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Low-energy constants and condensates from ALEPH hadronic τ decay data

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ABSTRACT

We determine the NLO chiral low-energy constant L_{10}^r, and combinations $C_{12}^r \pm C_{61}^r + C_{80}^r$, $C_{13}^r - C_{62}^r + C_{81}^r$, C_{61}^r, and C_{87}^r, of the NNLO chiral low-energy constants incorporating recently revised ALEPH results for the non-strange vector (V) and axial-vector (A) hadronic τ decay distributions and recently updated RBC/UKQCD lattice data for the non-strange $V-A$ two-point function. In the $\overline{\rm MS}$ scheme, at renormalization scale $\mu = 770$ MeV, we find $L_{10}^r = -0.00350(17)$, $C_{12}^r + C_{61}^r + C_{80}^r = 0.00237(16)$ GeV^{-2}, $C_{12}^r - C_{61}^r + C_{80}^r = -0.00056(15)$ GeV^{-2},

$C_{13}^r - C_{62}^r + C_{81}^r = 0.00046(9)$ GeV^{-2}, $C_{61}^r = 0.00146(15)$ GeV^{-2}, and $C_{87}^r = 0.00510(22)$ GeV^{-2}. With errors here at or below the level expected for contributions of yet higher order in the chiral expansion, the analysis exhausts the possibilities of what can be meaningfully achieved in an NNLO analysis. We also consider the dimension six and dimension eight coefficients in the operator product expansion in the $V - A$ channel.

I. INTRODUCTION

Recently, a revised version [1] of the ALEPH data [2] for the non-strange vector (V) and axial vector (A) spectral distributions obtained from measurements of hadronic τ decays became available. These results corrected a problem, uncovered in Ref. [3], in the publicly posted 2005 and 2008 versions of the correlations between different energy bins.[1] In Ref. [4] we analyzed these data in order to extract the strong coupling at the τ mass, $\alpha_s(m_\tau^2)$, as well as OPE condensates, following the strategy previously developed in Refs. [5, 6]. Thanks to the explicit modelling of duality violating (DV) contributions, this analysis provides a complete description of the V and A spectral functions as a function of the energy-squared s, including in the region $s > m_\tau^2$.

The resulting complete representation of the spectral functions allows one to construct the (subtracted) vacuum polarizations in both channels, and the unsubtracted vacuum polarization in the $V - A$ channel, as a function of s. Combining these representations with the analytic expressions derived from chiral perturbation theory (ChPT), which are known to next-to-next-to-leading order (NNLO) [7] allows us to extract the low-energy constant (LEC) C_{87}, and a linear combination of L_{10}, $C_{12} - C_{61} + C_{80}$ and $C_{13} - C_{62} + C_{81}$ from conventional chiral sum rules for the non-strange $V - A$ channel [8, 9], and $C_{12} + C_{61} + C_{80}$ and a linear combination of L_{10} and $C_{12} - C_{61} + C_{80}$ from flavor-breaking ($ud - us$) chiral sum rules [10] in the $V \pm A$ channels. These determinations employ as input existing values of L_5 and L_9, estimates existing from both phenomenology [11, 12] and the lattice [13, 14]. In order to disentangle further L_{10}, $C_{12} - C_{61} + C_{80}$ and $C_{13} - C_{62} + C_{81}$, we exploit the dependence of the coefficients of these LEC combinations appearing in the $V - A$ polarization on the pion and kaon masses using data for this polarization from the lattice [15, 16].[2]

The goal of this article is to update the analysis of Refs. [15–17], replacing the experimental data for the non-strange spectral functions, which previously came from OPAL [18], with the revised data from Ref. [1], and, at the same time, updating the lattice input of Ref. [16]. The expectation is that the errors on L_{10} and the accessible NNLO combinations

[1] The corrected data may be found at http://aleph.web.lal.in2p3.fr/tau/specfun13.html.

[2] We hope to revisit the determination of L_5 and L_9 from the ud and us V polarizations in the future, but it remains to be seen whether the errors from such an analysis are competitive with those of Refs. [11–14]. Here we will use values from the literature.

will decrease, because the ALEPH data are more precise, especially in the low-energy region. Improvements in the lattice data should also help in the process of disentangling L_{10} from the combinations $C_{12} - C_{61} + C_{80}$ and $C_{13} - C_{62} + C_{81}$, described previously in Ref. [15]. In the current analysis, we employ a slightly different input for L_5, choosing now to use the $2+1$-flavor estimate of Ref. [14], $L_5^r(\mu = 770\text{ MeV}) = 0.84(38) \times 10^{-3}$. This value is the $2+1$-flavor estimate adopted in Ref. [13], and straddles several nominally more precise, but not mutually consistent, estimates, including the result $L_5^r(\mu = 770\text{ MeV}) = 0.58(13) \times 10^{-3}$ of Ref. [11] used in our previous analysis.

While the main emphasis here is on the LECs of ChPT, we will also update our estimates for the operator product expansion (OPE) condensates $C_{6,V-A}$ and $C_{8,V-A}$, which are order parameters (in contrast to the analogous condensates in the $V+A$ channel). By their nature, these condensates are sensitive to the high-energy part of the spectral function, which is less well known. Therefore, it is interesting to compare the values for the condensates obtained from the OPAL and ALEPH data.

II. THEORY COMPENDIUM

In this section, we briefly collect the definitions and relations between quantities needed in order to present our results. More detailed overviews can be found in Refs. [15, 17] and references therein. We first consider the required sum-rule results, and then collect relevant results from ChPT.

A. Sum rules

The vacuum polarizations, $\Pi_T^{(J)}(Q^2)$, with $T = V$, $V \pm A$ and $J = 0, 1$, are the spin J scalar parts of the corresponding current-current two-point functions, $\Pi_{\mu\nu}^T(q)$, and are related to the corresponding spectral functions, $\rho_T^{(J)}(s)$, by finite-energy sum rules (FESRs) [19]. The flavor ud and us versions of these spectral functions can be obtained from experimental differential hadronic τ decay distribution data for energies up to the τ mass [20, 21]. Above the τ mass one needs a theoretical representation, and we will use the one obtained in Ref. [4].

First, let us consider the non-strange $V - A$ channel, for which the vacuum polarization obeys the unsubtracted dispersion relation

$$\begin{aligned}\Pi_{V-A}(Q^2) \;&\equiv\; \Pi^{(0+1)}_{V-A}(Q^2) = \int_0^\infty ds\; \frac{\rho_V(s)-\rho_A(s)}{s+Q^2}\;, \\ \Pi^{(0+1)}_{V-A}(Q^2) \;&\equiv\; -\frac{1}{3q^2}\left(g^{\mu\nu}-\frac{4q^\mu q^\nu}{q^2}\right)\Pi^{V-A}_{\mu\nu}(q)\;,\end{aligned} \tag{2.1}$$

where the Euclidean momentum-squared $Q^2 = -q^2$. Here ρ_V (ρ_A) is the non-strange, $I=1$ vector (axial) spectral function summing the angular momentum $J=1$ and $J=0$ contributions. Generalizing the definition of $\Pi_{V-A}(Q^2)$, we also define functions $\Pi^{(w)}_{V-A}$, to be used in the restricted sense employed below, involving additional polynomial weight factors $w(y)$:

$$\Pi^{(w)}_{V-A}(Q^2) = \int_0^\infty ds\; w(s/s_0)\;\frac{\rho_V(s)-\rho_A(s)}{s+Q^2} \tag{2.2}$$

