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Paul R. Anderson, Carmen Molina-París, and Dillon H. Sanders

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Paul R. Anderson,[1,] * Carmen Molina-París,[2,] † and Dillon H. Sanders[1,3]

[1]*Department of Physics, Wake Forest University,*
Winston-Salem, North Carolina, 27109, USA
[2]*Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK*
[3]*Department of Nuclear Engineering, North Carolina State University, Raleigh, NC 27695, USA*[‡]

The validity of the semiclassical approximation is investigated during the preheating phase in models of chaotic inflation using a modification of a criterion previously proposed for semiclassical gravity. If the modified criterion is violated then fluctuations of the two-point function for the quantum fields are large and the semiclassical approximation is not valid. Evidence is provided that the semiclassical approximation breaks down during the early stages of preheating, well before either scattering effects or backreaction effects are important.

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*Electronic address: anderson@wfu.edu
†Electronic address: carmen@maths.leeds.ac.uk
‡Electronic address: dhsander@ncsu.edu

The semiclassical approximation has been used to study the effects of quantized fields on a classical background field in a wide variety of scenarios including black hole evaporation [1], the decay of an electric field due to the Schwinger effect [2], heavy ion collisions in nuclear physics [3], and preheating in chaotic inflation [4, 5]. It is expected to be valid in cases where quantum effects are small, such as the initial stages of the evaporation of a solar mass black hole. At the opposite end of the spectrum is the case of preheating in models of chaotic inflation. Preheating occurs immediately after the inflationary phase and is a period in which the rate of particle production is extremely rapid, resulting in strong backreaction effects upon the inflaton field [6, 7]. It is not known whether the predictions of the semiclassical approximation can be trusted when quantum effects are so large.

The semiclassical backreaction equations for quantum fields coupled to a classical background field arise out of the one loop effective action for that field [8]. As such they would typically be expected to break down when backreaction effects are large and terms coming from higher loops may be important. One way around this is to use a large N expansion where N is the number of identical quantum fields. The semiclassical backreaction equations become exact in the limit $N \to \infty$. This expansion has been used in cases such as preheating [5, 9] where backreaction effects are significant.

For the semiclassical approximation to be valid, quantum fluctuations about the mean of whatever quantity couples the quantum fields to the classical background field(s) must be small. One way to characterize these fluctuations in semiclassical gravity is through the two-point function for the energy-momentum tensor. However, for the symmetric part of this two-point function there can be state-dependent divergences [10], and different renormalization schemes can yield different results when the points come together [11]. To overcome these difficulties a criterion was given in [12] that relates the validity of the semiclassical approximation in gravity to the stability of solutions to the linear response equation, which results when the semiclassical backreaction equation is perturbed about a solution to that equation. The linear response equation has a term which involves the perturbed energy-momentum tensor, so renormalization proceeds in the usual way and there are no state dependent divergences.

The criterion states that the large N semiclassical approximation in gravity will break down if any linearized gauge invariant quantity constructed from solutions to the linear response equations with finite non-singular initial data, grows without bound. It has been shown to be satisfied for massive and massless free scalar fields in flat space in the Minkowski vacuum state [12], and for conformally invariant free fields in the expanding part of de Sitter space, when spatially flat coordinates are used, the fields are in the Bunch-Davies state, and scalar perturbations are considered [13]. Tensor perturbations for conformally invariant free fields were investigated in [14] and it was found that they are bounded, so the criterion is satisfied in that case as well.

In both flat space and the expanding part of de Sitter space the character of the solution to the semiclassical backreaction equations does not change in time. However there are important situations such as preheating where the question of the validity of the semiclassical approximation becomes a time dependent one. There are two reasons for this in preheating. First, the damping of the inflaton field due to the backreaction effects of the produced particles is not uniform in time and in particular does not go on for an arbitrarily long period of time. Second, one of the approximations that is usually made when the semiclassical approximation is used for preheating is that interactions between the produced particles are neglected. This should be a good approximation at early times but interactions become important at later times [4].

