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Observer dependence of angular momentum in general relativity and its relationship to the gravitational-wave memory effect

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We define a procedure by which observers can measure a type of special-relativistic linear and angular momentum (P^a, J^{ab}) at a point in a curved spacetime using only the spacetime geometry in a neighborhood of that point. The method is chosen to yield the conventional results in stationary spacetimes near future null infinity.

We also explore the extent to which spatially separated observers can compare the values of angular momentum that they measure and find consistent results. We define a generalization of parallel transport along curves which gives a prescription for transporting values of angular momentum along curves that yields the correct result in special relativity. If observers use this prescription, then they will find that the angular momenta they measure are observer dependent, because of the effects of spacetime curvature. The observer dependence can be quantified by a kind of generalized holonomy. We show that bursts of gravitational waves with memory generically give rise to a nontrivial generalized holonomy: there is, in this context, a close relation between the observer dependence of angular momentum and the gravitational-wave memory effect.

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I. INTRODUCTION AND SUMMARY

A. Angular momentum in general relativity

An interesting nonlinear feature of dynamical asymptotically flat solutions in general relativity is that there is no canonical way to define a special-relativistic angular momentum at future null infinity. This result follows from the work of Bondi, van der Burg, and Metzner [1] and Sachs [2, 3]. They showed that the group of asymptotic symmetries of asymptotically flat spacetimes at future null infinity is not the Poincaré group, but is instead an infinite dimensional group now known as the Bondi-Metzner-Sachs (BMS) group. Its structure is similar to that of the Poincaré group: rather than being a semidirect product of the Lorentz group with a four-parameter abelian group of spacetime translations, it is a similar product of the conformal group on a 2-sphere (which is isomorphic to the universal covering group of the Lorentz group) with an infinite-dimensional commutative group called the supertranslations [4]. The translations are a four-parameter normal subgroup of the larger group of supertranslations, from which the Bondi energy-momentum [5] is defined. The supertranslations, however, make relativistic angular momentum (the charge associated with the Lorentz symmetries) behave differently in asymptotically flat spacetimes than in Minkowski space. In the latter, angular momentum depends only upon a choice of origin, which is a consequence of the four spacetime translations in the Poincaré group; in the former, angular momentum depends upon a smooth function on the 2-sphere that parameterizes the supertranslations in the BMS group. This property is typically called the supertranslation ambiguity of angular momentum (e.g., [6]). It arises because there is no unique way to pick out a preferred Poincaré group with which to define a special-relativistic angular momentum.

Instead of special-relativistic linear and angular momentum, one has an infinite set of conserved charges, one associated with each generator of the BMS group [7, 8]. These charges can be computed from a surface integral over any cross section of future null infinity, and the difference between the values of a charge at two different cross sections is given by the integral of a 3-form (or "flux") over the region of future null infinity between the two cross sections. These BMS charges transform covariantly under BMS transformations, just as special relativistic linear and angular momentum transform under Poincaré transformations, and they include the Bondi four-momentum.

Some researchers have argued that it is necessary or desirable to give a definition of a preferred finite set of conserved charges, that would be more similar to the familiar conserved charges of special relativity [9–13]. An alternative philosophy, which we espouse, is that all of the BMS charges are physically relevant, and that one should try to understand more deeply their physical nature, starting with operational prescriptions by which they can be measured by asymptotic observers.

The purpose of this paper is an initial attempt to understand how BMS charges can be measured. For simplicity, we take a "bottom-up" approach: we suppose that observers who are unaware of the BMS group attempt to measure conserved charges, and ask how the charges they measure are related. [This is analogous to Newtonian observers who are ignorant of special relativity making measurements in Minkowski spacetime; their observations of Newtonian quantities are inconsistent because of Lorentz contraction, etc.] Similarly, here we find that the charges measured by different observers using special-relativistic methods are inconsistent due to spacetime curvature. One of our goals is to characterize or interpret the inconsistencies between different observers' measurements, and relate the inconsistencies to other measurable quantities.

For simplicity, we will restrict attention to measurements made in stationary regions, and focus on how measurements made in two successive stationary regions can be compared to one another.

More specifically, our main results are as follows:

- We give a local, operational definition of an "angular momentum" that can be measured by an observer at a point in a curved spacetime, using only information contained in the geometry in a neighborhood of that point. The result is a pair of tensors (P^a, J^{ab}) at that spacetime point. This prescription is chosen to give the expected result in stationary spacetime regions near future null infinity. See Sec. II for details.

- We define a method by which two observers at two different points in a curved spacetime can compare the values of angular momenta that they measure. The philosophy we adopt is to imagine observers who assume the validity of special relativity, and who make measurements based on this assumption. We devise a method of comparison based on a generalization of parallel transport, which reduces to the correct method in flat spacetimes. In curved spacetimes, the method of comparison will be curve dependent, and, in general, inconsistencies will arise when observers attempt to compare values of angular momenta. Therefore, from this point of view, angular momentum inevitably becomes observer dependent in curved spacetimes. See Sec. III for details.

- We identify a simple physical mechanism that accounts for and explains the observer dependence in simple cases. Specifically, two observers who measure the change in angular momentum of a given source can disagree on that change, since they disagree on where they believe the source used to be. They disagree on the source's original location because of the gravitational-wave memory effect, the permenent relative displacement of observers due to the passage of a burst of waves [14–16]. The memory effect has, unbeknownst to the observers, displaced them by different amounts. This argument is given in more detail in Sec. I B below.

- We argue that the close relation between gravitational-wave memory and the observer dependence of angular momentum is in fact very general, by using covariant methods and looking at a number of examples (Secs. III and IV). While the connection between gravitional-wave memory and angular-momentum ambiguity has often been noted, our analysis shows the explicit and precise form of the relationship in a general context. In addition, there is a close relation between gravitational-wave memory and the BMS supertranslation that relates the shear-free cuts of a stationary spacetime before a burst of gravitational waves to those after the burst, as noted by, for example, Strominger and Zhiboedov [17]. Therefore, our examples in Sec. III and IV also highlight the role of BMS supertranslations in the observer dependence of angular momentum in these simple contexts.

B. Universality of observer dependence of angular momentum

As discussed above, if different observers attempt to measure angular momentum in general relativity using special-relativistic methods, they will disagree on the results. In other words, angular momentum becomes observer dependent. In this section, we will show that this observer dependence is a universal and local feature of general relativity, independent of the choice of asymptotic boundary conditions. (However, this observer dependence does not necessarily imply the existence of ambiguities of the BMS type, as we discuss in Sec. V). We will compute the observer dependence explicitly in a specific simple case, and show that it is closely related to gravitational-wave memory.

Consider two observers A and B (Alice and Bob) in a flat region of spacetime, who are at rest with respect to one another and share a common inertial frame $(t, \mathbf{x})$. Suppose that they both measure the angular momentum of a nearby particle. The observer A will obtain the result

$$\mathbf{J}_A = \mathbf{S} + (\mathbf{x}_p - \mathbf{x}_A) \times \mathbf{p}, \tag{1.1}$$

where $\mathbf{S}$ is the intrinsic angular momentum of the particle, $\mathbf{p}$ is its momentum in the observer's common inertial frame, $\mathbf{x}_p$ is the location of the particle, and $\mathbf{x}_A$ is A's location. Here we assume that A measures the angular momentum about her own location. Similarly, observer B will measure an angular momentum about his own location, and obtain the result

$$\mathbf{J}_B = \mathbf{S} + (\mathbf{x}_p - \mathbf{x}_B) \times \mathbf{p}. \tag{1.2}$$

If A and B compare their measurements, they will find a difference given by

$$\mathbf{J}_A - \mathbf{J}_B = -(\mathbf{x}_A - \mathbf{x}_B) \times \mathbf{p}, \tag{1.3}$$

which is consistent with their measured relative displacement $\mathbf{x}_A - \mathbf{x}_B$.

Now suppose that a gravitational-wave burst of finite duration is incident on the observers. Thus the spacetime consists of a flat region, followed by a gravitational-wave pulse, followed by a subsequent flat region. We can adopt transverse-traceless (TT) coordinates (T, X^i) to describe the entire relevant spacetime region—before, during, and after the burst of waves—which we chose to coincide with the inertial-frame coordinates (t, x^i) before the burst. In the TT coordinates and in the linearized approximation, the metric is

$$ds^2 = -dT^2 + [\delta_{ij} + h_{ij}(T-Z)]dX^i dX^j, \qquad (1.4)$$

where for simplicity we have specialized to a burst propagating in the $+Z$ direction. At late times, the metric perturbation becomes constant, $h_{ij}(T-Z) \to h_{ij}^\infty =$ (constant), while at early times h_{ij} vanishes, and $T = t$, $X^i = x^i$.

We now extend the definition of the coordinates (t, x^i) to the region after the burst by defining

$$t = T, \quad x^i = \left[\delta_{ij} + \frac{1}{2}h_{ij}^\infty\right] X^j. \qquad (1.5)$$

These coordinates are then inertial coordinates after the burst. Now the observers A and B are freely falling, which implies that their TT coordinate locations X_A^i and X_B^i are conserved. Hence their relative displacement in the (t, x^i) inertial frame after the burst is

$$\mathbf{x}'_A - \mathbf{x}'_B = \left[\mathbf{1} + \frac{1}{2}\mathbf{h}^\infty\right] \cdot (\mathbf{x}_A - \mathbf{x}_B). \qquad (1.6)$$

This is the standard formula for gravitational-wave memory. Here $\mathbf{x}'_\mathbf{A}$ and $\mathbf{x}'_\mathbf{B}$ are the locations of A and B in the inertial-frame coordinates after the burst.

In the spacetime region after the burst has passed, the observers A and B can again measure the angular momentum of the particle, in the new inertial frame (t, x^i). The observer A obtains the result

$$\mathbf{J}'_A = \mathbf{S}' + (\mathbf{x}'_p - \mathbf{x}'_A) \times \mathbf{p}\,, \qquad (1.7)$$

where primes denote quantities as measured after the burst. We imagine that the particle's spin and location may have changed in the intervening period, but for simplicity we assume that its momentum $\mathbf{p}$ has not. A similar formula applies to the observer B, and once again if A and B compare their measurements, they will find a difference given by

$$\mathbf{J}'_A - \mathbf{J}'_B = -(\mathbf{x}'_A - \mathbf{x}'_B) \times \mathbf{p}, \qquad (1.8)$$

which is consistent with their measured relative displacement after the burst $\mathbf{x}'_A - \mathbf{x}'_B$. So far, there is no observer-dependence.

Next, we assume that observer A wishes to compute the change in angular momentum of the particle between early and late times. This is given by

$$\delta\mathbf{J}_A = \mathbf{J}'_A - \mathbf{J}_A + \delta\mathbf{x}_A \times \mathbf{p}. \qquad (1.9)$$

Here $\delta\mathbf{x}_A$ is the change in A's location between early and late times, and the third term is necessary to transform the original angular-momentum measurement to her new location, so that she is subtracting angular momenta as measured about the same point. However, as far as observer A is concerned, $\delta\mathbf{x}_A$ vanishes, since she is an inertial observer sitting at the origin of her inertial frame. In particular, she is unaware of the effects of the gravitational-wave burst. [More generally, if the observer were accelerated by non-gravitational forces, she could measure $\delta\mathbf{x}_A$ using an accelerometer carried with her. In the present context, the accelerometer reading would be zero.]

Inserting the assumption $\delta\mathbf{x}_A = 0$ into Eq. (1.9) and subtracting a similar equation for B finally yields

$$\delta\mathbf{J}_B - \delta\mathbf{J}_A = (\mathbf{x}_B - \mathbf{x}_A) \times \mathbf{p} - (\mathbf{x}'_B - \mathbf{x}'_A) \times \mathbf{p}\,. \qquad (1.10)$$

Using the gravitational-wave-memory formula (1.6) simplifies this to

$$\delta\mathbf{J}_B - \delta\mathbf{J}_A = -\frac{1}{2}[\mathbf{h}^\infty \cdot (\mathbf{x}_B - \mathbf{x}_A)] \times \mathbf{p}\,. \qquad (1.11)$$

Thus, A and B disagree on the change in angular momentum, by an amount which is proportional to the gravitational-wave memory. Essentially what has happened is that the two observers disagree on where the particle used to be, because they have been displaced relative to one another by the gravitational-wave memory effect, and they assume there is no such relative displacement.

The result (1.11) will be re-derived by a more formal and covariant computation in Sec. IV A below.

C. Covariant description of angular momentum's observer dependence: Methods of this paper

While the example of the previous section was intuitively useful and suggestive of the generality of the phenomenon, it is important to have a covariant method for comparing angular momenta at different times and as measured by different observers. There are, however, several subtle aspects of how to define angular momentum and how to compare values between different observers in curved spacetime. The remainder of this paper is devoted to articulating a procedure that treats these issues and allows observers to compare angular momentum covariantly. We now give a brief sketch of the approach taken in this paper and summarize the organization of this paper's sections.

Section II contains a local operational definition of a linear and angular momentum (P^a, J^{ab}) that can be measured by individual observers. For simplicity we will refer to this pair simply as "angular momentum". Section II A defines the mathematical space in which the local angular momentum lives: the dual space of the space of Poincaré transformations from the tangent space at a point in spacetime to itself. The next part, Sec. II B, defines a prescription whereby an angular momentum (a particular element of this dual space) can be obtained from local measurements of the Riemann tensor and its derivatives. Section II C shows that the prescription for measuring angular momentum yields the expected value in stationary spacetimes near future null infinity, Sec. II D describes the accuracy and errors of the algorithm in more general spacetimes, Sec. II E notes that the algorithm is not unique, and Sec. II F focuses on the accuracy with which the center-of-mass worldline can be measured.