with $0 < s_0 \le m_\tau^2$. In what follows, we will use the weights

$$w_k(y) = (1-y)^k\;, \quad k=1,2\;. \tag{2.3}$$

In evaluating the integral in Eq. (2.2), we will use the ALEPH experimental data for the spectral functions for $s \le s_0$, and the duality-violating (DV) part

$$\rho_V(s)-\rho_A(s) \approx \rho_V^{\rm DV}(s)-\rho_A^{\rm DV}(s)\;, \qquad s \ge s_0\;, \tag{2.4}$$

above s_0, with $\rho_V^{\rm DV}(s)-\rho_A^{\rm DV}(s)$ equal to $1/\pi$ times the imaginary part of $\Pi^{\rm DV}_{V-A}(Q^2)$, defined, in turn, from

$$\Pi_{V-A}(Q^2) = \Pi^{\rm OPE}_{V-A}(Q^2) + \Pi^{\rm DV}_{V-A}(Q^2)\;, \tag{2.5}$$

for $|Q^2| \ge s_0$, where the OPE part has the form

$$\Pi^{\rm OPE}_{V-A}(Q^2) = \sum_{k=1}^\infty \frac{C_{2k,V/A}}{(Q^2)^k}\;, \tag{2.6}$$

with $C_{2k,V/A}$ the OPE coefficients. We will always assume that we can choose s_0 smaller than m_τ^2, but large enough that the separation (2.5) into OPE and DV parts makes sense. We use a model for the DV parts of the spectral functions:

$$\rho_{V/A}(s) = e^{-\delta-\gamma s}\sin(\alpha+\beta s), \tag{2.7}$$

with α, β, γ, and δ parameters which differ in the V and A channels. The form of this *ansatz* was motivated in Ref. [22], and it was shown in Refs. [4, 5] that this model can be used to successfully parametrize the resonance structure in the data for $s \gtrsim 1.4$ GeV^2. (For earlier work and further discussion, see also Ref.[23].) Here we will fix the values of the V and A DV parameters using the results of the FOPT fit of Table IV of Ref. [4] with $s_{\rm min} = 1.55$ GeV^2.[3]

In what follows we will denote by $\overline{\Pi}_A$, $\overline{\Pi}_A^{(w)}$ and $\overline{\rho}_A$ the versions of Π_A, $\Pi_A^{(w)}$ and ρ_A from which the pion pole contribution has been subtracted. Analogous pion-pole-subtracted versions of the $V \pm A$ polarizations and spectral functions are denoted $\overline{\Pi}_{V\pm A} = \Pi_V \pm \overline{\Pi}_A$, $\overline{\Pi}_{V\pm A}^{(w)} = \Pi_V^{(w)} \pm \overline{\Pi}_A^{(w)}$ and $\overline{\rho}_{V\pm A} = \rho_V \pm \overline{\rho}_A$.

For $Q^2 < 4m_\pi^2$, $\overline{\Pi}_{V-A}(Q^2)$ admits a Taylor expansion, and we can thus define the intercept and slope at $Q^2 = 0$,

$$\overline{\Pi}_{V-A}(Q^2) = -8L_{10}^{\rm eff} - 16C_{87}^{\rm eff} Q^2 + O(Q^4) \ . \tag{2.8}$$

Employing FESRs for the weights (2.3) and analytic results for the OPE coefficients $C_{2,V-A}$ and $C_{4,V-A}$ [24, 25], it follows that [5]

$$\begin{aligned} -8L_{10}^{\rm eff} &= \overline{\Pi}_{V-A}(0) = \overline{\Pi}_{V-A}^{(w_1)}(0) + \frac{2f_\pi^2}{s_0} \qquad (2.9)\\ &= \overline{\Pi}_{V-A}^{(w_2)}(0) + \frac{4f_\pi^2}{s_0}\left[1 - \frac{17}{16\pi^2}\left(\frac{\alpha_s(s_0)}{\pi}\right)^2 \frac{m_u(s_0)m_d(s_0)}{f_\pi^2} - \frac{m_\pi^2}{2s_0}\left(1 + \frac{4}{3}\frac{\alpha_s(s_0)}{\pi}\right)\right] \ . \end{aligned}$$

Since the terms proportional to $\alpha_s(s_0)$ lead to effects smaller than the experimental errors, we will omit these terms from the actual analysis leading to our results for $L_{10}^{\rm eff}$ and $C_{87}^{\rm eff}$.

In addition to the information obtained from the flavor ud V-A channel, further constraints can be obtained from inverse-moment FESRs (IMFESRs) for the flavor-breaking differences

$$\Delta\Pi_T(Q^2) \equiv \Pi_{ud;T}^{(0+1)}(Q^2) - \Pi_{us;T}^{(0+1)}(Q^2) \ , \tag{2.10}$$

defined in Ref. [17]. This analysis represents an extension of the V channel IMFESR analysis employed in obtaining a first determination of the NNLO LEC C_{61}^r in Ref. [10]. Note that the $\Delta\Pi_T(Q^2)$ are finite and satisfy unsubtracted dispersion relations. The IMFESRs of Ref.

[3] Essentially identical results are obtained if one employs instead the results of the corresponding CIPT fit.

[17] have the forms

$$\begin{aligned}
\Delta\Pi_V(0) &= \int_{s_{\rm th}}^{s_0} ds\, \frac{w(s/s_0)}{s}\, \Delta\rho_V(s) + \frac{1}{2\pi i}\oint_{|z|=s_0} dz\, \frac{w(z/s_0)}{z}\, [\Delta\Pi_V(-z)]^{\rm OPE}\ , \\
\Delta\overline{\Pi}_{V\pm A}(0) &= \int_{s_{\rm th}}^{s_0} ds\, \frac{w(s/s_0)}{s}\, \Delta\overline{\rho}_{V\pm A}(s) \pm \left[\frac{f_K^2}{s_0}\, f^w_{\rm res}(y_K) - \frac{f_\pi^2}{s_0}\, f^w_{\rm res}(y_\pi)\right] \\
&\qquad + \frac{1}{2\pi i}\oint_{|z|=s_0} dz\, \frac{w(z/s_0)}{z}\, [\Delta\Pi_{V\pm A}(-z)]^{\rm OPE}\ ,
\end{aligned} \tag{2.11}$$

in which $s_{\rm th}$ is the continuum threshold $4m_\pi^2$, $y_\pi = m_\pi^2/s_0$, $y_K = m_K^2/s_0$, and

$$f(y) = \frac{2}{y}\,(w(0) - w(y))\ . \tag{2.12}$$

As long as we retain the exact $\Delta\Pi(-z)$ in the contour integrals of Eq. (2.11), the full right-hand sides are necessarily independent of the weight choice w, provided we restrict our attention to $w(y)$ all having $w(0) = 1$. In Eq. (2.11) we dropped the DV term from the split (2.5), and kept only the OPE contribution. It is reasonable to do so, because the only weights we will use will be triply pinched, *i.e.*, contain a factor $(z - s_0)^3$, which suppresses DVs strongly. In our analysis, we will use the weights

$$\begin{aligned}
\hat{w}(y) &= (1-y)^3\ , \\
w_{\rm DK}(y) &= (1-y)^3\left(1 + y + \frac{1}{2}y^2\right) = 1 - 2y + \frac{1}{2}\left(y^2 + y^3 + y^4 - y^5\right)\ .
\end{aligned} \tag{2.13}$$

The IMFESR with $w_{\rm DK}$ was first considered in Ref. [10].