The above criterion needs to be modified in such cases in order to take the time dependence of the background field into account. Since the criterion is related to the stability of the solutions it is useful to consider situations in which there is an instability but the system is only observed for a finite amount of time. For such systems one will find that perturbations grow rapidly during the allotted time. However, they obviously cannot grow without bound and further, if the perturbations are small enough initially then they will not have time to grow to a large enough size to become significant. Thus it is the rapid growth of a perturbation rather than its size which indicates an instability.

The criterion is also stated only for gravity and needs to be modified in a straight-forward way to cover other cases. We propose the following: The large N semiclassical approximation will break down if any linearized gauge invariant quantity constructed from solutions to the linear response equation with finite non-singular initial data, grows rapidly for some period of time. By linear response equation we mean the equation that is obtained by perturbing the semiclassical backreaction equation about one of its solutions.

In this paper we continue an investigation begun in [15], where we adapted the criterion in [12] to check

the validity of the semiclassical approximation in models of preheating in chaotic inflation[1], in which rapid damping of the inflaton field occurs, and found evidence that quantum fluctuations are large in between the two periods of rapid damping. Here we study in great detail the relationship between solutions to the linear response equation and quantum fluctuations, and use the results to relate the size of the fluctuations to the particle production rate. We also include a case in which there is no rapid damping. We find evidence that quantum fluctuations are large and the semiclassical approximation breaks down whenever the particle production rate is high, including during the early stages of preheating when scattering effects ignored in our model and backreaction effects on the inflaton field are small.

We consider a model of chaotic inflation for which the inflaton field ϕ is coupled to N identical massless scalar fields ψ_i with a coupling of the form $\sum_{i=1}^{N} g^2 \phi^2 \psi_i^2$. Full backreaction effects for this coupling have been investigated in detail in Refs. [4, 5, 18] (although not all of these were in the context of the large N expansion or for massless quantum fields). After a standard rescaling of the coupling constant g [5], the problem reduces to the coupling of the inflaton field to a single scalar field ψ, and the semiclassical backreaction equation for the inflaton field is $\Box\phi - (m^2 + g^2\langle\psi^2\rangle)\phi = 0$. As in [5] we work in a flat space background and consider only homogeneous and isotropic solutions for ϕ.

The mass of the inflaton field can be scaled out of the equations using $t \to mt$ and $\phi \to \phi/m$, with similar changes of variable for other relevant quantities. (See [5] for details.) The result is

$$\ddot{\phi} + (1 + g^2\langle\psi^2\rangle)\phi = 0 \,, \tag{1a}$$

$$\langle\psi^2\rangle = \frac{1}{2\pi^2} \int_0^\epsilon dk k^2 \left(|f_k(t)|^2 - \frac{1}{2k} \right)$$

$$+ \frac{1}{2\pi^2} \int_\epsilon^\infty dk k^2 \left(|f_k(t)|^2 - \frac{1}{2k} + \frac{g^2\phi^2}{4k^3} \right)$$

$$- \frac{g^2\phi^2}{8\pi^2} \left[1 - \log\left(\frac{2\epsilon}{M} \right) \right] \,, \tag{1b}$$

$$\ddot{f}_k + (k^2 + g^2\phi^2)f_k = 0 \,. \tag{1c}$$

Note that $\langle\psi^2\rangle$ is independent of the positive constant ϵ and that M is a mass scale which typically enters when computing renormalized quantities for massless fields [5].

The linear response equation can be derived as in [12] by taking a second variation of the effective action. The result is

$$(\Box - m^2 - g^2\langle\psi^2\rangle)\delta\phi - g^2\delta\langle\psi^2\rangle\phi = 0 \,, \tag{2a}$$

$$\delta\langle\psi^2\rangle = -ig^2 \int d^4x' \, \phi(x')\delta\phi(x')\theta(t - t') \langle[\psi^2(x), \psi^2(x')]\rangle + \delta\langle\psi^2\rangle_{\rm SD} \,. \tag{2b}$$

Here $\delta\langle\psi^2\rangle_{\rm SD}$ comes from a variation in the state of the quantum field.

