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Rotating black holes in Einstein-Dilaton-Gauss-Bonnet gravity with finite coupling

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Among various strong-curvature extensions of General Relativity, Einstein-Dilaton-Gauss-Bonnet gravity stands out as the only nontrivial theory containing quadratic curvature corrections while being free from the Ostrogradsky instability to any order in the coupling parameter. We derive an approximate stationary and axisymmetric black hole solution of this gravitational theory in closed form, which is of fifth order in the black hole spin and of seventh order in the coupling parameter of the theory. This extends previous work that obtained the corrections to the metric only to second order in the spin and at the leading order in the coupling parameter, and allows us to consider values of the coupling parameter close to the maximum permitted by theoretical constraints. We compute some quantities which characterize this solution, such as the dilaton charge, the moment of inertia and the quadrupole moment, and its geodesic structure, including the innermost stable circular orbit and the epicyclic frequencies for massive particles. The latter provides a valuable tool to test General Relativity against strong-curvature corrections through observations of the electromagnetic spectrum of accreting black holes.

I. INTRODUCTION

Observational and experimental tests of General Relativity (GR) [1] have mostly probed the weak-field/slow-motion regimes of the theory, while a number of strong-field, relativistic GR predictions still remain elusive and difficult to verify [2, 3]. Furthermore, a series of long lasting problems in Einstein's theory – such as the accelerate expansion of the Universe, dark matter, the nature of curvature singularities and the quest for an ultraviolet completion of GR – have motivated strong efforts to develop extended theories of gravity which would modify GR in its most extreme regimes while conforming with current weak-field observations [4].

Black holes (BHs) are genuine strong-field predictions of GR and have no analog in Newtonian theory. Thus, they are natural candidates to test gravity in the strong-field regime. Future networks of electromagnetic detectors [5–8] and ground-based gravitational-wave detectors [9, 10] will allow us to measure some crucial properties of BHs, such as their shadows, the location of the event horizon and of the innermost stable circular orbit (ISCO). This information will be instrumental to test the *Kerr hypothesis*, according to which all stationary astrophysical BHs are uniquely described by the Kerr family, and are thus characterized by only two parameters: their mass and angular momentum (see e.g. Ref. [11] and references therein).

In recent years, several modified theories of gravity have been proposed. They can be divided in various categories, each one lifting some of the fundamental principles (Lorentz invariance, weak and strong equivalence principles, massless spin-2 mediators, etc...) upon which Einstein's theory is uniquely built [4]. From this and other classifications, it emerges that one of the simplest and best motivated ways to modify GR consists of including a fundamental scalar field which is non-minimally coupled to the metric tensor. In order to modify the strong-curvature regime, it is natural to couple this scalar field, in the gravitational action, to terms quadratic in the curvature tensor. Such couplings can also be interpreted as the first terms in the expansion in all possible curvature invariants, as suggested by low-energy effective string theories [12]. Generally, a quadratic curvature term in the action leads to field equations of third (or higher) order, which are subject to Ostrogradsky's instability [13]. Therefore, these theories should be considered as effective, i.e., truncations of a theory with further terms in the action, which are neglected in the perturbative regime.

It should also be mentioned that quadratic curvature terms are crucial, not only to modify the strong-curvature regime of GR, but also to affect the behavior of stationary BHs; indeed, standard scalar-tensor theories (in which one or more scalar fields are included in the gravitational sector of the action) satisfy the so-called no-hair theorems, i.e., stationary, vacuum BHs are the same as in GR [14–16] (but see [17–19] for possible violations of these theorems). When quadratic curvature terms are included in the action, instead, stationary BH solutions are different.

We shall consider a member of this family of modified gravity theories, Einstein-Dilaton-Gauss-Bonnet (EDGB) theory, in which a scalar field (the dilaton) is coupled to the Gauss-Bonnet invariant [12, 20]:

$$\mathcal{R}^2_{\rm GB} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \qquad (1)$$

in the action. EDGB gravity is one of the best motivated alternatives to GR. Indeed, it is the only theory of gravity with quadratic curvature terms in the action, whose field equations are of second differential order for *any* coupling, and not just in the weak-coupling limit which is assumed in the effective-field-theory approach [4]. As a consequence, EDGB gravity is ghost-free, i.e. it avoids the Ostrogradsky instability [13]. Furthermore, as mentioned above, the higher-curvature coupling – which modifies the strong-curvature regime of gravity – violates the hypothesis of the BH no-hair theorems, so that BH solutions in EDGB gravity are different from those predicted by GR and provide the ideal arena for genuine strong-field tests of the Kerr hypothesis. Finally, the EDGB term naturally arises in low-energy effective string theories [21].

In this work, we construct an analytical, perturbative solution of EDGB theory, which describes a slowly-rotating BH endowed with a scalar field. To this aim, we extend the formalism developed in [22, 23] up to fifth order in the BH (dimensionless) spin parameter $\chi = J/M^2$, where J and M are the angular momentum and the Arnowitt-Deser-Misner mass of the solution, respectively.

Analytical BH solutions of EDGB theory in the small-coupling limit have been investigated in [24, 25], where stationary, spherically-symmetric configurations where found. Approximate, stationary and axisymmetric solutions to linear and quadratic order in the BH spin were obtained in [26] and [27], respectively. Both these works considered a weak-field expansion of the coupling between the scalar field and the Gauss-Bonnet invariant $\mathcal{R}^2_{\rm GB}$ in terms of a dimensionful coupling constant α. Exact numerical solutions were constructed to zeroth [20] and first order [28] in the spin, and also for arbitrary values of the angular momentum [29, 30]. Although exact in α, such solutions are of limited practical use (for instance for Monte Carlo data analysis) because they require a numerical integration for each set of parameters. On the other hand, numerical solutions are necessary in regimes where the slow-spin expansion does not converge and are therefore complementary to our analysis.

Our results extend the study carried out so far. In particular, we go beyond the analysis of Ref. [27], where a BH solution was obtained to second order both in the spin and in the coupling parameter. Indeed, we compute the metric tensor and the scalar field up to $\mathcal{O}\left(\zeta^7, \chi^5\right)$, where $\zeta \equiv \alpha/M^2$ and α is the EDGB coupling constant. We use this expansion to derive the main features of the solution, such as the geometry of the event horizon and of the ergoregion. Furthermore, we study the geodesic structure of this solution, by computing the ISCO and the epicyclic frequencies (see e.g. Refs. [31–33]) consistently with our approximation scheme. We compare these quantities with those obtained in [34], where a numerical solution was derived, which is exact in the coupling parameter (i.e., with no perturbative expansion in ζ) and approximate to linear order in the BH spin. We find relative errors at most of the order of 1% for the maximum value of ζ allowed by theoretical constraints for the existence of BH solutions, $\zeta \lesssim 0.691$ [20], and much smaller for less extreme couplings.

The results of this paper can be useful to devise tests of GR in the strong-field regime through astrophysical observations of BHs. For instance, we have shown [34] that observations of quasi-periodic oscillations of accreting BHs, with the sensitivity of recently proposed large-area X-ray space telescopes (e.g. [6, 7]), allow us to set constraints on the parameter space of EDGB theory, thus probing the strong-field regime of gravity (see also Ref. [35] for a recent study). However, since BH solutions in EDGB theory (for finite α) were only known at first order[1] in the spin parameter χ, in [34] we only considered BHs with very slow rotation rate, for which the deviations from GR are expected to be small.

This paper is organized as follows. In Section II we derive our solution of the EDGB field equations, describing rotating BHs up to $\mathcal{O}\left(\zeta^7, \chi^5\right)$. In Section III we study this solution, computing its geometrical properties, the location of the ISCO, the azimuthal and epicyclic frequencies. We also estimate the accuracy of our approximation in the determination of these quantities and how our results improve on the existing literature. In particular, we discuss how the spin correction to the azimuthal and epicyclic frequencies can affect possible tests of GR based on observations of accreting BHs, such as those discussed in [34]. Finally, in Section IV we draw our conclusions.