The quantities $\overline{\Pi}_{V-A}(0)$, $\overline{\Pi}'_{V-A}(0)$, $\Delta\Pi_V(0)$ and $\Delta\overline{\Pi}_{V\pm A}(0)$ can be obtained from hadronic τ-decay data, which yield the spectral functions $\rho_{V/A}(s)$, either directly, for $s < m_\tau^2$, or, where it is needed for $s > m_\tau^2$ in the $V-A$ channel analysis, indirectly through Eq. (2.4), using the values of the DV parameters obtained from the fits to these same data, described in Ref. [4]. In addition, one needs the experimental values of m_π, f_π,[4] m_K and f_K. For a detailed discussion of the OPE contributions to Eq. (2.11), we refer to Ref. [17].[5] A key point is that the numerical contributions from the OPE terms to $\Delta\Pi_V(0)$ and $\Delta\overline{\Pi}_{V\pm A}(0)$ are very small. That implies that even if these OPE contributions are not very well known, and one has therefore to include very conservative estimates of their errors in the total error budget, the impact on our final errors is minor.

[4] Our normalization is such that $f_\pi = 92.21(14)$ MeV.

[5] See, in particular, Sec. III.B of that reference.

As already noted above, we have dropped DV contributions in Eq. (2.11). The reason is that these are very suppressed for the weights (2.13), which are triply pinched, and moreover suppress the large-s region by an additional factor $1/s$. Since the s_0 dependence from both the OPE and DVs is non-trivial, our treatment of the OPE and the omission of DVs can be tested by checking the s_0 independence of $\Delta\Pi_V(0)$, $\Delta\overline{\Pi}_{V+A}(0)$ and $\Delta\overline{\Pi}_{V-A}(0)$ produced using the right-hand side of the corresponding IMFESR.

B. Chiral perturbation theory

The motivation for considering the quantities L_{10}^{eff}, C_{87}^{eff}, $\Delta\Pi_V(0)$, and $\Delta\overline{\Pi}_{V\pm A}(0)$ is that they all depend on NLO and NNLO LECs of the chiral effective theory, and thus yield information on the QCD values for these LECs if sufficiently accurate data (from experiment or lattice QCD) are available. Here we will collect the relevant NNLO ChPT expressions needed in order to connect the quantities defined in the previous subsection to the LECs of ChPT.

The representation of $\overline{\Pi}_{V-A}(Q^2)$ to NNLO in ChPT has the form

$$\begin{aligned}\overline{\Pi}_{V-A}(Q^2) \;=\; & -8\Big(1-4(2\mu_\pi+\mu_K)\Big)L_{10}^r + 32m_\pi^2\,(C_{12}^r - C_{61}^r + C_{80}^r) \\ & +32(2m_K^2+m_\pi^2)\,(C_{13}^r - C_{62}^r + C_{81}^r) - 16C_{87}^r Q^2 + R_{\pi K}(\mu, Q^2; L_9^r)\;,\end{aligned} \tag{2.14}$$

where the explicit expression for $R_{\pi K}(\mu, Q^2; L_9^r)$ can be reconstituted from the results of Ref. [7].[6] The subscript πK indicates that R depends also on m_π, m_K and f_π, in addition

[6] Since the value of L_9^r is well known [12], we treat the loop contribution proportional to this LEC as known, and we thus include this contribution in $R_{\pi K}(\mu, Q^2; L_9^r)$.

to the explicitly shown arguments.[7] $L_{10}^{\rm eff}$ and $C_{87}^{\rm eff}$ are then given by:

$$\begin{aligned}
\overline{\Pi}_{V-A}(0) &= -8L_{10}^{\rm eff} \\
&= -8L_{10}^r\Big(1-4(2\mu_\pi+\mu_K)\Big)+16(2\mu_\pi+\mu_K)L_9^r \qquad (2.15a)\\
&\quad +32m_\pi^2\,(C_{12}^r-C_{61}^r+C_{80}^r)+32(2m_K^2+m_\pi^2)\,(C_{13}^r-C_{62}^r+C_{81}^r)\\
&\quad +\hat{R}_{\pi K}(\mu,0)\ ,\\
\overline{\Pi}'_{V-A}(0) &= -16C_{87}^{\rm eff} \qquad (2.15b)\\
&= -16C_{87}^r+\frac{1}{4\pi^2 f_\pi^2}\left(1-\log\frac{\mu^2}{m_\pi^2}+\frac{1}{3}\log\frac{m_K^2}{m_\pi^2}\right)L_9^r\\
&\quad +\left.\frac{\partial \hat{R}_{\pi K}(\mu,Q^2)}{\partial Q^2}\right|_{Q^2=0}\ ,\\
\mu_P &= \frac{m_P^2}{32\pi^2 f_\pi^2}\log\frac{m_P^2}{\mu^2}\ , \qquad (2.15c)
\end{aligned}$$

where $\hat{R}_{\pi K}(\mu,Q^2)$ is the part of $R_{\pi K}(\mu,Q^2;L_9^r)$ independent of L_9^r. Here the superscript r denotes the values of LECs renormalized in the $\overline{\rm MS}$ scheme at scale μ, which we will take to be $\mu=770$ MeV in what follows.

We will also need the NNLO ChPT expressions for $\Delta\Pi_T(0)$, $T=V$, $V\pm A$, but only at physical values of m_π, m_K and f_π. We therefore give the expressions in terms of the LECs with the numerical values of the coefficients for the NLO LECs $L_{5,9,10}^r$ evaluated at $m_\pi=139.57$ MeV, $M_K=495.65$ MeV,[8] $f_\pi=92.21$ MeV, and $\mu=770$ MeV:

$$\begin{aligned}
\Delta\Pi_V(0) &= 0.00775-0.7218\,L_5^r+1.423\,L_9^r+1.062\,L_{10}^r+32(m_K^2-m_\pi^2)C_{61}^r\ ,\\
\Delta\overline{\Pi}_{V+A}(0) &= 0.00880-0.7218\,L_5^r+1.423\,L_9^r+32(m_K^2-m_\pi^2)\,[C_{12}^r+C_{61}^r+C_{80}^r]\ ,\\
\Delta\overline{\Pi}_{V-A}(0) &= 0.00670-0.7218\,L_5^r+1.423\,L_9^r+2.125\,L_{10}^r\\
&\qquad -32(m_K^2-m_\pi^2)\,[C_{12}^r-C_{61}^r+C_{80}^r]\ . \qquad (2.16)
\end{aligned}$$

Equations (2.14), (2.15) and (2.16) give access to combinations involving the LECs L_{10}^r, C_{87}^r, and the linear combinations

$$\begin{aligned}
\mathcal{C}_0^r &\equiv 32m_\pi^2\,(C_{12}^r-C_{61}^r+C_{80}^r)\ ,\\
\mathcal{C}_1^r &\equiv 32\left(m_\pi^2+2m_K^2\right)(C_{13}^r-C_{62}^r+C_{81}^r)\ . \qquad (2.17)
\end{aligned}$$

[7] In Ref. [15] we denoted this term simply as $R(Q^2;L_9^r)$.

[8] This is the average of the charged and neutral kaon masses. Taking just the charged or neutral value has no impact on our final results.