The linear response equation can also be derived by perturbing the semiclassical backreaction equation about one of its solutions. We illustrate this for homogeneous and isotropic perturbations. The equation for the inflaton field (1a) and the mode equation (1c) are perturbed in the usual way, keeping quantities that are first order in $\delta\phi$ and δf_k. The perturbed mode equation is then solved in terms of the solutions to (1) with the result:

$$\delta f_k = A_k f_k + B_k f_k^* + 2g^2 i \int_0^t dt' \, \phi(t')\delta\phi(t')f_k(t') \left[f_k^*(t)f_k(t') - f_k(t)f_k^*(t') \right] \,. \tag{3}$$

The coefficients A_k and B_k are related to a change of state and are fixed by the initial values of δf_k and its first derivative. Such a change in state (as pointed out in [19] for semiclassical gravity) must occur if the original state is a second order or higher adiabatic state [8].

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[1] A different type of backreaction problem that is also relevant to inflation is the generation of density perturbations during the inflationary phase. The backreaction in this case involves the generation of fluctuations in the gravitational field due to quantum fluctuations of a scalar field. The resulting quantum to classical transition has been described in terms of squeezed states in Refs. [16, 17]. In the type of backreaction problem we are considering for preheating, quantum fluctuations are averaged over when the quantity $\langle\phi^2\rangle$ is computed. This quantity is then coupled to the classical inflaton field as shown in Eq. (1a).

The linear response equation in this case is

$$\delta\ddot{\phi} + (1 + g^2\langle\psi^2\rangle)\delta\phi + g^2\phi\delta\langle\psi^2\rangle = 0 , \tag{4a}$$

$$\delta\langle\psi^2\rangle = \frac{1}{2\pi^2}\int_0^\epsilon dk k^2 (f_k\delta f_k^* + f_k^*\delta f_k)$$

$$+ \frac{1}{2\pi^2}\int_\epsilon^\infty dk k^2 \left(f_k\delta f_k^* + f_k^*\delta f_k + \frac{g^2\phi\,\delta\phi}{2k^3}\right)$$

$$- \frac{g^2\phi\,\delta\phi}{4\pi^2}\left[1 - \log\left(\frac{2\epsilon}{M}\right)\right] . \tag{4b}$$

For the fourth order adiabatic states used in [5], we find that $A_k = 0$ to linear order. An explicit expression for B_k can easily be obtained but we will not display it here.

If one can find solutions to (1) then it is easy to generate approximate solutions to (2a). One simply takes two solutions, ϕ_1 and ϕ_2, which have nearly the same values at the initial time $t = 0$, and evolves them numerically in time. If we define the difference between the solutions to be $\delta\phi_e \equiv \phi_2 - \phi_1$, then $\delta\phi_e$ satisfies the equation:

$$\delta\ddot{\phi}_e + (1 + g^2\langle\psi^2\rangle_1)\delta\phi_e + g^2(\langle\psi^2\rangle_2 - \langle\psi^2\rangle_1)(\phi_1 + \delta\phi_e) . \tag{5}$$

The linear response equation (4a) in this case is

$$\delta\ddot{\phi} + (1 + g^2\langle\psi^2\rangle_1)\delta\phi + g^2\,\phi_1\,\delta\langle\psi^2\rangle[\phi \to \phi_1] = 0 . \tag{6}$$

Here, and below, the square brackets after $\delta\langle\psi^2\rangle$ give instructions as to how this quantity, which is given in (4b), is to be evaluated. Note that the first two terms in these equations have the same form. Thus, $\delta\phi_e$, which is a solution to (5), is also an approximate solution to (6) so long as the amplitude of the oscillations of ϕ_1 is much larger than the amplitude of oscillations of $\delta\phi_e$ and $\delta\langle\psi^2\rangle[\phi \to \phi_1, \delta\phi \to \delta\phi_e] \approx \langle\psi^2\rangle_2 - \langle\psi^2\rangle_1$.

