One-loop quantum electrodynamic correction to the gravitational potentials on de Sitter spacetime

C. L. Wang and R. P. Woodard

Phys. Rev. D **92**, 084008 — Published 2 October 2015

DOI: 10.1103/PhysRevD.92.084008

UFIFT-QG-14-07

One-loop Quantum Electrodynamic Correction to the Gravitational Potentials on de Sitter

C. L. Wang* and R. P. Woodard†

Department of Physics, University of Florida, Gainesville, FL 32611

ABSTRACT

We compute the one loop photon contribution to the graviton self-energy on de Sitter background and use it to solve the linearized Einstein equation for a point mass. Our results show that a co-moving observer sees a logarithmic spatial running Newton's constant. Equivalently a static observer reports a secular suppression of the Newtonian potential.

PACS numbers: 04.62.+v, 98.80.Cq, 04.60.-m

* e-mail: clwang@ufl.edu

† e-mail: woodard@phys.ufl.edu

1 Introduction

From Newton's law of universal gravitation to Einstein's general relativity, our understanding of gravity has developed profoundly. However, efforts to incorporate quantum mechanics in gravity have been unsuccessful, at least partly due to the fact that quantum gravitational effects are unobservably weak at low energies [1]. This unsatisfactory situation has been transformed by the accumulation of large cosmological data sets on primordial perturbations which are predicted by the theory of inflation to arise from the gravitational response to quantum fluctuations of matter and from quantum fluctuations in gravitational radiation [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. These quantum gravitational effects are hugely enhanced during primordial inflation and then fossilize so that they can survive to much later times [13].

Electromagnetism provided humanity's first and simplest example of a relativistic unified field theory. It is therefore natural to wonder how electromagnetism affects quantum gravity during inflation. Of course Einstein + Maxwell is not perturbatively renormalizable, even at one loop order [14, 15], and that does mean we must accept some limitations on what can be studied with it. We use Einstein + Maxwell in the standard sense of effective field theory, absorbing ultraviolet divergences order-by-order with BPHZ (Bogoliubov, Parsiuk, Hepp and Zimmermann) counterterms [16]. Once this is done, the renormalized results from loops of massless gravitons and photons engender non-local and ultraviolet finite contributions to the effective action which are unique predictions of the theory that cannot be changed by its still unknown ultraviolet completion [17, 18]. A famous example is the computation by Bjerrum-Bohr of the one graviton loop correction to the Coulomb potential on flat space background [19]. We work in the same theoretical context but on de Sitter background, and with the slightly different goal of inferring the one photon loop corrections to the Newtonian potential.

One way of studying quantum weak-field corrections to a classical theory is to first compute the corresponding one-particle-irreducible (1PI) 2-point function. This is then used to quantum-correct the linearized effective field equations. For our problem we must compute the one photon loop contribution to the graviton self-energy $-i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}\big](x;x')$. We can then infer quantum corrections to graviton mode functions, or to the gravitational response to sources, from the linearized effective field equation for the graviton field $h_{\mu\nu}(x)$,

$$\mathcal{D}^{\mu\nu\rho\sigma}\kappa h_{\rho\sigma}(x) - \int d^4x' \big[{}^{\mu\nu}\Sigma^{\rho\sigma}\big](x;x')\kappa h_{\rho\sigma}(x') = \mathcal{T}^{\mu\nu}_{\rm lin}(x) \equiv -\kappa \frac{\delta S_{\rm matter}}{\delta h_{\mu\nu}(x)}\bigg|_{h=0} , \qquad (1)$$

Here $\mathcal{D}^{\mu\nu\rho\sigma}$ is the Lichnerowicz operator in de Sitter background, $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity, and $\mathcal{T}^{\mu\nu}_{\rm lin}(x)$ is the linearized stress tensor density of whatever source is desired. In our case this source is that of a static point mass. The same framework (with zero source) has been employed to investigate corrections to graviton mode functions from massless, minimally coupled (MMC) scalars [20, 21]. There have also been studies of the effect of a loop of massless, conformally invariant scalars on the graviton mode function [22, 23] and on the background geometry [24].

This paper contains six sections of which the first is this Introduction. Section 2 gives those of the Feynman rules for Maxwell + Einstein which are needed for our computation, as well as some facts about the background geometry and the way we represent the tensor structure of the graviton self-energy. In section 3 we derive the primitive, dimensionally regulated result for the one photon loop contribution to the graviton self-energy. Section 4 introduces appropriate BPHZ couterterms for this theory and gives the result of renormalization. In section 5, we convert our in-out graviton self-energy to the retarded one of the Schwinger-Keldysh formalism, and use it to solve (1) for a stationary point mass. In section 6 we propose a physical interpretation of our result.

2 Feynman rules

2.1 Preliminary clarifications

This work is based on quantum gravitational perturbation theory in de Sitter space which is known to be the unique maximally symmetric solution to the Einstein field equation with a positive cosmological constant Λ. But we will not work on the full D-dimensional de Sitter manifold rather on a sub-manifold known as "cosmological patch" which is homogeneous, isotropic and spatially flat. The invariant line element for this geometry is,

$$ds^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x} \; , \tag{2}$$

where $a(t)$ is the scale factor measuring the expansion of the universe. The Hubble parameter is define as $H(t) \equiv \dot{a}(t)/a(t)$. It is convenient to use open conformal coordinates $x^\mu = (\eta, x^i)$ with,

$$-\infty < \eta < 0 \;\; , \;\; -\infty < x^i < +\infty \;\; , \;\; \text{for} \;\; i = 1, 2, \cdots, D-1 \; . \tag{3}$$

where the conformal time coordinate η is related to the normal time coordinate t through the relation, $dt \equiv a d\eta$. The metric in open conformal coordinates is conformally flat,

$$ds^2 = a^2(\eta) \left(-d\eta^2 + d\vec{x} \cdot d\vec{x}\right) = a^2(\eta) \eta_{\mu\nu} dx^\mu dx^\nu \; . \tag{4}$$

The Hubble parameter H is constant in de Sitter, and the scale factor in terms of conformal time coordinate is $a(\eta) = -\frac{1}{H\eta}$. We then define the graviton field $h_{\mu\nu}(x)$ by conformally transforming the full metric $g_{\mu\nu}(x)$ and then subtracting off

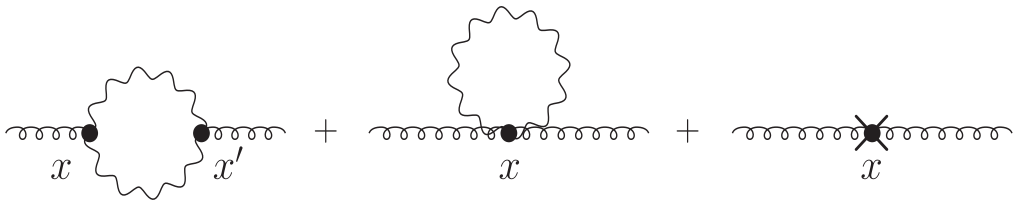

Figure 1: Photon contributions to the one loop graviton self-energy. Photon propagators are wavy and graviton propagators are curly.

the background,

$$g_{\mu\nu}(x) \equiv a^2(\eta)\Big[\eta_{\mu\nu} + \kappa h_{\mu\nu}(x)\Big] \equiv a^2(\eta)\widetilde{g}_{\mu\nu}(x) \; , \tag{5}$$

where $\eta_{\mu\nu}$ is the D-dimensional Minkowski metric with spacelike signature, $h_{\mu\nu}$ is the graviton field whose indices are raised and lowered with the Minkowski metric. And we will make great use of the de Sitter invariant length function $y(x;x')$ while representing our propagators on de Sitter,

$$y(x;x') \equiv a(\eta)a(\eta')H^2\Big[\|\vec{x}-\vec{x}'\|^2 - (|\eta-\eta'| - i\delta)^2\Big] \; , \tag{6}$$

2.2 Primitive diagrams

All possible propagators and interaction vertices are derived from the primitive Lagrangian density $\mathcal{L}$,

$$\mathcal{L} = \mathcal{L}_{\rm GR} + \mathcal{L}_{\rm EM} + \mathcal{L}_{\rm GRfix} + \mathcal{L}_{\rm EMfix} + \mathcal{L}_{\rm BPHZ} \; . \tag{7}$$

Here the gravitational Lagrangian density of low energy effective field theory is,

$$\mathcal{L}_{\rm GR} = \frac{1}{\kappa^2}\Big[R - (D-2)\Lambda\Big]\sqrt{-g} \; , \tag{8}$$

where the factor of $(D-2)$ multiplying Λ makes the pure gravitational field equation take the form of $R_{\mu\nu} = \Lambda g_{\mu\nu}$ in any dimension. And we define the cosmological constant as $\Lambda \equiv (D-1)H^2$. The Lagrangian density for electromagnetism takes the familiar form of Maxwell's theory,

$$\mathcal{L}_{\rm EM} = -\frac{1}{4}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\sqrt{-g} \; , \tag{9}$$

- **Propagators**

To get the graviton propagator, we expand the Lagrangian density (8) with the full metric (5) and extract a presumably irrelevant surface term. Then, the quadratic part takes the form [26],

$$\mathcal{L}_{\rm GR} - \text{Surface Terms} = \left(\frac{D}{2}-1\right) H a^{D-1} \sqrt{-\widetilde{g}}\, \widetilde{g}^{\rho\sigma} \widetilde{g}^{\mu\nu} h_{\nu 0} \partial_\mu h_{\rho\sigma}$$
$$+ a^{D-2} \sqrt{-\widetilde{g}}\, \widetilde{g}^{\alpha\beta} \widetilde{g}^{\rho\sigma} \widetilde{g}^{\mu\nu} \left\{ \frac{1}{2} \partial_\mu h_{\alpha\rho} \partial_\beta h_{\nu\sigma} - \frac{1}{2} \partial_\rho h_{\alpha\beta} \partial_\nu h_{\sigma\mu} + \frac{1}{4} \partial_\rho h_{\alpha\beta} \partial_\sigma h_{\mu\nu} - \frac{1}{4} \partial_\mu h_{\alpha\rho} \partial_\nu h_{\beta\sigma} \right\} . \quad (10)$$

We fix the gauge by adding [26],

$$\mathcal{L}_{\rm GRfix} = -\frac{1}{2} a^{D-2} \eta^{\mu\nu} F_\mu F_\nu \quad , \quad F_\mu \equiv \eta^{\rho\sigma} \left(\partial_\sigma h_{\mu\rho} - \frac{1}{2} \partial_\mu h_{\rho\sigma} + (D-2) H a \delta^0_\sigma h_{\mu\rho} \right) \; . \quad (11)$$

This gauge fixing term (11) is nice because it allow us to express the graviton propagator as a sum of scalar propagators multiplied by constant tensor factors,

$$i\big[{}_{\mu\nu}\Delta_{\alpha\beta}\big](x;x') \equiv \sum_{I=A,B,C} \Big[{}_{\mu\nu}T^I_{\alpha\beta}\Big] \times i\Delta_I(x;x') \; . \quad (12)$$

The various tensor factors are,

$$\Big[{}_{\mu\nu}T^A_{\rho\sigma}\Big] = 2\,\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - \frac{2}{(D-3)} \overline{\eta}_{\mu\nu} \overline{\eta}_{\rho\sigma} \; , \quad (13)$$
$$\Big[{}_{\mu\nu}T^B_{\rho\sigma}\Big] = -4 \delta^0_{(\mu} \overline{\eta}_{\nu)(\rho} \delta^0_{\sigma)} \; , \quad (14)$$
$$\Big[{}_{\mu\nu}T^C_{\rho\sigma}\Big] = \frac{2}{(D-2)(D-3)} \Big[(D-3)\delta^0_\mu \delta^0_\nu + \overline{\eta}_{\mu\nu}\Big] \Big[(D-3)\delta^0_\rho \delta^0_\sigma + \overline{\eta}_{\rho\sigma}\Big] \; , \quad (15)$$

where $\overline{\eta}^{\mu\nu} \equiv \eta^{\mu\nu} + \delta^\mu_0 \delta^\nu_0$ is the spatial part of the Minkowski metric. Each of those scalar propagators obey the associated propagator equation,

$$\mathcal{D}_I i\Delta_I(x;x') = i\delta^D(x-x') \qquad , \qquad I = A, B, C \; . \quad (16)$$

where we define three scalar kinetic operators as follows,

$$\mathcal{D}_A \equiv \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \right) \equiv \sqrt{-g}\; \square \; , \quad (17)$$

and,

$$\mathcal{D}_B \equiv \mathcal{D}_A - (D-2) H^2 \sqrt{-g} \qquad , \qquad \mathcal{D}_C \equiv \mathcal{D}_A - 2(D-3) H^2 \sqrt{-g} \; . \quad (18)$$

Note that the $i\Delta_A(x;x')$ is just the propagator of a MMC scalar. Thanks to the previous work [27], all the scalar propagators have been worked out,

$$\begin{aligned} i\Delta_A(x;x') &= i\Delta_{\rm cf}(x;x') \\ &\quad + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Bigg\{\frac{D}{D-4}\frac{\Gamma^2(\frac{D}{2})}{\Gamma(D-1)}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} - \pi\cot\Big(\frac{\pi}{2}D\Big) + \ln(aa')\Bigg\} \\ &\quad + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=1}^{\infty}\Bigg\{\frac{1}{n}\frac{\Gamma(n{+}D{-}1)}{\Gamma(n{+}\frac{D}{2})}\Big(\frac{y}{4}\Big)^{n} - \frac{1}{n{-}\frac{D}{2}{+}2}\frac{\Gamma(n{+}\frac{D}{2}{+}1)}{\Gamma(n{+}2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2}\Bigg\} \ , \end{aligned} \tag{19}$$

$$\begin{aligned} i\Delta_B(x;x') &= B(y) = i\Delta_{\rm cf}(x;x') \\ &\quad - \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=0}^{\infty}\Bigg\{\frac{\Gamma(n+D-2)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^{n} - \frac{\Gamma(n+\frac{D}{2})}{\Gamma(n+2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2}\Bigg\} \ , \end{aligned} \tag{20}$$

$$\begin{aligned} i\Delta_C(x;x') &= C(y) = i\Delta_{\rm cf}(x;x') \\ &\quad + \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\sum_{n=0}^{\infty}\Bigg\{(n+1)\frac{\Gamma(n+D-3)}{\Gamma(n+\frac{D}{2})}\Big(\frac{y}{4}\Big)^{n} \\ &\qquad\qquad - \Big(n-\frac{D}{2}+3\Big)\frac{\Gamma(n+\frac{D}{2}-1)}{\Gamma(n+2)}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2}\Bigg\} \ , \end{aligned} \tag{21}$$

where $i\Delta_{\rm cf}(x;x')$ is the de Sitter invariant propagator of a conformally coupled scalar,

$$i\Delta_{\rm cf}(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-1\right)\left(\frac{4}{y}\right)^{\frac{D}{2}-1} \ . \tag{22}$$

Note that the infinite series in each of these scalar propagators vanish in $D = 4$ dimensions, which means they are only needed when multiplied by some divergent pieces.