In addition, the LECs L_5^r, C_{61}^r and the linear combination $C_{12}^r + C_{61}^r + C_{80}^r$ appear in Eq. (2.16). Our goal is to use the ALEPH hadronic τ-decay data to extract L_{10}^r, C_{61}^r, C_{87}^r and $C_{12}^r \pm C_{61}^r + C_{80}^r$, using the known values of L_5^r and L_9^r. However, L_{10}^r appears in a linear combination with $\mathcal{C}_0^r$ and $\mathcal{C}_1^r$ that does not depend on Q^2, and we thus need other information in order to disentangle L_{10}^r, $\mathcal{C}_0^r$ and $\mathcal{C}_1^r$ from each other. This can be done by using lattice data for different values of the meson masses, and such data are available for $T = V - A$ [16].[9]

We thus will consider, following Ref. [15],

$$\Delta\overline{\Pi}_{V-A}^{\mathrm{L,E}}(Q^2) \equiv \overline{\Pi}_{V-A}^{\mathrm{L,E}}(Q^2) - \overline{\Pi}_{V-A}(Q^2) \ , \tag{2.18}$$

the difference between the non-strange pion-pole-subtracted $V - A$ correlator $\overline{\Pi}_{V-A}^{\mathrm{L,E}}(Q^2)$ evaluated on a lattice ensemble E with π and K masses and decay constants different from the physical ones, and the same correlator, $\overline{\Pi}_{V-A}(Q^2)$, evaluated at the same Q^2, for the physical quark mass case. The latter is obtained from the dispersive representation using spectral functions obtained from the hadronic τ decay data. It then follows that in terms of LECs this difference can be expressed as

$$\Delta\overline{\Pi}_{V-A}^{\mathrm{L,E}}(Q^2) = \Delta R_{\pi K}^{\mathrm{L,E}}(\mu, Q^2; L_9^r) \ + \ \delta_{10}^{\mathrm{L,E}}\, L_{10}^r \ + \ \delta_0^{\mathrm{L,E}}\mathcal{C}_0^r \ + \ \delta_1^{\mathrm{L,E}}\mathcal{C}_1^r \ , \tag{2.19}$$

where $\Delta R^{\mathrm{L,E}}(\mu, Q^2; L_9^r)$ and the Q^2-independent coefficients $\delta_{10,0,1}^{\mathrm{L,E}}$ are known in terms of the lattice and physical meson masses and decay constants, and the chiral renormalization scale μ. Evaluating Eq. (2.18) using lattice and dispersive results, Eq. (2.19) yields a constraint on L_{10}^r, $\mathcal{C}_0^r$ and $\mathcal{C}_1^r$ for each ensemble, E, and each Q^2. As explained in more detail below, different choices of Q^2 for fixed E provide self-consistency checks on the use of the lattice data.

III. INPUT DATA

We will evaluate $\Pi_{V-A}(Q^2)$ and $\Pi_{V-A}^{(w_k)}(Q^2)$ using ALEPH experimental data [1] for the spectral functions $\rho_V(s)$ and $\overline{\rho}_A(s)$ for $s \le s_0 = s_{\mathrm{switch}}$,[10] and approximating the difference $\rho_V(s) - \overline{\rho}_A(s)$ by Eq. (2.4) for $s \ge s_0 = s_{\mathrm{switch}}$, with values for the DV parameters from the

[9] No such lattice analysis has been performed yet for the us channel, which is why we will use known values for L_5^r and L_9^r here.

[10] For details on the handling of the ALEPH data, see Ref. [4].

combined V and A channel fits of Ref. [4]. We will choose $s_{\rm switch}$ to be the upper end of an ALEPH bin, obtaining, in the notation adopted in Ref. [4],

$$\begin{aligned}\overline{\Pi}^{(w)}_{V-A}(Q^2) \;=\; & \sum_{\texttt{sbin}<s_{\rm switch}} \int_{\texttt{sbin}-\texttt{dsbin}/2}^{\texttt{sbin}+\texttt{dsbin}/2} ds\; w\left(\frac{s}{s_{\rm switch}}\right)\; \frac{\rho_V(\texttt{sbin})-\overline{\rho}_A(\texttt{sbin})}{\texttt{sbin}+Q^2} \\ & + \int_{s_{\rm switch}}^{\infty} ds\; w\left(\frac{s}{s_{\rm switch}}\right)\; \frac{\rho_V^{\rm DV}(s)-\rho_A^{\rm DV}(s)}{s+Q^2}\;. \qquad (3.1)\end{aligned}$$

Here we will choose $s_{\rm switch} = 1.55\ \mathrm{GeV}^2$, the value of $s_{\rm min}$ which produced the best fit to the weighted spectral integrals in our extraction of α_s in Ref. [4]. Since we are only interested in the behavior of $\overline{\Pi}^{(w)}_{V-A}(Q^2)$ at values of $Q^2 \ll 1.55\ \mathrm{GeV}^2$, the right-hand side of Eq. (3.1) is very insensitive to the precise choice of $s_{\rm switch}$, and varying it within the range of $s_{\rm min}$ values for which we obtained good fits in Ref. [4] has no effect on either the values or errors we will obtain for the LECs below.

We have fully propagated all errors and correlations in the results we will report on below. In particular, the DV parameter values used in Eq. (3.1) are correlated with the data, and we have computed these correlations using the linear error propagation method summarized in the appendix of Ref. [6] (see, in particular, Eq. (A.4) of that reference, which can be used to express the parameter-data covariances in terms of the data covariance matrix).

For the us data needed in order to evaluate the strange spectral integrals in Eq. (2.11) we will use exactly the same treatment and input as used in Ref. [17]. The $K\pi$ data employed is that of Refs. [27, 28], the $K^-\pi^+\pi^-$ data that of Refs. [29, 30] and the $\bar{K}^0\pi^-\pi^0$ data that of Ref. [31]. It is only for these exclusive strange modes that high-statistics B-factory distribution results exist. For other, higher multiplicity strange modes, we are thus forced to employ the older ALEPH results with much lower statistics [26]. For a detailed discussion of the treatment of all the us data, the reader is referred to Sec. III.C of Ref. [17].

We collect the values of all other external input parameters:

$$\begin{aligned}
m_\pi &= 139.57 \text{ MeV} , \\
m_K &= 495.65 \text{ MeV} , \\
m_\eta &= 547.85 \text{ MeV} , \\
f_\pi &= 92.21(14) \text{ MeV} , \\
f_K &= 110.5(5) \text{ MeV} , \\
L_5^r(\mu = 770 \text{ MeV}) &= 0.84(38) \times 10^{-3} , \qquad [14] , \\
L_9^r(\mu = 770 \text{ MeV}) &= 5.93(43) \times 10^{-3} , \qquad [12] .
\end{aligned} \tag{3.2}$$

For the value of L_5^r we choose the value reported of Ref. [14], which has a larger error than the value of Ref. [11] we used in Ref. [15]. The comparison with other values in Ref. [13] shows that the error quoted in Eq. (3.2) above covers these other values, and we thus consider its use to represent a conservative choice.

IV. RESULTS

In this section, we present the results of our analysis, dividing the presentation into several parts. We first present our values for $L_{10}^{\rm eff}$ and $C_{87}^{\rm eff}$, which are based purely on non-strange τ-decay data, then derive additional constraints employing also the lattice data, and finally use the us spectral data to obtain further constraints via the flavor-breaking IMFESRs. The output is a determination of L_{10}^r, C_{61}^r, C_{87}^r, $C_{12}^r - C_{61}^r + C_{80}^r$ and $C_{13} - C_{62} + C_{81}$, in terms of the known values for L_5^r and L_9^r. In the final subsection, we switch gears, and consider what we can learn from the ALEPH data about the OPE coefficients $C_{6,V-A}$ and $C_{8,V-A}$.