Given the structure of Eq. (6) it is possible to go further and separate out the part of the perturbation driven by $\delta\langle\psi^2\rangle$. For simplicity we choose the starting values for ϕ_2 and ϕ_1 such that $\phi_2(0) = \phi_{20}$, $\phi_1(0) = \phi_{10}$, and $\dot{\phi}_2(0) = \dot{\phi}_1(0) = 0$. Then let

$$\delta\phi_e = \phi_2 - \phi_1 = c\phi_1 + \delta\phi_c , \tag{7}$$

with $c = (\phi_{20} - \phi_{10})/\phi_{10}$. Substituting into (6) and using (1a) one finds that if $\delta\phi_e$ is an approximate solution to (6), then $\delta\phi_c$ is an approximate solution to the equation

$$\delta\ddot{\phi}_c + (1 + g^2\langle\psi^2\rangle_1)\delta\phi_c + g^2\phi_1\delta\langle\psi^2\rangle[\phi \to \phi_1, \delta\phi \to \delta\phi_c, A_k \to 0, B_k \to 0]$$
$$= -g^2\phi_1\delta\langle\psi^2\rangle|[\phi \to \phi_1, \delta\phi \to c\phi_1] . \tag{8}$$

The term on the right hand side is a source term which depends on $\delta\langle\psi^2\rangle$ in (4b) evaluated with $\phi = \phi_1$ and $\delta\phi = c\phi_1$. Since the initial conditions are $\delta\phi_c(0) = \delta\dot{\phi}_c(0) = 0$, at early times the growth of $\delta\phi_c$ is driven by the source term.

Given that the point of our criterion is that the semiclassical approximation breaks down when quantum fluctuations are significant, for the specific case of preheating the criterion can be further revised to state that the semiclassical approximation breaks down if either of the quantities $\delta\phi_e$ or $\delta\phi_c$ grow significantly for some period of time. The reason it is important to include $\delta\phi_c$ is because $\delta\phi_e \sim c\cos(t)$ at early times before backreaction effects due to the quantum fields are important. However, quantum fluctuations can still be significant at this time.

An important question is, what does the breakdown of the semiclassical approximation mean if backreaction effects are small? If the breakdown is due to quantum fluctuations then there will be a sensitivity to initial conditions which will make it virtually impossible to determine in a detailed manner the damping of the inflaton field using the semiclassical approximation even at early times when there is only a small amount of damping.

As an illustration it is interesting to first look at a toy model in which $\langle\psi^2\rangle$ in (1a) is replaced by the last term in (1b) with ϵ/M chosen so that this term is equal to $-\phi^2/g^2$ and the resulting equation for ϕ is $\ddot{\phi} + (1 - \phi^2)\phi = 0$. Then $g^2\delta\langle\psi^2\rangle = -2\phi^2$ and the source term for $\delta\phi_c$ is $2c\phi^3$. The solutions for ϕ are stable
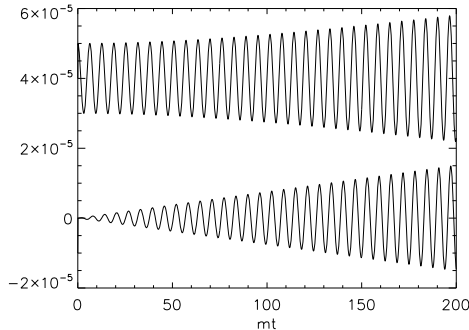
FIG. 1: Plotted are $\delta\phi$ (upper curve) and $\delta\phi_c$ for the toy model described in the text with $\phi(0) = 10^{-1}$ and $\delta\phi(0) = 10^{-5}$. The upper curve has been offset by 4×10^{-5}.

for the starting values $0 < \phi(0) < 1$ and $\dot{\phi}(0) = 0$. In this case it is easy to solve the linear response equation directly. The results for $\phi(0) = 10^{-1}$ and $\delta\phi(0) = 10^{-5}$ are shown in Fig. 1. One sees that over the range shown there is linear growth in the amplitude of $\delta\phi_c$ while the amplitude of $\delta\phi$ does not grow significantly initially. This pattern of early growth of $\delta\phi_c$ is also seen in the solutions to the full set of backreaction equations.