Unlike gravitons, photons show no physical breaking of de Sitter invariance. So we could in principle employ an exact de Sitter invariant gauge to make photon propagator manifestly de Sitter invariant [28]. However, a noncovariant gauge fixing $\mathcal{L}_{\rm EMfix}$ [26, 29] results in a much simpler form of the photon propagator,

$$i\big[{}_\mu\Delta_\nu\big](x;x') = \overline{\eta}_{\mu\nu} \times aa' i\Delta_B(x;x') - \delta^0_\mu\delta^0_\nu \times aa' i\Delta_C(x;x') \ . \tag{23}$$

Here and henceforth we adopt a short hand notation for the conformal time dependence of the scale factor,

$$a \equiv a(\eta) \ \ , \ \ a' \equiv a(\eta') \ . \tag{24}$$

- **Vertices**

Note that the action of electromagnetism, $S_{\rm EM} = \int d^D x \mathcal{L}_{\rm EM}$, contains interactions between two photons and any number of gravitons. So we obtain the 3-point and 4-point interaction vertices by functionally differentiating $S_{\rm EM}$ once and twice with respect to the graviton field, and then setting the graviton field to zero,

$$\frac{\delta S_{\rm EM}}{\delta h_{\mu\nu}(x)}\bigg|_{h=0} = \kappa a^{D-4} V^{\alpha\kappa\beta\lambda\mu\nu} \partial_\kappa A_\alpha(x) \partial_\lambda A_\beta(x) \; , \tag{25}$$

$$\frac{\delta^2 S_{\rm EM}}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')}\bigg|_{h=0} = \kappa^2 a^{D-4} U^{\alpha\kappa\beta\lambda\mu\nu\rho\sigma} \partial_\kappa A_\alpha(x) \partial_\lambda A_\beta(x) \delta^D(x-x') \; . \tag{26}$$

The tensor factors for the 3-point and 4-point vertices are [30],

$$V^{\alpha\kappa\beta\lambda\mu\nu} = \eta^{\mu\nu} \eta^{\alpha[\beta} \eta^{\lambda]\kappa} + 4\eta^{\mu)[\alpha} \eta^{\kappa][\beta} \eta^{\lambda](\nu} \; , \tag{27}$$

$$\begin{aligned} U^{\alpha\kappa\beta\lambda\mu\nu\rho\sigma} = & \Big(\frac{1}{4}\eta^{\mu\nu}\eta^{\rho\sigma} - \frac{1}{2}\eta^{\mu(\rho}\eta^{\sigma)\nu}\Big)\eta^{\alpha[\beta}\eta^{\lambda]\kappa} \\ & + \eta^{\mu\nu}\eta^{\rho)[\alpha}\eta^{\kappa][\beta}\eta^{\lambda](\sigma} + \eta^{\rho\sigma}\eta^{\mu)[\alpha}\eta^{\kappa][\beta}\eta^{\lambda](\nu} \\ & + \eta^{\alpha(\mu}\eta^{\nu)(\rho}\eta^{\sigma)[\beta}\eta^{\lambda]\kappa} + \eta^{\kappa(\mu}\eta^{\nu)(\rho}\eta^{\sigma)[\lambda}\eta^{\beta]\alpha} \\ & + \eta^{\alpha(\mu}\eta^{\nu)[\beta}\eta^{\lambda](\rho}\eta^{\sigma)\kappa} + \eta^{\kappa(\mu}\eta^{\nu)[\lambda}\eta^{\beta](\rho}\eta^{\sigma)\alpha} \\ & + \eta^{\alpha[\beta}\eta^{\lambda](\mu}\eta^{\nu)(\rho}\eta^{\sigma)\kappa} + \eta^{\kappa[\lambda}\eta^{\beta](\mu}\eta^{\nu)(\rho}\eta^{\sigma)\alpha} \; . \end{aligned} \tag{28}$$

- **Diagram Expressions**

Now using the Feynman rules stated in section 2, we can write down the formal expressions for the first two diagrams in Fig. 1. The leftmost diagram, also known as the "3pt function", is constructed from two 3-point vertices,

$$\begin{aligned} -i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm 3pt}\big](x;x') = & \frac{1}{2}\left(-i\kappa\right) a^{D-4} V^{\alpha\kappa\gamma\theta\mu\nu} \partial_\kappa \partial'_\lambda i\big[{}_\alpha\Delta_\beta\big](x;x') \\ & \times \left(-i\kappa\right) a'^{D-4} V^{\beta\lambda\delta\phi\rho\sigma} \partial_\theta \partial'_\phi i\big[{}_\gamma\Delta_\delta\big](x;x') \; . \end{aligned} \tag{29}$$

The middle diagram, also known as the "4pt function", is constructed from a single 4-point vertex,

$$-i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm 4pt}\big](x;x') = \left(-i\kappa^2\right) a^{D-4} U^{\alpha\kappa\beta\lambda\mu\nu\rho\sigma} \partial_\kappa \partial'_\lambda i\big[{}_\alpha\Delta_\beta\big](x;x') \delta^D(x-x') \; . \tag{30}$$

The rightmost diagram denotes a series of couterterms which will be explained in section 4.1.

2.3 Structure Functions

The graviton self-energy possesses 100 components in $D=4$ dimensions. We could report all of them, but the symmetries will relate these components such that they can be expressed using a few structure functions. This idea is actually inspired by experience in flat space. For example the Poincaré invariance makes the vacuum polarization of photons take the form with a single structure function $\Pi\big((x-x')^2\big)$,

$$i\big[{}^{\mu}\Pi^{\nu}_{\rm flat}\big](x;x') = (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})\,\partial_\rho\partial'_\sigma i\Pi\big((x-x')^2\big) \; . \tag{31}$$

And the graviton self-energy takes the form,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm flat}\Big](x;x') = \Pi^{\mu\nu}\Pi^{\rho\sigma}F_0\big((x-x')^2\big) + \bigg(\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{\Pi^{\mu\nu}\Pi^{\rho\sigma}}{D-1}\bigg)\,F_2\big((x-x')^2\big) \; , \tag{32}$$

where we define the projection operator as $\Pi^{\mu\nu} \equiv \partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2$. We usually call F_0 and F_2 as the spin zero and spin two part. The de Sitter background for inflation is spatially homogeneous and isotropic. Then if these are the only isometries we take to be manifest in our representation for the graviton self-energy, it turns out that we shall need four structure functions. And guided by the previous studies [31, 32] about a noncovariant representation of the vacuum polarization on de Sitter,

$$i\big[{}^{\mu}\Pi^{\nu}\big](x;x') = (\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})\,\partial_\rho\partial'_\sigma F(x;x') + (\overline{\eta}^{\mu\nu}\overline{\eta}^{\rho\sigma} - \overline{\eta}^{\mu\sigma}\overline{\eta}^{\nu\rho})\,\partial_\rho\partial'_\sigma G(x;x') \; , \tag{33}$$

which makes interpretation of physics much easier, here we will adopt a similar noncovariant representation of the graviton self-energy [33],

$$\begin{aligned} -i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}\big](x;x') &= \mathcal{F}^{\mu\nu}(x)\times\mathcal{F}^{\rho\sigma}(x')\Big[F_0(x;x')\Big] \\ &\quad + \mathcal{G}^{\mu\nu}(x)\times\mathcal{G}^{\rho\sigma}(x')\Big[G_0(x;x')\Big] + \mathcal{F}^{\mu\nu\rho\sigma}\Big[F_2(x;x')\Big] + \mathcal{G}^{\mu\nu\rho\sigma}\Big[G_2(x;x')\Big] \; . \end{aligned} \tag{34}$$

where two scalar projector $\mathcal{F}^{\mu\nu}$ and $\mathcal{G}^{\mu\nu}$ are,

$$\begin{aligned} \mathcal{F}^{\mu\nu} &= \partial^\mu\partial^\nu + 2(D-1)aH\delta^{(\mu}_0\partial^{\nu)} + (D-2)(D-1)a^2H^2\delta^\mu_0\delta^\nu_0 \\ &\qquad -\eta^{\mu\nu}\left[\partial^2 + (D-1)aH\partial_0 + (D-1)a^2H^2\right] \; , \end{aligned} \tag{35}$$

$$\begin{aligned} \mathcal{G}^{\mu\nu} &= \overline{\partial}^\mu\overline{\partial}^\nu + 2(D-2)aH\delta^{(\mu}_0\overline{\partial}^{\nu)} + (D-2)(D-1)a^2H^2\delta^\mu_0\delta^\nu_0 \\ &\qquad -\overline{\eta}^{\mu\nu}\left[\nabla^2 + (D-2)aH\partial_0 + (D-2)a^2H^2\right] \; , \end{aligned} \tag{36}$$

where $\overline{\partial}^\mu \equiv \partial^\mu + \delta^\mu_0\partial_0$. And two tensor projectors $\mathcal{F}^{\mu\nu\rho\sigma}$ and $\mathcal{G}^{\mu\nu\rho\sigma}$ are,

$$\mathcal{F}^{\mu\nu\rho\sigma} = \mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu}(x)\times\mathcal{C}_{\kappa\lambda\theta\phi}{}^{\rho\sigma}(x')\times\eta^{\alpha\kappa}\eta^{\beta\lambda}\eta^{\gamma\theta}\eta^{\delta\phi} \; , \tag{37}$$

$$\mathcal{G}^{\mu\nu\rho\sigma} = \mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu}(x)\times\mathcal{C}_{\kappa\lambda\theta\phi}{}^{\rho\sigma}(x')\times\overline{\eta}^{\alpha\kappa}\overline{\eta}^{\beta\lambda}\overline{\eta}^{\gamma\theta}\overline{\eta}^{\delta\phi} \; . \tag{38}$$

where we define the second order differential operator $\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu}$ by expanding the Weyl tensor of the conformally transformed metric to the linear order of graviton field,

$$\widetilde{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu} \quad \Rightarrow \quad \widetilde{C}_{\alpha\beta\gamma\delta} \equiv \mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu}\times\kappa h_{\mu\nu} + O(\kappa^2h^2) \; . \tag{39}$$

And explicit expressions for $\mathcal{F}^{\mu\nu\rho\sigma}$ and $\mathcal{G}^{\mu\nu\rho\sigma}$ can be found in the Appendix of [33].

It's worth mentioning that the de Sitter representation (34) must be consistent with its corresponding flat space limit (32). This means that all terms in $G_0(x;x')$ and $G_2(x;x')$ must contain at least one factor of H^2 so as to vanish in the limit that H vanishes with the co-moving time $t = \ln(a)/H$ held fixed. Hence, the leading divergences are contained in $F_0(x;x')$ and $F_2(x;x')$. $G_0(x;x')$ and $G_2(x;x')$ are correspondingly less divergent.

Now it leaves the question, how do we find the structure functions? In general, suppose that after substituting the photon propagators in our formal expressions given by equations (29,30), doing index contractions and acting derivatives, we obtain a primitive result for $-i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}\big](x;x')$. Since this result can be expressed in the form suggested by equation (34) as we explained previously, we can extract out those four structure functions by looking at particular components. The procedure will be

- First, we take the trace on one index group of $-i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}\big]$, say ρ and σ. This will cause the spin two part to drop out. Then we shall look at the components $-i\big[{}^{0i}\Sigma^{\rho\sigma}\big] \times \eta_{\rho\sigma}$ and $-i\big[{}^{jk}\Sigma^{\rho\sigma}\big] \times \eta_{\rho\sigma}$ $(j \neq k)$ to obtain two linearly independent equations to solve for $F_0(x;x')$ and $G_0(x;x')$.
- Next, we look at the components $-i\big[{}^{0i}\Sigma^{jk}\big]$ and $-i\big[{}^{jk}\Sigma^{0i}\big]$, and always assume that $i \neq j \neq k \neq i$. This will also allow us to derive two linearly independent equations to solve for $F_2(x;x')$ and $G_2(x;x')$.

More details about how this procedure works will be given when we carry out the explicit calculation in section 3.2 and 3.3.

3 One Loop Graviton Self-energy

3.1 Primitive Results of 3pt and 4pt Functions

In this section, we will substitute our vertex tensor factors (27-28) into the formal expressions (29-30) and then perform naive contractions. Before diving into the tensor algebra, it is worth noticing the fact that in both of the diagram expressions (29-30) the doubly differentiated photon propagators inherit an anti-symmetry from the vertex tensor factors through index contractions. So we will have a cleaner, simpler analysis if expressing the computation in terms of a 4-index object which is the correlator of two field strengths,

$$i\big[{}_{\alpha\beta}\Delta_{\gamma\delta}\big](x;x') = \langle\Omega|F_{\alpha\beta}(x)F_{\gamma\delta}(x')|\Omega\rangle = 4\partial_{[\alpha}\partial'_{[[\gamma} i\big[{}_{\beta]}\Delta_{\delta]]}\big](x;x') \; . \tag{40}$$

Here the single and double square brackets indicate anti-symmetrization on the index pairs $\alpha \leftrightarrow \beta$ and $\gamma \leftrightarrow \delta$, respectively.