A. Effective LECs

We first give the values of $L_{10}^{\rm eff}$ and $C_{87}^{\rm eff}$, which follow directly from Eq. (2.9) and the evaluation of Eq. (3.1). For the three different weights we find

$$L_{10}^{\rm eff} = -6.482(64) \times 10^{-3} , \qquad (\text{from } \overline{\Pi}_{V-A}(0)) , \tag{4.1a}$$

$$= -6.486(64) \times 10^{-3} , \qquad (\text{from } \overline{\Pi}_{V-A}^{(w_1)}(0)) , \tag{4.1b}$$

$$= -6.446(50) \times 10^{-3} , \qquad (\text{from } \overline{\Pi}_{V-A}^{(w_2)}(0)) . \tag{4.1c}$$

These values are consistent with those found employing OPAL data in Ref. [15], but more precise, with errors about a factor two smaller. Our results are also compatible with the determination of Ref. [33] which was also based on the revised data set but again, our results have errors smaller by a factor of about two. The contribution from the DV part of the spectral function in Eq. (3.1) ranges from 3% for the first estimate in Eq. (4.1) to about 1% for the third estimate. Their size is thus comparable with the quoted errors, suggesting that the uncertainty in the DV part due to the use of a model for DVs, Eq. (2.7), is negligible. From the slope of $\overline{\Pi}_{V-A}(Q^2)$ at zero we find

$$C_{87}^{\rm eff} = 8.38(18) \times 10^{-3}\ \mathrm{GeV}^2\ . \tag{4.2}$$

In this case, the contribution from the DV part is about 1%. These results are in good agreement with those in Refs. [32, 33]. However, our errors are somewhat smaller, and our analysis employs versions of the DV contributions fitted individually in the V and A channels, as required by the data, avoiding the simplifications employed in Ref. [32, 33] (see also the discussion in Ref. [15]).

From our best value for $L_{10}^{\rm eff}$, Eq. (4.1c), and using Eq. (2.15a), we find the constraint

$$\begin{aligned} L_{10}^r &= 0.6576 L_{10}^{\rm eff} + 0.001161 - 0.1712 L_9^r + 0.08220(\mathcal{C}_0^r + \mathcal{C}_1^r) \\ &= -0.004094(33)_{\rm ALEPH}(74)_{L_9^r} + 0.08220(\mathcal{C}_0^r + \mathcal{C}_1^r)\ . \end{aligned} \tag{4.3}$$

Together with information from the lattice on the combinations $\mathcal{C}_0^r$ and $\mathcal{C}_1^r$, this constraint will yield an estimate for L_{10}^r. Likewise, Eq. (2.15b) leads to the constraint [15]

$$C_{87}^{\rm eff} = C_{87}^r + 0.292 L_9^r + 0.00155\ \mathrm{GeV}^{-2}\ , \tag{4.4}$$

and, with Eqs. (4.2) and (3.2), this leads to the estimate

$$C_{87}^r = 5.10(22) \times 10^{-3}\ \mathrm{GeV}^{-2}\ . \tag{4.5}$$

B. Constraints using the lattice

As shown in Ref. [16], and noted already above, useful independent constraints on L_{10}^r, $\mathcal{C}_0^r$ and $\mathcal{C}_1^r$ can be obtained by considering the difference of the lattice-ensemble, E, and physical-mass, continuum results for $\overline{\Pi}_{V-A}(Q^2)$, evaluated at the same Q^2. [11] In what follows, we will use the superscript *phys* to specify quantities evaluated with physical values of the masses and decay constants.

Recasting the NNLO representation, Eq. (2.19), in the form

$$\delta_{10}^{\mathrm{L,E}}\, L_{10}^r + \delta_0^{L,E} \mathcal{C}_0^r + \delta_1^{\mathrm{L,E}} \mathcal{C}_1^r = \Delta\overline{\Pi}_{V-A}^{\mathrm{L,E}}(Q^2) - \Delta R_{\pi K}^{\mathrm{L,E}}(\mu, Q^2; L_9^r) \equiv T^{\mathrm{L,E}}(Q^2) , \tag{4.6}$$

one notes that the left-hand side is explicitly Q^2-independent, while the right-hand side is the difference of two Q^2-dependent terms. For values of Q^2 for which the NNLO representation is reliable, the $T^{\mathrm{L,E}}(Q^2)$ for fixed ensemble E but different Q^2 should be compatible within errors, thus providing non-trivial self-consistency checks. We will denote the average of the $T^{\mathrm{L,E}}(Q^2)$ for an ensemble satisfying these self-consistency tests by $T^{\mathrm{L,E}}$.

The explicit expression for $\Delta R_{\pi K}^{\mathrm{L,E}}(\mu, Q^2; L_9^r) \equiv R_{\pi K}^{\mathrm{L,E}}(\mu, Q^2; L_9^r) - R_{\pi K}^{phys}(\mu, Q^2; L_9^r)$ follows from the results of Ref. [7], but is very lengthy and hence not given here. The explicit expressions for the mass-dependent constants appearing on the LHS of Eq. (4.6) are

$$\begin{aligned}
\delta_{10}^{\mathrm{L,E}} &= 32 \left[(2\mu_\pi + \mu_K)^{\mathrm{L,E}} - (2\mu_\pi + \mu_K)^{phys} \right] , \\
\delta_0^{\mathrm{L,E}} &\equiv \frac{[m_\pi^2]^{\mathrm{L,E}}}{[m_\pi^2]^{phys}} - 1 , \\
\delta_1^{\mathrm{L,E}} &\equiv \frac{[m_\pi^2 + 2m_K^2]^{\mathrm{L,E}}}{[m_\pi^2 + 2m_K^2]^{phys}} - 1 .
\end{aligned} \tag{4.7}$$

The lattice data we employ in forming the lattice-minus-continuum constraints are obtained using the $n_f = 2+1$ domain wall fermion ensembles of the RBC/UKQCD collaboration. Details of the underlying simulations, including the values of the π and K masses and decay constants in lattice units, may be found in Refs. [34, 35], with updated values of the lattice spacings, a, obtained after incorporating results from the new physical point ensembles, given in Ref. [36]. The latter are required to convert the former to physical units.

We have used the following criteria in deciding on the choice of ensembles and Q^2 values to be employed. First, we restrict our attention to ensembles with $m_\pi < 350$ MeV. Second, we require the ensemble to have sufficiently many Q^2 points in the expected range of validity of

[11] Note that we are focussed here on $\overline{\Pi}_{V-A}(Q^2)$ for Q^2 in the chiral regime. For such low Q^2, lattice artifacts associated with the lattice cutoff $1/a$ play no role. In addition, even at much higher Q^2, it is known that there is a strong cancellation between such V and A channel discretization effects when one forms the $V - A$ difference [39].

the representation, Eq. (2.19), that meaningful self-consistency tests can be performed. With the errors at the lowest Q^2 point turning out to be very large for all ensembles considered, this means that a minimum of two additional such Q^2, or three in total, are required. Finally, we identify the range of validity of the representation Eq. (2.19) as follows. We first note that the supplemented NNLO representation of $\overline{\Pi}_{V-A}(Q^2)$, discussed in more detail in Refs. [15, 37], and the corresponding NNLO representation given above, both yield the same representation of the lattice-continuum difference $\Delta\overline{\Pi}^{\mathrm{L,E}}_{V-A}(Q^2)$. The supplemented form of $\overline{\Pi}_{V-A}(Q^2)$ incorporates resonance-induced NNNLO contributions analogous to those already encoded in the NNLO contribution proportional to C^r_{87} through the inclusion of an additional analytic term CQ^4. The inclusion of such a term was shown to extend the reliability of the supplemented version of the representations to significantly larger Q^2 in both the $V-A$ and V correlator cases [15, 37]. Here we investigate the supplemented NNLO fit by fixing L^{eff}_{10} and C^{eff}_{87} to the values given in Eqs. (4.1) and (4.2), and fitting the additional effective mass-independent NNNLO LEC, C, to the dispersive results for $\overline{\Pi}_{V-A}(Q^2)$ in the window $0 < Q^2 < Q^2_{\mathrm{max}}$. The range of validity of the supplemented form is then identified as the largest Q^2_{max} for which such a fit is successful within errors. The results of this exploration show that the supplemented form can be employed up to ~ 0.25 GeV^2, but not beyond. We thus restrict our attention further to ensembles having at least three Q^2 in the region below 0.25 GeV^2.