Section III describes how to compare angular momentum at different spacetime points. It defines a transport law in Sec. III A—which will be called the *affine transport*—that can be used for comparing angular momentum at two different spacetime points. Section III B explains in detail how to compare angular momentum at two points using the affine transport. When the curve is a closed loop, the transport law defines a *generalized holonomy* operation, the basic properties of which are given in Sec. III C. When the generalized holonomy reduces to the identity, it indicates that there is a consistent (observer-independent) notion of angular momentum for different observers along the curve; when it does not, it provides a notion of the size of the observer dependence in angular momentum between different observers along the curve. Section III D shows that the generalized holonomy contains four independent pieces. When the closed curve is generated by portions of worldlines of two freely falling observers, connected by spatial geodesics, each of the four pieces can be interpreted as a kind of gravitational-wave memory. In particular, for nearby geodesics, the generalized holonomy contains the usual gravitational-wave memory.

Section IV gives two examples of the generalized holonomy for an idealized spacetime consisting of a region of flat Minkowski space followed by a burst of linearized gravitational waves with memory that propagates away leaving a second flat Minkowski spacetime region. The first half of the section, Sec. IV A, reproduces the nearly Newtonian argument of Sec. I B using the language of the generalized holonomy. The next half of the section, Sec. IV B, examines the more general example of a gravitational wave expanded in symmetric trace-free multipoles that is emitted radially outward from a pointlike source. The paper concludes in Sec. V.

Throughout this paper, we use units in which $G = c = 1$, and we use the conventions of Misner, Thorne, and Wheeler [18] for the metric and curvature tensors. We use Latin letters from the beginning of the alphabet for general spacetime indices and Greek letters for those associated with specific coordinate systems. Latin letters from the middle of the alphabet (starting at i) will be reserved for spatial indices, and a 0 will denote a time index in the latter context.

II. OPERATIONAL DEFINITION OF THE ANGULAR MOMENTUM OF A SOURCE AS MEASURED BY A LOCAL OBSERVER

In this section, we describe a method by which an observer in the vicinity of some source of gravity can attempt to measure the angular momentum of that source, by using only information about the geometry of spacetime in the observer's vicinity. Specifically, we describe an algorithm by which an angular momentum can be constructed from the Riemann tensor and its gradients at the observer's location. The algorithm we propose, moreover, is not unique, and the angular momenta obtained will differ from one observer to another. However, in a certain limit (Sec. II C below), the angular momenta will become observer independent and characterize the source. In more general situations, the nonuniqueness of the algorithm will be unimportant, and the angular momenta will be observer dependent. In these situations, the nature of this observer dependence will be physically interesting, as discussed in the remaining sections of this paper.

While the literature on angular momentum in general relativity is extensive and well developed (see, e.g., [19]), our approach here introduces a new perspective, in that it focuses on a local and operational definition of a quantity that observers can measure. Our procedure can be applied at any point in any spacetime (subject to a small number of local assumptions), and yields the expected result in the limit of large distances from a source in an isolated, linearized, stationary, vacuum spacetime.

A measurement of the general type considered here, where the angular momentum of a source is extracted from measurements of the geometry of spacetime, has been carried out once in the history of physics: the measurement of the spin of the Earth to $\sim 20\%$ by Gravity Probe B [20] [1].

We start in Sec. II A by defining a vector space that can be interpreted as the space of angular momenta for an observer at a given point $\mathcal{P}$ in a curved spacetime. We give the general algorithm for measuring angular momentum in Sec. II B. In Sec. II C we explain the motivation for this algorithm, namely that it gives the expected result in stationary linearized spacetimes near future null infinity. We discuss the physical interpretation of the measured angular momentum in general spacetimes in Sec. II D, the non-uniqueness of the algorithm in Sec. II E, and the accuracy of the center-of-mass measurement in Sec. II F.

A. Definition of a linear space of angular momenta at a given point in spacetime

At a point $\mathcal{P}$ in a spacetime (M, g_{ab}), let $T_{\mathcal{P}}(M)$ denote the tangent space. Let $G_{\mathcal{P}}$ be the Poincaré group that

[1] Our measurement procedure is not quite the same as that used by Gravity Probe B. While both extract angular momentum information from the spacetime geometry, our procedure uses the curvature tensors in an infinitesimal region about a spacetime point, while Gravity Probe B uses information about the geometry in the vicinity of an entire orbit in addition to information about asymptotic inertial frames provided by the direction to a guide star.

acts on $T_\mathcal{P}(M)$, that is, the space of affine maps from $T_\mathcal{P}(M)$ to itself that preserve the metric. Since $G_\mathcal{P}$ is a Lie group, it has an associated Lie algebra $\mathcal{G}_\mathcal{P}$ that consists of infinitesimal Poincaré transformations. The corresponding dual space $\mathcal{G}^*_\mathcal{P}$, the space of linear maps from $\mathcal{G}_\mathcal{P}$ to the real numbers, is the space of linear and angular momenta at the event $\mathcal{P}$.

To see this explicitly, consider an affine coordinate system x^a on $T_\mathcal{P}(M)$. Such a coordinate system is associated with a choice of basis vectors $\vec{e}_a$ and a fixed vector $\vec{x}_0$ such that the coordinates x^a of a vector $\vec{x}$ are given by $\vec{x} = \vec{x}_0 + x^a \vec{e}_a$. In this coordinate system, the maps in $G_\mathcal{P}$ have the usual form of a Poincaré transformation: $x^a \to \Lambda^a{}_b x^b + \kappa^a$. Here $\Lambda^a{}_b$ is a Lorentz transformation and κ^a is a translation. The infinitesimal versions of these maps in $\mathcal{G}_\mathcal{P}$ have the same form, but with infinitesimal κ^a and with $\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b$, where ω_{ab} is an infinitesimal antisymmetric tensor. Now consider the dual space, $\mathcal{G}^*_\mathcal{P}$. A general linear map from $\mathcal{G}_\mathcal{P}$ to real numbers can be written as

$$(\kappa_a, \omega_{ab}) \to P^a \kappa_a - \frac{1}{2} J^{ab} \omega_{ab} \tag{2.1}$$

for some vector P^a and some antisymmetric tensor J^{ab}. Therefore, elements of $\mathcal{G}^*_\mathcal{P}$ can be parameterized in terms of pairs of tensors (P^a, J^{ab}), a linear momentum and an angular momentum. The angular momentum J^{ab} transforms under changes of origin in $T_\mathcal{P}(M)$ as angular momentum should: for $\vec{x}_0 \to \vec{x}_0 + \vec{\delta x}$, $J^{ab} \to J^{ab} + 2P^{[a}\delta x^{b]}$. The angular momentum J^{ab} would be interpreted by an observer at $\mathcal{P}$ as angular momentum about a point which is "displaced from $\mathcal{P}$ by an amount $\vec{x}_0$", even though such a displacement is ambiguous in general relativity.

B. Definition of the general prescription for measuring an angular momentum

In this section, we define a prescription for how an observer at an event $\mathcal{P}$ can measure an element of the dual space $\mathcal{G}^*_\mathcal{P}$ of linearized Poincaré transformations on the tangent space at $\mathcal{P}$. The prescription requires several assumptions about the geometry near $\mathcal{P}$, as discussed further below, and therefore it is applicable only in certain situations.

The steps of the prescription are as follows:

1. Measure all the components of the Riemann tensor R_{abcd} and of its gradient $\nabla_a R_{bcde}$ at the event $\mathcal{P}$. The electric pieces of the Riemann tensor in the observers frame can be measured by monitoring the relative acceleration of test masses using the geodesic deviation equation. Similarly, the magnetic pieces can be measured by monitoring the relative angular velocity of gyroscopes induced by frame dragging [21]. By repeating these measurements at nearby spacetime points, the observer can in principle also measure the components of the gradient $\nabla_a R_{bcde}$.

2. Compute the curvature invariants
$$K_1 \equiv R_{abcd} R^{abcd}, \tag{2.2a}$$
$$\mathcal{K}_1 \equiv \nabla_a R_{bcde} \nabla^a R^{bcde}. \tag{2.2b}$$
We assume that $K_1 > 0$ and $\mathcal{K}_1 > 0$. Then, compute quantities M and r using
$$M = \frac{15\sqrt{5} K_1^2}{4\mathcal{K}_1^{3/2}}, \tag{2.3a}$$
$$r = \sqrt{\frac{15 K_1}{\mathcal{K}_1}}. \tag{2.3b}$$

3. Repeat the above measurements and computations at nearby[2] spacetime points, thus measuring the gradient $\nabla_a r$ of the quantity r.

4. Assuming that the vector $\nabla_a r$ is spacelike, define the unit vector n^a in the direction of $\nabla_a r$ by $n^a = N^{-1} \nabla_a r$ where $N = \sqrt{\nabla^a r \nabla_a r}$. Compute the quantity
$$y^a = -r n^a \tag{2.4}$$
which the observer interprets as the displacement vector from her own location to the center-of-mass worldline of the source.

5. Compute the symmetric tensor H_{ab} from
$$H_{ab} = R_{acbd} n^c n^d. \tag{2.5}$$
Compute the eigenvectors ζ^a and eigenvalues λ of this matrix from $H_{ab}\zeta^b = \lambda \zeta_a$. From the definition (2.5), one of the eigendirections will be $\zeta^a = n^a$ with corresponding eigenvalue $\lambda = 0$. We assume that there is at least one eigenvector with a strictly positive eigenvalue, and we we denote the eigendirection corresponding to the largest eigenvalue by t^a. It follows that this vector is orthogonal to n_a, $t^a n_a = 0$.

6. Assuming that the vector t^a is timelike, define a unit, future-directed timelike vector u^a by $u^a = N^{-1} t^a$ where $N^2 = -t_a t^a$ and the sign of N is chosen so that u^a is future directed. The linear momentum is then given by $P^a = M u^a$.

7. Compute the curvature invariant
$$K_2 \equiv \frac{1}{2} \epsilon_{abcd} R^{ab}{}_{ef} R^{cdef}. \tag{2.6}$$
From this compute a spin vector S^a by
$$S^a = \frac{r^7 K_2}{288 M^2} n^a + \frac{1}{3} r^4 \epsilon^{abcd} u_b n_c H_{de} u^e. \tag{2.7}$$

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[2] Equivalently, measure the Riemann tensor and its first two derivatives at $\mathcal{P}$; the quantity $\nabla_a r$ can then be expressed in terms of these using Eqs. (2.2) and (2.3).

8. Compute the angular momentum J^{ab} by
$$J^{ab} = \epsilon^{abcd} u_c S_d + y^a P^b - y^b P^a. \tag{2.8}$$
Finally from (P^a, J^{ab}) compute an element of $\mathcal{G}^*_{\mathcal{P}}$ using the definition (2.1) specialized to $\vec{x}_0 = 0$.

Although the procedure is somewhat lengthy, these eight steps define a method for computing an element of $\mathcal{G}^*_{\mathcal{P}}$ from the Riemann tensor and its derivatives at a point $\mathcal{P}$.

C. Motivation for the prescription: stationary linearized spacetimes near future null infinity

We now explain the motivation for the choice of prescription described in the last subsection: it is designed to give the expected answer in a certain limit. Specifically, we consider spacetimes that are stationary and free of matter in the neighborhood of an observer, and for which the sources are sufficiently distant from the observer that the metric can be described by a linearized multipolar expansion. For these distant sources, the dominant terms in the multipolar expansion will be the mass monopole and the current dipole or spin, with the remaining multipoles being negligible. In this situation, the measured P^a and J^{ab} coincide with the conserved charges of the spacetime to a good approximation, as we now show. This requirement does not fix the prescription uniquely, but we shall argue in Sec. II E below that the nonuniqueness is not significant.

