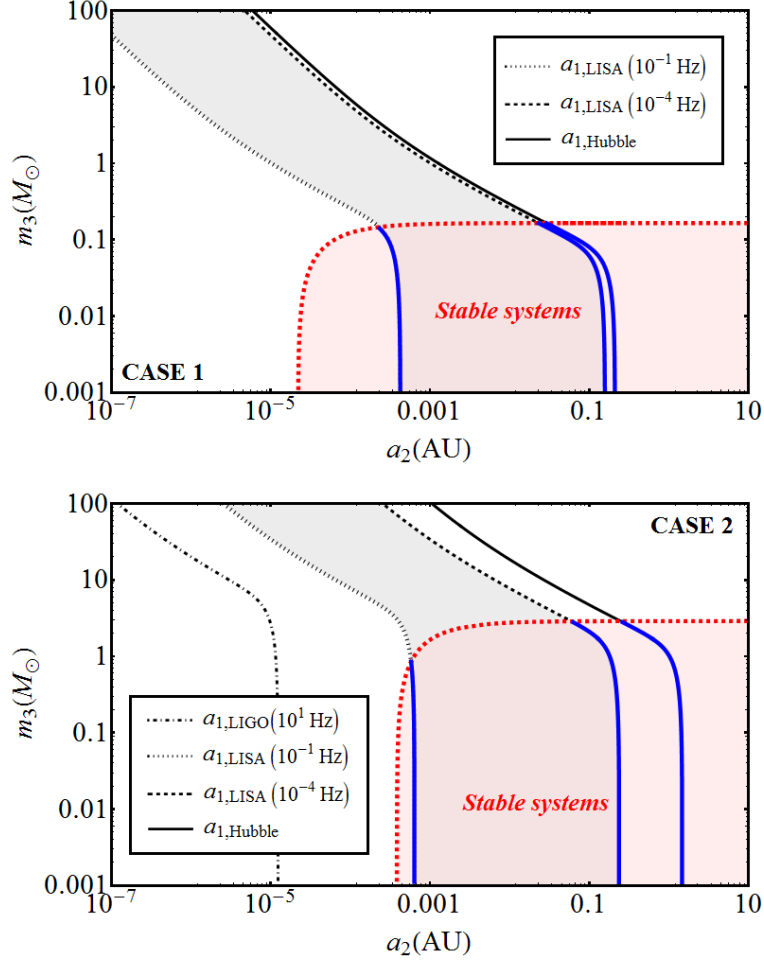| Parameter	System	m_1 ($M_\odot$)	m_2 ($M_\odot$)	$a_{1,\text{Hubble}}$ (AU)	$a_{1,\text{LISA}}(10^{-4}\text{Hz})$ (AU)	$a_{1,\text{LISA}}(10^{-1}\text{Hz})$ (AU)	$a_{1,\text{LIGO}}(10^1\text{Hz})$ (AU)
Case 1	WD-WD	0.8	0.4	1.0×10^{-2}	7.8×10^{-3}	7.8×10^{-5}	$\sim$
Case 2	BH-BH	14	7	8.4×10^{-2}	2.0×10^{-2}	2.0×10^{-4}	9.5×10^{-6}

FIG. 2. The mass of the third body as a function of the semi-major axis of the outer binary when the apsidal precession resonance occurs, for different values of semi-major axis of the inner binary a_1. The upper panel shows the result for WD-WD binaries, while the lower panel shows that for BH-BH binaries. The masses of m_1 and m_2 are listed in Table 1. The red region corresponds to stable systems given by the criterion (see Eq.39). We use blue curves to mark the stable systems for different values of a_1.

regardless of the type of binaries, we focus on two types of inner compact binaries, with masses given in Table 1.