Three RBC/UKQCD ensembles satisfy the above criteria, two coarse ensembles with $1/a = 1.379(7)$ GeV and $m_\pi = 172$ and 250 MeV, and one intermediate ensemble with $1/a = 1.785(5)$ GeV and $m_\pi = 340$ MeV [36].[12] These are labelled $E = 1$, 2 and 3 in what follows. The coarse ensembles have eight points each below 0.25 GeV^2, the intermediate ensemble has four. The results for the $V-A$ correlators for all three of these ensembles pass the self-consistency tests discussed above.

The lattice-continuum constraints for the first two cases were obtained previously in Ref. [16]. While the results of Ref. [36] lead to a small shift in the value of $1/a$ for these ensembles, this change affects the constraints only through the values of the Q^2 at which the dispersive representation of the physical-mass correlator is evaluated, the values of a^2Q^2

[12] The corresponding m_K values are 495, 512 and 594 MeV, and the f_π values are 92.6, 98.5 and 105.3 MeV (in our normalization, in which $f_\pi = 92.2$ MeV for physical masses), for ensembles 1, 2 and 3, respectively.

being fixed by the lattice size. The small resulting Q^2 changes turn out to have no impact on the resulting $T^{\rm L,E}$ averages for these ensembles to the number of significant figures previously quoted. The resulting average $T^{\rm L,E}$ are thus those given in Ref.[16],

$$\begin{aligned} T^{\rm L,1} &= 0.0007(17)\ , \\ T^{\rm L,2} &= 0.0039(21)\ . \end{aligned} \tag{4.8}$$

The reader is referred to Ref. [16] for the figures showing the Q^2-stability of $T^{\rm L,1}(Q^2)$ and $T^{\rm L,2}(Q^2)$.

The third ensemble has a relative error on af_π a factor of 2 smaller than that for the two coarse ensembles, and hence a smaller uncertainty in the pion-pole subtraction involved in forming $\overline{\Pi}^{\rm L,E}_{V-A}(Q^2)$ from the directly measured unsubtracted version. In order to improve further the associated constraint, the statistics for this ensemble were increased using multiple time sources. This increase in statistics was greatly aided by the use of the HDCG algorithm of Ref. [38], employed in performing the propagator inversions. The results for $T^{L,3}(Q^2)$ in the low-Q^2 region produced by the improved data for this ensemble are shown in Fig. 1, and obviously display good Q^2-independence within errors.

The covariances of the corresponding lattice-minus-continuum differences for different Q^2 (equal to the sum of the covariances of the corresponding lattice and dispersive results) are strongly dominated by the lattice contributions. With the covariance matrix available, the average constraint value, $T^{\rm L,3}$, can be obtained by a standard, fully correlated χ^2 fit to the four $T^{\rm L,3}(Q^2)$ with $Q^2 < 0.25$ GeV^2 available for this ensemble. The result is

$$T^{\rm L,3} = 0.00625(48)\ . \tag{4.9}$$

The error here reflects only the errors on (and correlations among) the lattice results at different Q^2 and dispersive results at different Q^2. Additional errors due to the uncertainties on the input quantities $L_5^r(\mu)$ and $L_9^r(\mu)$, which are 100% correlated with the analogous uncertainties entering the $\overline{\Pi}_{V-A}(0)$ and flavor-breaking IMFESR constraints, are handled by running the analysis several times with different input $L_5^r(\mu)$ and $L_9^r(\mu)$ spanning the quoted error range.

C. Constraints using IMFESRs

In this subsection, we evaluate $\Delta\overline{\Pi}_T(0)$ with $T = V, A$, and $V \pm A$ from Eq. (2.11), for

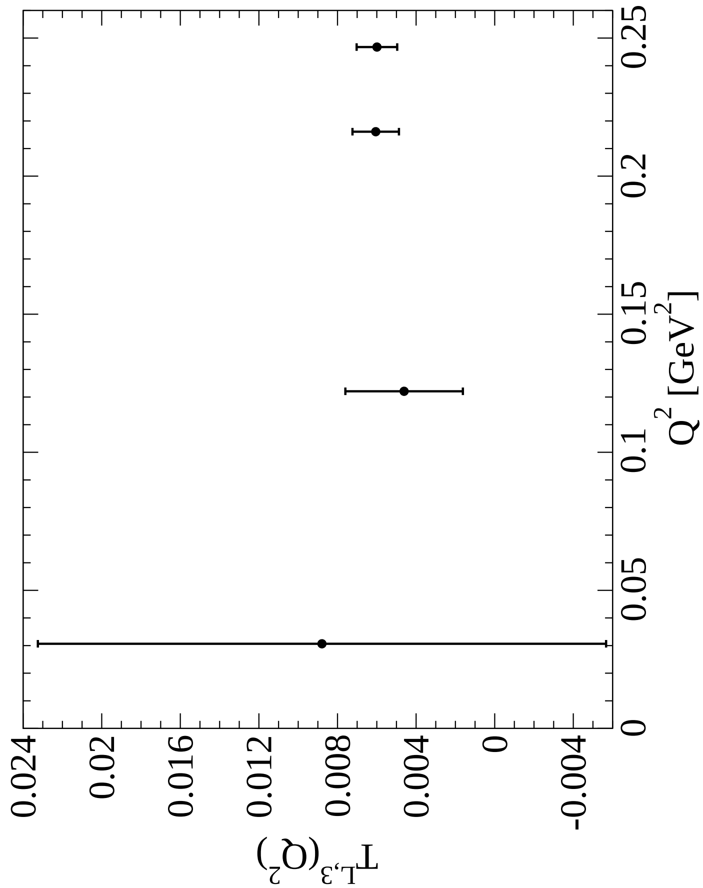

FIG. 1: *Low-Q^2 region results for $T^{L,3}(Q^2)$ generated using the improved RBC/UKQCD $n_f = 2+1$ data for this ensemble noted in the text*

values of s_0 between 2.15 GeV^2 and m_τ^2. Of course, $\Delta\overline{\Pi}_T(0)$ is independent of s_0, implying that evaluating the right-hand side of the expressions in Eq. (2.11) and checking for s_0 independence provides a self-consistency check on the use of the OPE, and the assumption that DVs can be neglected. The values we find are, using $w(y) = w_{\rm DK}(y)$ in Eq. (2.11),

$$\begin{aligned} \Delta\Pi_V(0) &= 0.0224(9) \ , \qquad (4.10) \\ \Delta\overline{\Pi}_A(0) &= 0.0113(8) \ , \\ \Delta\overline{\Pi}_{V+A}(0) &= 0.0338(10) \ , \\ \Delta\overline{\Pi}_{V-A}(0) &= 0.0111(11) \ . \end{aligned}$$

The results, only two of which are independent, are, of course, correlated. The quoted errors take into account the experimental errors in the ALEPH data, the uncertainties in the estimates of the OPE contribution, and the small residual s_0-dependence observed as s_0 is varied over the analysis window noted above. As a further check of the self-consistency of the values in Eq. (4.10), we have rerun the analysis using $w(y) = \hat{w}(y)$ in place of $w_{\rm DK}(y)$, and find results compatible with those obtained using $w_{\rm DK}(y)$ to well within the errors quoted in Eq. (4.10). The analysis method leading to the values (4.10) is identical to that of Ref. [15], to which we refer for a detailed discussion.