We start by writing down a Poincaré covariant expression for the metric for stationary linearized spacetimes, keeping only the first two multipoles. This metric can be written as $ds^2 = (\eta_{\alpha\beta} + h_{\alpha\beta}) dx^\alpha dx^\beta$, where we have specialized to Lorentzian coordinates x^α for the background metric, and indices are raised and lowered with $\eta_{\alpha\beta}$. Let the four-momentum of the source be $\hat{P}^\alpha = \hat{M}\hat{u}^\alpha$, where $\hat{u}^a$ is the four velocity and $\hat{M}$ is the rest mass. (We use a hatted notation for these quantities to distinguish them from the quantities, defined in the previous subsection, that the observer measures.) Let the intrinsic angular momentum of the source be $\hat{S}^\alpha$ with $\hat{S}^\alpha \hat{u}_\alpha = 0$, and let $\hat{z}^\alpha$ be a point on the center-of-mass worldline of the source. Let x^α be the point at which we want to evaluate the metric perturbation $h_{\alpha\beta}$. We define the projection tensor
$$\hat{p}_{\alpha\beta} = \eta_{\alpha\beta} + \hat{u}_\alpha \hat{u}_\beta \tag{2.9}$$
and define the distance $\hat{r}$ by $\hat{r}^2 = \hat{p}_{\alpha\beta}(x^\alpha - \hat{z}^\alpha)(x^\beta - \hat{z}^\beta)$. Finally, we define the unit vector $\hat{n}^\alpha$ by
$$\hat{n}_\alpha = \nabla_\alpha \hat{r} = \frac{1}{\hat{r}} \hat{p}_{\alpha\beta}(x^\beta - \hat{z}^\beta). \tag{2.10}$$

In terms of these quantities, the total angular momentum $\hat{J}^{\alpha\beta}$ about the point x^α is is
$$\hat{J}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} \hat{u}_\gamma \hat{S}_\delta + \hat{y}^\alpha \hat{P}^\beta - \hat{y}^\beta \hat{P}^\alpha, \tag{2.11}$$
where $\hat{y}^\alpha = \hat{p}^\alpha{}_\beta(\hat{z}^\beta - x^\beta) = -\hat{r}\hat{n}^\alpha$ is a vector which points from the field point x^α to the center-of-mass worldline. The metric perturbation is
$$h_{\alpha\beta}(x^\alpha) = \frac{2\hat{M}}{\hat{r}}(\eta_{\alpha\beta} + 2\hat{u}_\alpha \hat{u}_\beta) - \frac{4}{\hat{r}^2} \hat{u}_{(\alpha} \epsilon_{\beta)\gamma\delta\epsilon} \hat{S}^\gamma \hat{n}^\delta \hat{u}^\epsilon \,, \tag{2.12}$$
which is equivalent to the stationary limit of the linearized metric perturbation in [22], after one makes the substitution that $u_\alpha = (dt)_\alpha + v_i \delta^i{}_\alpha$, where (t, x^i) are harmonic coordinates. Finally, the Riemann tensor is
$$R_{\alpha\beta\gamma\delta} = \frac{1}{2}(h_{\alpha\delta,\beta\gamma} + h_{\beta\gamma,\alpha\delta} - h_{\alpha\gamma,\beta\delta} - h_{\beta\delta,\alpha\gamma}) \,, \tag{2.13}$$
where
$$\begin{aligned} h_{\alpha\delta,\beta\gamma} &= \frac{2\hat{M}}{\hat{r}^3}(\eta_{\alpha\delta} + 2\hat{u}_\alpha \hat{u}_\delta)(3\hat{n}_\beta \hat{n}_\gamma - \hat{p}_{\beta\gamma}) \\ &- \frac{12}{\hat{r}^4} \hat{u}_{(\alpha} \epsilon_{\delta)\lambda\mu\nu} \hat{S}^\lambda \hat{u}^\nu [(5\hat{n}_\beta \hat{n}_\gamma - \hat{p}_{\beta\gamma})\hat{n}^\mu - 2\hat{n}_{(\beta} \hat{p}_{\gamma)}{}^\mu] \,. \end{aligned} \tag{2.14}$$

We now compute the angular momentum that an observer at x^α would measure in this spacetime, using the algorithm described in the last subsection. The curvature invariants (2.2) are given by
$$K_1 = \frac{48\hat{M}^2}{\hat{r}^6}\left[1 + O(\epsilon)\right] \,, \tag{2.15a}$$
$$\mathcal{K}_1 = \frac{720\hat{M}^2}{\hat{r}^8}\left[1 + O(\epsilon)\right] \,, \tag{2.15b}$$
where for ease of notation we have defined
$$O(\epsilon) \equiv O\left(\frac{\hat{S}^2}{\hat{M}^2 \hat{r}^2}\right) + O\left(\frac{\hat{M}}{\hat{r}}\right). \tag{2.16}$$
Note that correction terms linear in the spin are forbidden by parity considerations. Computing M and r using Eqs. (2.3) yields
$$M = \hat{M}[1 + O(\epsilon)], \qquad r = \hat{r}[1 + O(\epsilon)]. \tag{2.17}$$
Similarly by evaluating the gradient of r according to steps 3 and 4, we find
$$n^\alpha = \hat{n}^\alpha \left[1 + O(\epsilon)\right], \qquad y^\alpha = \hat{y}^\alpha \left[1 + O(\epsilon)\right]. \tag{2.18}$$

Next, we evaluate the symmetric tensor (2.5) using the expression (2.13) for the Riemann tensor. The result is
$$\begin{aligned} H_{\alpha\beta} &= -\frac{\hat{M}}{\hat{r}^3}(\eta_{\alpha\beta} + 3\hat{u}_\alpha \hat{u}_\beta - \hat{n}_\alpha \hat{n}_\beta) \\ &+ \frac{6}{\hat{r}^4} \hat{u}_{(\alpha} \epsilon_{\beta)\gamma\delta\epsilon} \hat{S}^\gamma \hat{n}^\delta \hat{u}^\epsilon \,. \end{aligned} \tag{2.19}$$
Because of the symmetries of the Riemann tensor, the tensor $H_{\alpha\beta}$ is symmetric and has n^α as an eigenvector with its corresponding eigenvalue being identically zero. The three remaining eigenvectors at leading order in an

expansion in $1/\hat{r}$ are $\hat{u}^\alpha$, $\epsilon_{\alpha\beta\gamma\delta}\hat{S}^\beta\hat{n}^\gamma\hat{u}^\delta$, and a third vector that is orthogonal to those two as well as $\hat{n}^\alpha$. The eigenvalues associated with these eigenvectors are (again at leading order in an expansion in $1/\hat{r}$) $2\hat{M}/\hat{r}^3$, and a repeated eigenvalue equal to $-\hat{M}/\hat{r}^3$ for the latter two, respectively. Therefore, if we follow step 5 and choose the normalized eigenvector corresponding to the largest eigenvalue, we obtain $u^\alpha = \hat{u}^\alpha[1 + O(\epsilon)]$. It follows that

$$P^\alpha = \hat{P}^\alpha[1 + O(\epsilon)]. \tag{2.20}$$

Next, from Eqs. (2.13) and (2.14), the curvature invariant (2.6) is given by

$$K_2 = \frac{288\hat{M}^2}{\hat{r}^7}(\hat{S}^\alpha\hat{n}_\alpha)[1 + O(\epsilon)]\ . \tag{2.21}$$

Inserting this equation and the expression (2.19) for $H_{\alpha\beta}$ into the formula (2.7) for the intrinsic angular momentum, we determine

$$S^\alpha = \hat{S}^\alpha[1 + O(\epsilon)]. \tag{2.22}$$

Thus, the algorithm successfully recovers the linear momentum and intrinsic angular momentum of the spacetime. Also, from Eqs. (2.8), (2.11) and (2.18) we find that $J^{\alpha\beta} = \hat{J}^{\alpha\beta}[1 + O(\epsilon)]$, so that the algorithm yields the total angular momentum of the source about the observer's location x^α.

D. Physical interpretation of the measured linear and angular momenta in more general contexts

In the previous subsections, we showed that an observer that is sufficiently distant from a stationary source of gravity can measure that source's linear and angular momentum to a good approximation, using just the spacetime geometry in the vicinity of the observer. The measurement procedure required several assumptions about that spacetime geometry: (i) the curvature invariants (2.2) needed to be positive; (ii) the vector $\nabla_a r$ needed to be spacelike; (iii) the tensor H_{ab} needed to have at least one strictly positive eigenvalue; and (iv) the corresponding eigenvector needed to be timelike. These assumptions are satisfied for linearized stationary spacetimes described by just two multipoles at sufficiently large $\hat{r}$. By continuity, therefore, they will also be satisfied in regions of spacetimes that are sufficiently close to this case. We now discuss in more detail how the measurement procedure applies to these more general situations and spacetimes.

There are a number of physical effects that can make the spacetime geometry measured by observers differ from the idealized case discussed above of asymptotic regions in linearized stationary spacetimes with two multipoles. The effects that we consider include nonlinearities, higher-order multipoles, non-isolated systems, and nonstationarity. We now estimate the size of these effects in more general contexts, and thereby determine both when we might expect the assumptions listed above to break down, and also when the algorithm yields physically sensible results. The various effects are:

- *Nonlinearities:* Our analysis above assumed that the spacetime could be described as a linear perturbation about Minkowski spacetime. For an isolated, stationary source in an asymptotically flat spacetime, there will be corrections to the metric arising from nonlinearities. These nonlinearities will give corrections to the metric perturbation $h_{\alpha\beta}$ that scale[3] as

$$O\left(\frac{\hat{M}^2}{\hat{r}^2}\right), \qquad O\left(\frac{\hat{M}\hat{S}}{\hat{r}^3}\right), \qquad O\left(\frac{\hat{S}^2}{\hat{r}^4}\right). \tag{2.23}$$

  The form of these corrections can be found, for example, from the leading nonlinear terms in the post-Newtonian expansion of the metric (given in, e.g., [23]). These corrections will be small[4] compared to the leading-order terms $\sim \hat{M}/\hat{r}$ and $\sim \hat{S}/\hat{r}^2$ in the metric perturbation (2.12), as long as $\hat{r}$ is large compared to $\hat{M}$, $\sqrt{\hat{S}}$ and $\hat{S}^{2/3}\hat{M}^{-1/3}$. For sufficiently large $\hat{r}$, therefore, the effects of nonlinearities can be neglected.

- *Higher-order multipoles:* Our analysis in Sec. II C above included only the mass and spin and neglected higher-order mass and current multipoles. However, as is well known, the effect of these multipoles will be small at sufficiently large $\hat{r}$. The dominant correction to the metric perturbation in the parity-even sector will be

$$\delta h_{\alpha\beta} \sim \frac{Q}{\hat{r}^3}, \tag{2.24}$$

  where Q is the mass quadrupole. Using the estimate $Q \sim \hat{M}\mathcal{L}^2$, where $\mathcal{L}$ is the size of the source, we see that this correction will be small compared to $\hat{M}/\hat{r}$ in the regime

$$\hat{r} \gg \mathcal{L}. \tag{2.25}$$

[3] This will be true in suitable coordinates, for which the limit $\hat{M} \to 0$ of the metric at fixed $\hat{S}/\hat{M}$ is the Minkowski metric in Minkowski coordinates (for example, Cartesian Kerr-Schild coordinates in the Kerr spacetime). For more general coordinates (such as Boyer-Lindquist coordinates), other terms can occur that are larger than some of the terms in Eq. (2.23) [e.g., $\hat{S}^2/(\hat{M}^2\hat{r}^2)$]. These larger terms are gauge effects, and they can be ignored for the argument given here.

[4] An exception is the term $\sim \hat{M}^2/\hat{r}^2$ which will be comparable to the $\sim \hat{S}/\hat{r}^2$ term in the metric (2.12) when $\hat{S} \sim \hat{M}^2$. One might expect that this term would give rise to fractional corrections of order unity to the measured momentum and angular momentum; the corrections, however, are suppressed, because the $\hat{S}/\hat{r}^2$ term is parity odd while the $\hat{M}^2/\hat{r}^2$ term is parity even.

Similarly, in the parity-odd sector, the dominant correction will be

$$\delta h_{\alpha\beta} \sim \frac{\mathcal{S}}{\hat{r}^3}, \tag{2.26}$$

where $\mathcal{S} \sim \hat{S}\mathcal{L}$ is the current quadrupole. This correction will be small compared to the spin term in Eq. (2.12) whenever $\hat{r} \gg \mathcal{L}$. Therefore, in the regime (2.25), corrections to the measured linear and angular momentum P^α and $J^{\alpha\beta}$ will be small.

We note that in the context of linearized gravity, it is possible in principle to measure P^α and $J^{\alpha\beta}$ accurately even in the regime $\hat{r} \sim \mathcal{L}$, by using measurement procedures more sophisticated than those envisaged in this paper. As is well known, in linearized gravity the charges P^α and $J^{\alpha\beta}$ can be extracted unambiguously from the metric perturbation using surface integrals [18]. Therefore, a family of observers distributed over the surface of a sphere, who make measurements of the spacetime geometry in their vicinity and compare notes in a suitable way, can measure P^α and $J^{\alpha\beta}$ with high accuracy. In this paper, we will not need to consider such nonlocal measurement procedures, because the issues we want to explore are all present in the regime (2.25) in which our local measurement procedure is sufficient.

- *Non-isolated systems:* So far we have considered observers near isolated sources in asymptotically flat spacetimes. Suppose, however, that there are also distant sources, or that the spacetime is not asymptotically flat. In linearized gravity, the effect of distant sources can be quantified in terms of the tidal tensor $\mathcal{E}_{ij}$ (the electric components of the associated Riemann tensor). The corresponding fractional corrections to the linear momentum measured by observers using the procedure of Sec. II B will be of order $\sim \mathcal{E}\hat{r}^3/\hat{M}$. Similarly the fractional corrections to the angular momentum will be of order $\sim \mathcal{B}\hat{r}^4/\hat{S}$, where $\mathcal{B}_{ij}$ is the magnetic tidal tensor. These effects limit the accuracy and utility of our measurement method of Sec. II B above. Within the context of linearized gravity, it is possible to circumvent this difficulty using the nonlocal measurement method discussed above, which uses the angular dependence to disentangle the effects of the locally produced curvature $\sim \hat{M}/\hat{r}^3$ from the curvature $\mathcal{E}_{ij}$ produced by distant sources.