In Fig. 2, we plot m_3 as a function of a_2 for given inner binary masses (m_1, m_2) and several values of semi-major axis a_1. Each point of the curve corresponds to a resonant triple system, i.e., resonance occurs if the inner binary passes through at such separation (a_1). Here $a_{1,\text{Hubble}}$

is given by Eq.(35), $a_{1,\text{LISA}}$ (10^{-4}Hz) is the semi-major axis of the binary corresponding GW frequency entering the LISA band (see Eq.36) and similarly for $a_{1,\text{LISA}}$ (10^{-1}Hz) and $a_{1,\text{LIGO}}$ (10Hz, LIGO band). We see that the curves of $a_{1,\text{LISA}}$ and $a_{1,\text{LIGO}}$ are on the left side of curve $a_{1,\text{Hubble}}$. This implies that the resonance can happen within the Hubble time and the corresponding
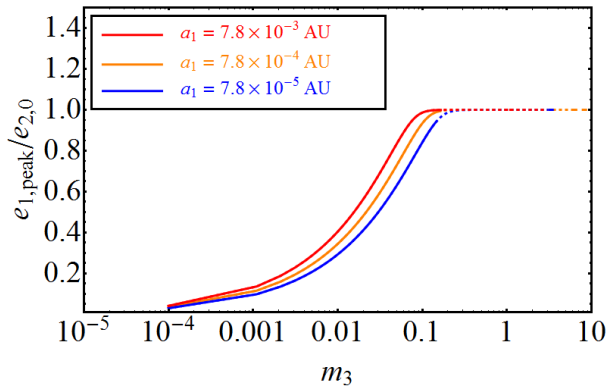
FIG. 3. The peak eccentricity in units of the initial outer binary eccentricity as a function of m_3 for the different values of a_1. The dashed lines correspond to unstable systems (see Eq.39). In this example, $e_{1,\text{peak}}$ increases, approaching $e_{2,0}$, as the external perturber mass m_3 increases.

resonant GW frequencies enter the LISA band or LIGO band.

However, the triple system must satisfy the stability criterion given by [40]:

$$\frac{a_2}{a_1} > 2.8\left(1 + \frac{m_3}{m_1 + m_2}\right)^{2/5}\frac{(1 + e_2)^{2/5}}{(1 - e_2)^{6/5}}\left(1 - \frac{0.3i}{180°}\right). \tag{39}$$

This further constrains the "existence" of the third body. Thus, only those triple systems in the "stable region" of Fig.2 are realistic.

Before studying resonant excitation of eccentricity during GW-induced orbital decay, we consider a few examples to illustrate the dependence of the peak values of the excited eccentricity on the mass of the third body. We focus on WD-WD binaries (Case 1) and choose three values of a_1. Combing Eqs.(33) and (38), setting the initial eccentricities as $e_{1,0} = 0.0001$, $e_{2,0} = 0.1$, we plot the ratio $e_{1,\text{peak}}/e_{2,0}$ as a function of m_3 in Fig. 3. We see that $e_{1,\text{peak}}$ approaches $e_{2,0}$ as m_3 increases.

V. EFFECTS OF THE GRAVITATIONAL RADIATION

In this section, we carry out calculations for the evolution of coplanar systems, including the dissipative effect of gravitational radiation. During the decay of the inner orbit, the system can encounter the apsidal precession resonance for eccentricity excitation. However, it is not clear whether the peak eccentricity $e_{1,\text{peak}}$ can be achieved as the system may pass through the resonance too quickly. Also the analytical result in Section III is restricted to small eccentricities, whereas the numerical calculations here apply to general eccentricities.

Due to gravitational radiation, the orbital semi-major

axis of the inner binary decays according to (e.g., [25])

$$\dot{a}_1|_{\text{GW}} = -\frac{64}{5}\frac{G^3}{c^5}\frac{m_1m_2(m_1 + m_2)}{a_1^3(1 - e_1^2)^{7/2}}\left(1 + \frac{73}{24}e_1^2 + \frac{37}{96}e_1^4\right). \tag{40}$$

Gravitational radiation also induces a secular change of e_1:

$$\dot{e}_1|_{\text{GW}} = -\frac{304}{15}\frac{G^3}{c^5}\frac{m_1m_2(m_1 + m_2)e_1}{a_1^4(1 - e_1^2)^{5/2}}\left(1 + \frac{121}{304}e_1^2\right). \tag{41}$$

Now consider an initially circular WD-WD binary ($m_1 = 0.8\text{M}_\odot$, $m_2 = 0.4\text{M}_\odot$) with the semi-major axis $a_{1,0} = 1.2 \times 10^{-3}\text{AU}$, and a small perturber ($m_3 = 0.01\text{M}_\odot$) orbits the center mass of the inner binary on a wider orbit ($a_2 = 1.27 \times 10^{-2}\text{AU}$). We set up the outer binary with two different initial eccentricities, $e_{2,0} = 0.1$ and $e_{2,0} = 0.5$. Combining Eqs.(15), (40) with (41), we integrate Eqs. (11)-(14) until the inner binary mergers. For each case, we also calculate the evolution of the peak GW frequencies (Eq.36). The results are shown in Fig.4.

