Beyond six parameters: Extending ΛCDM

Eleonora Di Valentino, Alessandro Melchiorri, and Joseph Silk

Phys. Rev. D **92**, 121302 — Published 22 December 2015

DOI: 10.1103/PhysRevD.92.121302

Beyond six parameters: extending ΛCDM

Eleonora Di Valentino,[1] Alessandro Melchiorri,[2] and Joseph Silk[1, 3, 4, 5]
[1] *Institut d'Astrophysique de Paris (UMR7095: CNRS & UPMC- Sorbonne Universities), F-75014, Paris, France*
[2] *Physics Department and INFN, Università di Roma "La Sapienza", Ple Aldo Moro 2, 00185, Rome, Italy*
[3] *AIM-Paris-Saclay, CEA/DSM/IRFU, CNRS, Univ. Paris VII, F-91191 Gif-sur-Yvette, France*
[4] *Department of Physics and Astronomy, The Johns Hopkins University Homewood Campus, Baltimore, MD 21218, USA*
[5] *BIPAC, Department of Physics, University of Oxford, Keble Road, Oxford OX1 3RH, UK*

Cosmological constraints are usually derived under the assumption of a 6 parameters Λ-CDM theoretical framework or simple one-parameter extensions. In this paper we present, for the first time, cosmological constraints in a significantly extended scenario, varying up to 12 cosmological parameters simultaneously, including the sum of neutrino masses, the neutrino effective number, the dark energy equation of state, the gravitational waves background and the running of the spectral index of primordial perturbations. Using the latest Planck 2015 data release (with polarization) we found no significant indication for extensions to the standard Λ-CDM scenario, with the notable exception of the angular power spectrum lensing amplitude, $A_{\rm lens}$ that is larger than the expected value at more than two standard deviations even when combining the Planck data with BAO and supernovae type Ia external datasets. In our extended cosmological framework, we find that a combined Planck+BAO analysis constrains the value of the r.m.s. density fluctuation parameter to $\sigma_8 = 0.781^{+0.065}_{-0.063}$ at 95% c.l., helping to relieve the possible tensions with the CFHTlenS cosmic shear survey. We also find a lower value for the reionization optical depth $\tau = 0.058^{+0.040}_{-0.043}$ at 95 % c.l. respect to the one derived under the assumption of Λ-CDM. The scalar spectral index n_S is now compatible with a Harrison-Zeldovich spectrum to within 2.5 standard deviations. Combining the Planck dataset with the HST prior on the Hubble constant provides a value for the equation of state $w < -1$ at more than two standard deviations while the neutrino effective number is fully compatible with the expectations of the standard three neutrino framework.

PACS numbers: 98.80.-k 95.85.Sz, 98.70.Vc, 98.80.Cq

INTRODUCTION

In the past twenty years, measurements of the Cosmic Microwave Background (CMB, hereafter) anisotropy angular power spectrum have witnessed one of the most impressive technological advances in experimental physics. Following the first detection of CMB temperature anisotropies at large angular scales by the COBE satellite in 1992 [1], the angular power spectrum has been measured with increasing precision by balloon-borne experiments such as BOOMERanG [2], MAXIMA [3] and by ground-based experiments as DASI [4], showing the unambiguous presence of a "first peak" and subsequent oscillations in the angular power spectrum at intermediate angular scales ($\theta \sim 0.2^o$). The spectacular measurements obtained by the WMAP satellite mission [5] have not only confirmed the presence of these acoustic oscillations but also provided the first precise measurement of the cross temperature-polarization angular spectrum and the first constraints on the epoch of reionization. The very small-scale part of the angular temperature power spectrum, and especially the damping tail, has been accurately determined by experiments as ACT [6] and SPT [7]. This impressive progress in the measurement of the CMB anisotropies temperature angular spectrum has culminated with the cosmic-variance limited measurements of the Planck experiment that has now also provided exquisite results on the polarization and cross temperature-polarization spectra.