  When nonlinearities are included, however, there is an unavoidable ambiguity: the linear and angular momenta of individual objects cannot be defined in general. We can estimate the ambiguities from nonlinearities using the fact that different definitions of the "mass of an object" in post-1-Newtonian theory differ by a quantity of order the tidal-interaction energy, $Q_{ij}\mathcal{E}_{ij}$, where $Q_{ij} \sim \hat{M}\mathcal{L}^2$ is a mass quadrupole. Therefore, objects of mass $\hat{M}$, size $\mathcal{L}$ and separated by distances $\sim \mathcal{D}$ have an uncertainty or ambiguity in their masses of order[5]

$$\Delta\hat{M}/\hat{M} \sim \hat{M}\mathcal{L}^2/\mathcal{D}^3 \ . \tag{2.27}$$

  The measurement method discussed in Sec. II B above will be subject to this ambiguity; however, in many situations the ambiguity will be negligible.

- *Nonstationary systems:* For dynamical, radiating sources, it is immediately clear that our measurement procedure will not be applicable in general. The reason is that the Weyl tensor for radiated gravitational waves falls off at large $\hat{r}$ as $1/\hat{r}$, whereas the static piece of the Weyl tensor associated with the mass and spin falls off as $1/\hat{r}^3$. Therefore, at sufficiently large $\hat{r}$, if an observer measures the Riemann tensor and its derivatives at her location, her result will be dominated by the radiative pieces of the metric, and the measurement method of Sec. II B above will fail.

  As discussed in the introduction, however, the measurement method can still yield interesting information about dynamical systems, for an *intermittently stationary* spacetime (by which we mean a spacetime which is stationary at early times and again at late times). Observers can apply the measurement procedure at early and at late times and then attempt to compare their results. This scenario is discussed in detail in the remaining sections of the paper.

  As an aside, we note that we can classify nonstationary systems into two types. The first is what we will call *asymptotically linear* systems, that is, systems for which the linear approximation is valid[6] at sufficiently large $\hat{r}$. For these systems, one can define unambiguous linear and angular momenta using surface integrals, and they can be measured using the nonlocal measurement procedure discussed above. Our local measurement procedure can work for such systems, but only if $\mathcal{L} \ll \hat{r} \ll \lambda$, where λ is the wavelength of the radiation. The second type of system, *asymptotically nonlinear* systems, are those for which the linear approximation is not valid at large $\hat{r}$. These are the systems for which the BMS asymptotic symmetry group is most relevant. Neither our local measurement procedure, nor the

[5] This estimate is valid for generic sources which have a nonvanishing intrinsic quadrupole moment. It is not valid for spherically symmetric sources whose intrinsic quadrupole vanishes. For such sources, the scaling of the mass ambiguity can be estimated from the quadrupoles $Q \sim \hat{M}\mathcal{L}^5/\mathcal{D}^3$ induced by tidal interactions; it is of order $\Delta\hat{M}/\hat{M} \sim \hat{M}\mathcal{L}^5/\mathcal{D}^6$.

[6] Here the assumption is that the linear approximation is valid in a neighborhood of some two-sphere which encloses the source, not the weaker assumption the linear approximation is valid in a neighborhood of some observer.

nonlocal measurement procedure based on surface integrals of linearized theory, apply to systems in this regime.[7]

E. Nonuniqueness of the measurement algorithm

The algorithm discussed above is not uniquely determined by the requirement that it give the correct answer in linearized stationary spacetimes with two multipoles, because the information about the linear and angular momentum of the spacetime is encoded redundantly in the Riemann tensor and its first two derivatives at any point. Therefore, there are several methods that can be used to extract these momenta. For example, Eq. (2.4) could be replaced by $y^a = -\nabla^a r^2/2$, which would give the same result to leading order.

In stationary linearized spacetimes with two multipoles, there is a unique and accepted definition of the linear and angular momentum of the spacetime; therefore, any nonuniqueness or ambiguities in the measurement procedure must vanish in this limit as the measurement is taken at large distances from the source. More specifically, this implies that the effects of these ambiguities all scale as $1/r$ as $r \to \infty$ (or as $1/v$, where v is a null coordinate with goes to infinity at future null infinity). Most importantly, they are small compared to the observer dependence of angular momentum that we discuss in the remainder of the paper (that characterized by generalized holonomies, which we show gives rise to finite effects in the limit $v \to \infty$).

F. Accuracy of measurement of the center-of-mass worldline

The procedure discussed above allows an observer to measure the angular momentum of the spacetime about his own location to an accuracy of $\epsilon = \hat{M}/\hat{r}$:

$$J^{\alpha\beta} = \hat{J}^{\alpha\beta}\left[1 + O(\epsilon)\right]. \tag{2.28}$$

In particular, the displacement vector y^α from the observer to the center-of-mass worldline [cf. Eq. (2.8) above] will be measured with this accuracy:

$$y^\alpha = \hat{y}^\alpha \left[1 + O\left(\frac{\hat{M}}{\hat{r}}\right)\right]. \tag{2.29}$$

However, $\hat{y}^\alpha$ is of order $\hat{r}$, and, therefore, the error in the measurement is of order

$$\delta y^\alpha \sim \hat{M}. \tag{2.30}$$

This error is large: it is of the same order as the maximum displacements caused by gravitational-wave memory effects[8].

III. AFFINE TRANSPORT AND GENERALIZED HOLONOMY: PROPERTIES AND APPLICATION TO ANGULAR MOMENTUM

We now turn to the question of how two observers at different locations in a curved spacetime can compare values of linear and angular momentum. The philosophy we adopt is to imagine that the observers attempt to compare values using the same methods they would use in special relativity (i.e., in the absence of gravity).

The first part of this section introduces a curve-dependent transport law, which we call affine transport, and which serves as the basis for our method of comparing angular momentum. The next subsection describes how the affine transport can be used to compare values of the angular momentum defined at different spacetime points. The final subsection describes the affine transport around a closed curve, which we call the generalized holonomy, and it explains its relation to the inevitable observer dependence of angular momentum in curved spacetimes.

A. Definition of an affine transport law

In this section, we define a transport law that can be used to transport vectors along curves, and which is a generalization of parallel transport. Let $\mathcal{C}$ be a curve between the spacetime points $\mathcal{P}$ and $\mathcal{Q}$, and let the curve have tangent vector $\vec{k}$. Next, define a map $\chi_{\mathcal{C}}$ from $T_{\mathcal{P}}(M)$ to $T_{\mathcal{Q}}(M)$ through the solution of the differential equation

$$\nabla_{\vec{k}}\vec{\xi} = \alpha\vec{k}\,. \tag{3.1}$$

[7] Except to the extent that measurements before and after the non-stationarity can probe effects of the non-stationarity, as we discuss in the remainder of this paper.

[8] It is possible, however, to modify the measurement method to increase the accuracy as follows. Modify the definitions of M and r in Eqs. (2.3) to

$$M = \frac{15\sqrt{5}K_1^2}{4\mathcal{K}_1^{3/2}}\left[1 - \frac{15\sqrt{3}(K_1)^{3/2}}{4\mathcal{K}_1}\right], \tag{2.31a}$$

$$r = \sqrt{\frac{15K_1}{\mathcal{K}_1}}\left[1 - \frac{5\sqrt{3}(K_1)^{3/2}}{4\mathcal{K}_1}\right], \tag{2.31b}$$

and leave the rest of the measurement algorithm unaltered. Then the fractional error in measurement of y^α is decreased to $O(\hat{M}^2/\hat{r}^2)$, and the errors in y^α vanish as the observers approach future null infinity. This modified algorithm is derived from the expressions for the curvature invariants of the Kerr spacetime in Boyer-Lindquist coordinates, and therefore yield the Boyer-Lindquist radial coordinate value at the observers location.

Here α is a dimensionless constant. Namely, starting from an initial condition $\vec{\xi}_{\mathcal{P}}$ in $T_{\mathcal{P}}(M)$, we solve the differential equation to obtain the value $\vec{\xi}_{\mathcal{Q}}$ of $\vec{\xi}$ at $\mathcal{Q}$. The image of $\vec{\xi}_{\mathcal{P}}$ under the map $\chi_{\mathcal{C}}$ is then defined to be $\vec{\xi}_{\mathcal{Q}}$. Since we are not aware of a name for this specific transport law, we will call it the *affine transport* of the vector $\vec{\xi}$ along the curve $\mathcal{C}$ with tangent $\vec{k}$. This map satisfies six important properties that are listed below:

1. It is independent of the choice of parameterization along the curve (which follows because both sides of the equation are linear in the tangent to the curve $\vec{k}$).

2. When two curves are composed, the composition of maps is equivalent to the map on the composed curve (i.e., if $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ then $\chi_{\mathcal{C}} = \chi_{\mathcal{C}_1} \circ \chi_{\mathcal{C}_2}$).

3. For a fixed curve, $\mathcal{C}$, Eq. (3.1) is a linear differential equation in $\vec{\xi}$. The solution for given initial data, therefore, can be expressed as the sum of two terms: the first term is the solution of the homogeneous differential equation (parallel transport) with the same initial data, and the second is the solution of the inhomogeneous differential equation with zero initial data. The complete solution is
$$\xi^{\bar{a}}_{\mathcal{Q}} = \Lambda_{\mathcal{P}\mathcal{Q}}{}^{\bar{a}}{}_a \xi^a_{\mathcal{P}} + \alpha \Delta \xi^{\bar{a}}_{\mathcal{P}\mathcal{Q}} \,, \tag{3.2}$$
where $\Lambda_{\mathcal{P}\mathcal{Q}}{}^{\bar{a}}{}_a$ denotes the parallel transport operation from $\mathcal{P}$ to $\mathcal{Q}$ and $\Delta\xi^{\bar{a}}_{\mathcal{P}\mathcal{Q}}$ is the inhomogeneous solution for $\alpha = 1$. The notation here is that overlined indices are associated with the point $\mathcal{Q}$ and indices without extra adornment are associated with $\mathcal{P}$.

4. It follows that (unlike parallel transport), affine transport does *not* preserve the norm of the transported vector.

5. For geodesic curves, one can show that the inhomogeneous part of the solution $\Delta\vec{\xi}_{\mathcal{P}\mathcal{Q}}$ is just the tangent to the curve at the point $\mathcal{Q}$ (i.e., $\alpha\vec{k}_{\mathcal{Q}}$). [9]

6. Finally, for curves in a flat spacetime, $\Delta\vec{\xi}_{\mathcal{P}\mathcal{Q}}$ is just the vectorial displacement $\vec{\mathcal{Q}} - \vec{\mathcal{P}}$ in any inertial coordinate system. In particular, it vanishes for closed curves in a flat spacetime, because, as we show later, it is only nontrivial in the presence of spacetime curvature.

[9] The calculation which shows this is short: Let $\lambda \in [0,1]$ be an affine parameter along a geodesic curve with $\lambda = 0$ corresponding to $\mathcal{P}$ and $\lambda = 1$ be the point $\mathcal{Q}$, and denote the directional derivative along the geodesic by $\nabla_{\vec{k}} = D/D\lambda$. For a point $\mathcal{P}'$ between $\mathcal{P}$ and $\mathcal{Q}$, one can confirm that $\xi^{\bar{a}}_{\mathcal{P}'} = \Lambda_{\mathcal{P}\mathcal{P}'}{}^{\bar{a}}{}_a \xi^a_{\mathcal{P}} + \alpha\lambda k^{\bar{a}}_{\mathcal{P}'}$ is the solution along the geodesic curve, because $\nabla_{\vec{k}} \Lambda_{\mathcal{P}\mathcal{P}'}{}^{\bar{a}}{}_a \xi^a_{\mathcal{P}} = 0$ and $\nabla_{\vec{k}}\vec{k} = 0$. Evaluating the expression at $\lambda = 1$, one finds that the inhomogeneous part of the solution is $\alpha\vec{k}_{\mathcal{Q}}$.

We will use these properties frequently in the calculations in the remainder of this paper.

B. Application to a curve-dependent definition of angular-momentum transport

Using the affine transport law, we can define a method of comparing the local values of angular momentum at two different spacetime points, by transporting angular momenta from one spacetime point to another, in a curve-dependent manner.

We define a map from $\mathcal{G}^*_{\mathcal{P}}$ to $\mathcal{G}^*_{\mathcal{Q}}$ that depends on a choice of curve $\mathcal{C}$ that joins these two points. Because the elements of $\mathcal{G}^*_{\mathcal{P}}$ act on maps in $\mathcal{G}_{\mathcal{P}}$ [those from the tangent space $T_{\mathcal{P}}(M)$ to itself], a natural map is one based on the affine transport of elements in $\mathcal{G}_{\mathcal{P}}$ to elements of $\mathcal{G}_{\mathcal{Q}}$. Namely, for $h_{\mathcal{P}} \in \mathcal{G}_{\mathcal{P}}$, the corresponding $h_{\mathcal{Q}} \in \mathcal{G}_{\mathcal{Q}}$ is defined by
$$h_{\mathcal{P}} = \chi^{-1}_{\mathcal{C}} \circ h_{\mathcal{Q}} \circ \chi_{\mathcal{C}} \,, \tag{3.3}$$
where $\chi_{\mathcal{C}}$ is the affine transport along $\mathcal{C}$. For an element $q_{\mathcal{P}} \in \mathcal{G}^*_{\mathcal{P}}$, therefore we define the corresponding element $q_{\mathcal{Q}} \in \mathcal{G}^*_{\mathcal{Q}}$ by
$$q_{\mathcal{Q}}(h_{\mathcal{Q}}) = q_{\mathcal{P}}(\chi^{-1}_{\mathcal{C}} \circ h_{\mathcal{Q}} \circ \chi_{\mathcal{C}}) = q_{\mathcal{P}}(h_{\mathcal{P}}) \,. \tag{3.4}$$
In order to recover the correct transformation properties of angular momentum under displacements with this definition, it is necessary to choose the value
$$\alpha = -1 \tag{3.5}$$
of the parameter in the definition (3.1) of the function $\chi_{\mathcal{C}}$, which we now show by writing this mapping from the angular-momentum space $\mathcal{G}^*_{\mathcal{P}}$ to the angular-momentum space $\mathcal{G}^*_{\mathcal{Q}}$ in a more explicit notation.