The left panel of Fig.4 corresponds to the case of $e_{2,0} = 0.1$. The eccentricity of the inner binary undergoes small-amplitude oscillations at the early state, due to the quadrupole potential from the outer binary. During the gradual orbital decay, $\dot{\varpi}_1$ becomes comparable with $\dot{\varpi}_2$, the resonance starts to operate when a_1 is close to 0.0011AU and forces e_1 to achieve $e_{1,\text{peak}}$. After that, gravitational radiation gradually reduces e_1, circularizing the inner binary before the final merger. Corresponding to the excitation of e_1, the outer binary eccentricity e_2 decreases a lower value because of the resonance.

To compare the numerical results (including gravitational radiation) with the analytical solution presented in Section III, we calculate the maximum of e_1 and the minimum of e_2 attained for different values of a_1 based on Eq. (28) and Eq. (29). The results are shown as dashed curves in Fig. 4. We see that the maximum e_1 attained in our numerical calculation agrees approximately with the analytic $e_{1,\text{peak}}$, with the agreement better for small $e_{2,0}$. It is easy to notice that, in the numerical calculations, the time that the system spends near $e_{1,\text{peak}}$ is longer than the one in the analytical solution. Also, the peak e_1 is reached at somewhat smaller a_1 compared to analytical prediction; this difference arises from the finite value of $e_{2,0}$. Although this finite $e_{2,0}$ is unsatisfied with the linear approximation, it seems that the "Apsidal Precession Resonance" still occurs and drives the eccentricity to the evident excitation, as well as the GWs frequency.