Despite this impressive progress on the experimental side, it is interesting to note that the constraints on cosmological parameters are still presented (as in the latest Planck 2015 data release, [8]) under the assumption of a simple Λ-CDM model, based on the variation of just 6 cosmological parameters. While this model still provides a good fit to the data, it is the same model used, for example, in the analysis of the BOOMERanG 1998 data (see [2]), i.e. more than fifteen years ago. While this "minimal" approach is justified by the good fit to the data that the Λ-CDM provides we believe that it does not do adequate justice to the high quality of the most recent datasets. In light of the new precise data, some of the assumptions made in the 6 parameters approach are indeed not anymore fully justified. For example, fixing the total neutrino mass to zero or to some small value is completely arbitrary since we know that neutrinos must have masses and that current cosmological datasets are sensitive to variations in the absolute neutrino mass scale of order ~ 100 meV. At the same time, considering that a cosmological constant offers difficulties in any theoretical interpretation, it seems reasonable to incorporate in any analysis a possible dynamical dark energy component. This is certainly plausible (and even preferred if one wants to address the "Why Now ?" problem), and indeed fixing the dark energy equation of state to -1 is not favoured by any theoretical argument. Most inflationary models predict a sizable contribution of gravitational waves. Given the progress made in the search for B-mode polarization, especially by the recent combined BICEP2+Planck analysis [20], it is an opportune moment to allow any such contribution to be directly constrained by the data, without assuming a null contribution as in the 6-parameter model. A similar argument can be made for the running of the scalar spectral index $dn_s/dlnk$. Moreover, the neutrino effective number, N_{eff} could be easily different from the standard expected value of 3.046. Even assuming the standard three neutrino framework, non-standard decoupling, inflationary reheating, dark matter decay and many other physical process could alter its value. Finally, the Planck 2015 release still hints for an anomalous value for the lensing amplitude $A_{\rm lens}$ [21]. While this parameter is purely phenomenological, one should clearly consider it and check if the cosmology obtained is consistent with other datasets. The goal of this *Letter* is to constrain cosmological parameters in this extended parameter space.

METHOD

As discussed in the introduction, besides the six parameters of the "standard" Λ-CDM model, i.e. the Hubble constant H_0, the baryon $\Omega_b h^2$ and CDM energy densities $\Omega_c h^2$, the primordial amplitude and spectral index of scalar perturbations A_s and n_s (at pivot scale $k_0 = 0.05 hMpc^{-1}$), and the reionization optical depth τ, we also consider variations in 6 additional parameters: the total mass for the 3 standard neutrinos, Σm_ν, the dark energy equation of state w assumed constant with redshift, the tensor/scalar ratio of amplitude r at pivot scale $k_0 = 0.05 hMpc^{-1}$, the running of the scalar spectral index $dn_s/dlnk$, at pivot scale $k_0 = 0.05 hMpc^{-1}$, the amplitude of the lensing signal in the CMB angular spectra, $A_{\rm lens}$ as defined in [21], the effective number of relativistic neutrinos, $N_{\rm eff}$. In what follows, we refer to this model as eCDM (extended Cold Dark Matter).

We let all these parameters vary freely simultaneously in a range of external, conservative, priors listed in Table I.

We produce constraints on these cosmological parameters by making use of several, recent, datasets. Firstly, we use the full Planck 2015 release on temperature and polarization CMB angular power spectra. This dataset includes the large angular scale temperature and polarization measured by the Planck LFI experiment and the small-scale temperature and polarization spectra measured by Planck HFI. We refer to this dataset simply as Planck [11]. We also include information on CMB lensing from Planck trispectrum detection (see [9]). We refer to this dataset as *lensing*. We add baryonic acoustic oscillation data from 6dFGS [12], SDSS-MGS [13], BOSSLOWZ [14] and CMASS-DR11 [14] surveys as in [8]. We refer to this dataset as BAO. We impose a constraint on the Hubble constant from the Hubble Space Telescope [15] dataset. Recently this constraint has been criticized in [16] where a more conservative value was suggested, a choice adopted in the recent Planck analysis [8]. For reasons that will appear below, we choose to use the less conservative [15] determination. We refer to this dataset as HST. We use luminosity distances of supernovae type IA from the "Joint Light-curve Analysis" derived from the SNLS and SDSS catalogs [17]. We refer to this dataset as JLA. We add weak lensing galaxy data from the CFHTlenS [18] survey with the priors and conservative cuts to the data as described in [8]. We refer to this dataset as WL. We consider redshift space distortions from [19] with the prescription give in [8]. We refer to this dataset as RSD. Finally, we include upper limits on CMB polarization B modes as recently placed by a common analysis of Planck, BICEP2 and Keck Array