Now let's evaluate the field strength correlator (40). Starting with our photon propagator expression (23), we want to first express the tensor factors in terms of the derivatives of $y(x;x')$ and $u(x;x') \equiv \ln(aa')$,

$$
\begin{aligned}
aa'\overline{\eta}_{\mu\nu} &= \frac{1}{2H^2}\Big[-\partial_\mu\partial'_\nu y + \partial_\mu y \times \partial'_\nu u + \partial_\mu u \times \partial'_\nu y + (2-y)\partial_\mu u \times \partial'_\nu u\Big] \; , \qquad (41) \\
aa'\delta^0_\mu\delta^0_\nu &= \frac{1}{2H^2}\Big[2\partial_\mu u \times \partial'_\nu u\Big] \; . \qquad (42)
\end{aligned}
$$

And there is a very useful relation between $B(y)$ and $C(y)$,

$$
C(y) = \frac{1}{2}(2-y)B(y) + \frac{k}{(D-3)} \qquad \text{where} \qquad k \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \; . \qquad (43)
$$

Substituting expressions (41-43) into (23) allows one to express our propagator as,

$$
i\big[{}_\mu\Delta_\nu\big](x;x') = -\frac{1}{2H^2}\Big(\partial_\mu\partial'_\nu y - \partial_\mu y\partial'_\nu u - \partial_\mu u\partial'_\nu y\Big)B(y) - \frac{k}{(D-3)H^2}\Big(\partial_\mu u\partial'_\nu u\Big) \; . \qquad (44)
$$

The next step is to act two derivatives on (44) and anti-symmetrize to form the field strength correlator (40). Notice the fact that $\partial_\mu u = Ha\delta^0_\mu$ is independent of x'^μ and that $\partial'_\nu u = Ha'\delta^0_\nu$ is independent of x^μ. It follows that only the de Sitter invariant 1st term of expression (44) contributes to the field strength correlator,

$$
i\big[{}_{\alpha\beta}\Delta_{\gamma\delta}\big](x;x') = -\frac{2}{H^2}\Big(D_{[\alpha}D'_{[[\gamma}B(y)\Big)\Big(D_{\beta]}D'_{\delta]]}y\Big) \; , \qquad (45)
$$

where D_α and D'_γ are the covariant derivatives associated with the de Sitter background metrics $\overline{g}_{\mu\nu}(x) \equiv a^2\eta_{\mu\nu}$ and $\overline{g}_{\rho\sigma}(x') \equiv a'^2\eta_{\rho\sigma}$ respectively. It also turns out useful to work out the coincidence limit of the field strength correlator. Starting with (45) and using the fact that,

$$
\lim_{x\to x'}\Big(D_\alpha D'_\gamma B(y)\Big) = -2H^2a'^2B'(0)\eta_{\alpha\gamma} \qquad \text{where} \qquad B'(0) \equiv \lim_{y\to 0}\Big(\frac{d}{dy}B(y)\Big) \; . \qquad (46)
$$

Hence the coincident field strength correlator is,

$$
\lim_{x\to x'} i\big[{}_{\alpha\beta}\Delta_{\gamma\delta}\big](x;x') = -8H^2a'^4B'(0)\eta_{\alpha[\gamma}\eta_{\delta]\beta} \; . \qquad (47)
$$

The final identity we like to derive for future convenience is based on the fact there are of course local delta function contributions hidden in the expression (45) because of the doubly differentiated scalar propagator. And one can show that,

$$
D_\mu D'_\nu B(y) = \Big\{D_\mu D'_\nu B(y)\Big\}_{\rm naive} + \frac{1}{a^{D-2}}\delta^0_\mu\delta^0_\nu i\delta^D(x-x') \; , \qquad (48)
$$

where $\{\dots\}_{\rm naive}$ denotes whatever we got by naively acting derivatives. Then, the result (48) allows us to segregate out the local delta function contributions in the field strength correlator,

$$
i\big[{}_{\alpha\beta}\Delta_{\gamma\delta}\big](x;x') = \Big\{i\big[{}_{\alpha\beta}\Delta_{\gamma\delta}\big](x;x')\Big\}_{\rm naive} + \frac{4}{a^{D-4}}\delta^0_{[\beta}\eta_{\alpha][\gamma}\delta^0_{\delta]}i\delta^D(x-x') \; . \qquad (49)
$$

Now we are all set to compute the primitive results of the first two diagrams in Fig. (1) by substitution of (27-28) in (29-30) and naive index contractions. First, it's easy to see that in terms of the field strength correlator (40) the 4pt and 3pt functions can be rewritten as,

$$-i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm 4pt}\big](x;x') = -\frac{1}{2}\kappa^2 a'^{D-4} i\delta^D(x-x')\times$$
$$\Bigg\{\bigg(\frac{1}{8}\eta^{\mu\nu}\eta^{\rho\sigma} - \frac{1}{4}\eta^{\mu(\rho}\eta^{\sigma)\nu}\bigg)\, i\big[{}^{\alpha\beta}\Delta_{\alpha\beta}\big](x';x') - \frac{1}{2}\eta^{\mu\nu} i\big[{}^{\rho\beta}\Delta^{\sigma}{}_{\beta}\big](x';x') - \frac{1}{2}\eta^{\rho\sigma} i\big[{}^{\mu\beta}\Delta^{\nu}{}_{\beta}\big](x';x')$$
$$+2\eta^{\nu)(\rho} i\big[{}^{\sigma)\beta}\Delta^{(\mu}{}_{\beta}\big](x';x') - i\big[{}^{\mu(\rho}\Delta^{\sigma)\nu}\big](x';x')\Bigg\}\ , \quad (50)$$

$$-i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm 3pt}\big](x;x') = -\frac{1}{2}\kappa^2 (aa')^{D-4}\times$$
$$\Bigg\{\frac{1}{16}\eta^{\mu\nu}\eta^{\rho\sigma} i\big[{}^{\alpha\beta}\Delta^{\gamma\delta}\big](x;x')\times i\big[{}_{\alpha\beta}\Delta_{\gamma\delta}\big](x;x') - \frac{1}{4}\eta^{\mu\nu} i\big[{}^{\alpha\beta}\Delta^{\gamma\rho}\big](x;x')\times i\big[{}_{\alpha\beta}\Delta_{\gamma}{}^{\sigma}\big](x;x')$$
$$-\frac{1}{4}\eta^{\rho\sigma} i\big[{}^{\alpha\mu}\Delta^{\gamma\delta}\big](x;x')\times i\big[{}_{\alpha}{}^{\nu}\Delta_{\gamma\delta}\big](x;x') + i\big[{}^{\alpha\mu}\Delta^{\gamma(\rho}\big](x;x')\times i\big[{}_{\alpha}{}^{\nu}\Delta_{\gamma}{}^{\sigma)}\big](x;x')\Bigg\}\ . \quad (51)$$

Note that while getting (50-51) we raise or lower indices only with the Minkowski metric which just plays with the relative minus sign between the temporal and spatial components.

- **Local Contributions**

Note that all terms in (50) are proportional to delta function. So, applying the coincident identity of the field strength correlator (47), we immediately obtain the contributions from our 4pt functions which are purely local,

$$-i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm 4pt}\big]_{\rm local}(x;x') = -\kappa^2 H^2 a'^D B'(0)\Bigg\{\frac{1}{2}\big(D^2-9D+12\big)\,\eta^{\mu(\rho}\eta^{\sigma)\nu}$$
$$-\frac{1}{4}\big(D^2-9D+16\big)\,\eta^{\mu\nu}\eta^{\rho\sigma}\Bigg\} i\delta^D(x-x')\ . \quad (52)$$

Obviously, the expression (51) will contain both local and non-local contributions, and we first deal with the local terms. So, applying the identity (49) and ignoring all the naive terms for the time being, we can also easily obtain the local contributions from our 3pt function,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{3\rm pt}\Big]_{\rm local}(x;x') = -\kappa^2 H^2 a'^D B'(0)\Bigg\{(D-5)\,\eta^{\mu\nu}\eta^{\rho\sigma} + 4\eta^{\mu(\rho}\eta^{\sigma)\nu}$$
$$+2(D-4)\,\eta^{\mu\nu}\delta^\sigma_0\delta^\rho_0 + 2(D-4)\,\eta^{\rho\sigma}\delta^\mu_0\delta^\nu_0 - 4(D-4)\,\delta^{(\mu}_0\eta^{\nu)(\rho}\delta^{\sigma)}_0\Bigg\} i\delta^D(x-x')\ . \quad (53)$$

Adding (52) and (53) together simply gives the entire local contributions from our 3pt and 4pt functions,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big]_{\rm local}(x;x') = (D-4)\kappa^2 H^2 a'^D B'(0)\Bigg\{\frac{1}{4}(D-9)\eta^{\mu\nu}\eta^{\rho\sigma}$$
$$-\frac{1}{2}(D-5)\eta^{\mu(\rho}\eta^{\sigma)\nu} - 2\eta^{\mu\nu}\delta^\sigma_0\delta^\rho_0 - 2\eta^{\rho\sigma}\delta^\mu_0\delta^\nu_0 + 4\delta^{(\mu}_0\eta^{\nu)(\rho}\delta^{\sigma)}_0\Bigg\} i\delta^D(x-x')\ . \quad (54)$$

From our explicit expression of $B(y)$ (20) we know that $B'(0)$ is finite in $D = 4$ dimensions,

$$B'(0) = -\frac{H^{D-2}}{4(4\pi)^{\frac{D}{2}}}\frac{\Gamma(D-1)}{\Gamma(\frac{D}{2}+1)}\ . \quad (55)$$

So, we can simply set $D = 4$ and conclude that the net contribution of (54) vanishes.

- **Non-local Contributions**

Now let's focus on the non-local contributions which are nothing but all the naive terms, that produce no delta functions, in our 3pt function. Please note that (51) is NOT written in a manifestly covariant form because we have raised indices using the Minkowski inverse metric instead of the inverse background metric. That's merely for the purpose of naive index contraction, but we can easily recover the covariance by putting back the appropriate factors of a or a'. Also note that the tensor structure of the naive terms in our field strength correlator (45) can be expressed in terms of covariant basis furnished by derivatives of $y(x;x')$,

$$\Big\{i\big[{}_{\alpha\beta}\Delta_{\gamma\delta}\big](x;x')\Big\}_{\rm naive} = \frac{2}{H^2}\Big[\big(D_{[\alpha}y\big)\big(D_{\beta]}D'_{[\gamma}y\big)\big(D'_{\delta]}y\big)B''(y)$$
$$-\big(D_\alpha D'_{[\gamma}y\big)\big(D'_{\delta]}D_\beta y\big)B'(y)\Big]\ . \quad (56)$$

And one can show there are some very useful contraction identities of these basis tensors [29],

$$\overline{g}^{\mu\nu}(x)\,(D_\mu y)\,(D_\nu y) = H^2\big(4y-y^2\big) = \overline{g}^{\rho\sigma}(x')\,\big(D'_\rho y\big)\,(D'_\sigma y)\ , \tag{57}$$

$$\overline{g}^{\mu\nu}(x)\,(D_\mu y)\,\big(D_\nu D'_\rho y\big) = H^2(2-y)\,\big(D'_\rho y\big)\ , \tag{58}$$

$$\overline{g}^{\rho\sigma}(x')\,\big(D'_\rho y\big)\,(D_\mu D'_\sigma y) = H^2(2-y)\,(D_\mu y)\ , \tag{59}$$

$$\overline{g}^{\mu\nu}(x)\,\big(D_\mu D'_\rho y\big)\,(D_\nu D'_\sigma y) = 4H^4\overline{g}_{\rho\sigma}(x') - H^2\,\big(D'_\rho y\big)\,(D'_\sigma y)\ , \tag{60}$$

$$\overline{g}^{\rho\sigma}(x')\,\big(D_\mu D'_\rho y\big)\,(D_\nu D'_\sigma y) = 4H^4\overline{g}_{\mu\nu}(x) - H^2\,(D_\mu y)\,(D_\nu y)\ . \tag{61}$$

We remind the reader that $\overline{g}^{\mu\nu} = a^{-2}\eta^{\mu\nu}$ and $\overline{g}^{\rho\sigma} = a'^{-2}\eta^{\rho\sigma}$ are the inverse background metrics on de Sitter. Next step is to substitute our result (56) for the naive terms in the expression (51), and then make use of the contraction identities (57-61). It is straightforward to express all the naive terms in our 3pt function as a linear combination of the de Sitter invariant basis tensors,

$$\begin{aligned}
-i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}\big](x;x') = (aa')^{D+2}\Big\{&\big(D^\mu D'^{(\rho}y\big)\,\big(D'^{\sigma)}D^\nu y\big)\,\alpha(y)\\
+\big(D^{(\mu}y\big)\big(D^{\nu)}D'^{(\rho}y\big)\big(D'^{\sigma)}y\big)\beta(y) &+ (D^\mu y)(D^\nu y)(D'^\rho y)(D'^\sigma y)\gamma(y) + H^4\overline{g}^{\mu\nu}(x)\overline{g}^{\rho\sigma}(x')\delta(y)\\
+H^2\Big[\overline{g}^{\mu\nu}(x)\,\big(D'^{(\rho}y\big)\,\big(D'^{\sigma)}y\big) &+ \overline{g}^{\rho\sigma}(x')\,\big(D^{(\mu}y\big)\,\big(D^{\nu)}y\big)\Big]\epsilon(y)\Big\}\ .
\end{aligned} \tag{62}$$