VI. RESONANCE IN THE INCLINED SYSTEMS

A. Without Gravitational Radiation

In this section, we extend our calculations to the general cases of mutually inclined inner/outer binaries. Since no simple analytical result can be derived, we sample a_1
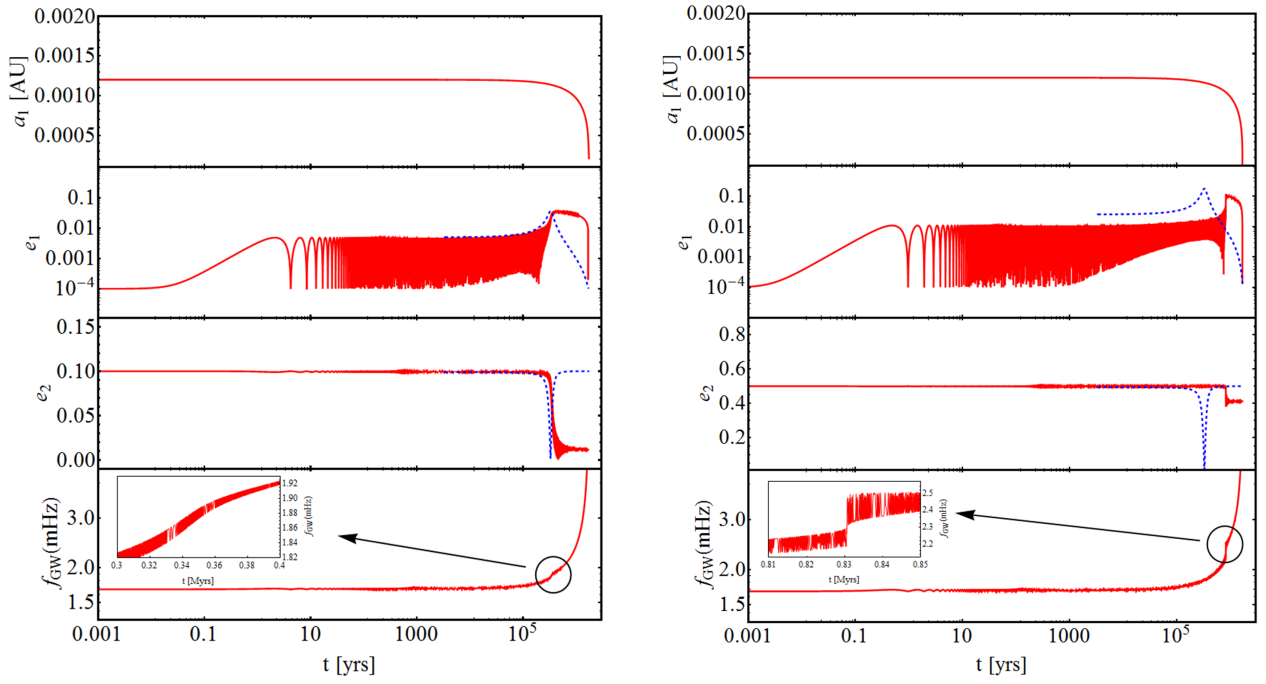
FIG. 4. The evolution of an inner merging WD-WD binary with a third body orbiting the mass center of the inner binary. The masses of the WDs are $m_1 = 0.8 M_\odot$, $m_2 = 0.4 M_\odot$, respectively. The initial semi-major axis of the binary is $a_{1,0} = 1.2 \times 10^{-3}$AU, the mass of the third body is $m_3 = 0.01 M_\odot$, the semi-major axis of the outer orbit is $a_2 = 1.27 \times 10^{-2}$AU. The eccentricity of the inner binary and the longitude of the periapsis are initialized as $e_1 = 0.0001$, $\varpi_{1,0} = \varpi_{2,0} = 0$, respectively. The outer binary is set up with two different initial eccentricities, $e_{2,0} = 0.1$ for the left panel and $e_{2,0} = 0.5$ for the right panel. We numerically integrate Eqs. (11)-(14) with orbital decay (Eqs.15, 40 and 41). Different panels show: (top) the inner binary semi-major axis a_1 from numerical results (red) and analytical solution (dashed blue); (top middle) the eccentricity of the inner binary e_1 from numerical integrations (red) and Eq.(28) (dashed blue); (top bottom) the eccentricity of the outer binary e_2 from numerical integrations (red) and Eq.(29) (dashed blue); and (bottom) the the peak GW frequency, also zoomed-in near the resonance. Clearly, resonance occurs in both cases. The analytical result for e_{lim} agrees with the numerical result for small $e_{2,0}$.
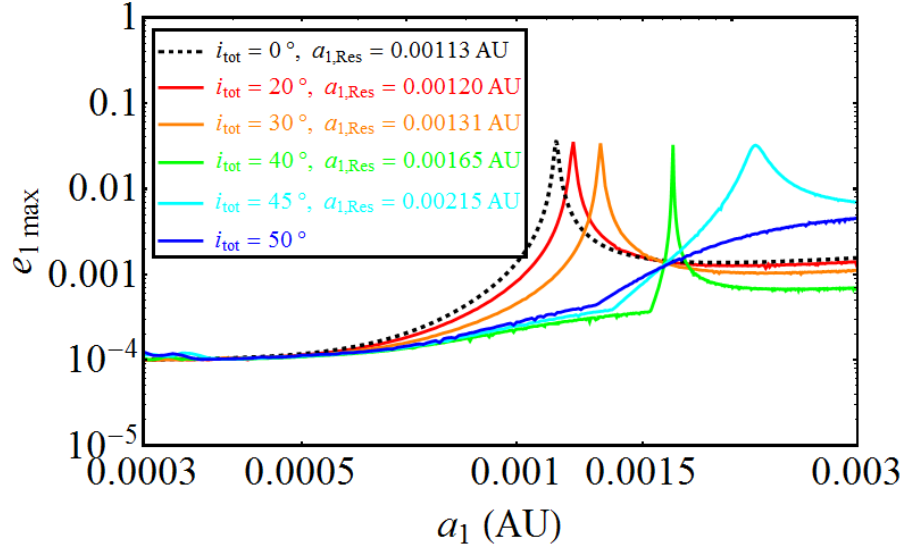