Parameter	Prior
$\Omega_{\rm b}h^2$	$[0.005, 0.1]$
$\Omega_{\rm cdm}h^2$	$[0.001, 0.99]$
$\Theta_{\rm s}$	$[0.5, 10]$
τ	$[0.01, 0.8]$
n_s	$[0.8, 1.1]$
$\log[10^{10}A_s]$	$[2, 4]$
$\sum m_\nu$ (eV)	$[0, 3]$
w	[-3.5,0.5]
$\frac{dn_s}{dlnk}$	[-0.5,0.5]
r	[0,0.5]
$N_{\rm eff}$	[0.05,10]
$A_{\rm lens}$	[0,10]

TABLE I: External priors on the cosmological parameters assumed in this paper.

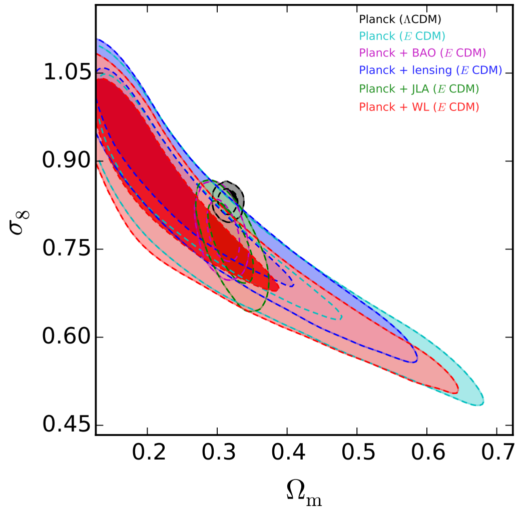

FIG. 1: Constraints at 68% and 95% confidence levels on the σ_8 vs Ω_m plane under the assumption of $e\Lambda$CDM and different datasets. Black contours are the constraints under ΛCDM.