II. SPINNING BLACK HOLES IN EINSTEIN-DILATON-GAUSS-BONNET THEORY

In this Section we derive the spacetime metric and scalar field, describing rotating BHs in EDGB theory, up to $\mathcal{O}\left(\zeta^7, \chi^5\right)$.

A. EDGB gravity

Einstein-Dilaton-Gauss-Bonnet theory is defined by the following action [12, 20]:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[R - \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{\alpha e^{\Phi}}{4}\mathcal{R}^2_{\rm GB}\right] \ , \quad (2)$$

where $g < 0$ is the metric determinant, Φ is a scalar field coupled to the Gauss-Bonnet invariant (1) and $\alpha > 0$ is the coupling constant [20]. Since we are interested in BH solutions, in the action above we have neglected matter fields. We use geometric units $G = c = 1$: with this choice, the scalar field Φ is dimensionless and α has the dimensions of a length squared.

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[1] As mentioned above, a solution for finite spin and coupling is only known in numerical form [29, 30], and it is impractical for extensive studies of geodesic properties. However, numerical solutions are necessary to explore the high-spin regime, especially because EDGB BHs can violate the Kerr bound and can have $\chi > 1$ [29].

The field equations of EDGB gravity are found by varying the action (2) with respect to $g_{\mu\nu}$ and Φ:

$$G^\mu{}_\nu = \frac{1}{2}\partial^\mu\Phi\partial_\nu\Phi - \frac{1}{4}g^\mu{}_\nu\partial_\alpha\Phi\partial^\alpha\Phi - \alpha\mathcal{K}^\mu{}_\nu\,, \quad (3)$$

$$\mathcal{S} \equiv \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\Phi) + \frac{\alpha}{4}e^\Phi\mathcal{R}^2_{\rm GB} = 0\,, \quad (4)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor,

$$\mathcal{K}_{\mu\nu} = \frac{1}{8}\left(g_{\mu\rho}g_{\nu\lambda} + g_{\mu\lambda}g_{\nu\rho}\right)\epsilon^{k\lambda\alpha\beta} \times\nabla_\gamma\left(\epsilon^{\rho\gamma\mu\nu}R_{\mu\nu\alpha\beta}\partial_k e^\Phi\right)\,, \quad (5)$$

and $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor, with $\epsilon^{0123} = -(-g)^{-1/2}$. Note that – by virtue of the GB combination entering the action (2) – the equations are of second differential order, and therefore this theory is free from the Ostrogradsky instability [13]. Indeed, EDGB gravity is a particular case [36] of Horndeski gravity – the most general scalar-tensor theory with second-order field equations [37]. This special subcase is the only one known to date in which regular, stationary, asymptotically-flat, hairy BH solutions other than GR ones are found [38]. Furthermore, EDGB gravity can be obtained from the low-energy expansion of the bosonic sector of heterotic string theory [12, 21], in such case the coupling α is related to the string tension.

In order to simplify our notation, in the next sections we shall introduce the modified Einstein tensor $\tilde{G}^\mu{}_\nu = G^\mu{}_\nu - T^\mu{}_\nu$, where

$$T^\mu{}_\nu = \frac{1}{2}\partial^\mu\Phi\partial_\nu\Phi - \frac{1}{4}g^\mu{}_\nu\partial_\alpha\Phi\partial^\alpha\Phi - \alpha\mathcal{K}^\mu{}_\nu\,, \quad (6)$$

is the effective stress-energy tensor for the dilaton.

B. Static BH solutions

Since the EDGB coupling constant has the dimensions of the inverse of the curvature tensor, it is natural to expect that in this theory the strongest deviations from GR will come from physical systems involving high curvature, such as BHs, neutron stars and the early Universe. We focus here on BH solutions and, in particular, on rotating BH geometries that are obtained through a slow-rotation expansion around a static background solution.

The exact BH background solution (first derived in [20]) is described by the static, spherically-symmetric line element

$$ds^2 = -e^{\Gamma(r)}dt^2 + e^{-\Lambda(r)}dr^2 + r^2d\Omega^2\,, \quad (7)$$

and by a spherically-symmetric scalar field, $\Phi = \phi(r)$. The field equations (3)-(4) supplied by the metric ansatz (7) reduce to a set of differential equations for the scalar field and for the functions Γ and Λ. Indeed, Eq. (4) yields

$$\phi'' + \phi'\left(\frac{\Gamma'-\Lambda'}{2} + \frac{2}{r}\right) = \frac{\alpha e^\phi}{2r^2}\left(\Gamma'\Lambda'e^{-\Lambda} + + (1-e^{-\Lambda})\left[\Gamma'' + \frac{\Gamma'}{2}(\Gamma'-\Lambda')\right]\right)\,, \quad (8)$$

while the t-t, r-r and θ-θ components of $\tilde{G}^\mu{}_\nu = 0$ reduce to

$$\left[1 + \frac{\alpha e^\phi}{2r}\phi'(1-3e^{-\Lambda})\right]\Lambda' = \frac{\phi'^2 r}{4} + \frac{1-e^\Lambda}{r} + +\frac{\alpha e^\phi}{r}(1-e^{-\Lambda})(\phi''+\phi'^2)\,, \quad (9)$$

$$\left[1 + \frac{\alpha e^\phi}{2r}\phi'(1-3e^{-\Lambda})\right]\Gamma' = \frac{\phi'^2 r}{4} + \frac{e^\Lambda-1}{r}\,, \quad (10)$$

$$\Gamma'' + \left(\frac{\Gamma'}{2} + \frac{1}{r}\right)(\Gamma'-\Lambda') = -\frac{\phi'^2}{2} + \frac{\alpha e^{\phi-\Lambda}}{r}\times \times\left[\phi'\Gamma'' + \Gamma'(\phi''+\phi'^2) + \frac{\Gamma'\phi'}{2}(\phi'-3\Lambda')\right]\,. \quad (11)$$

Note that Eqs. (9)–(11) are not all independent and that the r-r component can be solved analytically, yielding

$$e^\Lambda = \frac{-\beta + \sqrt{\beta^2 - 4\gamma}}{2}\,, \quad (12)$$

where

$$\beta = \frac{\phi'^2 r^2}{4} - 1 - \Gamma'\left(r + \frac{e^\phi\phi'}{2}\right)\,, \quad \gamma = \frac{3}{2}\Gamma'\phi' e^\phi\,. \quad (13)$$

The remaining two independent equations can be written as

$$\phi'' = -\frac{d_1}{d} \quad , \quad \Gamma'' = -\frac{d_2}{d}\,, \quad (14)$$

where the radial functions d, d_1 and d_2 are given in Appendix A of [20]. The Arnowitt-Deser-Misner mass M and the dilatonic charge $\mathcal{D}$ can be read off the asymptotic behavior of the metric and of the dilaton field,

$$g_{tt} \to -1 + 2M/r + \ldots \quad (15)$$

$$\phi \to \mathcal{D}/r + \ldots \quad (16)$$

It turns out that for each value of M there is only one solution describing a static BH. In other words, the scalar field is a "secondary hair": the dilatonic charge $\mathcal{D}$ is not an independent parameter but is determined in terms of the BH mass M.

The field equations are invariant under the rescaling $\phi \to \phi + \hat{\phi}$ and $r \to re^{\hat{\phi}/2}$ (or equivalently $M \to Me^{\hat{\phi}/2}$ and $\mathcal{D} \to \mathcal{D}e^{\hat{\phi}/2}$) where $\hat{\phi}$ is a constant. We fix this freedom by requiring that $\phi \to 0$ at spatial infinity; this means that at infinity, the Gauss-Bonnet invariant appears in the action (2) multiplied by the constant $\alpha/4$.