To do so, let us represent elements of the algebra $h_{\mathcal{P}}$ as pairs
$$h_{\mathcal{P}} \leftrightarrow (\kappa^{\mathcal{P}}_a, \omega^{\mathcal{P}}_{ab}) \tag{3.6}$$
and the map between algebras at different points $\mathcal{P}$ and $\mathcal{Q}$, $\chi_{\mathcal{C}}$, as
$$\chi_{\mathcal{C}} \leftrightarrow (\alpha\Delta\xi^{\bar{a}}_{\mathcal{P}\mathcal{Q}}, \Lambda_{\mathcal{P}\mathcal{Q}}{}^{\bar{a}}{}_a) \,, \tag{3.7}$$
where the quantities in this equation are exactly those appearing in Eq. (3.2). The transformation rule for the algebra elements (3.3) then has the representation in terms of these pairs as
$$(\kappa^{\mathcal{P}}_{\bar{a}}, \omega^{\mathcal{P}}_{\bar{a}\bar{b}}) = (\Lambda_{\mathcal{P}\mathcal{Q}}{}^{\bar{a}}{}_a \kappa^{\mathcal{Q}}_{\bar{a}} + \alpha \Lambda_{\mathcal{Q}\mathcal{P}}{}^{\bar{a}}{}_a \Lambda_{\mathcal{Q}\mathcal{P}}{}^{\bar{b}}{}_b \omega^{\mathcal{Q}}_{\bar{a}\bar{b}} \Delta\xi^b_{\mathcal{P}\mathcal{Q}}, \Lambda_{\mathcal{Q}\mathcal{P}}{}^{\bar{a}}{}_a \Lambda_{\mathcal{Q}\mathcal{P}}{}^{\bar{b}}{}_b \omega^{\mathcal{Q}}_{\bar{a}\bar{b}}) \,, \tag{3.8}$$
which respects the multiplication rule for a semidirect-product structure. In the above expression, we have used the notation $\Delta\xi^a_{\mathcal{P}\mathcal{Q}} = \Lambda_{\mathcal{P}\mathcal{Q}\bar{a}}{}^a \Delta\xi^{\bar{a}}_{\mathcal{P}\mathcal{Q}}$ and the fact that

$\Lambda_{\mathcal{QP}}{}^{\bar{a}}{}_a$ and $\Lambda_{\mathcal{PQ}}{}^{\bar{a}}{}_a$ are related in accord with the typical notation for the inverse of the parallel propagator. Keeping in mind the representation of the maps $q_{\mathcal{P}}$ as

$$q_{\mathcal{P}}(h_{\mathcal{P}}) = P^a_{\mathcal{P}}\kappa^{\mathcal{P}}_a - \frac{1}{2}J^{ab}_{\mathcal{P}}\omega^{\mathcal{P}}_{ab} \tag{3.9}$$

given in (2.1), we can then take the definition of the transformation property of angular momentum in Eq. (3.4) above, substitute in the result of (3.8), and by equating the coefficients of $\omega^{\mathcal{Q}}_{\bar{a}\bar{b}}$ and $\kappa^{\mathcal{Q}}_{\bar{a}}$, we find that the angular momenta at the two points are related by

$$J^{\bar{a}\bar{b}}_{\mathcal{Q}} = \Lambda_{\mathcal{QP}}{}^{\bar{a}}{}_a\Lambda_{\mathcal{QP}}{}^{\bar{b}}{}_b\left(J^{ab}_{\mathcal{P}} + 2\alpha\Delta\xi^{[a}_{\mathcal{PQ}}P^{b]}_{\mathcal{P}}\right)\,, \tag{3.10}$$

and the corresponding momenta are related by

$$P^{\bar{a}}_{\mathcal{Q}} = \Lambda_{\mathcal{PQ}}{}^{\bar{a}}{}_a P^a_{\mathcal{P}}\,. \tag{3.11}$$

We, therefore, see that to have the usual transformation law for angular momentum

$$J^{\bar{a}\bar{b}}_{\mathcal{Q}} = \Lambda_{\mathcal{QP}}{}^{\bar{a}}{}_a\Lambda_{\mathcal{QP}}{}^{\bar{b}}{}_b\left(J^{ab}_{\mathcal{P}} - 2\Delta\xi^{[a}_{\mathcal{PQ}}P^{b]}_{\mathcal{P}}\right)\,, \tag{3.12}$$

we must choose $\alpha = -1$. An alternative and simpler formulation of the transport law given by Eqs. (3.12) and (3.11) is discussed in Appendix A.

If we decompose the angular momentum J^{ab} into an intrinsic spin S^a and a displacement vector y^a using the definitions

$$y^a = -\frac{1}{M^2}J^{ab}P_b\,, \tag{3.13a}$$

$$S^a = \frac{1}{2M}\epsilon_{bcd}{}^a P^b J^{cd}\,, \tag{3.13b}$$

then from Eqs. (3.11) and (3.12), the fact that $P^aP_a = -M^2$, and that $y^aP_a = 0$, we can show after some algebra that the spin is parallel transported just like the linear momentum,

$$S^{\bar{a}}_{\mathcal{Q}} = \Lambda_{\mathcal{PQ}}{}^{\bar{a}}{}_a S^a_{\mathcal{P}}\,, \tag{3.14}$$

while the displacement vector transforms as

$$y^{\bar{a}}_{\mathcal{Q}} = \Lambda_{\mathcal{QP}}{}^{\bar{a}}{}_a\left(y^b_{\mathcal{P}} - \Delta\xi^a_{\mathcal{PQ}}\right)\,. \tag{3.15}$$

Additional properties of the affine transport for closed curves are discussed next.

C. Generalized holonomy: a measure of observer dependence of angular momentum

For closed curves starting from a point $\mathcal{P}$, the affine transport around the curve defines a *generalized holonomy*, a map from the tangent space at $\mathcal{P}$ to itself. For flat spacetimes, the generalized holonomy is always the identity map. Specializing the result (3.12) to closed curves (when $\mathcal{Q}$ is the same point as $\mathcal{P}$), yields the mapping

$$J^{ab} \to \Lambda^a{}_c\Lambda^b{}_d(J^{cd} - 2\Delta\xi^{[c}P^{d]})\,. \tag{3.16}$$

Thus, if there is a nontrivial holonomy of parallel transport or a nonzero inhomogeneous solution, then the observers along the curve will find that the angular momentum is observer dependent. The extent to which a generalized holonomy is nontrivial is a measure of how much spacetime curvature is an obstruction to separated observers arriving at a consistent definition of angular momentum.

As a simple example of this generalized holonomy, consider an infinitesimal quadrilateral starting from a point $\mathcal{P}$ with legs given by ϵu^a and ϵv^a where ϵ is small. The quadrilateral is traversed first in the direction of u^a, then v^a, then $-u^a$, then $-v^a$. If we start at $\mathcal{P}$ with some initial vector ξ^a, and solve the transport equation (3.1) around the loop, a relatively straightforward calculation shows that the homogeneous part of the solution is

$$\xi^a + \epsilon^2 R^a{}_{bcd}\xi^b v^c u^d + O(\epsilon^3). \tag{3.17}$$

This is the usual expression for the holonomy around a small loop. The inhomogeneous part of the solution is

$$\Delta\xi^a = \tfrac{1}{2}\epsilon^3 R^a{}_{bcd}v^c u^d(u^b + v^b) + O(\epsilon^4). \tag{3.18}$$

A more detailed calculation is given in Ref. [24]. Thus, while the holonomy of parallel transport is proportional to the Riemann tensor contracted with the area of the quadrilateral, the generalized holonomy contains an additional term proportional to the Riemann tensor contracted with both the area and the perimeter of the region.

D. Relation between generalized holonomy and gravitational-wave memory

In this section, we give a precise and covariant definition of an observable that can be interpreted as a "gravitational-wave memory" generalized to an arbitrary spacetime. We then show that the generalized holonomy around a suitably constructed loop contains information about this covariant gravitational-wave memory, showing a very general relationship between these two observables.

However, we also show that the generalized holonomy contains additional information, and specifically contains three other independent pieces, each of which could arise as a kind of "memory" effect due to the passage of a burst of gravitational waves. The first is a difference in proper time measured by two observers (a gravitational redshift effect). The second is a relative boost of two initially comoving observers. The third is a relative rotation of the inertial frames of two observers. In the limit of nearby geodesics at large distances from a source emitting a burst of gravitational waves with memory, these effects reduce to a combination of more familiar notions of gravitational-wave memory arising from solutions of the equation of geodesic deviation and of differential frame dragging, and difference in proper time of nearby geodesics [25]. (We are also investigating the relationship between the new gravitational-wave memory of Pasterski *et al.*, [26], and the memory effects quantified by the generalized holonomy in [25].) We now describe a calculation that elucidates the relationship between the generalized holonomy and the ordinary memory plus differences in proper time, relative rotations, and relative boosts.

Consider two freely falling observers A and B in an arbitrary spacetime. We fix attention on an interval of A's worldline between two events $\mathcal{P}$ and $\mathcal{R}$, where A's proper time τ varies between τ_1 and τ_2, as illustrated in Fig. 1. We denote by $\vec{u}_A(\tau)$ the four-velocity of A along her worldline. We also introduce an orthonormal tetrad $\vec{e}_{\hat{\alpha}}(\tau)$ which is parallel transported along A's worldline, where $\vec{e}_{\hat{0}} = \vec{u}_A$.

At each point on A's worldline, there is unique spatial vector

$$\xi_B^{\hat{i}}(\tau)\vec{e}_{\hat{i}}(\tau) \tag{3.19}$$

such that the exponential map evaluated on this vector is a point $z_B(\tau)$ on B's worldline.[10] Or, equivalently, $(\tau, \xi_B^{\hat{i}}(\tau))$ gives the location of B's worldline in Fermi normal coordinates centered on A's worldline. We denote by $\vec{u}_B(\tau)$ the four-velocity of observer B at the point $z_B(\tau)$. We denote by $\mathcal{Q}$ and $\mathcal{S}$ the initial and final point $z_B(\tau_1)$ and $z_B(\tau_2)$ on B's worldline (see Fig. 1 below). Finally, we let $\vec{f}_{\hat{\alpha}}(\tau)$ be the orthonormal tetrad at $z_B(\tau)$ obtained by parallel transporting $\vec{e}_{\hat{\alpha}}(\tau)$ from the corresponding point on A's worldline along the spatial geodesic with initial tangent (3.19).[11]

We assume that the observers A and B are initially comoving, in the sense that $\vec{f}_{\hat{0}}(\tau_1)$ is B's four-velocity at $\mathcal{Q}$. We define a closed loop $\mathcal{C}$ by starting at $\mathcal{R}$, traveling along A's worldline back to $\mathcal{P}$, traveling along the spatial geodesic with initial tangent (3.19) to $\mathcal{Q}$, traveling along B's worldline to $\mathcal{S}$, and then back to $\mathcal{R}$ along the spatial geodesic whose final tangent at $\mathcal{R}$ is the vector (3.19) at $\tau = \tau_2$.

The inhomogeneous part of the generalized holonomy about the loop $\mathcal{C}$ is given by

$$\begin{aligned}\vec{\Delta\xi} = \left[\xi_B^{\hat{i}}(\tau_1) - \xi_B^{\hat{i}}(\tau_2)\right]\vec{e}_{\hat{i}} + (\Delta\tau_B - \Delta\tau_A)\vec{w}_B \\ + \xi_B^{\hat{i}}(\tau_1)\left(\mathbf{\Lambda}\cdot\vec{e}_{\hat{i}} - \vec{e}_{\hat{i}}\right).\end{aligned} \tag{3.20}$$

Here $\Delta\tau_A = \tau_2 - \tau_1$ is the interval of A's proper time between $\mathcal{P}$ and $\mathcal{R}$, and $\Delta\tau_B$ is the interval of B's proper time between $\mathcal{Q}$ and $\mathcal{S}$. The quantity $\Lambda^a{}_b$ is the usual holonomy around the loop $\mathcal{C}$. Finally $\vec{w}_B$ is the four-vector at $\mathcal{R}$ obtained by parallel transporting B's four-velocity $\vec{u}_B(\tau_2)$ at $\mathcal{S}$ along the spatial geodesic to $\mathcal{R}$. Equivalently, it can be obtained by acting with the holonomy around the loop on A's four-velocity at $\mathcal{R}$, $w_B^a = \Lambda^a{}_b u_A^b(\tau_2)$.

The first term in the generalized holonomy (3.20) can be interpreted as (a generalization of) the gravitational-wave memory effect. It is the change in the relative displacement of the observers A and B, as seen by A in her Fermi normal coordinates, when A and B are initially comoving. The second term depends on the difference in the proper times measured by A and B along the corresponding segments of their worldlines. It also depends on the boost that relates the final velocity of B to that of A. Finally, the third term depends on the holonomy $\Lambda^a{}_b$ around the loop, which in general will consist of a spatial rotation together with the aforementioned boost.