And those five scalar functions in (62) are given by,

$$\begin{aligned}
\alpha(y) \equiv \kappa^2\Big\{&-\frac{1}{8}\left(4y-y^2\right)^2\big[B''(y)\big]^2 - \Big[2(D-2) - \frac{1}{2}\left(4y-y^2\right)\Big]\big[B'(y)\big]^2\\
&-\frac{1}{2}(2-y)\left(4y-y^2\right)\big[B'(y)B''(y)\big]\Big\}\ ,
\end{aligned} \tag{63}$$

$$\begin{aligned}
\beta(y) \equiv \kappa^2\Big\{&\frac{1}{4}(2-y)\left(4y-y^2\right)\big[B''(y)\big]^2 - (2-y)\big[B'(y)\big]^2\\
&-\Big[2(D-4) + \left(4y-y^2\right)\Big]\big[B'(y)B''(y)\big]\Big\}\ ,
\end{aligned} \tag{64}$$

$$\begin{aligned}
\gamma(y) \equiv \kappa^2\Big\{&-\Big[\frac{1}{2}(D-2) - \frac{1}{8}\left(4y-y^2\right)\Big]\big[B''(y)\big]^2 - \frac{1}{2}\big[B'(y)\big]^2\\
&+\frac{1}{2}(2-y)\big[B'(y)B''(y)\big]\Big\}\ ,
\end{aligned} \tag{65}$$

$$\delta(y) \equiv \kappa^2 \Biggl\{ -\frac{1}{8}(D-5)\bigl(4y-y^2\bigr)^2 \bigl[B''(y)\bigr]^2 - \Bigl[\bigl(D^2-9D+16\bigr) - \frac{1}{2}(D-5)\bigl(4y-y^2\bigr)\Bigr]\bigl[B'(y)\bigr]^2$$

$$-\frac{1}{2}(D-5)(2-y)\left(4y-y^2\right)\left[B'(y)B''(y)\right]\Biggr\} , \qquad (66)$$

$$\epsilon(y) \equiv \kappa^2 \Biggl\{ \frac{1}{4}(D-4)\left(4y-y^2\right)\left[B''(y)\right]^2 - (D-4)\bigl[B'(y)\bigr]^2$$

$$+(D-4)(2-y)\bigl[B'(y)B''(y)\bigr]\Biggr\} . \qquad (67)$$

Since our local contributions (54) vanishes at one-loop order, the non-local terms (62) complete the one-loop photon contributions to graviton self-energy.

3.2 Spin Zero Structure Functions

In this section, we will follow the procedure introduced in section 2.3 to extract two spin zero structure functions $F_0(x;x')$ and $G_0(x;x')$. We start with the trace of expression (62),

$$-i\bigl[{}^{\mu\nu}\Sigma^{\rho\sigma}\bigr](x;x') \times \eta_{\rho\sigma} = a^2 H^2 (aa')^D \Biggl\{ H^2 \overline{g}^{\mu\nu}(x)\Bigl[4\alpha(y) + D\delta(y) + \left(4y-y^2\right)\epsilon(y)\Bigr]$$

$$+ \left(D^\mu y\right)\left(D^\nu y\right)\Bigl[-\alpha(y) + (2-y)\beta(y) + \left(4y-y^2\right)\gamma(y) + D\epsilon(y)\Bigr]\Biggr\} . \qquad (68)$$

Now recall that our strategy is to look at the components $-i\bigl[{}^{0i}\Sigma^{\rho\sigma}\bigr] \times \eta_{\rho\sigma}$ and $-i\bigl[{}^{jk}\Sigma^{\rho\sigma}\bigr] \times \eta_{\rho\sigma}$ $(j \neq k)$ to infer equations that F_0 and G_0 satisfy. One can see that time derivative on $y(x;x')$ gives,

$$\partial_0 y = aH\left(y - 2 + 2\frac{a'}{a}\right) , \qquad (69)$$

which leads to the components we need of $\left(D^\mu y\right)\left(D^\nu y\right)$,

$$\left(D^0 y\right)\left(D^i y\right) = -a^{-4}\left(\partial_0 y\right)\left(\partial_i y\right) = -a^{-4} \times aH\left(y - 2 + 2\frac{a'}{a}\right)\left(\partial_i y\right) , \qquad (70)$$

$$\left(D^j y\right)\left(D^k y\right) = a^{-4} \times \left(\partial_j y\right)\left(\partial_k y\right) . \qquad (71)$$

As explained in the work [33], now we need to recognize two important scalar sources $S_1(x;x')$ and $S_2(x;x')$ by writing,

$$-i\bigl[{}^{0i}\Sigma^{\rho\sigma}\bigr](x;x') \times \eta_{\rho\sigma} \equiv a^{D-2}a'^{D} \times \partial_i S_1(x;x') , \qquad (72)$$

$$-i\bigl[{}^{jk}\Sigma^{\rho\sigma}\bigr](x;x') \times \eta_{\rho\sigma} \equiv a^{D-2}a'^{D} \times \partial_j \partial_k S_2(x;x') . \qquad (73)$$

If we define the indefinite integral of a function $f(y)$ as $I[f(y)] \equiv \int^y dz f(z)$, and another scalar function $F(y)$ as,

$$F(y) \equiv I^2\Big[-\alpha(y) + (2-y)\beta(y) + \left(4y - y^2\right)\gamma(y) + D\epsilon(y)\Big] \; , \tag{74}$$

one can see that,

$$-i\left[{}^{0i}\Sigma^{\rho\sigma}\right](x;x') \times \eta_{\rho\sigma} = a^{D-2}a'^{D}\partial_i\Big\{-aH^3 I\Big[(y-2)F''(y)\Big] - 2a'H^3F'(y)\Big\} \; , \tag{75}$$

$$-i\left[{}^{jk}\Sigma^{\rho\sigma}\right](x;x') \times \eta_{\rho\sigma} = a^{D-2}a'^{D}\partial_j\partial_k\Big\{H^2F(y)\Big\} \; . \tag{76}$$

It follows that,

$$S_1(x;x') = H^2(-\partial_0 + aH)F(y) \; , \tag{77}$$

$$S_2(x;x') = H^2F(y) \; . \tag{78}$$

Then, substituting (77-78) into (56-57) of [33] allows us to express the spin zero structure functions as,

$$F_0(x;x') = -\frac{(aa')^{D-2}}{(D-1)}\left(\frac{H^2}{\Box' + DH^2}\right)F(y) \; , \tag{79}$$

$$G_0(x;x') = 0 \; , \tag{80}$$

where $\Box'$ denotes the invariant scalar d'Alembertian evaluated at x'^{μ}, i.e., $\Box' \equiv \overline{g}^{\rho\sigma}(x')D'_\rho D'_\sigma = a'^{-D}\partial_\rho\left(a'^{D-2}\eta^{\rho\sigma}\partial_\sigma\right)$. Now, let's first work out what explicitly the scalar source $F(y)$ is. Substituting (63-65) and (67) into (74) gives,

$$F(y) = \kappa^2(D-2)(D-4)I^2\Big[-\left(B'\right)^2 + (2-y)\left(B'B''\right) + \frac{1}{4}\left(4y-y^2\right)\left(B''\right)^2\Big] \; . \tag{81}$$

Then, substituting the explicit expression of $B(y)$(20) leads to the following expansion of $F(y)$,

$$F(y) = \frac{(D-4)^2}{16(D-1)}\frac{\kappa^2H^{2D-4}\Gamma^2(\frac{D}{2})}{(4\pi^D)}\left\{(D-2)\left(\frac{4}{y}\right)^{D-1} - (D-4)\left(\frac{4}{y}\right)^{D-2} + \ldots\right\} \; , \tag{82}$$

where we have neglected terms that are integrable and vanish in $D = 4$ dimensions. Similar terms will also be regarded as irrelevant in subsequent expansions.

Next step is of course to recover the structure function $F_0(x;x')$ from expression (79) by employing the Green's function for $(\Box' + DH^2)$. But note that electromagnetism is conformally invariant in $D = 4$ dimensions which means our graviton self-energy shall be traceless in that dimensions, therefore the spin zero structure function $F_0(x;x')$ is expected to contain a factor of $(D-4)$, as we see in our result of source term $F(y)$ (82). However, there happens to be a factor of $(D-4)^2$ in $F(y)$ which causes absolute zero one-loop contributions to $F_0(x;x')$, since the most divergence we can have at this order is $(D-4)^{-1}$. To sum up, both of the spin zero structure functions $F_0(x;x')$ and $G_0(x;x')$ vanish at one-loop order.

3.3 Spin Two Structure Functions

In this section, we intend to extract the spin two structure functions $F_2(x;x')$ and $G_2(x;x')$ from the components $-i\Big[{}^{0i}\Sigma^{jk}\Big](x;x')$ and $-i\Big[{}^{jk}\Sigma^{0i}\Big](x;x')$ with $i \neq j \neq k \neq i$, [33]

$$-i\Big[{}^{0i}\Sigma^{jk}\Big](x;x') \equiv (aa')^{D-2}\partial_i\partial_j\partial_k S_3(x;x') \; , \tag{83}$$

$$-i\Big[{}^{jk}\Sigma^{0i}\Big](x;x') \equiv (aa')^{D-2}\partial_i\partial_j\partial_k S_4(x;x') \; . \tag{84}$$

This means we will need the following identities,

$$\Big(D^0 y\Big)\Big(D^i y\Big) = -a^{-4} \times aH\left(y - 2 + 2\frac{a'}{a}\right)(\partial_i y) \; , \tag{85}$$

$$\Big(D'^0 y\Big)\Big(D'^i y\Big) = a'^{-4} \times a'H\left(y - 2 + 2\frac{a}{a'}\right)(\partial_i y) \; , \tag{86}$$

$$\Big(D^0 D'^i y\Big) = (aa')^{-2} \times aH\,(\partial_i y) \quad , \quad \Big(D'^0 D^i y\Big) = -(aa')^{-2} \times a'H\,(\partial_i y) \; . \tag{87}$$

Applying these identities to the desired components of (62) gives,

$$-i\Big[{}^{0i}\Sigma^{jk}\Big](x;x') = (aa')^{D-2}\partial_i\partial_j\partial_k\left\{-\frac{1}{2}HaI^3\Big[\beta(y)\Big] - (\partial_0 - 3Ha)\,I^4\Big[\gamma(y)\Big]\right\} , \tag{88}$$

$$-i\Big[{}^{jk}\Sigma^{0i}\Big](x;x') = (aa')^{D-2}\partial_i\partial_j\partial_k\left\{\frac{1}{2}Ha'I^3\Big[\beta(y)\Big] + (\partial_0' - 3Ha')\,I^4\Big[\gamma(y)\Big]\right\} . \tag{89}$$

Then, comparing the results (88-89) with expressions (83-84) gives another two scalar sources,

$$S_3(x;x') = -\frac{1}{2}HaI^3\Big[\beta(y)\Big] - (\partial_0 - 3Ha)\,I^4\Big[\gamma(y)\Big] \; , \tag{90}$$

$$S_4(x;x') = \frac{1}{2}Ha'I^3\Big[\beta(y)\Big] + (\partial_0' - 3Ha')\,I^4\Big[\gamma(y)\Big] \; . \tag{91}$$

Substituting expressions (90-91) into (66-67) of [33] and taking into account that in our particular case both $F_0(x;x')$ and $G_0(x;x')$ vanish, we find that $F_2(x;x')$ and $G_2(x;x')$ are determined by the following two independent equations,

$$(\partial_0' - \partial_0)\left\{\frac{(D-3)}{(D-1)}F_2 - \frac{(D-3)}{(D-1)(D-2)^2}G_2 - (aa')^{D-2}I^4\Big[\gamma(y)\Big]\right\}$$
$$= (a - a')H(aa')^{D-2} \times \left\{-\frac{1}{2}I^3\Big[\beta(y)\Big] + (D+1)I^4\Big[\gamma(y)\Big]\right\} , \tag{92}$$

$$(\partial_0' + \partial_0)\left\{\frac{(D-3)D}{(D-1)(D-2)}F_2 + \frac{(D-3)}{(D-1)(D-2)^2}G_2 + (aa')^{D-2}I^4\Big[\gamma(y)\Big]\right\}$$
$$= (a + a')H(aa')^{D-2} \times \left\{-\frac{1}{2}I^3\Big[\beta(y)\Big] + (D+1)I^4\Big[\gamma(y)\Big]\right\} . \tag{93}$$

To solve equation (92), consider acting $(\partial_0' - \partial_0)$ on $(aa')^{D-2}f(y)$,

$$(\partial_0' - \partial_0)(aa')^{D-2}f(y) = (a-a')H(aa')^{D-2}\Big[(4-y)f'(y) - (D-2)f(y)\Big] \; . \tag{94}$$