From the value for $\Delta\overline{\Pi}_{V+A}(0)$ and Eq. (2.16) we find, using the values of L_5^r and L_9^r in Eq. (3.2),

$$\begin{aligned} C_{12}^r + C_{61}^r + C_{80}^r &= 0.00237(13)_{\Delta\overline{\Pi}}(4)_{L_5^r}(8)_{L_9^r}\ \mathrm{GeV}^{-2} \\ &= 0.00237(16)\ \mathrm{GeV}^{-2}\ , \end{aligned} \tag{4.11}$$

where the subscript $\Delta\overline{\Pi}$ refers to the error coming from the value for $\Delta\overline{\Pi}_{V+A}(0)$ in Eq. (4.10), and on the second line we have combined errors in quadrature. The sum rules for $\Delta\overline{\Pi}_{V-A}(0)$ and $\Delta\Pi_V(0)$ provide the following two constraints on L_{10}^r and other combinations of NNLO LECs:

$$\begin{aligned} 2.125L_{10}^r - 32(m_K^2 - m_\pi^2)(C_{12}^r - C_{61}^r + C_{80}^r) &= -0.0034(11)_{\Delta\overline{\Pi}}(3)_{L_5^r}(6)_{L_9^r} \\ &= -0.0034(13)\ , \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} 1.062L_{10}^r + 32(m_K^2 - m_\pi^2)C_{61}^r &= 0.0071(9)_{\Delta\overline{\Pi}}(3)_{L_5^r}(6)_{L_9^r} \\ &= 0.0068(11)\ . \end{aligned} \tag{4.13}$$

Employing the constraints Eqs. (4.3), (4.12) and the $E = 1, 2, 3$ versions of Eq. (4.6), with Eqs. (4.8) and (4.9) as input on the right-hand side and Eq. (4.7) as input on the left-hand side, we find

$$\begin{aligned} L_{10}^r &= -0.00350(11)(13)_{L_5,L_9} = -0.00350(17)\ , \\ \mathcal{C}_0^r &= -0.00035(9)(4)_{L_5,L_9} = -0.00035(10)\ , \\ \mathcal{C}_1^r &= 0.0075(13)(8)_{L_5,L_9} = 0.0075(15)\ . \end{aligned} \tag{4.14}$$

Here the first error quoted comes primarily from the errors on the lattice data,[13] while the

[13] Lattice errors in the differences of Eq. (2.18) dominate over the errors coming from the τ spectral data.

second error comes from the errors in L_5^r and L_9^r. The value of $\mathcal{C}_0^r$ translates directly into

$$C_{12}^r - C_{61}^r + C_{80}^r = -0.00056(15)\ \mathrm{GeV}^{-2}\ , \tag{4.15}$$

the value of $\mathcal{C}_1^r$ into

$$C_{13}^r - C_{62}^r + C_{81}^r = 0.00046(9)\ \mathrm{GeV}^{-2}\ , \tag{4.16}$$

while the value of L_{10}^r in Eq. (4.14) together with Eq. (4.13) leads to the estimate

$$C_{61}^r = 0.00146(15)\ \mathrm{GeV}^{-2}\ . \tag{4.17}$$

For completeness, the result for the NNLO LEC combination entering the flavor-breaking A IMFESR is

$$C_{12}^r + C_{80}^r = 0.00090(9)\ \mathrm{GeV}^{-2}\ . \tag{4.18}$$

The use of spectral functions determined from the revised version of the ALEPH τ decay data, together with the new lattice results, leads to an improvement in the errors on all extracted quantities with respect to our previous work [15–17]. For L_{10}^r and the combination $C_{13}^r - C_{62}^r + C_{81}^r$ the improvement is more significant (a reduction by a factor of about 2.5). Other errors are reduced by a factor of about 1.5. Our result for L_{10}^r is compatible with that of Ref. [9] but has an uncertainty a factor of about 2 smaller. In addition to being smaller, our error also results from an analysis in which the relevant combinations of NNLO LECs are determined, from data, as part of the analysis. In contrast, the error on L_{10}^r in Ref. [9] is dominated by that assumed for the N_c-suppressed combination $C_{13}^r - C_{62}^r + C_{81}^r$. This estimate, which was based on $1/N_c$ arguments and a comparison with the non-N_c suppressed combination $C_{12}^r - C_{61}^r + C_{80}^r$, turns out to have been rather non-conservative due to the presence of significant cancellation in non-N_c-suppressed combination $C_{12}^r - C_{61}^r + C_{80}^r$. One should also bear in mind that the result of Ref. [9] was based on the uncorrected version of the ALEPH data, which, due to the missing unfolding contributions, had bin-to-bin correlations which were significantly underestimated. We do not comment on the relation of our result for L_{10}^r to those of earlier NLO determinations since it is now known that NNLO contributions, neglected in those determinations, are, in fact, non-negligible.

D. $V - A$ condensates

In addition to the LECs extracted in the previous subsections, $\overline{\Pi}_{V-A}(Q^2)$ also provides information on the OPE coefficients $C_{6,V-A}$ and $C_{8,V-A}$. Adapting Eq. (4.20) of Ref. [15] to the case of the ALEPH data, following the notation of Ref. [4], used also in Eq. (3.1) above, these two coefficients are given by

$$
\begin{aligned}
C_{6,V-A} &= \sum_{\mathtt{sbin}<s_{\rm switch}} \int_{\mathtt{sbin-dsbin/2}}^{\mathtt{sbin+dsbin/2}} ds\ (s-s_{\rm switch})^2 \left(\rho_V(\mathtt{sbin}) - \overline{\rho}_A(\mathtt{sbin})\right) \qquad (4.19a)\\
&\quad -2f_\pi^2\left(m_\pi^2 - s_{\rm switch}\right)^2 + \int_{s_{\rm switch}}^{\infty} ds\ (s-s_{\rm switch})^2 \left(\rho_V^{\rm DV}(s) - \rho_A^{\rm DV}(s)\right)\ ,\\
C_{8,V-A} &= -\sum_{\mathtt{sbin}<s_{\rm switch}} \int_{\mathtt{sbin-dsbin/2}}^{\mathtt{sbin+dsbin/2}} ds\ (s-s_{\rm switch})^2(s+2s_{\rm switch}) \left(\rho_V(\mathtt{sbin}) - \overline{\rho}_A(\mathtt{sbin})\right)\\
&\quad +2f_\pi^2\left(m_\pi^2 - s_{\rm switch}\right)^2 \left(m_\pi^2 + 2s_{\rm switch}\right)\\
&\quad -\int_{s_{\rm switch}}^{\infty} ds\ (s-s_{\rm switch})^2(s+2s_{\rm switch}) \left(\rho_V^{\rm DV}(s) - \rho_A^{\rm DV}(s)\right)\ . \qquad (4.19b)
\end{aligned}
$$