As noted in [20] static BH solutions in EDGB gravity exist only if

$$e^{\phi_{\rm h}} \leq \frac{r_{\rm h}}{\alpha\sqrt{6}} \,, \tag{17}$$

where $\phi_{\rm h}$ is the value of the scalar field computed at the horizon $r_{\rm h}$. As shown in [28], by requiring that $\phi \to 0$ at spatial infinity, Eq. (17) can be recast in the form

$$0 \leq \frac{\alpha}{M^2} \lesssim 0.691 \,. \tag{18}$$

Thus, smaller BHs would correspond to a more stringent bound on α.

Presently, the tightest observational bound on the EDGB coupling parameter (obtained by the orbital decay of X-ray binaries) is $\alpha \lesssim 47M_\odot^2$ [39]. As discussed in [34], this upper bound is weaker than the theoretical constraint (17) for BHs with $M \lesssim 8.2M_\odot$. For such BHs, the entire range (18) is phenomenologically allowed.

Solutions of Eqs. (14) has been solved numerically in Ref. [20], while an analytical, static BH solution has been derived to second order in α/M^2 in Refs. [24, 25].

C. Spinning BH solutions

To describe slowly-rotating BH solutions we extend the approach developed by Hartle [22, 23], in which spin corrections to the static solutions are introduced within a perturbative framework. The procedure described in this section is generic and can be applied also to other theories and different spinning solutions.

Let us start with the most general solution for a stationary, axially-symmetric spacetime[2] which is given by

$$\begin{aligned} ds^2 = -H^2dt^2 + Q^2dr^2 + r^2K^2[d\theta^2 + \\ + \sin^2\theta(d\varphi - Ldt)^2] \,, \end{aligned} \tag{19}$$

where H, Q, K and L are functions of (r,θ). The ansatz (19) can be expanded perturbatively in the spin around the static solution

$$\begin{aligned} ds^2 =& -e^{\Gamma}[1+2h(r,\theta)]dt^2 + e^{-\Lambda}[1+2m(r,\theta)]dr^2 \\ &+ r^2[1+2k(r,\theta)][d\theta^2+\sin^2\theta(d\varphi-\hat{\omega}(r,\theta)dt)^2] \,, \end{aligned} \tag{20}$$

where the functions $\hat{\omega}$, h, m, and k can be expanded in a complete basis of orthogonal functions accordingly to their symmetry properties as

$$\hat{\omega} = \sum_{n=1,3,5,\ldots}^{N_\chi - q} \sum_{l=1,3,5,\ldots}^{n} \chi^n \omega_l^{(n)}(r) S_l(\theta) \,, \tag{21}$$

$$h = \sum_{n=2,4,\ldots}^{N_\chi - p} \sum_{l=0,2,4,\ldots}^{n} \chi^n h_l^{(n)}(r) P_l(\cos\theta) \,, \tag{22}$$

$$m = \sum_{n=2,4,\ldots}^{N_\chi - p} \sum_{l=0,2,4,\ldots}^{n} \chi^n m_l^{(n)}(r) P_l(\cos\theta) \,, \tag{23}$$

$$k = \sum_{n=2,4,\ldots}^{N_\chi - p} \sum_{l=0,2,4,\ldots}^{n} \chi^n k_l^{(n)}(r) P_l(\cos\theta) \,, \tag{24}$$

(25)

where P_l are the Legendre polynomials, $S_l = -\frac{1}{\sin\theta}\frac{dP_l(\cos\theta)}{d\theta}$ (note that $P_0 = S_1 = 1$), and p (resp. q) is zero when the order N_χ of the spin expansion is even (resp. odd) whereas $p = 1$ (resp. $q = 1$) otherwise. The radial functions $\left(\omega_l^{(n)}, h_l^{(n)}, m_l^{(n)}, k_l^{(n)}\right)$ are of the order $\mathcal{O}(\chi^n)$. Note that, since the metric (20) is invariant under the rescaling $r \to f(r)$, the functions $k_0^{(n)}(r)$ can be set to zero without loss of generality [22, 23].

Because the dilaton field transforms as a scalar under rotation, we expand it as

$$\Phi(r) = \phi(r) + \sum_{n=2,4,\ldots}^{N_\chi - p} \sum_{l=0,2,4,\ldots}^{n} \chi^n \phi_l^{(n)}(r) P_l(\cos\theta) \tag{26}$$

where ϕ is the background static solution and the radial functions $\phi_l^{(n)}$ are of the order $\mathcal{O}(\chi^n)$.

1. $\mathcal{O}(\chi)$ corrections

Rotating BH solutions in EDGB gravity have been investigated to linear order in the spin angular momentum in Refs. [26, 28]. At this order the metric (20) reduces to the static case with a nonvanishing gravitomagnetic term described by $\hat{\omega}(r,\theta) = \omega_1^{(1)} \equiv \omega(r)$ [see Eq. (21)]:

$$\begin{aligned} ds^2 = -e^{\Gamma(r)}dt^2 + e^{-\Lambda(r)}dr^2 + r^2d\Omega^2 \ + \\ -2r^2\omega(r)\sin^2\theta dt d\varphi \,. \end{aligned} \tag{27}$$

From $\tilde{G}^t{}_\varphi = 0$, it is easy to show that ω satisfies the second-order equation [28]:

$$\begin{aligned} \omega'' + &\left[2r^2e^\Lambda - 2\alpha r\phi' e^\phi\right]^{-1} \\ &\left(-\alpha e^\phi\left[2\phi'' r + \phi'(6 + 2\phi' r - \Gamma' r - 3\Lambda' r)\right]\right. \\ &\left.-e^\Lambda r\left[-8 + r(\Gamma' + \Lambda')\right]\right)\omega' = 0 \,, \end{aligned} \tag{28}$$

where the coefficient of ω' depends on the nonspinning solution. The BH angular momentum can be read off the asymptotic behavior of the gyromagnetic term,

$$\omega(r) \to \frac{2J}{r^3} \,, \tag{29}$$

[2] We also assume equatorial symmetry and invariance under $(t \to -t, \varphi \to -\varphi)$.

at large distance.

2. $\mathcal{O}(\chi^n)$ corrections: $n \geq 2$ and even

Replacing the metric ansatz (20) into the field equations and using the decomposition in Legendre polynomials, a set of ordinary differential equations can be obtained, at each order in the spin expansion. The equations are inhomogeneous with source terms given by the lower-order functions.

At each given order $n \geq 2$ with even n, the equations are found from $E_1 \equiv \tilde{G}_{tt} = 0$, $E_2 \equiv \tilde{G}_{rr} = 0$, $E_3 \equiv \tilde{G}_{\theta\theta} + (\sin\theta)^{-2}\tilde{G}_{\varphi\varphi} = 0$, and $E_4 \equiv \mathcal{S}$ for the scalar equation (4), each contracted with a Legendre polynomial,

$$\int_0^\pi d\theta \sin\theta P_l(\cos\theta) E_i(r,\theta) = 0\,, \tag{30}$$

where $i = 1, 2, 3, 4$ and $l = 0, 2, 4, ..., n$. Due to the symmetry properties of the field equations and of the background, this procedure gives a set of purely radial, inhomogeneous, ordinary differential equations for $h_l^{(n)}$, $m_l^{(n)}$, $k_l^{(n)}$ and $\phi_l^{(n)}$ with $l = 0, 2, 4, .., n$ (we recall that $k_0^{(n)} = 0$).

3. $\mathcal{O}(\chi^n)$ corrections: $n \geq 3$ and odd

Similarly, at a given order $n \geq 3$ (with odd n) in the spin expansion, a set of radial equations for the gravitomagnetic terms can be obtained by contracting $\tilde{G}_{t\varphi} = 0$ with the (axisymmetric) vector spherical harmonics, namely

$$\int_0^\pi d\theta \sin\theta \frac{dP_l(\cos\theta)}{d\cos\theta}\tilde{G}_{t\varphi} = 0 \tag{31}$$

with $l = 1, 3, 5, ..., n$. Again, this procedure yields a set of purely radial, inhomogeneous, ordinary differential equations for $\omega_l^{(n)}$ with $l = 1, 3, 5, .., n$.