If we let,

$$(aa')^{D-2}f(y) \equiv \frac{(D-3)}{(D-1)}F_2 - \frac{(D-3)}{(D-1)(D-2)^2}G_2 - (aa')^{D-2}I^4\Big[\gamma(y)\Big] \; , \tag{95}$$

then, (92) becomes an 1st order linear differential equation,

$$f'(y) - \frac{(D-2)}{(4-y)}f(y) = \frac{1}{(4-y)}\Big\{-\frac{1}{2}I^3\Big[\beta(y)\Big] + (D+1)I^4\Big[\gamma(y)\Big]\Big\} \; , \tag{96}$$

which allow us to invert $(\partial_0' - \partial_0)$ in (92) and obtain an indefinite integral,

$$\begin{aligned} &\frac{(D-3)}{(D-1)}F_2(x;x') - \frac{(D-3)}{(D-1)(D-2)^2}G_2(x;x') - (aa')^{D-2}I^4\Big[\gamma(y)\Big] \\ &= \frac{(aa')^{D-2}}{(4-y)^{D-2}}\Bigg\{I\Bigg[(4-y)^{D-3}\Big\{-\frac{1}{2}I^3\Big[\beta(y)\Big] + (D+1)I^4\Big[\gamma(y)\Big]\Big\}\Bigg] + K_1\Bigg\} \; , \end{aligned} \tag{97}$$

where the integration constant K_1 is chosen so that there is no singularity at the antipdal point $y = 4$. Similarly, to solve equation (93), consider acting $(\partial_0' + \partial_0)$ on $(aa')^{D-2}f(y)$,

$$(\partial_0' + \partial_0)(aa')^{D-2}f(y) = (a+a')H(aa')^{D-2}\Big[yf'(y) + (D-2)f(y)\Big] \; , \tag{98}$$

which allow us to invert $(\partial_0' + \partial_0)$ in (93) and obtain another indefinite integral,

$$\begin{aligned} &\frac{(D-3)D}{(D-1)(D-2)}F_2(x;x') + \frac{(D-3)}{(D-1)(D-2)^2}G_2(x;x') + (aa')^{D-2}I^4\Big[\gamma(y)\Big] \\ &= \frac{(aa')^{D-2}}{y^{D-2}}\Bigg\{I\Bigg[y^{D-3}\Big\{-\frac{1}{2}I^3\Big[\beta(y)\Big] + (D+1)I^4\Big[\gamma(y)\Big]\Big\}\Bigg] + K_2\Bigg\} \; , \end{aligned} \tag{99}$$

where the integration constant K_2 is chosen so as to keep $G_2(x;x')$ less divergent than $F_2(x;x')$. Equations (97) and (99) simply imply that,

$$F_2(x;x') = \frac{(D-2)}{2(D-3)}\Bigg\{\frac{(aa')^{D-2}}{(4-y)^{D-2}}P(y) + \frac{(aa')^{D-2}}{y^{D-2}}Q(y)\Bigg\} \; , \tag{100}$$

$$\begin{aligned} G_2(x;x') = \frac{(D-1)(D-2)^2}{(D-3)}\Bigg\{&\frac{(D-3)}{(D-1)}F_2(x;x') \\ &-(aa')^{D-2}I^4\Big[\gamma(y)\Big] - \frac{(aa')^{D-2}}{(4-y)^{D-2}}P(y)\Bigg\} \; . \end{aligned} \tag{101}$$

where we have defined two functions of $y(x;x')$,

$$P(y) \equiv I\Bigl[(4-y)^{D-3}\Bigl\{-\frac{1}{2}I^3\Bigl[\beta(y)\Bigr]+(D+1)I^4\Bigl[\gamma(y)\Bigr]\Bigr\}\Bigr]+K_1 \; , \qquad (102)$$

$$Q(y) \equiv I\Bigl[y^{D-3}\Bigl\{-\frac{1}{2}I^3\Bigl[\beta(y)\Bigr]+(D+1)I^4\Bigl[\gamma(y)\Bigr]\Bigr\}\Bigr]+K_2 \; . \qquad (103)$$

Substituting the explicit expression of $B(y)$ (20) into (64-65) and working out various indefinite integrals, we obtain the following expansions,

$$I^3\Bigl[\beta(y)\Bigr] = \frac{\kappa^2 H^{2D-4}\Gamma^2(\frac{D}{2})}{(4\pi)^D}\Bigl\{-\frac{(3D-8)}{2(D-2)(D-1)}\Bigl(\frac{4}{y}\Bigr)^{D-2}+\dots\Bigr\} \; , \qquad (104)$$

$$I^4\Bigl[\gamma(y)\Bigr] = \frac{\kappa^2 H^{2D-4}\Gamma^2(\frac{D}{2})}{(4\pi)^D}\Bigl\{-\frac{D}{8(D-1)(D+1)}\Bigl(\frac{4}{y}\Bigr)^{D-2}+\dots\Bigr\} \; . \qquad (105)$$

As always, the neglected terms in these expansions are integrable and vanish in $D=4$ dimensions. Thanks to the factor of $(D-4)^2$ in the following result which is a little surprising,

$$-\frac{1}{2}I^3\Bigl[\beta(y)\Bigr]+(D+1)I^4\Bigl[\gamma(y)\Bigr] = \frac{\kappa^2 H^{2D-4}\Gamma^2(\frac{D}{2})}{(4\pi)^D}\Bigl\{-\frac{(D-4)^2}{8(D-2)(D-1)}\Bigl(\frac{4}{y}\Bigr)^{D-2}+\dots\Bigr\} \; , \qquad (106)$$

our analysis becomes a lot easier. As a result, we find,

$$P(y)=K_1 \quad , \quad Q(y)=K_2 \; . \qquad (107)$$

Note that we shall set $K_1=0$ to remove the singularity at the antipdal point $y=4$. Then, it follows that,

$$F_2(x;x') = \frac{(D-2)}{2(D-3)}\Bigl\{\frac{(aa')^{D-2}}{y^{D-2}}K_2\Bigr\} \; , \qquad (108)$$

$$G_2(x;x') = \frac{(D-1)(D-2)^2}{(D-3)}\Bigl\{\frac{(D-2)}{2(D-1)}\frac{(aa')^{D-2}}{y^{D-2}}K_2 + \frac{\kappa^2(H^2aa')^{D-2}\Gamma^2(\frac{D}{2})}{(4\pi)^D}\frac{D}{8(D-1)(D+1)}\Bigl(\frac{4}{y}\Bigr)^{D-2}\Bigr\} \; . \qquad (109)$$

Note that (108-109) suggest that both F_2 and G_2 are proportional to $1/y^{D-2}$. Recall G_2 should be less singular than F_2, therefore, the integration constant K_2 can only be chosen so that G_2 vanishes at this order,

$$\frac{(aa')^{D-2}}{y^{D-2}}K_2 = \frac{\kappa^2(H^2aa')^{D-2}\Gamma^2(\frac{D}{2})}{(4\pi)^D}\Bigl\{-\frac{D}{4(D-2)(D+1)}\Bigl(\frac{4}{y}\Bigr)^{D-2}\Bigr\} \; . \qquad (110)$$

It follows that,

$$F_2(x;x') = \frac{\kappa^2 (H^2 a a')^{D-2} \Gamma^2(\frac{D}{2})}{(4\pi)^D} \Biggl\{ -\frac{D}{8(D-3)(D+1)} \Bigl(\frac{4}{y}\Bigr)^{D-2} \Biggr\} , \qquad (111)$$

$$G_2(x;x') = 0 \; . \qquad (112)$$

4 Renormalization

4.1 Counterterms

To figure out what appropriate BPHZ counterterms are we are guided by two facts, first of which is that we are calculating 1PI graviton 2-point function. This means our counterterms should contain two graviton fields. The second fact is the superficial degree of divergence of 4 in our primitive diagrams at one loop order which means the counterterms should contain up to four derivatives. For quantum gravity at one loop order the necessary counterterms can be taken to be the Ricci curvature square R^2 and Weyl tensor square $C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}$ [34]. Note that we have obtained a perfectly de Sitter invariant representation of our graviton self-energy (62), thanks to the one-loop photon contribution being de Sitter invariant. As a result, there would be no additional counterterms.

Moreover, it has been shown in section (3.2-3.3) that the non-trivial contributions of photon to the graviton self-energy come from the spin two part. Thus, the only counterterm we really need is the Weyl tensor square counterterm which is defined by the following Lagrangian,

$$\Delta \mathcal{L}_2 \equiv c_2 C^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} \sqrt{-g} \; . \qquad (113)$$

Using the fact that the Weyl tensor with one upper index is conformally invariant,

$$C^{\alpha}{}_{\beta\gamma\delta} = \widetilde{C}^{\alpha}{}_{\beta\gamma\delta} \qquad \Rightarrow \qquad C_{\alpha\beta\gamma\delta} = a^2 \widetilde{C}_{\alpha\beta\gamma\delta} \; . \qquad (114)$$

We can rewrite the counterterm Lagrangian (113) as,

$$\begin{aligned} \Delta \mathcal{L}_2 &= c_2 a^{D-4} \widetilde{C}^{\alpha\beta\gamma\delta} \widetilde{C}_{\alpha\beta\gamma\delta} \sqrt{-\widetilde{g}} \\ &= c_2 a^{D-4} \eta^{\alpha\kappa} \eta^{\beta\lambda} \eta^{\gamma\theta} \eta^{\delta\phi} \left(\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu} \kappa h_{\mu\nu} \right) \left(\mathcal{C}_{\kappa\lambda\theta\phi}{}^{\rho\sigma} \kappa h_{\rho\sigma} \right) + O\left(\kappa^3 h^3 \right) \; . \qquad (115) \end{aligned}$$

Note we have used the linearized Weyl tensor operator $\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu}$ defined in section 2.3 to expand the counterterm Lagrangian up to quadratic order in graviton field. Then, one obtains the counterterm vertex by functionally differentiating i times counterterm action twice, and then setting the graviton field to zero,

$$\frac{i \delta^2 \Delta S_2}{\delta h_{\mu\nu}(x) \delta h_{\rho\sigma}(x')} \Bigg|_{h=0} = 2 c_2 \kappa^2 a^{D-4} \eta^{\alpha\kappa} \eta^{\beta\lambda} \eta^{\gamma\theta} \eta^{\delta\phi} \mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu} \mathcal{C}_{\kappa\lambda\theta\phi}{}^{\rho\sigma} \times i \delta^D(x-x') \; . \qquad (116)$$

By recognizing the tensor projector in (116),

$$\eta^{\alpha\kappa}\eta^{\beta\lambda}\eta^{\gamma\theta}\eta^{\delta\phi}\mathcal{C}_{\alpha\beta\gamma\delta}{}^{\mu\nu}(x)\mathcal{C}_{\kappa\lambda\theta\phi}{}^{\rho\sigma}(x)\times i\delta^D(x-x')\to\mathcal{F}^{\mu\nu\rho\sigma}\times i\delta^D(x-x')\ , \tag{117}$$

one can see that the contribution of the Weyl tensor square counterterm to our graviton self-energy is,

$$-i\Big[{}^{\mu\nu}\Delta\Sigma_2^{\rho\sigma}\Big](x;x')=2c_2\kappa^2a^{D-4}\mathcal{F}^{\mu\nu\rho\sigma}\times i\delta^D(x-x')\ . \tag{118}$$

4.2 Renormalizing the Non-integrability

In the previous section, we have derived our graviton self-energy in arbitrary D dimensions,

$$-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x')=\mathcal{F}^{\mu\nu\rho\sigma}\Big[F_2(x;x')\Big]\ , \tag{119}$$

where $F_2(x;x')$ is given by (111). One sees that our primitive result (119) contains the factor of y^{2-D} which diverges at coincidence ($x^\mu=x'^\mu$) with nonzero coefficients in $D=4$ dimensions. Recall that to quantum correct the linearized Einstein field equation (1) we will integrate $-i\big[{}^{\mu\nu}\Sigma^{\rho\sigma}\big]$ over the 4-dimensional measure, hence (119) is not integrable in that dimensions. However, only the terms that are at least as singular as y^{-2} need to be renormalized. Less singular terms are actually integrable, and therefore can be evaluated in $D=4$. To renormalize those non-integrable terms, the general procedure is,

- First, convert them into functions of Δx^2 using the definition $y\equiv aa'H^2\Delta x^2$.
- Second, extract derivatives with respect to the unprimed coordinate x^μ, for example,

$$\frac{1}{\Delta x^{2D-4}}=\frac{\partial^2}{2(D-3)(D-4)}\frac{1}{\Delta x^{2D-6}}\ , \tag{120}$$

$$\frac{1}{\Delta x^{2D-2}}=\frac{\partial^4}{4(D-2)^2(D-3)(D-4)}\frac{1}{\Delta x^{2D-6}}\ , \tag{121}$$

  in which the derivatives can be simply pulled out of the integration over primed coordinate x'^μ in the quantum corrected effective field equation (1). We keep doing this so as to get a less and less singular integrand until it becomes integrable in $D=4$, but the price we pay for it is to get a manifest divergence, the factor of $(D-4)^{-1}$.

- Third, localize the explicit divergence of $(D-4)^{-1}$ in the previous step by adding zero using the identity,

$$\partial^2\frac{1}{\Delta x^{D-2}}=\frac{i4\pi^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)}\delta^D(x-x')\ . \tag{122}$$

This means we can write,

$$\frac{\partial^2}{(D-4)}\frac{1}{\Delta x^{2D-6}} = \frac{i4\pi^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)}\frac{\mu^{D-4}\delta^D(x-x')}{(D-4)} + \frac{\partial^2}{(D-4)}\left\{\frac{1}{\Delta x^{2D-6}} - \frac{\mu^{D-4}}{\Delta x^{D-2}}\right\}, \quad (123)$$

where the factor of μ with inverse length dimension is added for dimensional consistency. Then we expand the terms in the curly bracket away from $D = 4$,

$$\frac{1}{\Delta x^{2D-6}} - \frac{\mu^{D-4}}{\Delta x^{D-2}} = -\frac{(D-4)}{2}\frac{\ln\left(\mu^2\Delta x^2\right)}{\Delta x^2} + O\Big[(D-4)^2\Big]. \quad (124)$$

So, we can write equation (123) as

$$\frac{\partial^2}{(D-4)}\frac{1}{\Delta x^{2D-6}} = \frac{i4\pi^{D/2}}{\Gamma\left(\frac{D}{2}-1\right)}\frac{\mu^{D-4}\delta^D(x-x')}{(D-4)} - \frac{\partial^2}{2}\left[\frac{\ln\left(\mu^2\Delta x^2\right)}{\Delta x^2}\right] + O(D-4). \quad (125)$$

Note that in equation (125) the logarithm term is integrable and delta function piece is local though singular in $D = 4$. This is what we mean by localizing the divergence.