FIG. 5. The maximum eccentricity $e_{1,\text{max}}$ as a function of a_1 for triple systems with different initial mutual inclinations. The triple system consists of a WD-WD binary ($m_1 = 0.8 M_\odot$, $m_2 = 0.4 M_\odot$) and a tertiary companion ($m_3 = 0.01 M_\odot$) with semi-major axis $a_2 = 1.27 \times 10^{-2}$AU. ($i_{\text{tot},0} < 50°$), there exists a distinct peak in $e_{1,\text{max}}$, while for large inclinations ($i_{\text{tot},0} \geq 50°$), the peak is erased by Lidov-Kozai oscillations. We denote $a_{1,\text{Res}}$ the location where $e_{1,\text{peak}}$ is achieved.
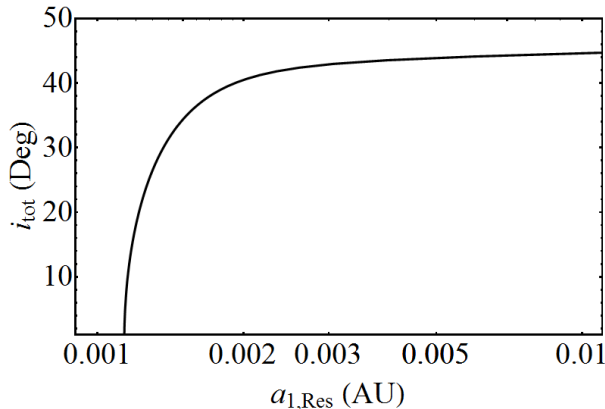
FIG. 6. The relationship between the inclination $i_{\rm tot}$ and $a_{1,\rm Res}$ in the case which is shown in Fig.5. This result is from Eqs.(42)-(43).

to determine the resonance location numerically for given outer binary parameters.

As an example, we consider a triple system consisting of a WD-WD inner binary ($m_1 = 0.8{\rm M}_\odot$, $m_2 = 0.4{\rm M}_\odot$) and a tertiary companion ($m_3 = 0.01{\rm M}_\odot$) with semi-major axis $a_2 = 1.27 \times 10^{-2}{\rm AU}$. The initial eccentricities of the two orbits are $e_{1,0} = 0.0001$ and $e_{2,0} = 0.1$. We set up the system by varying the initial semi-major axis $a_{1,0}$ between 0.0003AU and 0.003AU, and $i_{\rm tot,0}$ between 0° and 50°. For the each combination of the initial $a_{1,0}$ and $i_{\rm tot,0}$, we integrate Eqs.(15) and (A1)-(A9) in [35] (without evolving Eq.40) for a duration of $\sim 2t_K/\varepsilon_{\rm Oct}$, and record the maximum eccentricity $e_{1,\rm max}$ attained during the evolution. The results are shown in Fig.5. We see that for $i_{\rm tot} < 50^\circ$, there exists a distinct peak in $e_{1,\rm max}$ at the location $a_1 = a_{1,\rm Res}$. This resonant location increases with increasing $i_{\rm tot}$. Interestingly, the value of $e_{1,\rm peak}$ is almost independent of the inclination angle. For $i_{\rm tot} > 50^\circ$, the peak in the $e_{1,\rm max}$-a_1 curve becomes broader as the inner binary begins to experience Lidov-Kozai oscillations.

As discussed in Section III, the apsidal precession resonance occurs when $\dot{\varpi}_1 \sim \dot{\varpi}_2$. For inclined systems, we have (see Eqs.A6-A7 at the quadrupole level in [35])

$$\dot{\omega}_1 = \frac{3}{4}\left(\frac{a_1}{a_2}\right)^3 n_1 \frac{m_3}{m_1+m_2}\left(2-\cos i_{\rm tot}\right)+\dot{\omega}_{\rm GR}|_{e_1\to 0}, \quad (42)$$

$$\dot{\omega}_2 = \frac{3}{16}\left(\frac{a_1}{a_2}\right)^2 n_2 \frac{m_1 m_2}{(m_1+m_2)^2}\left(3-4\cos i_{\rm tot}+5\cos 2i_{\rm tot}\right), \quad (43)$$

where we have set $e_1 = e_2 = \omega_1 = 0$. Although we do not have an exact resonance condition for inclined systems, it is reasonable to assume that $\dot{\omega}_1 = \dot{\omega}_2$ provides a good estimate to the resonance radius. Equating Eq. (42) and Eq. (43), we obtain $a_{1,\rm Res}$ as a function of $i_{\rm tot}$. The result is shown in Fig.6. We see that, although this analytic resonant radius does not exactly

agree with the numerical value depicted in Fig.5, especially for $i_{\rm tot} \gtrsim 40^\circ$, it nevertheless reproduces the general feature of Fig.5. In particular, we see that $a_{1,\rm Res}$ shifts to larger values as $i_{\rm tot}$ increases, and the resonance disappears when $i_{\rm tot} \gtrsim 45^\circ$. Thus, the "Apsidal Precession Resonance" condition $\dot{\omega}_1 = \dot{\omega}_2$ provides a good description for eccentricity excitation in inclined systems.