D. Small-coupling approximation

The set of equations presented above provides a full description of the BH solution at any perturbative order in the spin, but generic (i.e., nonperturbative) in the EDGB coupling. However, such equations are cumbersome and it is impractical to solve them numerically. More importantly, the theoretical constraint (18) shows that the dimensionless coupling parameter has to be smaller than unity. This motivates a small-coupling approximation [26–28], in which the field equations are solved perturbatively in $\alpha/M^2 \ll 1$ to some desired order. Actually, because we are interested in the regime $\alpha/M^2 \lesssim 1$ (the maximum value[3] of this parameter is 0.691), we shall compute terms of relatively high order in this expansion.

To simplify the notation, we introduce the dimensionless parameter

$$\zeta = \frac{\alpha}{M^2}\,. \tag{32}$$

As a result of our approximation scheme, we expand all quantities, such as the metric functions and the scalar field, in terms of the two parameters ζ, χ. For example,

$$g_{\mu\nu} = g_{\mu\nu}^{(0,0)} + \sum_{i=1}^{N_\chi}\sum_{j=1}^{N_\zeta}\chi^i\zeta^j g_{\mu\nu}^{(i,j)}\,, \tag{33}$$

where the double superscript $^{(i,j)}$ denotes the order of the expansion in the BH spin parameter and in the EDGB coupling parameter, respectively; $g_{\mu\nu}^{(0,0)}$ is the Schwarzschild metric. In practice, using the spin decomposition previously discussed, we simply expand the set of radial variables $\vec{f} = \{\Gamma, \Lambda, \phi, \omega_l^{(n)}, m_l^{(n)}, h_l^{(n)}, k_l^{(n)}, \phi_l^{(n)}\}$ as

$$f = \sum_{j=0}^{N_\zeta}\zeta^j f^{(j)}\,, \tag{34}$$

where $f^{(j)}$ are radial functions which do not depend on the coupling parameter ζ. By replacing these expressions into the field equations derived in Sec. II C, and solving them order by order in ζ, we obtain the desired expansion for the metric tensor and the scalar field. Remarkably, this procedure yields an analytical solution. We compute the explicit solution up to $\mathcal{O}(\zeta^7, \chi^5)$, but the procedure can be straightforwardly extended to higher order both in ζ and in χ.

Solving the differential equations at each order in χ and ζ yields some integration constants, which are uniquely fixed by requiring that:

1. the metric is asymptotically flat, and the scalar field vanishes at spatial infinity;
2. there exists an event horizon, where perturbations are regular;
3. the *physical* mass and angular momentum of the BH are given by M and $M^2\chi$, as measured by an observer at spatial infinity. In particular, the bare mass of the $\mathcal{O}(\zeta^0)$ solution acquires some corrections to each order in ζ, which are reabsorbed in the physical mass M.

[3] Note that the constraint $\zeta \equiv \alpha/M^2 \lesssim 0.691$ is valid for nonspinning solutions at finite coupling. The precise value of the upper bound can be modified for large rotation rates [29]. Indeed, as discussed later in this section, the BH mass acquires $\mathcal{O}(\chi^2)$ corrections which can be reabsorbed in the definition of the mass. Nonetheless, the bound on α/M^2 emerges only from the nonperturbative BH solutions and does not appear in the small-coupling approximation (to any order in ζ).

We note that only one of the two integration constants appearing in the solution of the scalar field at each order in ζ is fixed by requiring regularity outside the horizon, while the metric is regular for each value of the remaining constants. Although this is not evident in the Schwarzschild coordinates adopted here, it can be nonetheless checked by computing some curvature invariants. However, the remaining integration constants can all be reabsorbed in the definitions of the physical mass and angular momentum, so that the final solution truncated at a given order depends only on two parameters, as in the Kerr case.

The explicit expressions of the metric tensor and of the scalar field up to $\mathcal{O}(\zeta^7, \chi^5)$ are quite long, and are available in a MATHEMATICA® notebook provided in the Supplemental Material. For completeness, the explicit Kerr metric to $\mathcal{O}(\chi^5)$ in the Hartle-Thorne coordinates is given in Appendix A.

III. GEOMETRICAL AND GEODESIC PROPERTIES OF THE SOLUTION

We here study the properties of the analytical solution we have derived. To this aim, we compute some geometrical and geodesic quantities which characterize the spinning EDGB BH solution to $\mathcal{O}(\zeta^7, \chi^5)$.

A. Event horizon, ergosphere, intrinsic curvature and dilaton charge

The event horizon is given by the largest root $r = r_{\rm h}$ of the equation (cf. e.g. [40]) $g_{\phi\phi}g_{tt} - g_{t\phi}^2 = 0$, which yields the following power expansion in terms of ζ and χ:

$$\frac{r_{\rm h}}{M} = \sum_{i=0}^{7} \zeta^i (a_i + b_i\chi + c_i\chi^2 + d_i\chi^3 + e_i\chi^4 + f_i\chi^5) \ . \tag{35}$$

where the coefficients $(a_i, b_i, c_i, d_i, e_i, f_i)$ are listed[4] in Table II of Appendix B. As in the Kerr case, the horizon radius $r_{\rm h}$ does not depend on the angular coordinates. Nonetheless, its intrinsic geometry – as computed by considering a spatial section $dt = 0$ at $r = r_{\rm h}$ – is nonspherical. Indeed

$$ds^2_{t={\rm const}, r=r_{\rm h}} = g_{\theta\theta}(r = r_{\rm h}, \theta) d\Omega^2 \ , \tag{36}$$

and since $g_{\theta\theta}$ explicitly depends on θ, the intrinsic geometry is non spherical. For the line element (36), the curvature radius is

$$R_{\rm intr} = \frac{2}{g_{\theta\theta}} - \frac{\cot\theta\, g'_{\theta\theta}}{g_{\theta\theta}^2} + \frac{g'^2_{\theta\theta}}{g_{\theta\theta}^3} - \frac{g''_{\theta\theta}}{g_{\theta\theta}^2} \ , \tag{37}$$

where (only in the above formula) a prime denotes differentiation with respect to θ, and for our solution is

$$M^2 R_{\rm intr} = \sum_{i=0}^{7} \zeta^i [l_i + \chi^2(m_i + n_i\cos 2\theta) + \\ + \chi^4(p_i + q_i\cos 2\theta + u_i\cos 4\theta)] \ ; \tag{38}$$

this is constant only when $\zeta = 0 = \chi$. Hereafter we adopt the same expansion of Eqs. (35) and (38) for other physical quantities. The numerical values of the coefficients of these expansions are given in Appendix B, whereas their exact form is provided in the supplementary MATHEMATICA® notebook.

The location of the ergosphere is given by the largest root of $g_{tt} = 0$:

$$\frac{r_{\rm ergo}}{M} = \sum_{i=0}^{7} \zeta^i [l_i + \chi^2(m_i + n_i\cos 2\theta) + \\ + \chi^4(p_i + q_i\cos 2\theta + u_i\cos 4\theta)] \ , \tag{39}$$

where the only nonvanishing spin corrections correspond to even powers of χ.

Finally, the dilaton charge $\mathcal{D}$ can be extracted from the leading-order, large-distance behavior of the dilaton field, $\Phi \to \mathcal{D}/r$, and reads

$$\frac{\mathcal{D}}{M} = \sum_{i=1}^{7} \zeta^i (a_i + b_i\chi + c_i\chi^2 + e_i\chi^4) \ , \tag{40}$$

where the coefficients d_i and f_i identically vanish.