Model Dataset	$\Omega_{\rm b}h^2$	$\Omega_{\rm c}h^2$	H_0	τ	n_s	σ_8	$\frac{dn_s}{dlnk}$	r	w	$\Sigma m_\nu[eV]$	$N_{\rm eff}$	$A_{\rm lens}$
Λ CDM Planck	$0.02226^{+0.00031}_{-0.00029}$	$0.1198^{+0.0028}_{-0.0028}$	$67.3^{+1.3}_{-1.3}$	$0.079^{+0.034}_{-0.035}$	$0.9646^{+0.0092}_{-0.0092}$	$0.831^{+0.026}_{-0.026}$	-	-	-	-	-	-
Λ CDM Planck+ BAO	$0.02229^{+0.00028}_{-0.00027}$	$0.1193^{+0.0021}_{-0.0020}$	$67.52^{+0.93}_{-0.93}$	$0.082^{+0.031}_{-0.032}$	$0.9662^{+0.0078}_{-0.0079}$	$0.832^{+0.025}_{-0.025}$	-	-	-	-	-	-
e CDM Planck	$0.02239^{+0.00060}_{-0.00056}$	$0.1186^{+0.0071}_{-0.0068}$	> 51.2	$0.058^{+0.040}_{-0.043}$	$0.967^{+0.025}_{-0.025}$	$0.81^{+0.24}_{-0.26}$	$-0.003^{+0.020}_{-0.019}$	< 0.183	$-1.32^{+0.98}_{-0.85}$	< 0.959	$3.08^{+0.57}_{-0.51}$	$1.21^{+0.27}_{-0.24}$
e CDM Planck+BAO	$0.02251^{+0.00056}_{-0.00052}$	$0.1185^{+0.0069}_{-0.0069}$	$68.4^{+4.3}_{-4.1}$	$0.058^{+0.041}_{-0.043}$	$0.972^{+0.024}_{-0.024}$	$0.781^{+0.065}_{-0.063}$	$-0.004^{+0.018}_{-0.018}$	< 0.187	$-1.04^{+0.20}_{-0.21}$	< 0.534	$3.11^{+0.52}_{-0.48}$	$1.20^{+0.19}_{-0.19}$
e CDM Planck+lensing	$0.02214^{+0.00053}_{-0.00052}$	$0.1176^{+0.0069}_{-0.0066}$	> 54.5	$0.058^{+0.040}_{-0.043}$	$0.959^{+0.024}_{-0.024}$	$0.85^{+0.21}_{-0.24}$	$-0.005^{+0.018}_{-0.018}$	< 0.178	$-1.45^{+0.96}_{-0.83}$	< 0.661	$2.93^{+0.51}_{-0.48}$	$1.04^{+0.16}_{-0.15}$
e CDM Planck+HST	$0.02239^{+0.00059}_{-0.00057}$	$0.1187^{+0.0072}_{-0.0070}$	$74.4^{+5.1}_{-5.1}$	$0.057^{+0.040}_{-0.045}$	$0.966^{+0.025}_{-0.025}$	$0.81^{+0.10}_{-0.11}$	$-0.003^{+0.020}_{-0.019}$	< 0.186	$-1.32^{+0.29}_{-0.31}$	< 0.957	$3.09^{+0.58}_{-0.55}$	$1.18^{+0.19}_{-0.18}$
e CDM Planck+JLA	$0.02242^{+0.00058}_{-0.00056}$	$0.1188^{+0.0071}_{-0.0067}$	$67.4^{+4.4}_{-4.2}$	$0.058^{+0.040}_{-0.043}$	$0.968^{+0.025}_{-0.025}$	$0.759^{+0.088}_{-0.089}$	$-0.004^{+0.020}_{-0.019}$	< 0.183	$-1.06^{+0.13}_{-0.14}$	< 0.854	$3.10^{+0.57}_{-0.54}$	$1.20^{+0.19}_{-0.17}$
e CDM Planck+WL	$0.02251^{+0.00056}_{-0.00055}$	$0.1188^{+0.0073}_{-0.0069}$	> 54.2	< 0.0835	$0.972^{+0.024}_{-0.024}$	$0.82^{+0.22}_{-0.25}$	$0.000^{+0.020}_{-0.019}$	< 0.197	$-1.41^{+0.98}_{-0.79}$	< 0.974	$3.16^{+0.58}_{-0.56}$	$1.24^{+0.23}_{-0.22}$
e CDM Planck+BAO-RSD	$0.02253^{+0.00052}_{-0.00050}$	$0.1184^{+0.0069}_{-0.0067}$	$68.6^{+4.2}_{-3.9}$	$0.056^{+0.038}_{-0.042}$	$0.972^{+0.023}_{-0.023}$	$0.774^{+0.055}_{-0.058}$	$-0.004^{+0.018}_{-0.018}$	< 0.188	$-1.05^{+0.17}_{-0.19}$	< 0.626	$3.12^{+0.51}_{-0.48}$	$1.22^{+0.18}_{-0.17}$
e CDM Planck+BKP	$0.02237^{+0.00057}_{-0.00056}$	$0.1186^{+0.0072}_{-0.0069}$	> 52.3	$0.058^{+0.039}_{-0.044}$	$0.966^{+0.026}_{-0.026}$	$0.81^{+0.23}_{-0.25}$	$-0.003^{+0.019}_{-0.018}$	< 0.101	$-1.31^{+0.96}_{-0.89}$	< 0.876	$3.07^{+0.57}_{-0.55}$	$1.20^{+0.24}_{-0.22}$

TABLE II: Constraints at 95% c.l. on the cosmological parameters assuming the standard 6-parameter ΛCDM model and the extended, 12-parameter, $e\Lambda$CDM model.