The first of these two expressions involves contributions proportional to $C_{2k,V-A}$ for $k = 1,\ 2,\ 3$, but the leading-order expressions [24, 25]

$$
\begin{aligned}
C_{2,V-A} &= -\frac{\alpha_s(\mu^2)}{\pi^3}\, m_u(\mu^2)m_d(\mu^2)\left(1 - \frac{\alpha_s(\mu^2)}{\pi}\left(\frac{17}{4}\log\frac{Q^2}{\mu^2} + c\right)\right) + \dots, \qquad (4.20a)\\
C_{4,V-A} &= -\frac{8}{3}\frac{\alpha_s}{\pi} f_\pi^2 m_\pi^2 + \dots\ , \qquad (4.20b)
\end{aligned}
$$

with c a numerical constant whose value is not required in what follows, suggest that the $D = 2$ and $D = 4$ terms are numerically negligible. A similar observation holds for the second of these expressions. The use of Eq. (4.19) to estimate $C_{6,V-A}$ and $C_{8,V-A}$ thus rests on the assumption that these estimates for the $D = 2$ and $D = 4$ contributions are sufficiently reliable, which is equivalent to assuming that the two Weinberg sum rules hold exactly. In this case, Eq. (4.19) leads to the estimates

$$
\begin{aligned}
\text{ALEPH}: \qquad C_{6,V-A} &= (-3.16 \pm 0.91) \times 10^{-3}\ \text{GeV}^6\ , \qquad (4.21)\\
C_{8,V-A} &= (-13.0 \pm 5.5) \times 10^{-3}\ \text{GeV}^8\ .
\end{aligned}
$$

These results correspond to the choice $s_{\rm switch} = 2.2$ GeV^2, which yields the smallest estimate for the errors on $C_{6,V-A}$ and $C_{8,V-A}$. We have checked that the central values remain stable as a function of $s_{\rm switch}$ as $s_{\rm switch}$ is varied between $s_{\rm min} = 1.55$ GeV^2 and m_τ^2. The results of Eq. (4.21) are in agreement within errors with those of Refs. [32, 33].

Let us compare these values with those we found from the OPAL data in Ref. [15]:

$$
\begin{aligned}
\text{OPAL}: \qquad C_{6,V-A} &= (-6.6 \pm 1.1) \times 10^{-3}\ \text{GeV}^6\ , \qquad (4.22)\\
C_{8,V-A} &= (5 \pm 5) \times 10^{-3}\ \text{GeV}^8\ .
\end{aligned}
$$

These values are compatible with the ALEPH ones, Eq. (4.21), only at the 2.4 σ level.

Instead of the weights employed in Eq. (4.19), one can use the weight s^2 to estimate $C_{6,V-A}$, and the weight s^3 to estimate $C_{8,V-A}$. These moments are less sensitive to dimension-four corrections than those of Eqs. (4.19). If we do so, we find the values

$$\begin{aligned} C_{6,V-A} &= (-4.4 \pm 5.2) \times 10^{-3}\ \mathrm{GeV}^6\ , \\ C_{8,V-A} &= (-8 \pm 18) \times 10^{-3}\ \mathrm{GeV}^8\ . \end{aligned} \tag{4.23}$$

These values have larger uncertainties because the moments employed do not suppress DVs but they are compatible within errors with those of Eq. (4.21) and (4.22). The smaller errors of Eq. (4.22), and Eq. (4.21) as well, on the other hand, rely on the *assumption* that $C_{2,V-A}$ and $C_{4,V-A}$ can be neglected, or, equivalently, the assumption that the first and second Weinberg sum rules are "exactly" satisfied by the spectral data, as opposed to satisfied only within experimental errors.[14] In principle, this assumption can be tested using the weights 1 and s on the right-hand side of Eq. (4.19) to obtain a direct determination of these coefficients. This yields values $C_{2,V-A} = (0.14 \pm 0.24) \times 10^{-3}$ GeV^2 and $C_{4,V-A} = (0.1 \pm 1.2) \times 10^{-3}$ GeV^4, compatible with the results of Eq. (4.20) but with errors which are sizeable. The present experimental data are thus compatible with values of $C_{2,4}$ larger than those obtained from Eq. (4.20), but also consistent with those expectations. The results are, in any case, consistent within errors with the assumption that the dimension-two and -four contributions can be neglected in Eq. (4.19).

In view of the discussion above, we conclude that current data lead to somewhat conflicting estimates for $C_{6,V-A}$ and $C_{8,V-A}$. This is not surprising, because these coefficients are sensitive to the large-s region of the data. In addition, we observe the contribution from the DV terms to the expressions on the right-hand side of Eq. (4.19), *i.e.*, the size of the DV contributions which must be subtracted to arrive at the values reported in Eq. (4.21), are non-negligible: about -0.28×10^{-3} GeV^6 and 3×10^{-3} GeV^8 for $C_{6,V-A}$ and $C_{8,V-A}$, respectively, which are consistent with the DV contributions found in Ref. [15] on the basis of OPAL data. We refrain from assigning individual errors to these DV contributions as

[14] We remind the reader that none of the three classical chiral sum rules (the two Weinberg sum rules and the DGMLY, or π electromagnetic mass difference, sum rule [40]) are imposed as part of the combined V, A channel fits of Ref. [4]. The results of these fits have, however, been tested, and shown to produce excellent agreement with all three of these sum rules.

they are highly correlated with the non-DV contributions. Note, however, that the ALEPH-OPAL condensate differences are significantly larger than the DV contributions, indicating that the discrepancy is dominated by the data parts of the relevant integrals.

V. CONCLUSION

Using the recently updated and corrected ALEPH results for the non-strange V and A hadronic τ decay distributions, existing results for the corresponding strange decay distributions, and updated lattice results for the non-strange $V-A$ polarization at heavier than physical meson masses, we have produced improved determinations of the NLO chiral LEC L_{10}^r and a number of NNLO LEC combinations. Those results are given in Eqs. (4.11), (4.14), (4.15), (4.16), (4.17) and (4.18). We have also used the non-strange ALEPH data to extract the dimension six and eight condensates appearing in the OPE representation of the $V-A$ polarization. These results are given in Eq. (4.21).

The improvements produced by using the new lattice results and the new ALEPH data in place of the old OPAL data reduce the fit component of the errors on L_{10}^r and the $1/N_c$-suppressed NNLO LEC combination $C_{13}^r - C_{62}^r + C_{81}^r$ by a factor of roughly 2.5 compared to our earlier analysis. The fit errors on the remaining NNLO LECs are about 2/3 of those of the previous analysis. It is worth noting that the improvement in the errors on $\Delta\Pi_V(0)$ and $\Delta\bar{\Pi}_{V\pm A}(0)$ found in Eqs. (4.10) relative to those of Ref. [17] is a result of the switch from OPAL ud data to the higher precision ALEPH ud data, a switch made possible by the release of the corrected ALEPH covariance matrix results. Taking, as in Ref. [15], 25% as an estimate of the expected reduction in size of contributions in going from one order to the next in the chiral expansion, we would expect the uncertainties from the neglect of NNNLO and higher contributions to be roughly 6% for L_{10}^r and 25% for the NNLO LECs. With the current fit errors, the total errors on all the LECs determined have reached these levels, suggesting that the optimal practical precision one can expect for an NNLO analysis has now been attained.

We find, in contrast, that using the higher precision ALEPH data produces essentially no improvement in the accuracy of the determination of the dimension six and eight $V-A$ OPE condensates. Our results for these quantities are in agreement with those of other ALEPH-based analyses, and show about 2.4σ discrepancies with the corresponding results obtained using the OPAL data. These discrepancies presumably reflect additional systematic uncertainties encountered in attempting to extract these small higher dimension contributions from existing data, and should be kept in mind if results based on one or the other of the two data sets are employed in other contexts.

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