B. Moment of inertia

The moment of inertia is defined as $I = J/\Omega_{\rm h}$, where J is the BH angular momentum and $\Omega_{\rm h}$ is the angular velocity at the horizon of locally nonrotating observers,

$$\Omega_{\rm h} = -\lim_{r\to r_{\rm h}} \frac{g_{t\varphi}}{g_{\varphi\varphi}} \ . \tag{41}$$

In our case we obtain

[4] For the sake of clarity, the coefficients shown in the appendix will be rounded to some numerical factors. The exact expressions are available in a supplemental MATHEMATICA® notebook.

$$
\begin{aligned}
\frac{I}{M^3} = \ & 4 - 0.2625000\zeta^2 - 0.1721966\zeta^3 - 0.1458764\zeta^4 - 0.1409996\zeta^5 - 0.1474998\zeta^6 - 0.1627298\zeta^7 + \\
& - \chi^2[1 - 0.2359276\zeta^2 - 0.2175544\zeta^3 - 0.2431079\zeta^4 - 0.2776072\zeta^5 - 0.3283860\zeta^6 + \\
& - 0.3984877\zeta^7] + \chi^4[0.25 - 0.1170266\zeta^2 - 0.04956483\zeta^3 + 0.01732049\zeta^4 + 0.09842336\zeta^5 + \\
& + 0.2055222\zeta^6 + 0.3503737\zeta^7] \ , \qquad (42)
\end{aligned}
$$

where again the only nonvanishing spin corrections correspond to even powers of χ.

C. Quadrupole moment

According to the BH no-hair theorems, the quadrupole moment (as well as the higher-order multipole moments) of any regular, stationary, asymptotically-flat BH in GR is uniquely determined by its mass M and angular momentum J [41–43]. A deformed Kerr geometry as the one just discussed, does not necessarily possess this unique no-hair property. Since the dilaton charge of this solution is not an independent parameter, the multipole moments of an EDGB BH can all be written in terms of M and J but the relations among them will change with respect to Kerr. The ζ-corrections to the BH quadrupole moment are thus relevant to test the Kerr hypothesis [2–4].

To compute the quadrupole moment, we follow the general approach described in [44], in which the multipole moments of an asymptotically-flat geometry are read off the asymptotic behavior of the metric. This approach requires the metric to be expressed in asymptotically-Cartesian and mass-centered (ACMC) coordinates. In particular, in order to extract the quadrupole moment the metric has to be ACMC-2, i.e. g_{tt} and g_{ij} $(i, j \neq t)$ should not contain any angular dependence up to $\mathcal{O}(1/r^2)$ terms. In our case, the coordinate transformation that enforces such property is

$$
\begin{aligned}
r &\to r + \frac{\chi^2 M^2}{2r}\left[1 + \frac{M}{r} - \frac{2M^2}{r^2} + \frac{M(6M-r)}{r^2}\cos^2\theta\right] , \\
\theta &\to \theta + \frac{\chi^2 M^3}{r^3}\sin\theta\cos\theta ,
\end{aligned}
$$

and does not involve the EDGB coupling ζ nor spin corrections higher than second order. In the new ACMC-2 coordinates, the g_{tt} component reads:

$$
\begin{aligned}
g_{tt} = -1 + \frac{2M}{r} + \frac{\sqrt{3}}{2r^3}[Q_{20}Y^{20} + \\
+ (l = 0 \text{ pole})] + \mathcal{O}\left(\frac{M^4}{r^4}\right) , \qquad (43)
\end{aligned}
$$

where Y_{20} is the $(l = 2, m = 0)$ spherical harmonic and Q_{20} is the $m = 0$ mass quadrupole moment. From our explicit solution we obtain, to order $\mathcal{O}(\zeta^7, \chi^5)$,

$$
\begin{aligned}
Q_{20} = -\sqrt{\frac{64\pi}{15}}\chi^2 M^3 \big[& \big(1 + 0.1061619\zeta^2 + 0.07524246\zeta^3 + 0.07459416\zeta^4 + 0.07756926\zeta^5 \ + \\
& + 0.08553316\zeta^6 + 0.09805643\zeta^7\big) - \chi^2\zeta^2\big(0.0308519 + 0.0408857\zeta + 0.0638894\zeta^2 + \\
& + 0.0866408\zeta^3 + 0.116314\zeta^4 + 0.154763\zeta^5\big)\big] \ . \qquad (44)
\end{aligned}
$$

Interestingly, the $\mathcal{O}(\chi^4)$ corrections to the quadrupole moment are proportional to ζ^2, i.e. they vanish in the GR limit. For $\zeta \sim 0.4$, the $\mathcal{O}(\zeta^2, \chi^2)$ correction to the quadrupole moment relative to the Kerr case is about 1.7%, whereas the $\mathcal{O}(\zeta^3, \chi^2)$ correction is approximately 0.5%. Finally, for $\zeta \sim 0.4$ and $\chi \sim 0.6$, the $\mathcal{O}(\zeta^2, \chi^4)$ correction is approximately 0.1%.

We remark that the quadrupole moment of spinning EDGB BHs has been computed numerically in [30]. Our solution has the advantage of giving this quantity in analytical form.

D. Geodesics and epicyclic frequencies

We shall now consider time-like geodesics in the slowly-rotating EDGB BH spacetime. We assume a minimally-coupled test particle and restrict to equatorial orbits, for which $\theta = \pi/2$ and $d\theta = 0$. We firstly compute stable circular orbits; then, by considering small perturbations of these orbits we derive the epicyclic frequencies ω_r and ω_θ (see e.g. Refs. [31–33]). For a stationary-axisymmetric spacetime, the ISCO corresponds to the radius at which the second derivative of the effective potential,

$$V(r) = \frac{1}{g_{rr}}\left(\frac{\mathcal{E}^2 g_{\varphi\varphi} + 2\mathcal{E}L g_{t\varphi} + L^2 g_{tt}}{g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}} - 1\right) , \quad (45)$$

vanishes. Here, we have introduced the particle specific energy and angular momentum, $\mathcal{E}$ and L [45], given by

$$\mathcal{E} = -\frac{g_{tt} + g_{t\varphi}\omega_\varphi}{\sqrt{-g_{tt} - 2g_{t\varphi}\omega_\varphi - g_{\varphi\varphi}\omega_\varphi^2}} , \quad (46)$$

$$L = \frac{g_{t\varphi} + g_{\varphi\varphi}\omega_\varphi}{\sqrt{-g_{tt} - 2g_{t\varphi}\omega_\varphi - g_{\varphi\varphi}\omega_\varphi^2}} , \quad (47)$$

where ω_φ is the azimuthal angular velocity

$$\omega_\varphi = \frac{-g_{t\varphi,r} + \sqrt{g_{t\varphi,r}^2 - g_{tt,r}g_{\varphi\varphi,r}}}{g_{\varphi\varphi,r}} . \quad (48)$$

Solving $V''(r) = 0$ order by order, we obtain the ISCO radius up to $O(\zeta^7, \chi^5)$:

$$\frac{r_{\rm ISCO}}{M} = \sum_{i=0}^{7} \zeta^i (a_i + b_i\chi + c_i\chi^2 + d_i\chi^3 + + e_i\chi^4 + f_i\chi^5) . \quad (49)$$

Orbits with radius $r > r_{\rm ISCO}$ are stable. Under a small perturbation, a massive particle orbiting in one of these stable, circular orbits oscillates with radial and vertical frequencies given by [31–33]

$$\omega_r^2 = \frac{(g_{tt} + \omega_\varphi g_{t\phi})^2}{2g_{rr}} \left.\frac{\partial^2 \mathcal{U}}{\partial r^2}\right|_l , \quad (50)$$

$$\omega_\theta^2 = \frac{(g_{tt} + \omega_\varphi g_{t\phi})^2}{2g_{\theta\theta}} \left.\frac{\partial^2 \mathcal{U}}{\partial \theta^2}\right|_l . \quad (51)$$

These are the epicyclic frequencies. Here $\mathcal{U} = g^{tt} - 2lg^{t\varphi} + l^2 g^{\varphi\varphi}$, with $l = L/\mathcal{E}$ being the ratio between the particle angular momentum and its energy [33]. The full expressions for ω_r, ω_θ, as well as for ω_φ as functions of (r, M, χ, ζ) and up to order $\mathcal{O}(\zeta^7, \chi^5)$ are available in a MATHEMATICA® notebook provided as supplemental material. We explicitly show here their values at the ISCO:

$$M\omega_\varphi\big|_{\rm ISCO} = \sum_{i=0}^{7} \zeta^i (a_i + b_i\chi + c_i\chi^2 + d_i\chi^3 + + e_i\chi^4 + f_i\chi^5) , \quad (52)$$

$$M\omega_\theta\big|_{\rm ISCO} = \sum_{i=0}^{7} \zeta^i (a_i + b_i\chi + c_i\chi^2 + d_i\chi^3 + + e_i\chi^4 + f_i\chi^5) , \quad (53)$$

whereas $\omega_r\big|_{\rm ISCO} = 0$ as in the Kerr case.