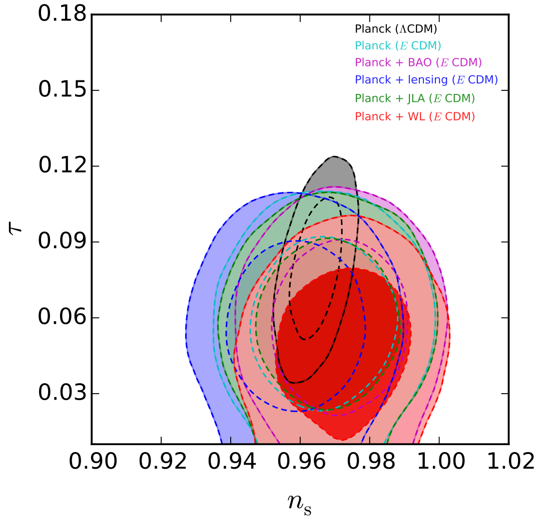

FIG. 2: Constraints at 68% and 95% confidence levels on the τ vs n_S plane under the assumption of $e\Lambda$CDM and different datasets. Black contours are the constraints under ΛCDM.

data [20]. We refer to this dataset as BKP.

We use the publicly available Monte Carlo Markov Chain package `cosmomc` [23] with a convergence diagnostic based on the Gelman and Rubin statistic. We use the July 2015 version which includes support for the Planck data release 2015 Likelihood [11] (see `http://cosmologist.info/cosmomc/`) and implements an efficient sampling using the fast/slow parameter decorrelations [24]. While in this paper we will focus the attention on cosmological parameters, we also vary foreground parameters using the same technique and parametrization described in [11] and [8].

RESULTS

The results of our analysis are reported in Table II where we also include, for comparison, the constraints obtained assuming the standard, 6 parameters in ΛCDM. The significant increase in the number of parameters produces, as expected, a relaxation in the constraints on the 6 ΛCDM parameters. Considering the allowable volume of the six-dimensional ΛCDM parameter space to be proportional to the square root of the determinant of the 6×6 parameter covariance, we find that moving from ΛCDM to $e\Lambda$CDM expands this volume by a factor ~ 63000. The parameters that are mostly affected are the Hubble constant and the r.m.s. amplitude of density fluctuations that are now practically undetermined from Planck measuments alone and have significantly larger errors with respect to ΛCDM even when external datasets such as BAO are included. The main reason for this relaxation is the inclusion in the analysis of the dark energy equation of state w, that introduces a geometrical degeneracy with the matter density and the Hubble constant. Moreover, marginalizing over the lensing amplitude $A_{\rm lens}$ removes the lensing information in the CMB spectra that could potentially break this geometrical degeneracy. In this respect, it is interesting to note that a Planck+HST analysis provides a value for the equation of state w less than the cosmological constant value -1 at more than 95% c.l.. Constraints on the baryon and cold dark matter densities, the scalar spectral index n_S and the optical depth τ are also much weaker, mainly due to degeneracies between these parameters and $A_{\rm lens}$ and $N_{\rm eff}$. Apart from the increase in the errors, it is interesting that parameters as σ_8 and the optical depth τ are shifted toward lower values respect to ΛCDM. These shifts are clear in Figures