B. With Gravitational Radiation

As in the section V, we now include gravitational radiation in the secular equations for inclined triples (Eqs. A1-A9 in [35] with Eqs. 15, 40 and 41), numerically evolving the system through the apsidal precession resonance.

We consider two examples with $i_{\rm tot,0} = 20^\circ$ and 30°. Since $a_{1,\rm Res}$ is shifted to a larger value compared to the $i_{\rm tot,0} = 0$ case, we set the initial semi-major axis as $a_{1,0} = 1.30 \times 10^{-3}{\rm AU}$ for $i_{\rm tot,0} = 20^\circ$ and $a_{1,0} = 1.35 \times 10^{-3}{\rm AU}$ for $i_{\rm tot,0} = 30^\circ$. Note that the initial mutual inclination of 20° corresponds to an inner and outer inclinations with respect to the total angular momentum of 2.05° and 17.95°. Similarly, for $i_{\rm tot,0} = 30^\circ$ we have $i_{1,0} = 3^\circ$ and $i_{2,0} = 27^\circ$. We also set the initial arguments of pericenter ($\omega_{1,0}, \omega_{2,0}$) and the ascending node ($\Omega_{1,0}$ and $\Omega_{2,0}$) of the inner and outer binaries to zero. The results are shown in Fig. 7. We find that the behaviors of e_1 and e_2, in particular the excitation of the inner binary eccentricity, are similar to the coplanar case. When the resonance is encountered during the orbital decay, e_1 increases while e_2 decreases. The peak eccentricity $e_{1,\rm peak}$ has a similar value as the coplanar case. The e_1 oscillation has a larger amplitude before the resonance for the higher $i_{\rm tot,0}$.

For systems with initial inclinations $i_{\rm tot,0} \geq 50^\circ$, we find that the inner binary does not undergo a distinct resonant eccentricity excitation. Instead, it experiences a gentle and modest eccentricity growth, followed by circularization directly due to GW emission.