E. Comparison with previous results

As a check, we can compare our results with those derived in [27], where the metric of the EDGB spinning BH was found to $\mathcal{O}(\zeta^2, \chi^2)$ in Boyer- Lindquist coordinates. A direct comparison of the metric coefficients is not possible, since the BH solutions have been derived on different charts. However, we can overcome this problem by computing the Kretschmann invariant $\mathcal{K} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, and evaluating it at a specific point. From our solution truncated at $\mathcal{O}(\zeta^2, \chi^2)$, we get

$$\begin{aligned}\mathcal{K}(r,\theta) = \ & 48\frac{M^2}{r^6} + \frac{144M^2}{r^8}\left[(1 - 8\cos^2\theta) + \frac{M}{r}\sin^2\theta + 2\frac{M^2}{r^2}(3\cos^2\theta - 1)\right] - \frac{\zeta^2}{r^4}\left[\frac{2M^3}{r^3} + \frac{M^4}{r^4} + 144\frac{M^5}{r^5} + \right.\\ & \left. + 14\frac{M^6}{r^6} + \frac{128}{5}\frac{M^7}{r^7} - 1680\frac{M^8}{r^8}\right] + \frac{\zeta^2\chi^2}{r^4}\left[\frac{M^3}{r^3} + \frac{54431}{1750}\frac{M^4}{r^4} + \frac{12846}{175}\frac{M^5}{r^5} + \frac{77047}{1225}\frac{M^6}{r^6} + \right.\\ & - \frac{348909}{350}\frac{M^7}{r^7} - \frac{304938}{175}\frac{M^8}{r^8} - \frac{28023}{35}\frac{M^9}{r^9} + \frac{359468}{35}\frac{M^{10}}{r^{10}} + \frac{53848}{5}\frac{M^{11}}{r^{11}} - 21984\frac{M^{12}}{r^{12}} + \\ & + \left(-\frac{80334}{875}\frac{M^4}{r^4} - \frac{19638}{175}\frac{M^5}{r^5} - \frac{234816}{1225}\frac{M^6}{r^6} + \frac{1448877}{350}\frac{M^7}{r^7} + \frac{711114}{175}\frac{M^8}{r^8} + \frac{92679}{35}\frac{M^9}{r^9} + \right.\\ & \left.\left. - \frac{2168052}{35}\frac{M^{10}}{r^{10}} - \frac{59544}{5}\frac{M^{11}}{r^{11}} + 65952\frac{M^{12}}{r^{12}}\right)\cos^2\theta\right] . \end{aligned} \quad (54)$$

Replacing the explicit expression for r_h in Eq. (35), we find that on the horizon

$$\mathcal{K}(r_{\rm h}, \pi/2) = \frac{3}{4M^4} + \frac{9\chi^2}{8M^4} + + \frac{\zeta^2}{M^4}\left[\frac{327}{1280} + \frac{404023\chi^2}{784000}\right] . \quad (55)$$

This results coincides with the Kretschmann scalar derived in [27] and evaluated at the event horizon on the equatorial plane in Boyer-Lindquist coordinates. Finally, we have verified the agreement between the expression for $M\omega_\varphi$ at the ISCO – which is also a gauge invariant quantity – obtained from the metric derived in Ref. [27], and the same expression obtained truncating the expression in Eq. (52) to $\mathcal{O}(\zeta^2, \chi^2)$.

F. Accuracy of the expansion

In this section we estimate the accuracy of our perturbative scheme. In particular, we estimate the truncation error arising from neglecting $\mathcal{O}(\zeta^8)$ terms in the expansion. To this aim, we compare our results with those obtained in Refs. [20, 28, 34], where a solution for slowly-rotating BHs in the EDGB theory has been derived at first order in χ, and is "exact" in ζ (i.e., with no perturbative expansion in ζ). To be consistent, we neglect terms of the order $\mathcal{O}(\chi^2)$ in Eqns. (35), (49), (52) and (53).

In Fig. 1, we compare the dilatonic charge computed in [20, 28] non-perturbatively in ζ, with the expression in Eq. (40), for $\chi = 0$, truncated at various orders of ζ. As expected, for $\zeta \ll 0.2$ higher-order corrections are negligible, but they contribute significantly as $\zeta \to 0.691$. To $\mathcal{O}(\zeta^7)$, the deviation from the "exact" result is about 1% for $\zeta \sim 0.6$ and is as large as 5% for $\zeta \sim 0.691$. In contrast, the $\mathcal{O}(\zeta^2)$ truncation differs by about 30% as ζ increases to its maximum value.

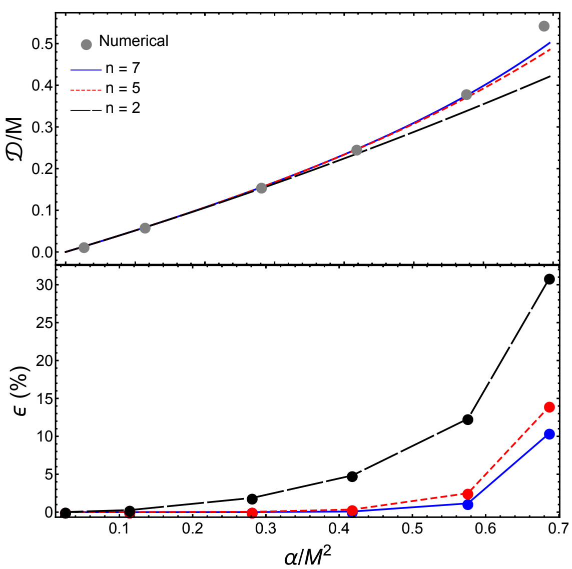

Figure 1. (color online). Top panel: dilatonic charge as a function of $\zeta = \alpha/M^2$ computed in [20, 28] (gray markers) compared with the expression in Eq. (40) truncated at $\mathcal{O}(\zeta^2)$, $\mathcal{O}(\zeta^5)$ and $\mathcal{O}(\zeta^7)$ and for vanishing spin. Bottom panel: relative discrepancy between the perturbative and nonperturbative estimates of the dilatonic charge, as a function of α for various truncations.

Likewise, for the set of quantities $f = \{r_h, r_{\rm ISCO}, M\omega_\varphi|_{\rm ISCO}, M\omega_\theta|_{\rm ISCO}\}$, we compute the relative error

$$\epsilon_n = \frac{f^{(n,1)}}{\bar{f}} - 1\,, \qquad (56)$$

where $\bar{f}$ represents the "exact" quantity (non-perturbative in ζ) [28, 34]. We estimate ϵ_n at various orders of approximation in ζ, for different values of the BH spin parameter. In Table I we show the largest relative errors obtained for all considered quantities, at different levels of accuracy, in the limiting case $\zeta = 0.691$ (left) and $\zeta = 0.576$ (right). We remark that $\zeta = 0.691$ is an extreme situation, since for slightly smaller values of ζ (i.e., $\zeta = 0.576$) the deviations are much smaller.