1 and 2 where we plot the 68% and 95% c.l. contour plots in the σ_8 vs Ω_m and τ vs n_S planes, respectively. This is mainly due to the anomalous value of $A_{\rm lens}$ and persists when external datasets as BAO, JLA, WL and RSD are included. Looking at the results in Table II, the value of $A_{\rm lens}$ is always different from the standard value at more than 95% c.l. when the Planck CMB dataset is combined with external datasets, with the only notable exception of the lensing information from the Planck trispectrum that pushes the value of $A_{\rm lens}$ back to agreement with unity. The nature of the Planck $A_{\rm lens}$ anomaly could be different from the lensing determination but since it also persists in our extended $e\Lambda$CDM scenario, this clearly deserves further investigation. Moreover, $A_{\rm lens}$ is the only parameter that hints at a tension with standard ΛCDM. Again, by looking at Table II, apart from $A_{\rm lesn}$, we see no evidence for "new physics": we just have (weaker) upper limits on the neutrino mass, the running of the spectral index is compatible with zero, the dark energy equation of state is compatible with $w = -1$ (expect when we use the HST prior), and the neutrino effective number is remarkably close to the standard value $N_{\rm eff} = 3.046$. It is impressive that even in a 12 parameter space, the neutrino effective number is still constrained with exquisite precision. This is mainly due to the inclusion of the Planck HFI small angular scale polarization data in the analysis. Removing this dataset, but keeping the low angular scale LFI polarization, we get a much weaker constraint from Planck+BAO of $N_{\rm eff} = 4.35^{+1.8}_{-1.6}$ at 95% c.l.. The Planck+BAO constraint on neutrino mass of $\Sigma m_{nu} < 0.534$ eV at 95% c.l. is significantly weaker with respect to the constraint $\Sigma m_{nu} < 0.174$ eV at 95% c.l.. obtained with the same dataset but assuming ΛCDM. The constraint on the tensor/scalar amplitude r is about a factor two larger than in ΛCDM. However, when the BKP dataset is included, the 95% c.l. upper limit of $r < 0.108$ is recovered. This clearly shows how a measurement of primordial B modes is crucial to constrain the tensor amplitude in a model-independent way. The inclusion of the BKP dataset affects only the constraint on the tensor amplltude and leaves the other constraints virtually unchanged.

CONCLUSIONS

In this *Letter* we have presented, for the first time, constraints on cosmological parameters in the framework of an "extended" cold dark matter model ($e\Lambda$CDM) that is based on 12 parameters instead of the usual 6 assumed in the ΛCDM model. In this extension some of the parameters usually well constrained under ΛCDM such as the Hubble constant and the amplitude of matter density fluctuations σ_8 are now unconstrained by CMB observations. Combining the CMB data with several other datasets reveals no statistically significant evidence for any tensions. More specifically, we have found no evidence for "new physics" beyond the standard ΛCDM model, i.e. there is no island of parameters in our extended theoretical framework that could be preferred to the standard ΛCDM territory. However $e\Lambda$ CDM prefers a lower value of σ_8 relative to that obtained for 6-parameter ΛCDM but still requires a slightly anomalous value of $A_{\rm lens} > 1$. The lower value of σ_8 in $e\Lambda$CDM brings the Planck data in more agreement with the results of the CFHTlenS survey [25]. This result motivates further studies that could explain the physical nature of the $A_{\rm lens} > 1$ anomaly.

The tension between the Planck and HST values of the Hubble parameter, in the $e\Lambda$CDM scenario is solved by a value of the dark energy equation of state $w < -1$ while the neutrino effective number remains compatible with the standard value of 3.04.

Of course, the number of parameters can be further extended by considering, for example, non-zero curvature, isocurvature primordial perturbations, features in the primordial spectrum, a varying (with redshift) dark energy equation of state, non-standard Big Bang Nucleosynthesis and a change in the primordial helium abundance Y_p, and so on. Further extensions however may be premature because of degeneracies. For example, most effects of varying curvature are degenerate with a variation in w. CDM isocurvature modes have a spectrum similar to a GW background and the $A_{\rm lens}$ parameter could account for undetected features in the angular spectrum. Moreover, a change in $N_{\rm eff}$ could account for a change in Y_p.

We find it impressive that despite the increase in the number of the parameters, some of the constraints on key parameters are relaxed but not significantly altered. The cold dark matter ansatz remains robust, the baryon density is compatible with BBN predictions, and the neutrino effective number is compatible with standard expectations. The excellent quality of the new data motivates our exploration beyond the overexploited territory of ΛCDM towards new and uncharted frontiers.