VII.   SUMMARY AND DISCUSSION

In this paper we have studied the effect of an external tertiary companion on compact binaries (consisting of WDs, NSs or BHs) undergoing orbital decay due to gravitational radiation. We find that, during the orbital decay, the binary eccentricity can be excited due to gravitational perturbation from the tertiary. The eccentricity excitation occurs when the system passes through an "apsidal precession resonance", when the pericenter precession rate of the inner binary, due to the combined effects of gravitational perturbation of the external companion and General Relativity, equals the precession rate of the outer binary. This eccentricity excitation can be thought of as "eccentricity exchange" between the inner and outer binaries. It requires that the outer companion be on an eccentric orbit. It also requires that the mutual inclination between the inner and outer orbits
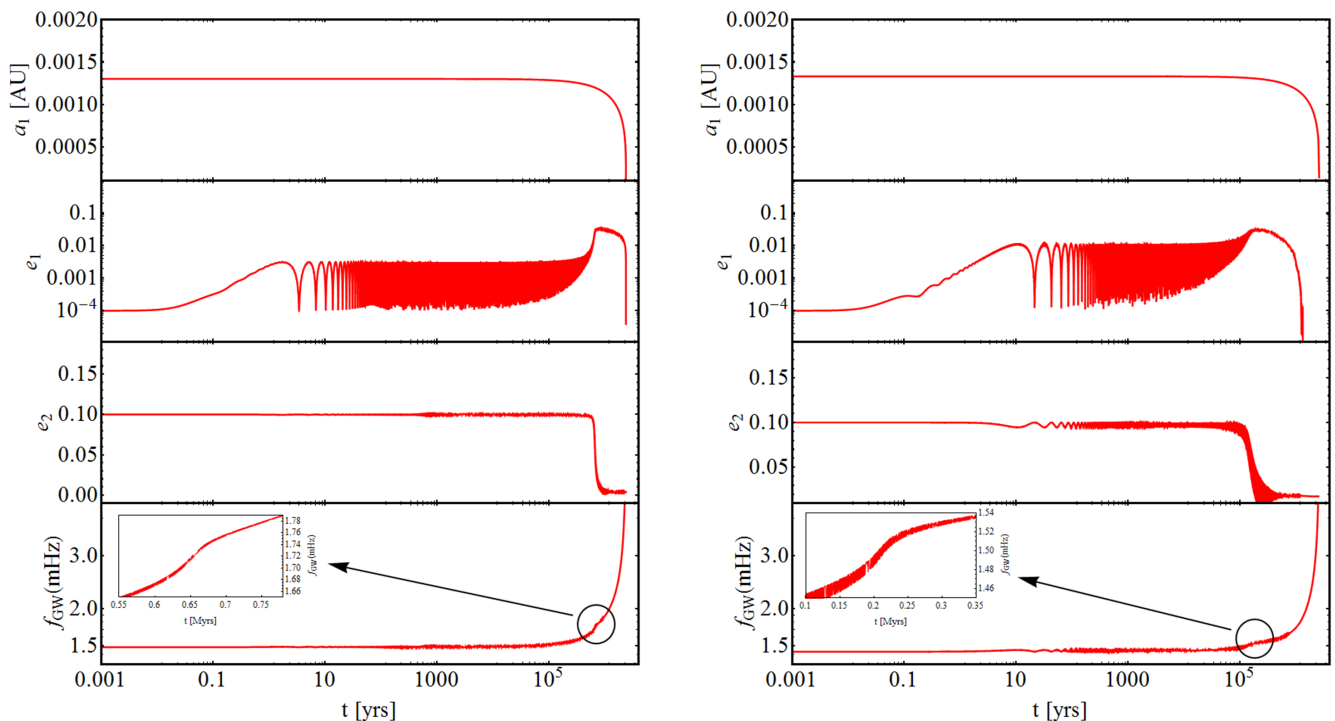
FIG. 7. Similar to Fig. 4, but for triple systems with finite initial mutual inclinations. The stellar masses and initial eccentricities the same as in the left panel of Fig. 4. Two examples are shown: In the left panels, we choose $i_{\rm tot,0} = 20°$ and $a_{1,0} = 1.30 \times 10^{-3}$AU; in the right panel, $i_{\rm tot,0} = 30°$ and $a_{1,0} = 1.35 \times 10^{-3}$AU. The qualitative behavior of the inclined systems is similar to the coplanar system (see Fig.4).

be less than $\sim 40°$, so that the resonance is not erased by Lidov-Kozai oscillations. Thus, as the triple system passes through the resonance (as a result of orbital decay of the inner binary), the outer binary "gives" part or all of its eccentricity to the inner binary.

Through analytical calculations (for coplanar systems with small eccentricities) and numerical integrations of the secular evolution equations, we have carried out a detailed analysis of the apsidal precession resonance and more importantly, we have examined the behavior of the system during resonance passage. For example, the value of the excited eccentricity, $e_{1,\rm peak}$, can be estimated by Eq.(33) and depends on the mass of the third body. In general, as the mass increases, $e_{1,\rm peak}$ becomes larger, but cannot exceed the initial eccentricity of the outer orbit (see Fig.3).

Obviously, the resonant excitation/exchange of eccentricity during binary orbital decay can significantly influence the gravitational wave signals from the binary. We have identified the parameter space where the resonance

occurs. For some system parameters (e.g., a WD binary with a brown dwarf tertiary), the resonance can happen when the binary emits GWs in the $10^{-4} - 10^{-1}$ Hz range (the sensitivity band of LISA). Our results indicate that the excitation of eccentricity is possible for a variety of compact binaries if there exists an appropriate tertiary companion that satisfies the resonant condition.

ACKNOWLEDGMENTS

[1] R. F. Webbink, Astrophys. J. **277**, 355 (1984).
[2] I. Iben, Jr., and A. V. Tutukov, Astrophys. J. Suppl. Ser. **54**, 335 (1984).
[3] D. Eichler, M. Livio, T. Piran, and D. N. Schramm, Nature **340**, 126 (1989).
[4] E. Berger, New Astron. Rev. **55**, 1 (2011).