Fig. 1 and Table I show that our analytical solution approaches the "exact" solution of [20, 28] as the value of n increases, i.e. when we consider more and more terms in the small-coupling expansion. In particular, for $r_{\rm ISCO}, M\omega_\varphi|_{\rm ISCO}, M\omega_\theta|_{\rm ISCO}$ the relative errors (for $n = 7$) are always smaller than 1% for any value of ζ, even for the maximum allowed value, $\zeta \sim 0.691$. For the horizon, the threshold above which $\epsilon_{n=7} > 0.01$ is lower, namely $\zeta \sim 0.55$.

G. Are spin corrections important?

The analysis presented in the Section III F shows that the metric expanded in powers of ζ, which we derived in a closed, analytic form, is a very good approximation of the "exact" numerical result: it reproduces the most relevant geodesic quantities within 1% for the maximum value $\zeta \sim 0.691$ and within 0.3% for $\zeta \sim 0.576$. It is therefore justified to adopt such higher-order perturbative expansion as a starting point to devise strong-field tests of gravity.

In Ref. [34] we studied the deviations of the azimuthal and epicyclic frequencies in a slowly-rotating EDGB BH to first order in the spin. However, deviations from the Kerr case should increase with higher values of the spin. Indeed, as the spin increases, the ISCO gets closer to the horizon, and therefore observables from orbits near the ISCO probe a region of higher curvature, where the deviations should be larger.

In Fig. 2, we confirm this claim by showing the deviations of the horizon and ISCO locations and, most importantly, of the azimuthal frequency ω_φ and angular epicyclic frequency ω_θ at the ISCO, relative to their values computed using the Kerr metric approximated at $\mathcal{O}(\chi^5)$, and as functions of ζ and χ. For a fixed value of ζ, the percentual errors are systematically larger as the spin increases, reaching up to 7% for $\chi = 0.6$. This large value of the spin parameter should be considered as an extrapolation. Indeed, our results neglects terms of the order $\mathcal{O}(\chi^6)$, which introduce corrections of roughly 5% for $\chi \sim 0.6$.

Our perturbative solution is also useful to estimate the convergence properties of the expansion. From the coefficients listed in Table II, we can compute the ratio of

	χ	$\epsilon_{n=2}(\%)$	$\epsilon_{n=4}(\%)$	$\epsilon_{n=6}(\%)$	$\epsilon_{n=7}(\%)$
$r_{\rm h}/M$	0	5.90	4.45	3.72	3.48
$r_{\rm ISCO}/M$	0	1.00	0.58	0.43	0.39
	0.05	1.11	0.65	0.49	0.44
	0.10	1.23	0.72	0.54	0.49
$M\omega_\varphi\vert_{\rm ISCO}$	0	1.36	0.79	0.59	0.53
	0.05	1.56	0.95	0.72	0.66
	0.10	1.88	1.22	0.98	0.91
$M\omega_\theta\vert_{\rm ISCO}$	0.05	1.53	0.92	0.69	0.63
	0.10	1.78	1.13	0.88	0.81

	χ	$\epsilon_{n=2}(\%)$	$\epsilon_{n=4}(\%)$	$\epsilon_{n=6}(\%)$	$\epsilon_{n=7}(\%)$
$r_{\rm h}/M$	0	1.33	0.52	0.24	0.17
$r_{\rm ISCO}/M$	0	0.32	0.093	0.038	0.026
	0.05	0.37	0.12	0.055	0.042
	0.1	0.42	0.14	0.074	0.059
$M\omega_\varphi\vert_{\rm ISCO}$	0	0.44	0.13	0.053	0.036
	0.05	0.56	0.23	0.14	0.13
	0.1	0.80	0.44	0.35	0.33
$M\omega_\theta\vert_{\rm ISCO}$	0.05	0.54	0.20	0.12	0.10
	0.1	0.71	0.36	0.27	0.25

Table I. Left: the relative error ϵ_n [cf. Eq. (56)] between different quantities listed in the first column, computed through the solution derived in [28], nonperturbative in ζ, and compared with our perturbative results truncated at $\mathcal{O}(\zeta^n)$. We consider the maximum value of ζ allowed for BH solutions in EDGB gravity, $\zeta = 0.691$, and different values of the BH spin parameter. Right: Same for $\zeta = 0.576$.

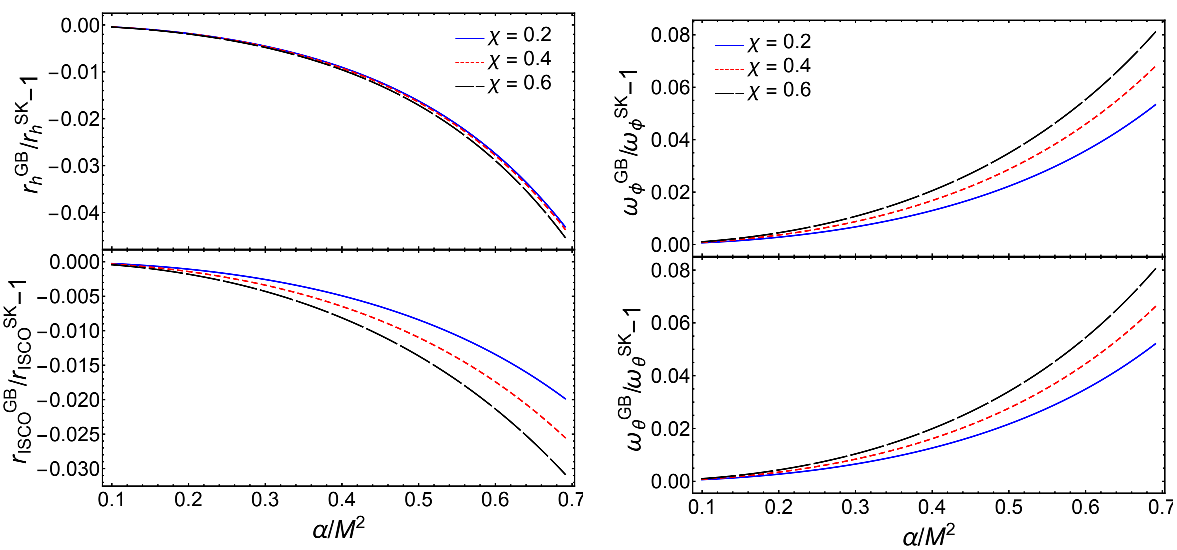

Figure 2. (color online). (Left panel) The percentual error in the horizon radius $r_{\rm h}$ and in the ISCO $r_{\rm ISCO}$ for our perturbative result [Eqs. (35) and (49)], relative to the Kerr solution expanded at $\mathcal{O}(\chi^5)$, as a function of the EDGB coupling parameter α. We consider three values of the BH spin parameter $\chi = (0.2, 0.4, 0.6)$. (Right panel) Same as the left panel, but for the epicyclic frequencies Eqns. (52)-(53) evaluated at the ISCO.

the $\mathcal{O}(\chi^n)$ and $\mathcal{O}(\chi^{n-1})$ corrections for a given quantity. For the angular epicyclic frequency $M\omega_\theta|_{\rm ISCO}$, this ratio is roughly $(0.41, 0.39, 0.37)$ for $n = (3, 4, 5)$, in the extreme case $\chi \sim 0.5$ and $\zeta \sim 0.5$. Therefore, the fifth-order spin correction is about 20% of the quadratic one. Other quantities show a similar behavior. Clearly, the convergence improves for smaller values of χ, whereas it is almost insensitive to the values of the EDGB coupling ζ.

Finally, we note that the percentual error of the horizon location is almost insensitive to the spin, whereas the epicyclic frequencies are much more sensitive to this parameter.