- Finally, use the technique introduced above to break our result of graviton self-energy into finite and divergent parts. Then we shall set appropriate counterterm coefficients so as to cancel all the divergent parts, and obtain the fully renormalized result.

Now, we are all set to renormalize the primitive result of $-i\left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right]$ (119), actually the primitive result of $F_2(x;x')$ (111) to be exact. Applying the identities (120) and (125) in (111) gives,

$$F_2(x;x') = \frac{\kappa^2}{640\pi^4}\partial^2\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] - \frac{\kappa^2\mu^{D-4}\Gamma(\frac{D}{2})}{128\pi^{\frac{D}{2}}}\frac{D(D-2)i\delta^D(x-x')}{(D-4)(D-3)^2(D+1)}. \quad (126)$$

Substituting (126) in (119) and comparing with (118), one sees that the local divergent part will be completely subtracted off by the counterterm contribution if we choose the arbitrary counterterm coefficient c_2 to be,

$$c_2 = \frac{\mu^{D-4}\Gamma(\frac{D}{2})}{256\pi^{\frac{D}{2}}}\frac{D(D-2)}{(D-4)(D-3)^2(D+1)}. \quad (127)$$

Now, using this counterterm coefficient c_2 we finally obtain the renormalized form of our graviton self-energy,

$$
\begin{aligned}
-i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm re}\Big](x;x') &= -i\Big[{}^{\mu\nu}\Sigma^{\rho\sigma}\Big](x;x') - i\Big[{}^{\mu\nu}\Delta\Sigma^{\rho\sigma}_{2}\Big](x;x') \\
&= \frac{\kappa^2}{80\pi^2}\ln(a)\mathcal{F}^{\mu\nu\rho\sigma} i\delta^4(x-x') + \mathcal{F}^{\mu\nu\rho\sigma}\Big[F_{2,\rm re}(x;x')\Big] \; . \qquad (128)
\end{aligned}
$$

And the non-local part of our renormalized spin two structure function is,

$$
F_{2,\rm re}(x;x') = \frac{\kappa^2}{640\pi^4}\partial^2\left[\frac{\ln(\mu^2\Delta x^2)}{\Delta x^2}\right] \; . \qquad (129)
$$

It is worth mentioning at this point that the spacetime dependence of $\delta^4(x-x')$ and $F_{2,\rm re}(\Delta x^2)$ in (128) allows us to easily reflect any derivatives on them, $\partial' \to -\partial$. Given the explicit expression for the tensor projector $\mathcal{F}^{\mu\nu\rho\sigma}$ in the appendix of [33], one can see that in $D=4$ dimensions it is reduced to the nice flat space form,

$$
\mathcal{F}^{\mu\nu\rho\sigma} \to \frac{1}{2}\left(\Pi^{\mu(\rho}\Pi^{\sigma)\nu} - \frac{1}{3}\Pi^{\mu\nu}\Pi^{\rho\sigma}\right) \quad , \quad \Pi^{\mu\nu} \equiv \partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2 \; . \qquad (130)
$$

Before closing this section, please note another fact that with the following two identities,

$$
\frac{1}{\Delta x^2} = \frac{\partial^2}{4}\ln\left(\mu^2\Delta x^2\right) \; , \qquad (131)
$$

$$
\frac{\ln\left(\mu^2\Delta x^2\right)}{\Delta x^2} = \frac{\partial^2}{8}\Big[\ln^2\left(\mu^2\Delta x^2\right) - 2\ln\left(\mu^2\Delta x^2\right)\Big] \; , \qquad (132)
$$

we can achieve a significant simplification by extracting another d'Alembertian from (129),

$$
F_{2,\rm re}(x;x') = \frac{\kappa^2}{5120\pi^4}\partial^4\Big[\ln^2\left(\mu^2\Delta x^2\right) - 2\ln\left(\mu^2\Delta x^2\right)\Big] \; . \qquad (133)
$$

This form of the structure function (133) will come in handy when used in Schwinger-Keldysh formalism.

5 Solving the Effective Field Equation

5.1 Schwinger-Keldysh Formalism

The in-out result of the graviton self-energy (128) seems to be ready for use in the linearized effective field equation (1), but that would lead to two problems:

- Causality – The in-out graviton self-energy (128) is nonzero for points x'^μ which lie in the future of x^μ, or at spacelike separation from it; and

- Reality – The in-out graviton self-energy (128) is not real.

Note that there is nothing wrong to correct graviton propagator with the in-out graviton self-energy for asymptotic scattering amplitudes in flat space. But the causality and reality require that in the effective field equations the appropriate 1PI 2-point function should be that of the Schwinger-Keldysh (S-K) formalism. This technique provides a way of computing causal and real vacuum expectation values that is almost as simple as the Feynman diagrams which produce in-out matrix elements [35, 36, 37, 38, 39, 40, 41, 42]. In our particular case, the photon propagator depends upon the $y(x;x')$ which is a function of the Lorentz interval $\Delta x^2(x;x')$. The four Schwinger-Keldysh propagators can be obtained by making the following replacements for the Lorentz interval [43],

$$\Delta x^2_{++} \equiv \Big\|\vec{x}-\vec{x}'\Big\|^2 - \Big(|\eta-\eta'|-i\delta\Big)^2 \quad , \quad \Delta x^2_{+-} \equiv \Big\|\vec{x}-\vec{x}'\Big\|^2 - \Big(\eta-\eta'+i\delta\Big)^2 ,$$
$$\Delta x^2_{--} \equiv \Big\|\vec{x}-\vec{x}'\Big\|^2 - \Big(|\eta-\eta'|+i\delta\Big)^2 \quad , \quad \Delta x^2_{-+} \equiv \Big\|\vec{x}-\vec{x}'\Big\|^2 - \Big(\eta-\eta'-i\delta\Big)^2 . \qquad (134)$$

To resolve the problem of reality and causality of the in-out formalism while used in effective field equation, what we should really be using in (1) is the retarded graviton self-energy $\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm R}\right]$ of the Schwinger-Keldysh formalism,

$$\left[{}^{\mu\nu}\Sigma^{\rho\sigma}\right](x;x') \rightarrow \left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm R}\right](x;x') \equiv \left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{++}\right](x;x') + \left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{+-}\right](x;x') \; . \qquad (135)$$

It's actually very simple to convert our in-out result (128) into S-K retarded one. We will get $\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{++}\right]$ and $\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{+-}\right]$ from (128) by replacing the Lorentz interval Δx^2 by Δx^2_{++} and Δx^2_{+-}. We will also drop the delta function terms in the $+-$ case, and introduce an overall minus sign. Using (133) we find that,

$$\begin{aligned}\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm R}\right](x;x') &\equiv \kappa^2\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm R}\right]^{(1)}(x;x') \\ &= -\frac{\kappa^2\ln(a)}{80\pi^2}\mathcal{F}^{\mu\nu\rho\sigma}\delta^4(x;x') + \kappa^2\mathcal{F}^{\mu\nu\rho\sigma}\Big[iF^{(1)}_{2,\rm R}(x;x')\Big] \; , \qquad (136)\end{aligned}$$

with the S-K retarded structure function for the non-local part,

$$iF^{(1)}_{2,\rm R}(x;x') = \frac{i\partial^4}{5120\pi^4}\Bigg\{\ln^2\left(\mu^2\Delta x^2_{++}\right) - \ln^2\left(\mu^2\Delta x^2_{+-}\right) - 2\Big[\ln\left(\mu^2\Delta x^2_{++}\right) - \ln\left(\mu^2\Delta x^2_{+}\right)\Big]\Bigg\} \; , \qquad (137)$$

Now using the identities,

$$\begin{aligned}\ln\left(\mu^2\Delta x^2_{++}\right) - \ln\left(\mu^2\Delta x^2_{+-}\right) &= 2i\pi\theta\left(\Delta\eta - \Delta x\right) \; , \qquad (138)\\ \ln^2\left(\mu^2\Delta x^2_{++}\right) - \ln^2\left(\mu^2\Delta x^2_{+-}\right) &= 4i\pi\theta\left(\Delta\eta - \Delta x\right)\ln\Big[\mu^2\left(\Delta\eta^2 - \Delta x^2\right)\Big] \; . \qquad (139)\end{aligned}$$

we can achieve a manifestly real and causal form for $iF^{(1)}_{2,\mathrm{R}}$,

$$iF^{(1)}_{2,\mathrm{R}}(x;x') = \frac{-\partial^4}{1280\pi^3}\Bigg\{\theta\left(\Delta\eta - \Delta x\right)\Big[\ln\big[\mu^2\left(\Delta\eta^2 - \Delta x^2\right)\big] - 1\Big]\Bigg\} \; . \tag{140}$$

Here and henceforth we define conformal time intervals $\Delta\eta \equiv \eta - \eta'$, spatial intervals $\Delta\vec{x} \equiv \vec{x} - \vec{x}'$ and its Euclidean norm $\Delta x \equiv \|\Delta\vec{x}\|$. And $\theta(\dots)$ denotes the Heaviside step function.

5.2 Perturbative Formulation

To get the correction to force law of gravity, we need to solve the linearized effective field equation (1). However, we have only solved the lowest loop result for the graviton self-energy (136),

$$\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\mathrm{R}}\right](x;x') = 0 + \kappa^2\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\mathrm{R}}\right]^{(1)}(x;x') + O(\kappa^4) \; . \tag{141}$$

Because the quantum corrections are only known to a finite order in κ^2, there is no alternative to making a similar expansion for the gravitational potentials,

$$h_{\rho\sigma}(x) = h^{(0)}_{\rho\sigma}(x) + \kappa^2 h^{(1)}_{\rho\sigma}(x) + O(\kappa^4) \; . \tag{142}$$

Substituting (141), (142) into (1) leads to the 0th and 1st order equations,

$$\mathcal{D}^{\mu\nu\rho\sigma}\kappa h^{(0)}_{\rho\sigma}(x) = \mathcal{T}^{\mu\nu}_{\mathrm{lin}}(x) \; , \tag{143}$$

$$\mathcal{D}^{\mu\nu\rho\sigma}\kappa h^{(1)}_{\rho\sigma}(x) = \mathcal{S}^{\mu\nu}(x) \; . \tag{144}$$

where we have the stress tensor density for the quantum corrections as,

$$\mathcal{S}^{\mu\nu}(x) \equiv \int d^4x' \left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\mathrm{R}}\right]^{(1)}(x;x')\kappa h^{(0)}_{\rho\sigma}(x') \; . \tag{145}$$

Note that we have regarded the matter source as 0th order, unless its stress tensor includes loop corrections from the 1PI 1-point function. And the Lichnerowicz operator in de Sitter is given by,

$$\mathcal{D}^{\mu\nu\rho\sigma} = \frac{1}{2}a^2\Big[\left(\eta^{\mu(\rho}\eta^{\sigma)\nu} - \eta^{\mu\nu}\eta^{\rho\sigma}\right)\partial^2 + \eta^{\mu\nu}\partial^\rho\partial^\sigma + \eta^{\rho\sigma}\partial^\mu\partial^\nu - 2\partial^{(\mu}\eta^{\nu)(\rho}\partial^{\sigma)}\Big]$$
$$+ Ha^3\Big[\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu(\rho}\eta^{\sigma)\nu}\right)\partial_0 - 2\eta^{\mu\nu}\delta^{(\rho}_0\partial^{\sigma)} + 2\delta^{(\rho}_0\eta^{\sigma)(\mu}\partial^{\nu)}\Big] + 3H^2a^4\eta^{\mu\nu}\delta^\rho_0\delta^\sigma_0 \; . \tag{146}$$

The force law of gravity is determined from the response of a point mass. So, consider a stationary point particle with mass M on de Sitter background. We can write the matter action as,

$$S_{\mathrm{matter}} \equiv -M\int d\tau\sqrt{-g_{\mu\nu}(q(\tau))\dot{q}^\mu(\tau)\dot{q}^\nu(\tau)}$$
$$= -M\int d\tau\Big\{a(q(\tau))\sqrt{1 - \kappa h_{00}(q(\tau))}\Big\} \; . \tag{147}$$

To get from the 1st line to the 2nd line in (147) we have used the full metric (5) and assumed the point particle has the world line $q^\mu \equiv \tau\delta^\mu_0$. So, the corresponding linearized stress tensor density takes the form,

$$\mathcal{T}^{\mu\nu}_{\rm lin}(x) \equiv -\kappa \frac{S_{\rm matter}[h]}{\delta h_{\mu\nu}(x)}\bigg|_{h=0} = -8\pi GM\delta^\mu_0\delta^\nu_0 \times a\delta^3(\vec{x}) \ . \qquad (148)$$

With this matter source, the solution to the 0th order equation (143) is given by [27],

$$\kappa h^{(0)}_{\rho\sigma} = \left(2\delta^0_\rho\delta^0_\sigma + \eta_{\rho\sigma}\right)\frac{2GM}{a\|\vec{x}\|} \ . \qquad (149)$$

Then, integrating $\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm R}\right]^{(1)}$ up against this tree order solution $h^{(0)}_{\rho\sigma}$ (149) simply gives the one-loop source to the 1st order equation (144) which is to be solved.