[5] S. Nissanke, M. Kasliwal, and A. Georgieva, Astrophys. J. **767**, 124 (2013).

[6] G. Nelemans, Classical Quantum Gravity **26**, 094030 (2009).

[7] https://www.elisascience.org/.

[8] http://www.ligo.org/.

[9] C. Cutler, T. A. Apostolatos, L. Bildsten, L. S. Finn, E. E. Flanagan, D. Kennefick, D. M. Markovic, A. Ori, E. Poisson, G. J. Sussman, and K. S. Thorne, Phys. Rev. Lett. **70**, 2984 (1993).

[10] M. Shibata, and K. Taniguchi, Phys. Rev. D **73**, 064027 (2006).

[11] F. Foucart, M. D. Duez, L. E. Kidder, M. A. Scheel, B. Szilagyi, and S. A. Teukolsky, Phys. Rev. D **85**, 044015 (2012).

[12] Y. Sekiguchi, K. Kiuchi, K. Kyutoku, and M. Shibata, Prog. Theo. Exp. Phys. **1** 01A304 (2012).

[13] A. Tokovinin, Astron. J. **147**, 87 (2014).

[14] M. L. Lidov, Planet. Space Sci. **9**, 719 (1962).

[15] Y. Kozai, Astron. J. **67**, 591 (1962).

[16] P. P. Eggleton, and L. Kiseleva–Eggleton, Astrophys. J. **562**, 1012 (2001).

[17] D. C. Fabrycky, and S. Tremaine, Astrophys. J. **669** 1298 (2007).

[18] M. Holman, J. Touma, and S. Tremain, Nature **386**, 254 (1997).

[19] K. A. Innanen, J. Q. Zheng, S. Mikkola, and M. J. Valtonen, Astron. J. **113**, 1915 (1997).

[20] Y. Wu, and N. Murray, Astrophys. J. **589**, 605 (2003).

[21] S. Naoz, W. M. Farr, Y. Lithwick, F. A. Rasio, J. Teyssandier, Nature **473**, 187 (2011).

[22] T. A. Thompson, Astrophys. J. **741**, 82 (2011).

[23] S. Prodan, N. Murray, and T. A. Thompson, arXiv:1305.2191.

[24] B. Katz, S. Dong, and R. Malhotra, Phys. Rev. Lett. **107**, 181101 (2011)

[25] O. Blaes, M. H. Lee, and A. Socrates, Astrophys. J. **578**, 775 (2002).

[26] M. C. Miller, and D.P. Hamilton, Astrophys. J. **576**, 894 (2002).

[27] L. Wen, Astrophys. J. **598**, 419 (2003).

[28] F. Antonini, N. Murray, and S. Mikkola, Astrophys. J. **781**, 45 (2014).

[29] N. I. Storch, K. R. Anderson, D. Lai, Science **345**, 1317 (2014)

[30] N. I. Storch, D. Lai, Mon. Not. R. Astr. Soc. **448**, 1821 (2015).

[31] R. S. Harrington, Astron. J. **73**, 190 (1968).

[32] C. Marchal, The Three-Body Problem (Elsevier, Amsterdam, 1990).

[33] E. B. Ford, B. Kozinsky, and F. A. Rasio, Astrophys. J. **535**, 385 (2000b).

[34] S. Naoz, W. M. Farr, Y. Lithwick, and F. A. Rasio, Mon. Not. R. Astr. Soc. **431**, 2155 (2013b).

[35] B. Liu, D. J. Muñoz, and D. Lai, Mon. Not. R. Astr. Soc. **447**, 747 (2015).

[36] R. A. Mardling, Mon. Not. R. Astr. Soc. **382**, 1768 (2007).

[37] S. Naoz, B. Kocsis, A. Loeb, N. Yunes, Astrophys. J. **773**, 187 (2013).

[38] C. D. Murray, and S. F. Dermott, Solar System Dynamics, Cambridge U. Press, NY (1999).

[39] Y. Wu, and P. Goldreich, Astrophys. J. **564**, 1024 (2002).

[40] R. A. Mardling, and S. J. Aarseth, Mon. Not. R. Astr. Soc. **321**, 398 (2001).

[41] P. C. Peters, Phys. Rev. **136**, 1224 (1964).

[42] R. A. Mardling, Mon. Not. R. Astr. Soc. **407**, 1048 (2010).