In [6] it was predicted that there are two qualitatively different types of solutions to the backreaction equation for ϕ. For one there is a relatively slow damping of the inflation field while for the other there is a period in which the inflaton field is rapidly damped. In [5] Eqs. (1) were solved numerically and it was found for a flat space background that rapid damping of the inflaton field occurs whenever $g^2\phi_0^2 \gtrsim 2$ for models in which the starting values are $\phi_0 = \phi(t = 0)$ and $\dot{\phi}(t = 0) = 0$. Rapid damping does not occur for significantly smaller values such as $g^2\phi_0^2 = 1$. Whenever rapid damping does occur it is observed to happen twice and there appears to be no significant damping after that. These effects are illustrated in the upper panels of Fig. 2 where the inflaton field is plotted as a function of time for $g^2\phi_0^2 = 1$ and $g^2\phi_0^2 = 10$.

Examination of the plots in Fig. 2 shows that in both cases $\delta\phi_e$ grows exponentially at about the time that a significant amount of damping of ϕ first occurs, while $\delta\phi_c$ grows exponentially starting at much earlier times. After $\delta\phi_c$ grows to be comparable in size to $\delta\phi_e$ the two quantities are nearly identical and cannot be distinguished on the scale of the plots. For $g^2\phi_0^2 = 1$ a small amount of damping of ϕ occurs very quickly followed by a much slower damping rate which goes on for a long time. During this latter period $\delta\phi$ grows approximately linearly in time. For $g^2\phi_0^2 = 10$ the exponential growth of $\delta\phi_e$ continues through the end of the second rapid damping period and then all growth appears to cease.

Using the detailed analysis of the particle production in [5] we find that the rate of growth of $\delta\phi_c$ appears to be closely tied to the overall particle production rate. It is exponential when the particle production rate is high, approximately linear in time during periods of slow damping when the rate is much smaller, and is negligible after the second rapid damping phase when the particle production rate is negligible (in cases where rapid damping occurs).

It is clear from the rapid growth of $\delta\phi_c$ at early times that our revised criterion for the validity of the semiclassical approximation during preheating is violated during the early stages of preheating, well before either scattering effects or backreaction effects are important. This has been shown explicitly for a flat space background. As pointed out in [6], the flat space approximation does not always give an accurate account of the details of the preheating process because the expansion of the universe can have a significant effect on the parametric amplification process. Nevertheless our results strongly suggest that during preheating whenever there is a period in which a lot of parametric amplification occurs, the semiclassical approximation breaks down.