5.3 The One Loop Source Term

The quantum source (145) consists both local and non-local parts,

$$\mathcal{S}^{\mu\nu}(x) = \mathcal{S}^{\mu\nu}_{\rm local}(x) + \mathcal{S}^{\mu\nu}_{\text{non-local}}(x) \ . \qquad (150)$$

First, let's look at the local part,

$$\mathcal{S}^{\mu\nu}_{\rm local}(x) \equiv \int d^4x' \left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm R}\right]^{(1)}_{\rm local}(x;x')\kappa h^{(0)}_{\rho\sigma}(x') \ , \qquad (151)$$

where $\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm R}\right]^{(1)}_{\rm local}(x;x')$ denotes,

$$\left[{}^{\mu\nu}\Sigma^{\rho\sigma}_{\rm R}\right]^{(1)}_{\text{non-local}}(x;x') \equiv -\frac{\ln(a)}{80\pi^2}\mathcal{F}^{\mu\nu\rho\sigma}\delta^D(x-x') \ . \qquad (152)$$

However, with (130), (149) and (152) it's easy to show that the local part of the one-loop source doesn't contribute to the 1st order equation,

$$\mathcal{S}^{\mu\nu}_{\rm local}(x) = 0 \quad \Longrightarrow \quad \mathcal{S}^{\mu\nu}(x) = \mathcal{S}^{\mu\nu}_{\text{non-local}}(x) \ . \qquad (153)$$

This is actually what we really expect, since the gravitational potential at a certain spacetime point must be generated by a source elsewhere. Therefore, only the non-local part of our graviton self-energy gives rise to non-trivial solutions.

- **Fundamental Integral**

To solve (144) with only the non-local part of the source, we need to first work out the corresponding one-loop source term, namely,

$$\mathcal{S}^{\mu\nu}(x) \equiv \mathcal{F}^{\mu\nu\rho\sigma} \int d^4x' \Big[iF^{(1)}_{2,\mathrm{R}}(x;x') \Big] \kappa h^{(0)}_{\rho\sigma}(x') \; . \tag{154}$$

Substituting (140) and (149) in (154) implies there is a fundamental integral to do,

$$F(\eta, r) \equiv \int d^4x' \left\{ \frac{\theta(\Delta\eta - \Delta x)}{a' \|\vec{x}'\|} \Big[\ln\Big[\mu^2 \left(\Delta\eta^2 - \Delta x^2 \right) \Big] - 1 \Big] \right\} . \tag{155}$$

After changing the spatial integration variable, $\vec{x}' \to \vec{r} \equiv \Delta\vec{x}$, letting $x \equiv \|\vec{x}\|$, and performing the angular integration under spherical polar coordinates, one sees that the integral (155) becomes,

$$\begin{aligned} &\frac{2\pi}{x} \int_0^{+\infty} \left(x + r - |x - r| \right) r dr \\ &\qquad \times \int_{\eta_i}^{0^-} \theta(\Delta\eta - r) \Big[\ln\Big[\mu^2 \left(\Delta\eta^2 - r^2 \right) \Big] - 1 \Big] (-H\eta') d\eta' \; , \end{aligned} \tag{156}$$

where $\eta_i \equiv -H^{-1}$ denotes the initial time. Now, let's first tackle the temporal integral. After using the Heaviside step function to fix the temporal integration limits, we find,

$$\begin{aligned} &\int_{\eta_i}^{0^-} \theta(\Delta\eta - r) \Big[\ln\Big[\mu^2 \left(\Delta\eta^2 - r^2 \right) \Big] - 1 \Big] (-H\eta') d\eta' \\ &\quad = \int_{\eta_i}^{\eta - r} \Big[\ln\Big[\mu^2 \left(\Delta\eta^2 - r^2 \right) \Big] - 1 \Big] (-H\eta') d\eta' \\ &\quad = 2H(\Delta\eta_i + \eta_i) r \ln(2\mu r) + H(2\Delta\eta_i + 3\eta_i - r)(\Delta\eta_i - r) \\ &\qquad - \frac{1}{2} H(\Delta\eta_i + 2\eta_i + r)(\Delta\eta_i + r) \ln\big[\mu(\Delta\eta_i + r) \big] \\ &\qquad\quad - \frac{1}{2} H(\Delta\eta_i + 2\eta_i - r)(\Delta\eta_i - r) \ln\big[\mu(\Delta\eta_i - r) \big] \; , \end{aligned} \tag{157}$$

where we have defined $\Delta\eta_i \equiv \eta - \eta_i$. Note that the result (157) will force the integration limit of the radial integral in (156) to be $0 < r < \Delta\eta_i$. Next step is to substitute (157) into (156), and we are left with a purely radial integration which is tedious but straightforward to do. Here, we simply give the full result of the fundamental integral (155),

$$F(\eta,x) = \frac{2\pi H}{3x} \times \Bigg\{ -(\Delta\eta_i+\eta_i)x^4\ln(2\mu x)$$
$$+\frac{1}{600}\Big[5(119\Delta\eta_i+440\eta_i)\Delta\eta_i^3 x + 1250(\Delta\eta_i+\eta_i)x^4 - 10(131\Delta\eta_i+220\eta_i)\Delta\eta_i x^3 - 261x^5\Big]$$
$$+\frac{1}{20}\Big[(\Delta\eta_i+10\eta_i)x - 2(\Delta\eta_i+5\eta_i)\Delta\eta_i + 3x^2\Big](\Delta\eta_i+x)^3\ln\big[\mu(\Delta\eta_i+x)\big]$$
$$+\frac{1}{20}\Big[(\Delta\eta_i+10\eta_i)x + 2(\Delta\eta_i+5\eta_i)\Delta\eta_i - 3x^2\Big](\Delta\eta_i-x)^3\ln\big[\mu(\Delta\eta_i-x)\big]\Bigg\} . \quad (158)$$

- **Acting Derivatives**

At this stage, we can rewrite the non-trivial part of our one-loop order term (154) as,

$$\mathcal{S}^{\mu\nu}(\eta,x) = -\frac{GM}{640\pi^3}\left(2\delta^0_\rho\delta^0_\sigma + \eta_{\rho\sigma}\right)\mathcal{F}^{\mu\nu\rho\sigma}\partial^4 F(\eta,x) . \quad (159)$$

Next is to act two d'Alembertians on $F(\eta,x)$ using the following differentiation formula,

$$\partial^4 F(\eta,x) = F^{(4,0)} - 2F^{(2,2)} + F^{(0,4)} - \frac{4}{x}F^{(2,1)} + \frac{4}{x}F^{(0,3)} , \quad (160)$$

where we have adopted a short hand notation of mixed partial differentiations,

$$F^{(m,n)} \equiv \frac{\partial^{m+n}}{\partial\eta^m\partial x^n}F(\eta,x) . \quad (161)$$

Then, it follows that two d'Alembertians on (158) gives gives,

$$G(\eta,x) \equiv \partial^4 F(\eta,x) = 16\pi H\left[1 - \frac{\eta}{x}\ln(2\mu x)\right] , \quad (162)$$

Now the remaining work is acting the operator $\left(2\delta^0_\rho\delta^0_\sigma + \eta_{\rho\sigma}\right)\mathcal{F}^{\mu\nu\rho\sigma}$ on $G(\eta,x)$. Note that the tensor projector $\mathcal{F}^{\mu\nu\rho\sigma}$ is traceless by design, therefore, what we really care about is the result of acting $\left(2\delta^0_\rho\delta^0_\sigma\right)\mathcal{F}^{\mu\nu\rho\sigma}$,

$$2\delta^0_\rho\delta^0_\sigma\mathcal{F}^{\mu\nu\rho\sigma} = \partial^\mu\partial^\nu\partial_0^2 - \frac{1}{3}\partial^\mu\partial^\nu\overline{\partial}^2 - 2\delta^{(\mu}_0\partial^{\nu)}\partial^2\partial_0 + \frac{1}{3}\eta^{\mu\nu}\partial^2\partial_0^2 + \left(\delta^\mu_0\delta^\nu_0 + \frac{1}{3}\eta^{\mu\nu}\right)\partial^4 . \quad (163)$$

Similarly, we can also derive a formula for this differential operation,

$$2\delta^0_\rho\delta^0_\sigma\mathcal{F}^{\mu\nu\rho\sigma}G(\eta,x) = \overline{\partial}^\mu\overline{\partial}^\nu\left(G^{(2,0)} - \frac{1}{3}G^{(0,2)} - \frac{2}{3x}G^{(0,1)}\right)$$
$$+\delta^{(\mu}_0\overline{\partial}^{\nu)}\left(-\frac{4}{3}G^{(1,2)} - \frac{8}{3x}G^{(1,1)}\right) + \delta^\mu_0\delta^\nu_0\left(\frac{2}{3}G^{(0,4)} + \frac{8}{3x}G^{(0,3)}\right)$$
$$+\overline{\eta}^{\mu\nu}\left(-\frac{1}{3}G^{(2,2)} + \frac{1}{3}G^{(0,4)} - \frac{2}{3x}G^{(2,1)} + \frac{4}{3x}G^{(0,3)}\right) . \quad (164)$$

Substituting the result of $G(\eta,x)$ (162) into (164) allows us to finally obtain the non-trivial part of our one-loop source term,

$$\mathcal{S}^{\mu\nu}(\eta,x) = \frac{GM}{40\pi^2}\Bigg\{\overline{\partial}^{\mu}\overline{\partial}^{\nu}\left(\frac{H\eta}{3x^3}\right) - \delta^{\mu}_{0}\delta^{\nu}_{0}\left(\frac{4H\eta}{x^5}\right) - \overline{\eta}^{\mu\nu}\left(\frac{2H\eta}{x^5}\right) + \delta^{(\mu}_{0}\overline{\partial}^{\nu)}\left(\frac{4H}{3x^3}\right)\Bigg\} \; . \quad (165)$$

Substituting (165) into (144) gives the following non-trivial components of our 1st order equation,

$$\mathcal{D}^{00\rho\sigma}\kappa h^{(1)}_{\rho\sigma}(\eta,\vec{x}) = -\frac{GM}{10\pi^2}\times\frac{H\eta}{x^5} \; , \quad (166)$$

$$\mathcal{D}^{0i\rho\sigma}\kappa h^{(1)}_{\rho\sigma}(\eta,\vec{x}) = \partial^i\left\{\frac{GM}{60\pi^2}\times\frac{H}{x^3}\right\} \; , \quad (167)$$

$$\mathcal{D}^{ij\rho\sigma}\kappa h^{(1)}_{\rho\sigma}(\eta,\vec{x}) = \partial^i\partial^j\left\{\frac{GM}{120\pi^2}\times\frac{H\eta}{x^3}\right\} + \delta^{ij}\left\{-\frac{GM}{20\pi^2}\times\frac{H\eta}{x^5}\right\} \; . \quad (168)$$

5.4 Quantum Corrected Potentials

We want to express the one-loop Newtonian potentials in Schwarzschild coordinates so that with a convenient gauge choice their non-zero components take the form,

$$\kappa h^{(1)}_{00}(\eta,\vec{x}) = f_1(\eta,\vec{x}) \quad , \quad \kappa h^{(1)}_{0i}(\eta,\vec{x}) = 0 \quad , \quad \kappa h^{(1)}_{ij}(\eta,\vec{x}) = f_2(\eta,\vec{x})\delta_{ij} \; . \quad (169)$$

Substituting our ansatz (169) into the (166-168) gives rise to the following equations,

$$-\frac{GM}{60\pi^2}\times\frac{H^3\eta^3}{x^3} = f_1 - f_2 \; , \quad (170)$$

$$-\frac{GM}{60\pi^2}\times\frac{H^3\eta^3}{x^3} = \eta f_2^{(1,0)} - f_1 \; , \quad (171)$$

$$-\frac{GM}{10\pi^2}\times\frac{H^3\eta^3}{x^5} = f_2^{(0,2)} + \frac{2}{x}f_2^{(0,1)} + \frac{3}{\eta}f_2^{(1,0)} - \frac{3}{\eta^2}f_1 \; , \quad (172)$$

$$-\frac{GM}{20\pi^2}\times\frac{H^3\eta^3}{x^5} = \frac{1}{2}f_1^{(0,2)} + \frac{1}{x}f_1^{(0,1)} - \frac{1}{\eta}f_1^{(1,0)} + \frac{3}{\eta^2}f_1 - \frac{1}{2}f_2^{(0,2)} - \frac{1}{x}f_2^{(0,1)} - \frac{2}{\eta}f_2^{(1,0)} + f_2^{(2,0)} \; . \quad (173)$$

We should remind readers of the notation for mixed partial differentiation (161). Now it seems that we have four equations (170-173) but only two field variables, f_1 and f_2. Actually, these equations are not completely independent. We can pick the easier ones to solve, then the remaining equations shall be automatically satisfied by the virtue of stress energy tensor conservation and Bianchi identity.