JS and EdV acknowledge support by ERC project 267117 (DARK) hosted by UPMC, and JS for support at JHU by National Science Foundation grant OIA-1124403 and by the Templeton Foundation. EdV has been supported in part by the Institute Lagrange de Paris. AM acknowledge support by the research grant Theoretical Astroparticle Physics number 2012CPPYP7 under the program PRIN 2012 funded by MIUR and by TASP, iniziativa specifica INFN.

---

[1] G. F. Smoot, C. L. Bennett, A. Kogut, *et al.*, Astrophys. J. **396** (1992) L1.

[2] P. de Bernardis *et al.* [Boomerang Collaboration], Nature

404 (2000) 955 [astro-ph/0004404].

[3] S. Hanany *et al.*, Astrophys. J. **545** (2000) L5 [astro-ph/0005123].

[4] N. W. Halverson *et al.*, Astrophys. J. **568** (2002) 38 [astro-ph/0104489].

[5] D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148** (2003) 175 [astro-ph/0302209].

[6] J. Dunkley *et al.*, Astrophys. J. **739** (2011) 52 [arXiv:1009.0866 [astro-ph.CO]].

[7] R. Keisler *et al.*, Astrophys. J. **743** (2011) 28 [arXiv:1105.3182 [astro-ph.CO]].

[8] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1502.01589 [astro-ph.CO].

[9] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1502.01591 [astro-ph.CO].

[10] P. A. R. Ade *et al.* [Planck Collaboration], arXiv:1502.01597 [astro-ph.CO].

[11] N. Aghanim *et al.* [Planck Collaboration], arXiv:1507.02704 [astro-ph.CO].

[12] F. Beutler *et al.*, Mon. Not. Roy. Astron. Soc. **416** (2011) 3017 [arXiv:1106.3366 [astro-ph.CO]].

[13] A. J. Ross *et al.*, Mon. Not. Roy. Astron. Soc. **449** (2015) 835 [arXiv:1409.3242 [astro-ph.CO]].

[14] L. Anderson *et al.* [BOSS Collaboration], Mon. Not. Roy. Astron. Soc. **441** (2014) 1, 24 [arXiv:1312.4877 [astro-ph.CO]].

[15] A. G. Riess *et al.*, Astrophys. J. **730** (2011) 119 [Astrophys. J. **732** (2011) 129] [arXiv:1103.2976 [astro-ph.CO]].

[16] G. Efstathiou, Mon. Not. Roy. Astron. Soc. **440** (2014) 2, 1138 [arXiv:1311.3461 [astro-ph.CO]].

[17] M. Betoule *et al.* [SDSS Collaboration], Astron. Astrophys. **568** (2014) A22 [arXiv:1401.4064 [astro-ph.CO]].

[18] C. Heymans *et al.*, Mon. Not. Roy. Astron. Soc. **427** (2012) 146 [arXiv:1210.0032 [astro-ph.CO]].

[19] L. Samushia *et al.*, Mon. Not. Roy. Astron. Soc. **439** (2014) 4, 3504 [arXiv:1312.4899 [astro-ph.CO]].

[20] P. A. R. Ade *et al.* [BICEP2 and Planck Collaborations], Phys. Rev. Lett. **114** (2015) 10, 101301 [arXiv:1502.00612 [astro-ph.CO]].

[21] E. Calabrese, A. Slosar, A. Melchiorri, G. F. Smoot and O. Zahn, Phys. Rev. D **77** (2008) 123531 [arXiv:0803.2309 [astro-ph]].

[22] E. Calabrese, A. Cooray, M. Martinelli, A. Melchiorri, L. Pagano, A. Slosar and G. F. Smoot, Phys. Rev. D **80** (2009) 103516 [arXiv:0908.1585 [astro-ph.CO]].

[23] A. Lewis and S. Bridle, Phys. Rev. D **66**, 103511 (2002) [astro-ph/0205436].

[24] A. Lewis, arXiv:1304.4473 [astro-ph.CO].

[25] N. MacCrann, J. Zuntz, S. Bridle, B. Jain and M. R. Becker, Mon. Not. Roy. Astron. Soc. **451** (2015) 2877 [arXiv:1408.4742 [astro-ph.CO]].