Given that the semiclassical approximation breaks down for preheating, one can ask what it should be replaced with. Since the breakdown happens before significant damping of the inflaton field occurs, one can legitimately solve the mode equation using solutions to the mode equation when the $g^2\langle\psi^2\rangle\phi$ term is neglected. Thus one can investigate the amount of particle production that occurs before backreaction effects become important and one can use that information to determine the time at which they become
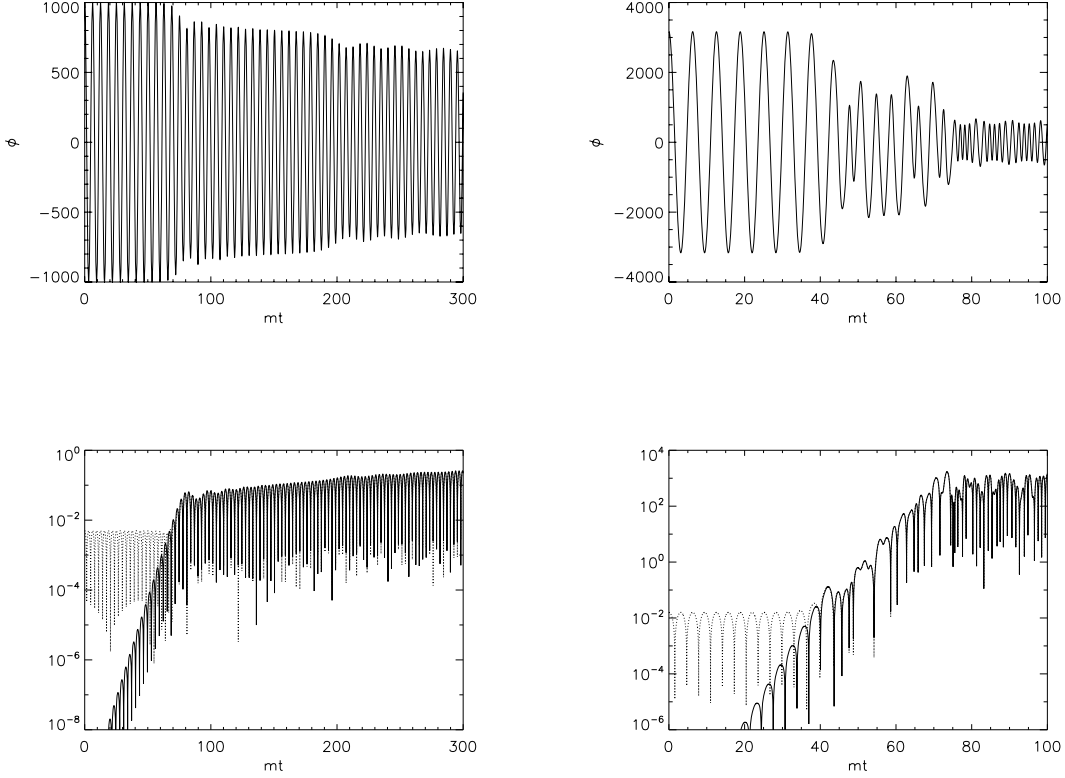
FIG. 2: Numerical solutions to the full set of equations (1) are shown for the inflaton field ϕ in the upper panels. In the lower panels $|\delta\phi_e|$ (dashed curves) and $|\delta\phi_c|$ (solid curves) are plotted. For each plot $g = 10^{-3}$. For the upper left panel $g^2\phi_0^2 = 1$ and the upper right one $g^2\phi_0^2 = 10$. In the lower left panel $\delta\phi_e$ is the difference between solutions to (1) with $g^2\phi_0^2 = 1 + 10^{-5}$ and $g^2\phi_0^2 = 1$. In the lower right panel $\delta\phi_e$ is the difference between solutions to (1) with $g^2\phi_0^2 = 10(1 + 10^{-5})$ and $g^2\phi_0^2 = 10$.

important. What one cannot do is to follow the detailed evolution of the inflaton field using the semiclassical approximation. Once backreaction effects become important a significant amount of particle production will have taken place. This should make it possible to compute the backreaction of the particles on the inflaton field by using a different type of approximation in which the classical equations of motion for the quantum fields are solved using random initial conditions [20–23]. This method has the additional advantage that it is possible to include interactions between the fields. Interestingly, in at least one case[2] the results of a calculation using this method are similar in nature to those obtained in [5] by solving the semiclassical backreaction equations. So even though the solutions to the semiclassical backreaction equations cannot be trusted in detail for preheating, they may still give a good qualitative picture of the initial and intermediate stages of the preheating process.

This is the first application that has been made of the criterion in [12] for the validity of the semiclassical approximation when particle production effects are significant. We think it likely, but cannot be certain, that our results generalize to similar situations. Thus the semiclassical approximation may never be valid, at least in terms of its detailed predictions, when there is a high rate of particle production.

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[2] Compare the top left plot of Fig. 2 with Fig. 1 in [21].

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