The equation (170) and (171) are much simpler to start with, so we first eliminate f_1 using these two and obtain an 1st order partial differential equations for f_2,

$$\frac{1}{\eta^2} f_2^{(1,0)} - \frac{1}{\eta^3} f_2 = -\frac{GM}{30\pi^2}\frac{H^3}{x^3} \;, \tag{174}$$

whose solution is,

$$f_2\big(\eta,\vec{x}\big) = -\frac{GM}{60\pi^2} \times \frac{H^3\eta^3}{x^3} + \eta \times C(x) \;. \tag{175}$$

Note that $C(x)$ is a integration "constant", but it may still depend on the Euclidean norm x. To fix this "constant", we can substitute (170) and (175) into (172) and obtain an 2nd order ordinary differential equation for $C(x)$,

$$C''(x) + \frac{2}{x} C'(x) = \frac{GM}{20\pi^2}\frac{H^3}{x^3} \;, \tag{176}$$

which can be regarded 1st order in $C'(x)$ and hence easily solved. Thus, we find,

$$C(x) = -\frac{H^2 GM}{20\pi^2} \times \frac{H}{x}\Big[1 + \ln{(Hx)}\Big] - \frac{A}{x} + B \;, \tag{177}$$

where A and B are really constants. To fix them we shall in principle make use (173). Well, as we have mentioned previously, the equation whichever is left should be satisfied automatically. And indeed, with the solutions given by (170) and (175) and (177) one can show that (173) is solved no matter what A and B are. Therefore, we can simply set $A = B = 0$. and obtain the formal solutions for f_1 and f_2,

$$f_1\big(\eta,\vec{x}\big) = -\frac{GM}{30\pi^2} \times \frac{H^3\eta^3}{x^3} - \frac{H^2GM}{20\pi^2} \times \frac{H\eta}{x}\Big[1 + \ln{(Hx)}\Big] \;, \tag{178}$$

$$f_2\big(\eta,\vec{x}\big) = -\frac{GM}{60\pi^2} \times \frac{H^3\eta^3}{x^3} - \frac{H^2GM}{20\pi^2} \times \frac{H\eta}{x}\Big[1 + \ln{(Hx)}\Big] \;. \tag{179}$$

Combining the tree order solutions (149) gives the final results of our one-loop Newtonian potentials,

$$\kappa^2\Big[\kappa h^{(1)}_{00}\big(\eta,\vec{x}\big)\Big] = \left\{\frac{\kappa^2}{60\pi^2(ax)^2} + \frac{\kappa^2H^2}{40\pi^2}\Big[1 + \ln{(Hx)}\Big] + O(\kappa^4)\right\} \times \Big[\kappa h^{(0)}_{00}\big(\eta,\vec{x}\big)\Big] \;, \tag{180}$$

$$\kappa^2\Big[\kappa h^{(1)}_{0i}\big(\eta,\vec{x}\big)\Big] = O(\kappa^4) \;, \tag{181}$$

$$\kappa^2\Big[\kappa h^{(1)}_{ij}\big(\eta,\vec{x}\big)\Big] = \left\{\frac{\kappa^2}{120\pi^2(ax)^2} + \frac{\kappa^2H^2}{40\pi^2}\Big[1 + \ln{(Hx)}\Big] + O(\kappa^4)\right\} \times \Big[\kappa h^{(0)}_{ij}\big(\eta,\vec{x}\big)\Big] \tag{182}$$

6 Discussion

We have computed the one photon loop contribution to the graviton self-energy on de Sitter, and used it to derive quantum corrections to the gravitational potentials of a stationary point mass. Our results (180-182) feature two sorts of terms which can be best be understood by expressing the full Newtonian potential as its classical value times a series of corrections, with the factors of $\hbar$ and c restored,

$$\Phi(\eta, \vec{x}) = -\frac{2GM}{ax}\left\{1 + \frac{4\hbar G}{15\pi c^3 (ax)^2} + \frac{2\hbar G H^2}{5\pi c^5}\Big[\ln\Big(\frac{Hx}{c}\Big) + 1\Big] + O(G^2)\right\}. \tag{183}$$

The fractional correction $4\hbar G/[15\pi c^3 (ax)^2]$ is just the de Sitter version of the flat space background correction first found in 1970 by Radkowski [59] and later confirmed by Capper, Duff and Halpern [60]. Getting this term right checks an important correspondence limit, however, it can only be significant at physical distances of $ax \sim \sqrt{\hbar G/c^3} \sim 10^{-35}\ m$. This is a small distance even with respect to the scales prevailing during primordial inflation. The more interesting part of (183) is the fractional correction $2\hbar G H^2/[5\pi c^5] \times \ln(Hx/c)$, which vanishes in the flat space limit. This represents a small logarithmic running of the Newton constant. At a fixed physical distance ax it represents a secular screening of the Newton constant because $x = ax/a$.

The logarithmic running correction in (183) might seem surprising because it is sourced by the gravitational vacuum polarization induced by photons, which is simply related to its value on flat space by the conformal invariance of classical electromagnetism. Indeed, the source terms we found in equations (166-168) are almost identical to those of flat space. The sources on the right hand side of (166) and (168) are just conformal re-scalings of the flat space result, and the new, "de Sitter" source on the right hand side of (167) is the bare minimum required by conservation in view of the time dependent conformal re-scaling. The reason this conformally re-scaled source can do more than "de Sitterize" response of flat background is that *the left hand side of the equation is different* because gravity is not conformally invariant. Gravity on de Sitter background responds far differently to sources than gravity does on flat background.

It is worthwhile working out the response seen by a static observer whose background geometry is,

$$ds^2 = -\left(1 - H^2 r^2\right) dt^2 + \frac{dr^2}{(1 - H^2 r^2)} + r^2 d\Omega^2\ . \tag{184}$$

The transformation to these coordinates is,

$$\eta = \frac{-H^{-1} e^{-Ht}}{\sqrt{1 - H^2 r^2}} \quad \Longrightarrow \quad d\eta = \frac{1}{a} \Bigl[dt - \frac{H r dr}{1 - H^2 r^2} \Bigr] \; , \tag{185}$$

$$x^i = \frac{r^i e^{-Ht}}{\sqrt{1 - H^2 r^2}} \quad \Longrightarrow \quad dx^i = \frac{1}{a} \Bigl[-H r^i dt + dr^i + \frac{H^2 r^i r dr}{1 - H^2 r^2} \Bigr] \; . \tag{186}$$

The potentials come from using transformations (185-186) on the full metric $g_{\mu\nu} \equiv a^2 [\eta_{\mu\nu} + \kappa h_{\mu\nu}]$,

$$\begin{aligned} ds^2_{\rm full} &\equiv a^2 \Bigl[-d\eta^2 + d\vec{x} \cdot d\vec{x} \Bigr] + \kappa h_{00} a^2 d\eta^2 + 2 \kappa h_{0i} a^2 d\eta dx^i + \kappa h_{ij} a^2 dx^i dx^j \; , && (187) \\ &\equiv -\Bigl[1 - H^2 r^2 \Bigr] dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 d\Omega^2 + \chi_{tt} dt^2 + 2 \chi_{0i} dt dr^i + \chi_{ij} dr^i dr^j \; . && (188) \end{aligned}$$

The final results are complicated by the fact that achieving the de Sitter-Schwarzchild form would require order GM modifications of the transformations (185-186). However, one important feature of the result is that all the factors of ax become the time independent static coordinate radius r, and the only time dependence resides in the logarithmic screening factor,

$$\ln(Hx) = \ln(Hr) - \ln(a) \; . \tag{189}$$

Acknowledgments

We are grateful for conversations and correspondence on this subject with M. Fröb, S. Park, T. Prokopec and E. Verdaguer. This work was partially supported by by NSF grant PHY-1506513 and by the Institute for Fundamental Theory at the University of Florida.

References

[1] R. P. Woodard, Rept. Prog. Phys. **72**, 126002 (2009) [arXiv:0907.4238 [gr-qc]].

[2] A. A. Starobinsky, JETP Lett. **30**, 682 (1979) [Pisma Zh. Eksp. Teor. Fiz. **30**, 719 (1979)].

[3] V. F. Mukhanov and G. V. Chibisov, JETP Lett. **33**, 532 (1981) [Pisma Zh. Eksp. Teor. Fiz. **33**, 549 (1981)].

[4] S. W. Hawking, Phys. Lett. B **115**, 295 (1982).

[5] A. H. Guth and S. Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).

[6] A. A. Starobinsky, Phys. Lett. B **117**, 175 (1982).

[7] J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D **28**, 679 (1983).

[8] V. F. Mukhanov, JETP Lett. **41**, 493 (1985) [Pisma Zh. Eksp. Teor. Fiz. **41**, 402 (1985)].

[9] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. **215**, 203 (1992).

[10] A. R. Liddle and D. H. Lyth, Phys. Rept. **231**, 1 (1993) [astro-ph/9303019].

[11] J. E. Lidsey, A. R. Liddle, E. W. Kolb, E. J. Copeland, T. Barreiro and M. Abney, Rev. Mod. Phys. **69**, 373 (1997) [astro-ph/9508078].

[12] L. M. Krauss and F. Wilczek, Phys. Rev. D **89**, 047501 (2014) [arXiv:1309.5343 [hep-th]].

[13] R. P. Woodard, arXiv:1407.4748 [gr-qc].

[14] S. Deser and P. van Nieuwenhuizen, Phys. Rev. Lett. **32**, 245 (1974).

[15] S. Deser and P. van Nieuwenhuizen, Phys. Rev. D **10**, 401 (1974).

[16] N. N. Bogoliubov and O. Parasiuk, Acta Math. **97** (1957) 227; K. Hepp, Commun. Math. Phys. **2** (1966) 301; W. Zimmermann, Commun. Math. Phys. **11** (1968) 1; **15** (1969) 208; in *Lectures on Elementary Particles and Quantum Field Theory*, ed. S. Deser, M. Grisaru and H. Pendleton (MIT Press, Cambridge, 1971), Vol. I.

[17] J. F. Donoghue, Phys. Rev. Lett. **72**, 2996 (1994) [gr-qc/9310024].

[18] J. F. Donoghue, Phys. Rev. D **50**, 3874 (1994) [gr-qc/9405057].

[19] N. E. J. Bjerrum-Bohr, Phys. Rev. D **66**, 084023 (2002) [hep-th/0206236].

[20] S. Park and R. P. Woodard, Class. Quant. Grav. **27**, 245008 (2010) [arXiv:1007.2662 [gr-qc]].

[21] S. Park and R. P. Woodard, Phys. Rev. D **84**, 124058 (2011) [arXiv:1109.4187 [gr-qc]].

[22] M. B. Fröb, A. Roura and E. Verdaguer, JCAP **1208**, 009 (2012) [arXiv:1205.3097 [gr-qc]].

[23] M. B. Fröb, A. Roura and E. Verdaguer, JCAP **1407**, 048 (2014) [arXiv:1403.3335 [gr-qc]].

[24] M. B. Fröb, D. B. Papadopoulos, A. Roura and E. Verdaguer, Phys. Rev. D **87**, no. 6, 064019 (2013) [arXiv:1301.5261 [gr-qc]].

[25] D. Glavan, S. P. Miao, T. Prokopec and R. P. Woodard, arXiv:1308.3453 [gr-qc].

[26] R. P. Woodard, gr-qc/0408002.

[27] N. C. Tsamis and R. P. Woodard, Commun. Math. Phys. **162**, 217 (1994).

[28] N. C. Tsamis and R. P. Woodard, J. Math. Phys. **48**, 052306 (2007) [gr-qc/0608069].

[29] E. O. Kahya and R. P. Woodard, Phys. Rev. D **72**, 104001 (2005) [gr-qc/0508015].

[30] K. E. Leonard and R. P. Woodard, Class. Quant. Grav. **31**, 015010 (2014) [arXiv:1304.7265 [gr-qc]].

[31] K. E. Leonard, T. Prokopec and R. P. Woodard, Phys. Rev. D **87**, no. 4, 044030 (2013) [arXiv:1210.6968 [gr-qc]].

[32] K. E. Leonard, T. Prokopec and R. P. Woodard, J. Math. Phys. **54**, 032301 (2013) [arXiv:1211.1342 [gr-qc]].

[33] K. E. Leonard, S. Park, T. Prokopec and R. P. Woodard, Phys. Rev. D **90**, 024032 (2014) [arXiv:1403.0896 [gr-qc]].

[34] G. 't Hooft and M. J. G. Veltman, Annales Poincare Phys. Theor. A **20**, 69 (1974).

[35] J. S. Schwinger, J. Math. Phys. **2**, 407 (1961).

[36] K. T. Mahanthappa, Phys. Rev. **126**, 329 (1962).

[37] P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. **4**, 1 (1963).

[38] P. M. Bakshi and K. T. Mahanthappa, J. Math. Phys. **4**, 12 (1963).

[39] L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].

[40] K. c. Chou, Z. b. Su, B. l. Hao and L. Yu, Phys. Rept. **118**, 1 (1985).

[41] R. D. Jordan, Phys. Rev. D **33**, 444 (1986).

[42] E. Calzetta and B. L. Hu, Phys. Rev. D **35**, 495 (1987).

[43] L. H. Ford and R. P. Woodard, Class. Quant. Grav. **22**, 1637 (2005) [gr-qc/0411003].

[44] S. P. Miao and R. P. Woodard, Class. Quant. Grav. **23**, 1721 (2006) [gr-qc/0511140].

[45] S. P. Miao and R. P. Woodard, Phys. Rev. D **74**, 024021 (2006) [gr-qc/0603135].

[46] S. P. Miao, arXiv:0705.0767 [hep-th].

[47] S. P. Miao and R. P. Woodard, Class. Quant. Grav. **25**, 145009 (2008) [arXiv:0803.2377 [gr-qc]].

[48] E. O. Kahya and R. P. Woodard, Phys. Rev. D **76**, 124005 (2007) [arXiv:0709.0536 [gr-qc]].

[49] E. O. Kahya and R. P. Woodard, Phys. Rev. D **77**, 084012 (2008) [arXiv:0710.5282 [gr-qc]].

[50] S. Park and R. P. Woodard, Phys. Rev. D **83**, 084049 (2011) [arXiv:1101.5804 [gr-qc]].

[51] P. J. Mora, N. C. Tsamis and R. P. Woodard, JCAP **1310**, 018 (2013) [arXiv:1307.1422, arXiv:1307.1422].

[52] A. A. Starobinsky, Lect. Notes Phys. **246**, 107 (1986).

[53] A. A. Starobinsky and J. Yokoyama, Phys. Rev. D **50**, 6357 (1994) [astro-ph/9407016].

[54] R. P. Woodard, Nucl. Phys. Proc. Suppl. **148**, 108 (2005) [astro-ph/0502556].

[55] N. C. Tsamis and R. P. Woodard, Nucl. Phys. B **724**, 295 (2005) [gr-qc/0505115].

[56] S. P. Miao and R. P. Woodard, Phys. Rev. D **74**, 044019 (2006) [gr-qc/0602110].

[57] T. Prokopec, N. C. Tsamis and R. P. Woodard, Annals Phys. **323**, 1324 (2008) [arXiv:0707.0847 [gr-qc]].

[58] H. Degueldre and R. P. Woodard, Eur. Phys. J. C **73**, no. 6, 2457 (2013) [arXiv:1303.3042 [gr-qc]].

[59] A. F. Radkowski, Annals Phys. bf 56, 319 (1970).

[60] D. M. Capper, M. J. Duff and L. Halpern, Phys. Rev. D **10**, 461 (1974).