We now turn to the derivation of the formula (3.20). The inhomogeneous part of the generalized holonomy can be obtained by solving the differential equation (3.1) with $\alpha = 1$ around the loop $\mathcal{C}$ starting with $\vec{\xi}_{\mathcal{R}} = 0$ at the initial point $\mathcal{R}$. The solution at the next point $\mathcal{P}$ can be obtained from the fifth property listed in Sec. III A above, that the inhomogeneous term for a geodesic is just the tangent to the geodesic. It is given by

$$\vec{\xi}_{\mathcal{P}} = -\Delta\tau_A\vec{u}_A(\tau_1). \tag{3.21}$$

We now solve the differential equation along the leg $\mathcal{PQ}$ of the loop. The solution at $\mathcal{Q}$ will be the sum of the parallel transport of the initial condition (3.21), together with an inhomogeneous term that is the tangent to the spatial geodesic. The result is

$$\vec{\xi}_{\mathcal{Q}} = -\Delta\tau_A\vec{u}_B(\tau_1) + \xi_B^{\hat{i}}(\tau_1)\vec{f}_{\hat{i}}(\tau_1). \tag{3.22}$$

Here the first term is the parallel transport term, and we have used the fact that the parallel transport of A's initial four-velocity is B's initial four-velocity. The second term is the tangent to the spatial geodesic at $\mathcal{Q}$, from the definitions (3.19) of $\xi_B^{\hat{i}}$ and of $\vec{f}_{\hat{\alpha}}$.

Next, we solve the differential equation along the segment $\mathcal{QS}$ of B's worldline. Since B parallel transports his own four-velocity, the result is

$$\vec{\xi}_{\mathcal{S}} = -\Delta\tau_A\vec{u}_B(\tau_2) + \xi_B^{\hat{i}}(\tau_1)\mathbf{\Gamma}\cdot\vec{f}_{\hat{i}}(\tau_1) + \Delta\tau_B\vec{u}_B(\tau_2). \tag{3.23}$$

Here $\Gamma^a{}_b$ is the parallel transport operator from $\mathcal{Q}$ to $\mathcal{S}$, and the last term is the inhomogeneous term, the tangent to B's worldline at $\mathcal{S}$. Finally we transport this result along the leg $\mathcal{SR}$ of the loop. When we parallel transport B's four-velocity $\vec{u}_B$, the result is the vector $\vec{w}_B$ defined above. Similarly, when we parallel transport the vector $\mathbf{\Gamma}\cdot\vec{f}_{\hat{i}}$, the result is the holonomy operator $\Lambda^a{}_b$ of the loop acting on the basis vector $\vec{e}_{\hat{i}}$ at $\mathcal{R}$. This is because $\vec{e}_{\hat{i}}$ is parallel transported along $\mathcal{RP}$, and because $\vec{f}_{\hat{i}}$ is obtained from $\vec{e}_{\hat{i}}$ by parallel transporting along $\mathcal{PQ}$. Thus we obtain

$$\vec{\xi}_{\mathcal{R}} = (\Delta\tau_B - \Delta\tau_A)\vec{w}_B + \xi_B^{\hat{i}}(\tau_1)\mathbf{\Lambda}\cdot\vec{e}_{\hat{i}} - \xi_B^{\hat{i}}(\tau_2)\vec{e}_{\hat{i}}, \tag{3.24}$$

[10] Uniqueness requires that B is sufficiently close to A to be inside a convex normal neighborhood.

[11] Note that the parameter τ need not be proper time along B's worldline, and the orthonormal tetrad $\vec{f}_{\hat{\alpha}}$ need not be parallel transported along B's worldline.

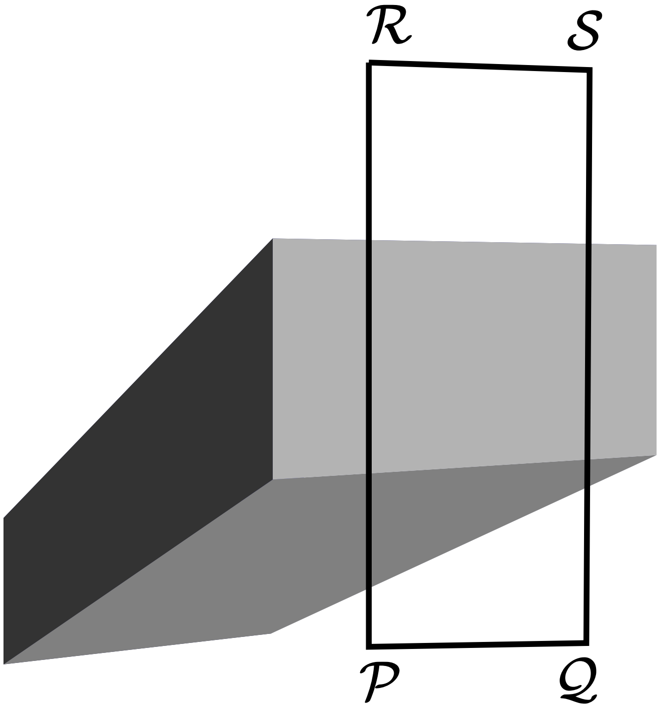

FIG. 1. Spacetime diagram of a burst of gravitational waves and the curve used to compute the generalized holonomy. The gray region represents the spacetime location of the gravitational waves, while the unshaded regions are Minkowski spacetimes before and after the burst. The curve bounded by $\mathcal{P}$ and $\mathcal{R}$ is the worldline of observer A, and that bordered by $\mathcal{Q}$ and $\mathcal{S}$ is that of B. The curves with endpoints $(\mathcal{P}, \mathcal{Q})$ and $(\mathcal{R}, \mathcal{S})$ are spacelike geodesics before and after the burst, respectively, which are just straight lines in the flat spacetime regions.

where the last term is the inhomogeneous term. This is equivalent to the formula (3.20).

IV. GENERALIZED HOLONOMY IN LINEARIZED GRAVITY

This section provides two related examples of the generalized holonomy. Both spacetimes consist of a flat Minkowski region followed by a burst of gravitational waves with memory, after which the spacetimes settle to a different Minkowski region. The first example treats a linearized plane wave, which reproduces the result of Sec. I B in a covariant language. The second example deals with a linearized pulse of waves heading radially outward from a pointlike source. This more general example gives an indication of the magnitude and the form of the disagreement that observers will have when measuring angular momentum.

A schematic spacetime diagram with the curve used to compute the generalized holonomy is depicted in Fig. 1. As discussed in the previous section, there are two freely falling observers A and B, and we consider a closed curve consisting of segments of their worldlines together with spatial geodesics that join the two worldlines. In this section, we additionally assume that spacetime is flat initially and at late times (the unshaded portions of the diagram), and that at intermediate times there is a burst of gravitational waves present (the gray shaded region). The two spacelike curves with endpoints bounded by $\mathcal{P}$ and $\mathcal{Q}$ before the burst, and $\mathcal{R}$ and $\mathcal{S}$ after are geodesics of Minkowski space (i.e., straight lines in some surface of constant time). The figure describes both the plane wave (Sec. IV A) and a local region of the radially propagating gravitational wave (Sec. IV B).

A. Generalized holonomy for a gravitational plane wave with memory

We consider a spacetime which is flat at early and at late times, and which contains a linearized plane wave at intermediate times. We use the conventions described in Sec. I B above: the metric is given by the expression (1.4) in global TT coordinates (T, X^i), with the metric perturbation being $h_{ij}(T-Z)$. This metric perturbation vanishes at early times but at late times asymptotes to the constant value h^{∞}_{ij}. We also introduce a coordinate system (t, x^i) which at early times is an inertial coordinate system and coincides with the TT coordinates (T, X^i), and which at late times is again inertial and related to the TT coordinates by Eq. (1.5).

The observers A and B are freely falling and are therefore stationary with respect to the TT coordinates, with $X^i = X^i_A =$ constant for A, and $X^i = X^i_B =$ constant for B. The inertial-frame locations of the observers at early times are

$$x^i_A = X^i_A, \quad x^i_B = X^i_B, \tag{4.1}$$

while at late times they are

$$x'^i_A = (\delta^i_j + \tfrac{1}{2} h^{\infty\,i}{}_j) X^j_A, \quad x'^i_B = (\delta^i_j + \tfrac{1}{2} h^{\infty\,i}{}_j) X^j_B, \tag{4.2}$$

as discussed in Sec. I B above.

To compute the generalized holonomy, it will be useful to recall results from Sec. III A. First, recall that the generalized holonomy along a curve composed of several segments is just the composition of the individual solutions to Eq. (3.1). Second, remember that the general solution can be written as the sum of a homogeneous solution (i.e., the usual holonomy) and an inhomogeneous solution (the part that is independent of the initial data), which allows the two solutions to be computed independently. Third, note that for geodesic curves the inhomogeneous part of the solution is proportional to the tangent to the curve at the endpoint. Thus, the generalized holonomy can be found by computing the affine transport in four steps ($\mathcal{P}$ to $\mathcal{R}$ to $\mathcal{S}$ to $\mathcal{Q}$ to $\mathcal{P}$), while computing the inhomogeneous and homogeneous parts separately.

1. Calculation of the inhomogeneous solution

- $\mathcal{P}$ to $\mathcal{R}$: We transport the initial vector $\vec{\xi}_{\mathcal{P}} = 0$ along the geodesic from $\mathcal{P}$ to $\mathcal{R}$ using the affine transport law (3.1) with $\alpha = 1$. The result is

$$\vec{\xi}_{\mathcal{R}} = (\delta t)\partial_T = (\delta t)\partial_t,$$

where δt is the interval of A's proper time between $\mathcal{P}$ and $\mathcal{R}$. In the second equation we have transformed from TT coordinates to the inertial coordinates.

- $\mathcal{R}$ to $\mathcal{S}$: Next, we use the vector $\vec{\xi}_{\mathcal{R}}$ as an initial condition for the affine transport along the straight line extending from $\mathcal{R}$ to $\mathcal{S}$ in the flat spacetime region after the burst. It is easiest to perform this computation in the inertial coordinates (t, x^i). The result is

$$\vec{\xi}_{\mathcal{S}} = (\delta t)\partial_t + (x'^i_B - x'^i_A)\frac{\partial}{\partial x^i}.$$

Next we use Eqs. (4.1) and (4.2) and transform back to the TT coordinates, giving

$$\vec{\xi}_{\mathcal{S}} = (\delta t)\partial_T + (x^i_B - x^i_A)\frac{\partial}{\partial X^i}.$$

(Here in the spatial components there was a cancellation between a factor of $\mathbf{1} + \frac{1}{2}\mathbf{h}^\infty$ and its inverse.)

- $\mathcal{S}$ to $\mathcal{Q}$: This part of the affine transport removes the timelike component of the vector, and it transforms the spatial part of $\xi^a_{\mathcal{S}}$ because the spatial vectors change under parallel transport. As a result, the outcome of the transport is

$$\vec{\xi}_{\mathcal{Q}} = (\delta^i_j + \tfrac{1}{2}h^{\infty\, i}{}_j)(x^j_B - x^j_A)\frac{\partial}{\partial X^i}.$$

- $\mathcal{Q}$ to $\mathcal{P}$: The affine transport takes place along a straight line in a flat spacetime region, and so its net effect is to add the corresponding displacement vector along the line. The final result at $\mathcal{P}$ gives the inhomogeneous piece of the general solution

$$\vec{\Delta\xi} = \vec{\xi}_{\mathcal{P}} = \tfrac{1}{2}h^{\infty\, i}{}_j(x^j_B - x^j_A)\frac{\partial}{\partial X^i}. \qquad (4.3)$$

2. Homogeneous solution and the generalized holonomy

It is not too difficult to see that the holonomy of parallel transport is the identity map,

$$\Lambda^a{}_b = \delta^a{}_b \qquad (4.4)$$

for the curve shown in Fig. 1 even though the spacetime has nontrivial curvature. It follows from the fact that the parallel transport is trivial in the flat regions of spacetime, and that it is identical on the two worldlines of the two different observers. Consequently, the inhomogeneous solution is the only relevant part of the generalized holonomy.

3. Relation to the memory effect and the observer dependence of angular momentum

In this example, the generalized holonomy is directly related to the change in proper distance between the two observers that arises from the solution to the equation of geodesic deviation (the usual physical effect of the gravitational-wave memory). This leads to an observer dependence in angular momentum which is given by $\delta J^{\alpha\beta} = 2\Delta\xi^{[\alpha}P^{\beta]}$, from Eqs. (3.12) and (4.4). Using the result (4.3) this corresponds to an observer dependence of the spatial angular momentum of the spatial angular momentum of $-\frac{1}{2}\varepsilon_{ijk}h^{\infty\, j}{}_l\delta x^l p^k$, where p^k is the spatial momentum and $\delta\mathbf{x} = \mathbf{x}_B - \mathbf{x}_A$. This is precisely the result (1.11) found in Sec. I B.

B. Generalized holonomy for a gravitational wave at large radius

For a gravitational wave propagating radially outward from a pointlike source, the computation of the generalized holonomy is very similar to that of the plane wave, but the expressions are somewhat lengthier. The linearized metric of this spacetime has the same form as that of Eq. (1.4), but the function $h_{\alpha\beta}(t - z)$ gets replaced by an outgoing wave solution in spherical polar coordinates. The most common form of this metric is given in Lorentz gauge—see, e.g., Eqs. (8.13a)–(8.13c) of [22]—which is often expressed as a sum of terms proportional to mass and current multipoles and the time derivatives of the multipoles. To compute the generalized holonomy, only the leading order terms in a series in $1/r$ will be needed. In addition, it will be most useful to express the metric perturbation in a TT gauge rather than Lorentz gauge.

1. Transverse-traceless metric perturbation

The quickest way to compute the TT metric perturbation is to compute the Riemann tensor and use the fact that the TT metric perturbation is related to the gauge-invariant Riemann tensor (at linear order in the metric perturbation) via the relation

$$R_{0i0j} = \ddot{h}^{\rm TT}_{ij}\,, \qquad (4.5)$$

where the pair of dots over $h^{\rm TT}_{ij}$ indicate taking two time derivatives. The metric perturbation can be found by integrating Eq. (4.5) twice with respect to time. In coordinates (u, r, θ, φ) where $u = t - r$, and starting from

Eqs. (8.13a)–(8.13c) of [22], the result is

$$h_{ij}^{\rm TT} = \frac{1}{r}\sum_{\ell=2}^{\infty}\left\{\frac{1}{\ell!}[4n_{(i}\overset{(\ell)}{\mathcal{I}}_{j)A_{\ell-1}}n^{A_{\ell-1}} - 2\overset{(\ell)}{\mathcal{I}}_{ijA_{\ell-2}}n^{A_{\ell-2}} - (\delta_{ij}+n_in_j)\overset{(\ell)}{\mathcal{I}}_{A_\ell}n^{A_\ell}] + \frac{4\ell}{(\ell+1)!}n^q\times [n_{(i}\varepsilon_{j)pq}\overset{(\ell)}{\mathcal{S}}_{pA_{\ell-1}}n^{A_{\ell-1}} - \varepsilon_{pq(i}\overset{(\ell)}{\mathcal{S}}_{j)pA_{\ell-2}}n^{A_{\ell-2}}]\right\} + O(1/r^2)\,. \tag{4.6}$$

Here $\mathcal{I}_{A_\ell} = \mathcal{I}_{A_\ell}(u)$ is an ℓ-pole mass moment and $\mathcal{S}_{A_\ell} = \mathcal{S}_{A_\ell}(u)$ is an ℓ-pole current moment, which are symmetric trace-free (STF) tensors with ℓ indices (the subscript A_ℓ is one notation used to represent ℓ spatial indices). The notation (ℓ) above the symbols for the moments means the $\ell^{\rm th}$ derivative with respect to u. The vector n^i is a unit radial vector (i.e., x^i/r) and n^{A_ℓ} is the tensor product of ℓ radial unit vectors.

2. Multipoles and coordinate change after the burst

As in the example of the plane wave, it will be assumed that before a retarded time $u \equiv t - r = 0$, all the multipoles vanish; they are dynamical between 0 and u_f; and after the retarded time u_f, the spacetime is again Minkowski, but some of the multipoles and their time derivatives can have nonzero constant values which correspond to the gravitational-wave memory. Interestingly, only certain multipoles can go to constant values and still have the spacetime be Minkowski (and, hence, stationary). Specifically, the $\ell^{\rm th}$ time derivative of the mass-multipole STF tensors $\mathcal{I}_{A_\ell}$ can take nonzero values whereas the equivalent time derivatives of the current multipoles $\mathcal{S}_{A_\ell}$ cannot asymptote to a nonzero value and still be Minkowski space. This seems to be closely related to the fact that there is no magnetic-type memory from physically realistic sources [27].

First, consider just the mass multipoles, and assume that the $\ell^{\rm th}$ time derivatives go to constant values. A short calculation can show that the generator of linearized gauge transformations below can remove the constant time derivatives of the mass multipoles after the burst of waves:

$$\Xi_0 = \sum_{\ell=2}^{\infty}\frac{\ell+2}{\ell(\ell!)}\overset{(\ell)}{\mathcal{I}}_{A_\ell}n^{A_\ell}\,, \tag{4.7a}$$

$$\Xi_i = -\sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left[\frac{1}{\ell-1}\overset{(\ell)}{\mathcal{I}}_{iA_{\ell-1}}n^{A_{\ell-1}} + \frac{1}{2}\overset{(\ell)}{\mathcal{I}}_{A_\ell}n^{A_\ell}n_i - \frac{(\ell+2)}{2r}(\overset{(\ell-1)}{\mathcal{I}}_{iA_{\ell-1}}n^{A_{\ell-1}} - \overset{(\ell-1)}{\mathcal{I}}_{A_\ell}n^{A_\ell}n_i)\right]. \tag{4.7b}$$

For the current multipoles, the only linearized gauge generator that can be constructed from the $\ell^{\rm th}$ time derivative of $\mathcal{S}_{A_\ell}$, the radial vectors n^i, and the antisymmetric tensor ε_{ipq} would be proportional to the following:

$$\Xi_i^{(\mathcal{S})} = \sum_{\ell=2}^{\infty}\varepsilon_{ipq}\overset{(\ell)}{\mathcal{S}}_{pA_{\ell-1}}n^{A_{\ell-1}}n^q\,. \tag{4.8}$$

A second quick calculation will show that this transformation does *not* make the spacetime flat. As a result, we will require that the multipoles satisfy the conditions

$$\overset{(\ell)}{\mathcal{I}}_{A_\ell} = \text{const.} \quad \text{and} \quad \overset{(\ell)}{\mathcal{S}}_{A_\ell} = 0\,, \tag{4.9}$$

when $u > u_f$.

With the metric determined by Eq. (4.6), subject to the condition (4.9), the gauge transformation (4.7) is sufficient to define Minkowski coordinates after the pulse of waves via the relation

$$y^\alpha = x^\alpha + \Xi^\alpha. \tag{4.10}$$

This also provides the necessary information to compute the generalized holonomy. As in the previous plane-wave example, we will split the calculation into the inhomogeneous and homogeneous parts, which are treated in the next subparts, respectively.

3. Calculation of the inhomogeneous solution

- $\mathcal{P}$ to $\mathcal{R}$: This is identical to the equivalent calculation involving the gravitational plane wave: the vector after affine transport is $\xi^\alpha_{\mathcal{R}} = (\delta t)u^\alpha$, where $\vec u = \partial_t$. As before, it is helpful to transform to the flat coordinates y^α after the pulse, defined by Eqs. (4.10) and (4.7). This introduces two new terms into the result: $\xi^{\alpha'}_{\mathcal{R}} = (\delta t - \Xi^{0'}_{\mathcal{R}})u^{\alpha'} - \Xi^{\alpha'}_{\mathcal{R},0'}\delta t$. Here we use primes to denote tensor components in the Minkowski coordinates, $x^{\alpha'} = y^\alpha$.
- $\mathcal{R}$ to $\mathcal{S}$: In the flat Minkowski space after the burst, the affine transport gives $\xi^{\alpha'}_{\mathcal{S}} = (\delta t - \Xi^{0'}_{\mathcal{R}})u^{\alpha'} - \Xi^{\alpha'}_{\mathcal{R},0'}\delta t + \delta^{\alpha'}{}_{i'}\delta y^{i'}$. Changing back to the x^α coordinates alters the spatial part of the vector so that $\xi^\alpha_{\mathcal{S}} = u^\alpha(\delta t + \delta\Xi^0) + \delta\Xi^\alpha{}_{,0}\delta t + \Xi^\alpha_{\mathcal{S},i}\delta x^i + \delta^\alpha{}_i(\delta x^i + \delta\Xi^i)$. Here $\delta\Xi^\alpha = \Xi^\alpha_{\mathcal{S}} - \Xi^\alpha_{\mathcal{R}}$ has been defined.
- $\mathcal{S}$ to $\mathcal{Q}$: Transporting back through the burst changes the spatial part of the vector to $\xi^\alpha_{\mathcal{Q}} = \delta\Xi^0u^\alpha + \delta\Xi^\alpha{}_{,0}\delta t + \frac{1}{2}(\Xi^\alpha_{\mathcal{S},i} - \Xi_i^{\mathcal{S},\alpha})\delta x^i + \delta^\alpha{}_i(\delta x^i + \delta\Xi^i)$, where the change occurred from the parallel transport of the affine frame back to the original point and where the fact that $h^\alpha{}_i = \Xi^\alpha{}_{,i} + \Xi_i{}^{,\alpha}$ was used to simplify the change in the spatial part of the vector.
- $\mathcal{Q}$ to $\mathcal{P}$: Along this flat geodesic in the Minkowski space prior to the burst, the affine transport adds the displacement vector δx^i to the result of $\xi^\alpha_{\mathcal{Q}}$. Thus, the complete inhomogeneous solution is

$$\Delta\xi^\alpha = \delta\Xi^0u^\alpha + \delta\Xi^\alpha{}_{,0}\delta t + \tfrac{1}{2}(\Xi^\alpha_{\mathcal{S},i} - \Xi_i^{\mathcal{S},\alpha})\delta x^i + \delta^\alpha{}_i\delta\Xi^i. \tag{4.11}$$

We will discuss the relationship between the terms that appear in $\Delta\xi^a_{\mathcal{P}}$ and the gravitational-wave memory in more detail below.

4. Homogeneous solution and the generalized holonomy

The calculation of the homogeneous part of the solution is simpler than that of the inhomogeneous portion above. The first set of nontrivial terms come from the parallel transport along the worldline extending from $\mathcal{P}$ to $\mathcal{R}$, and from the coordinate change at $\mathcal{R}$. For an arbitrary initial condition $\xi^\alpha_{(0)}$, this vector will be modified by an amount $-\frac{1}{2}(\Xi^\alpha_{\mathcal{R},\beta} - \Xi^{\mathcal{R},\alpha}_\beta)\xi^\beta_{(0)}$. There will be a similar contribution with the opposite sign involving quantities at the point $\mathcal{S}$ from the parallel transport along the worldline from $\mathcal{S}$ to $\mathcal{Q}$ and the coordinate change at $\mathcal{S}$. Thus, the part of the holonomy that differs from the identity is given by $\frac{1}{2}(\Xi^\alpha{}_{,\beta} - \Xi_\beta{}^{,\alpha})_{\mathcal{SR}}\xi^\beta_{(0)}$, where the subscript $\mathcal{SR}$ implies it is the difference of the values at the quantities at the coordinate points $\mathcal{S}$ and $\mathcal{R}$, transported back to $\mathcal{P}$.

From the expression for Ξ^α in the gauge transformation (4.7), it is possible to show that the generalized holonomy has a homogeneous piece in the form of a local infinitesimal Lorentz transformation that scales as $1/r$, and an inhomogeneous part that contains terms independent of δx and δt that are zeroth order in $1/r$ (and also terms that go as $1/r$, which we will not show), in addition to terms that scale as $\delta x/r$, and $\delta t/r$. For the inhomogeneous part, these terms will be labeled by $\Delta\xi^\alpha_{(1)}$, $\Delta\xi^\alpha_{(\delta x/r)}$, and $\Delta\xi^\alpha_{(\delta t/r)}$. The zeroth-order terms that are come from $-\delta\Xi^\alpha$, whereas the terms of order $\delta t/r$ and $\delta x/r$ come from the terms $\delta\Xi^\alpha{}_{,0}\delta t$ and $\frac{1}{2}(\Xi^\alpha_{\mathcal{S},i} - \Xi_i^{\mathcal{S},\alpha})\delta x^i$, respectively. These terms are

$$\Delta\xi^0_{(1)} = \sum_{\ell=2}^\infty \frac{\ell+2}{\ell(\ell!)}(\overset{(\ell)}{\mathcal{I}}_{A_\ell}\, n^{A_\ell})_{\mathcal{SR}} \,, \tag{4.12a}$$

$$\begin{aligned}\Delta\xi_i^{(1)} = &-\sum_{\ell=2}^\infty \frac{1}{\ell!}\left[\frac{1}{\ell-1}(\overset{(\ell)}{\mathcal{I}}_{iA_{\ell-1}}\, n^{A_{\ell-1}})_{\mathcal{SR}}\right.\\ &\left.+\frac{1}{2}(\overset{(\ell)}{\mathcal{I}}_{A_\ell}\, n^{A_\ell}n^i)_{\mathcal{SR}}\right] ,\end{aligned} \tag{4.12b}$$

$$\begin{aligned}\Delta\xi_i^{(\delta t/r)} = &-\delta t\sum_{\ell=2}^\infty \frac{\ell+2}{2(\ell!)}[(\overset{(\ell)}{\mathcal{I}}_{iA_{\ell-1}}\, n^{A_{\ell-1}}/r)_{\mathcal{SR}}\\ &-(\overset{(\ell)}{\mathcal{I}}_{A_\ell}\, n^{A_\ell}n_i/r)_{\mathcal{SR}}] ,\end{aligned} \tag{4.12c}$$

$$\begin{aligned}\Delta\xi^0_{(\delta x/r)} = &-\frac{\delta x^i}{r_{\mathcal{S}}}\sum_{\ell=2}^\infty \frac{\ell+2}{\ell!}[(\overset{(\ell)}{\mathcal{I}}_{iA_{\ell-1}}\, n^{A_{\ell-1}})_{\mathcal{S}}\\ &-(\overset{(\ell)}{\mathcal{I}}_{A_\ell}\, n^{A_\ell}n_i)_{\mathcal{S}}] ,\end{aligned} \tag{4.12d}$$

$$\Delta\xi_i^{(\delta x/r)} = \frac{2\delta x^j}{r_{\mathcal{S}}}\sum_{\ell=2}^\infty \frac{1}{\ell!}(n_{[i}\, \overset{(\ell)}{\mathcal{I}}_{j]A_{\ell-1}}\, n^{A_{\ell-1}})_{\mathcal{S}} \,. \tag{4.12e}$$

In the expression above, the subscript $\mathcal{SR}$ means to take the difference of the quantity within parentheses evaluated at the values of the coordinate points $\mathcal{S}$ and $\mathcal{R}$.

The local infinitesimal Lorentz transformation, which will be denoted as $\omega_{\alpha\beta} = \omega_{[\alpha\beta]}$ is strictly of order $1/r$ and can be written as

$$\begin{aligned}\omega_{i0} = &\sum_{\ell=2}^\infty \frac{\ell+2}{\ell!}[(\overset{(\ell)}{\mathcal{I}}_{iA_{\ell-1}}\, n^{A_{\ell-1}}/r)_{\mathcal{SR}} - (\overset{(\ell)}{\mathcal{I}}_{A_\ell}\, n^{A_\ell}n_i/r)_{\mathcal{SR}}]\\ &+O(1/r^2) \,,\end{aligned} \tag{4.13a}$$

$$\omega_{ij} = \sum_{\ell=2}^\infty \frac{2}{\ell!}(n_{[i}\, \overset{(\ell)}{\mathcal{I}}_{j]A_{\ell-1}}\, n^{A_{\ell-1}}/r)_{\mathcal{SR}} + O(1/r^2) \,. \tag{4.13b}$$

5. Relation to the memory effect and the observer dependence of angular momentum

The relation between the generalized holonomy and the physical effects associated with the gravitational-wave memory is somewhat more involved than it was for a gravitational plane wave. The term $\delta\Xi^i$ is a measure of the change in distance between the observers that occurs from the memory. In addition, the part $\delta\Xi^0$ gives information about the difference in proper time measured by the two observers that is a result of the memory of the gravitational-wave burst. The other term $\frac{1}{2}(\Xi^\alpha_{\mathcal{S},i} - \Xi_i^{\mathcal{S},\alpha})\delta x^i$ takes into account a boosting and rotation of the spatial displacement vector along the other observer's worldline from the wave's memory, and the part $\delta\Xi^\alpha{}_{,0}\delta t$ represents a relative change in the tangent to the observers' worldlines from the memory.

Because the inhomogeneous solution has a zeroth-order piece in $1/r$, the center of mass and the angular momentum will be have an observer dependence with a magnitude of order $P^0\Delta\xi^i_{(1)} + P^i\Delta\xi^0_{(1)}$ and $\epsilon_{ijk}\Delta\xi^j_{(1)}P^k$, respectively, where P^a is the four-momentum of the source. For separations for which δx is of order r, then the terms $\Delta\xi^a_{(\delta x/r)}$ will also have leading-order contributions to the observer dependence of the center of mass and of the angular momentum of the form $P^0\Delta\xi^i_{(\delta x/r)} + P^i\Delta\xi^0_{(\delta x/r)}$ and $\epsilon_{ijk}\Delta\xi^j_{(\delta x/r)}P^k$, respectively. Similarly, for times δt of order the light-travel time to the source (i.e., of order r), then there will be additional observer dependence from terms of the form $P^0\Delta\xi^i_{(\delta t/r)}$ and $\epsilon_{ijk}\Delta\xi^j_{(\delta t/r)}P^k$.

Equations (2.4) and (2.8) imply that the angular momentum tensor J^{ab} will have terms proportional to r at large radii: specifically, it is the orbital-like part of the angular momentum $2y^{[a}P^{b]} = 2rn^{[a}P^{b]}$ that has this scaling. When the angular momentum transforms by Eq. (3.16), the $1/r$ parts of the holonomy will induce a change in the angular momentum that is of order unity in a series in $1/r$. These terms will have the form $\delta J^{ab} = 2(\omega^a{}_c y^{[c}P^{b]} - \omega^a{}_c y^{[b}P^{c]})$. The lowest-order part of the four-momentum will still be unambiguous, and any observer dependence will be a relative $1/r$ effect.

V. CONCLUSIONS

In this paper, we noted that bursts of gravitational waves cause spatially separated observers to disagree on their changes in displacement, and therefore to disagree on their measured special-relativistic angular momenta of a source. This observer dependence of angular momentum is related to the gravitational-wave memory of the pulse of waves. We derived this phenomenon first in a simple context of linearized plane waves, and later in a more systematic and covariant framework.

We defined a procedure by which observers could measure a type of special-relativistic linear and angular momentum at their locations, from the spacetime geometry in their vicinity. The procedure gives the correct result when the spacetime is linear and stationary, and the measurement takes place near future null infinity. We estimated the errors in the procedure when the spacetime is nonlinear, dynamical, or the source is not isolated.

To compare angular momentum at different spacetime points, we defined a transport equation, the *affine transport*, which is a slight generalization of parallel transport. The transport around a closed curve, the *generalized holonomy*, consists of a Poincaré transformation, rather than a Lorentz transformation as for a normal holonomy. The generalized holonomy contains an inhomogeneous displacement term. The extent to which the generalized holonomy is nontrivial is a measure of how much spacetime curvature prevents different observers from arriving at a consistent definition of linear and angular momentum.

For two freely falling observers, who encounter a burst of gravitational waves, we showed that there are four independent observables that can be nontrivial when the burst has departed, and that can be considered to be types of "gravitational-wave memory." There is the usual displacement memory, a residual relative boost, a relative rotation, and a difference in elapsed proper time between the two observers. These four observables are all encoded in the generalized holonomy around a suitably defined closed loop in spacetime. Thus, we clarified and generalized the often-noted close relation between gravitational-wave memory and observer dependence of angular momentum.

Finally, we performed explicit computations in two different specific contexts that illustrate the relationships between generalized holonomy, observer dependence of angular momentum, and gravitational-wave memory. The first context was a plane gravitational wave with memory passing through flat spacetime, and the second was a outgoing linearized gravitational wave near future null infinity. The plane wave only showed the displacement memory effect, but the multipolar gravitational wave displayed all four of the physical observables associated with the memory.

Although our goal was to provide physical insight into the nature of the BMS group, the generalized holonomy tool does not quite achieve this goal: In Appendix B, we show that the generalized holonomy can be nontrivial for certain spacelike curves in the Schwarzschild spacetime, even as the curves tend to spatial infinity. Hence, observers along this curve would find their measured angular momentum to be observer dependent, even though angular momentum is well defined in Schwarzschild (as the BMS group has a preferred Poincaré subgroup in stationary spacetimes). Thus, our prescription for assessing observer dependence in angular momentum not only captures BMS/memory ambiguities in angular momentum, but also reflects other, more trivial effects of spacetime curvature on angular momentum measurements. Finding a method to isolate just the BMS ambiguities is a topic we will investigate in future work.

Because the affine transport law defines a way to compare other vectors in addition to the angular momentum at different spacetime points, it could find application to other problems. For example, if a burst of gravitational waves passes through a post-Newtonian spacetime, the momenta and angular momenta of the particles that enter into the post-Newtonian equations of motion could differ before and after the burst. The affine transport may be useful for deriving a prescription for matching the post-Newtonian spacetimes before and after the bursts, in a manner that allows one to compute the motion of an N-body system.

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Appendix A: Angular Momentum Transport Laws

In the body of this paper, we introduced a method of transporting a pair of tensors (P^a, J^{ab}) along a curve from one point to another in a curved spacetime. In this appendix, we show that the transport method is equivalent to solving the following simple set of differential equations along the curve:

$$k^a \nabla_a P^b = 0, \quad k^a \nabla_a J^{bc} = -2P^{[b}k^{c]}. \tag{A1}$$

The equivalence between the two methods was pointed out to us by Justin Vines [28].

Suppose we have a curve $x^\alpha = x^\alpha(\lambda)$ that joins at point $\mathcal{P}$ at $\lambda = 0$ to another point $\mathcal{Q}$ at $\lambda = 1$. We introduce an orthonormal basis of vectors $\vec{e}_{\hat{\alpha}}$ at $\mathcal{Q}$, and extend it along the curve by parallel transport. We decompose the four momentum and angular momentum on this basis as

$$P^a = P^{\hat{\alpha}} e^a_{\hat{\alpha}}, \qquad J^{ab} = J^{\hat{\alpha}\hat{\beta}} e^a_{\hat{\alpha}} e^b_{\hat{\beta}}. \tag{A2}$$

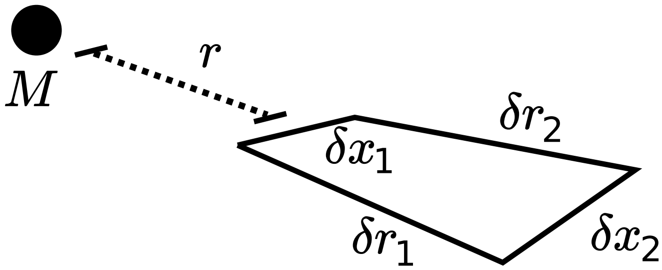

FIG. 2. Curve used to compute the generalized holonomy in the Schwarzschild spacetime of mass M at large radii. The lengths of the four segments that compose the curve, δr_1, δr_2, δx_1, and δx_2 are all of order r, where $r \gg M$ is the closest distance to the source along the segment labeled by δx_1. This curve has a nontrivial generalized holonomy as r goes to spatial infinity, even though the Schwarzschild spacetime has a well defined angular momentum.

The transport equations (A1), when written in terms of this basis, become

$$\frac{d}{d\lambda}P^{\hat{\alpha}} = 0, \tag{A3a}$$

$$\frac{d}{d\lambda}J^{\hat{\alpha}\hat{\beta}} = -P^{\hat{\alpha}}k^{\hat{\beta}} + P^{\hat{\beta}}k^{\hat{\alpha}}. \tag{A3b}$$

The first of these gives $P^{\hat{\alpha}}(\lambda) = P_0^{\hat{\alpha}} = \text{constant}$. We now make the ansatz for the angular momentum solution

$$J^{\hat{\alpha}\hat{\beta}}(\lambda) = J_0^{\hat{\alpha}\hat{\beta}} + P_0^{\hat{\alpha}}\chi^{\hat{\beta}}(\lambda) - P_0^{\hat{\beta}}\chi^{\hat{\chi}}(\lambda), \tag{A4}$$

for some vector $\chi^{\hat{\alpha}}(\lambda)$, where $J_0^{\hat{\alpha}\hat{\beta}}$ is the initial value of $J^{\hat{\alpha}\hat{\beta}}$ at $\lambda = 0$. Using this ansatz we see that the differential equation (A3b) will be satisfied if $\chi^{\hat{\alpha}}$ vanishes at $\mathcal{P}$ and satisfies

$$\frac{d}{d\lambda}\chi^{\hat{\alpha}} = -k^{\hat{\alpha}}. \tag{A5}$$

This differential equation coincides with the differential equation (3.1) that defines the generalized parallel transport, for the case $\alpha = -1$, the same case as was found in the body of the paper. By comparing with Eq. (3.2) we find

$$\chi^{\hat{\alpha}}(\mathcal{Q}) = -\Delta\xi^{\hat{\alpha}}. \tag{A6}$$

Substituting this result into the ansatz (A4) gives an expression for the angular momentum at $\mathcal{Q}$ which agrees with Eq. (3.12), establishing the result.

Appendix B: Generalized holonomy of a spacelike curve in the Schwarzschild spacetime

In this appendix, we compute the generalized holonomy of certain curves on in the Schwarzschild spacetime, and show that the generalized holonomy *does not* become the identity in the limit when the curves asymptote to future null infinity. As discussed in the body of the paper, this property implies that we cannot use the generalized holonomy as a tool to diagnose whether a given spacetime admits a well defined angular momentum (since in stationary spacetimes the BMS group has a preferred Poincaré subgroup and so normal angular momentum is well defined).

The specific curve we consider is shown in Fig. 2. Letting M denote the mass of the spacetime and $r \gg M$ be the distance to the closest point along the segment labeled by δx_1, we will assume that the lengths of the four sides of the curve, δx_1, δr_1, δx_2, and δr_2 are all of order r (or equivalently, the area enclosed by the curve is of order r^2). The curvature at the loop will scale as M/r^3. Because the holonomy associated with parallel transport scales as the curvature times the area, when the vector transported has magnitude of order r (such as the displacement vector y^a), the vector will undergo changes of order M. Similarly, because the inhomogeneous part of the generalized holonomy scales as curvature times the area to the three-halves power, the vector $\Delta\xi^a$ will also be of order M. Even as r approaches spatial infinity, this estimate suggests that there will be nontrivial generalized holonomy and observer dependence in angular momentum.

We did, in fact, compute the exact generalized holonomy in the case in which δr_1 and δr_2 are radial curves, and δx_1 and δx_2 are two coordinate lines between the end points of these two curves, respectively. The precise answer, while not particularly insightful, does indeed scale as M as r goes to spatial infinity. We conclude that the generalized holonomy is not specifically linked to BMS ambiguities in angular momentum, and is more generally a diagnostic of when spacetime curvature prevents observers from consistently measuring and comparing a type of special-relativistic angular momentum.

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