IV. CONCLUDING REMARKS

With the advent of precision measurements of the spectrum of accreting compact objects, it is of utmost importance to devise tests of gravity that use these measurements to probe the geometry near compact objects. To this aim, we have considered a specific modified theory – namely EDGB gravity– as a case-study. This theory has some appealing theoretical features, for example it is free from instabilities and circumvents the BH no-hair theorems. Furthermore, it modifies GR precisely in the strong-curvature regions, while passing all current solar-system and binary-pulsar tests [4].

Spinning BHs in this theory have been studied in the past, both numerically [20, 28–30] and analytically [24–

27] to leading order in the coupling parameter. Numerical solutions have the advantage of being general, but they are impractical for some applications, for example for Monte Carlo simulations spanning a high-dimensional parameter space. Approximate analytical solutions can be very useful for this purpose, although they are usually perturbative.

Here, as a first step to develop precision tests of gravity based on geodesic motion near stationary BHs, we have constructed an analytical, approximate solution of EDGB theory describing a deformed Kerr BH. The solution is valid to fifth order in the spin and to seventh order in the coupling parameter, thus extending previous solutions that are valid only to quadratic order in the coupling and the spin. With the analytical solution at hand, it is straightforward to compute various quantities of interest. We presented the corrections to the horizon and ergoregion location, moment of inertia and quadrupole moment relative to the Kerr metric, as well as the charge of the dilaton field that characterizes this solution. For a given value of the coupling ζ, the solution depends only on the mass M and on the dimensionless angular momentum χ, while the dilaton charge is fixed in terms of M. In addition, we have computed some geodesic quantities, namely the ISCO location and the azimuthal and epicyclic frequencies as functions of M, χ, ζ and the orbital radius r.

When truncated at first order in the BH spin, our solution reproduces the most relevant geodesic quantities obtained in [20, 28] with a numerical approach, within 1% for the maximum value $\zeta \sim 0.691$, and within 0.3% for $\zeta \sim 0.576$. The accuracy of the solution grows dramatically for smaller values of the coupling. These results indicate that our perturbative solution is a good approximation of the exact numerical results.

In a future publication we will extend the analysis of Ref. [34], which studied how observations of quasi-periodic oscillations in the spectrum of accreting BHs can be used to constrain EDGB theory in the strong-field regime, to larger values of the BH spin. An similar analysis can also performed for other tests based on stationary BHs, for example those based on the broadened iron line (e.g. [46, 47]) or the continuum fitting method (e.g. [48], see also [35]). On the technical side, our perturbative approach is generic: it can be applied to any order in ζ and in χ, as well as to other modified-gravity theories and different spinning solutions.

ACKNOWLEDGMENTS

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Appendix A: Kerr metric in the Hartle-Thorne approximation

In this appendix we show the form of the BH solution in the Hartle-Thorne approximation for $\alpha = 0$, i.e. the slowly-rotating Kerr BH in GR, up to the fifth order in the BH spin angular momentum.

$$g_{tt} = \; 1-\frac{2M}{r}+J^2\left[\left(\frac{2}{Mr^3}-\frac{2}{r^4}-\frac{4M}{r^5}\right)P_2(\cos\theta)+\frac{2\cos 2\theta}{r^4}\right]+J^4\left\{\frac{2}{5M^2r^6}-\frac{12M}{5r^9}+\frac{11}{5Mr^7}+\right.$$
$$+\frac{6}{5r^8}+\left[\frac{146}{7r^8}-\frac{16}{7M^2r^6}+\frac{44M}{7r^9}+\frac{46}{7Mr^7}-\cos(2\theta)\left(\frac{4}{Mr^7}+\frac{8}{r^8}\right)\right]P_2(\cos\theta)+$$
$$-\left(\frac{8}{5Mr^7}+\frac{24}{5r^8}\right)\cos(2\theta)+\sin^2(\theta)\left(\frac{8}{3M^2r^6}-\frac{8}{15Mr^7}-\frac{48}{5r^8}\right)S_3(\theta)+$$
$$\left.+\left(\frac{66}{35M^2r^6}-\frac{2}{M^3r^5}-\frac{192M}{35r^9}+\frac{316}{35Mr^7}-\frac{48}{7r^8}\right)P_4(\cos\theta)\right\}, \tag{A1}$$

$$g_{rr} = -\frac{r}{r-2M}+\frac{2J^2}{r^2(r-2M)}\left[\frac{1}{rM}\frac{(r-5M)}{(r-2M)}P_2(\cos\theta)+1\right]+\frac{J^4}{(2M-r)^3}\left[\frac{152M^2}{5r^7}+\frac{9}{5M^2r^3}+\right.$$
$$-\frac{264M}{5r^6}-\frac{59}{5Mr^4}+\frac{196}{5r^5}+\left(-\frac{1464M^2}{7r^7}+\frac{52}{7M^2r^3}+\frac{1496M}{7r^6}-\frac{242}{7Mr^4}-\frac{106}{7r^5}\right)P_2(\cos\theta)+$$
$$\left.+\left(\frac{2}{M^3r^2}+\frac{2112M^2}{35r^7}-\frac{358}{35M^2r^3}-\frac{4512M}{35r^6}-\frac{8}{35Mr^4}+\frac{2616}{35r^5}\right)P_4(\cos\theta)\right], \tag{A2}$$

$$g_{\theta\theta} = -r^2+J^2\left(\frac{2}{Mr}+\frac{4}{r^2}\right)P_2(\cos\theta)-J^4\left[\left(\frac{4}{7M^2r^4}+\frac{26}{7Mr^5}+\frac{68}{7r^6}\right)P_2(\cos\theta)+\left(\frac{2}{M^3r^3}+\right.\right.$$
$$\left.\left.+\frac{162}{35M^2r^4}+\frac{24}{35Mr^5}+\frac{24}{35r^6}\right)P_4(\cos\theta)\right], \tag{A3}$$

$$g_{\varphi\varphi} = g_{\theta\theta}\sin\theta^2 , \tag{A4}$$

$$g_{t\varphi} = \frac{2J}{r}-J^3\left[\frac{4}{5Mr^4}+\frac{12}{5r^5}+\left(\frac{4}{Mr^4}+\frac{8}{r^5}\right)P_2(\cos\theta)+\left(\frac{2}{3M^2r^3}-\frac{2}{15Mr^4}-\frac{12}{5r^5}\right)S_3(\theta)\right]+$$
$$+J^5\left\{\frac{24}{5Mr^8}+\frac{72}{5r^9}-\frac{6}{7M^3r^6}-\frac{73}{35M^2r^7}-\left(\frac{2}{15M^2r^7}+\frac{2}{Mr^8}+\frac{28}{5r^9}\right)S_3(\theta)+\left[\frac{96}{35M^2r^7}+\right.\right.$$
$$\left.+\left(\frac{4}{3M^3r^6}+\frac{12}{5M^2r^7}-\frac{16}{3Mr^8}-\frac{48}{5r^9}\right)S_3(\theta)+\frac{108}{7Mr^8}+\frac{1016}{35r^9}\right]P_2(\cos\theta)+\left(\frac{4}{M^3r^6}+\right.$$
$$\left.\left.+\frac{324}{35M^2r^7}+\frac{48}{35Mr^8}+\frac{48}{35r^9}\right)P_4(\cos\theta)+\frac{2}{5}\left(\frac{1}{M^4r^5}+\frac{1}{7M^3r^6}-\frac{44}{7M^2r^7}+\frac{8}{r^9}\right)S_5(\theta)\right\}. \tag{A5}$$

Appendix B: Coefficients of the small coupling

In the following table we show the numerical coefficients of various analytic expansions presented in the previous sections, as function of the BH spin angular momentum and the EDGB coupling parameter. For sake of clarity all the coefficients are rounded to the seventh digit. The exact expressions are available in a supplemental MATHEMATICA® notebook.

	$r_{\rm h}/M$	$r_{\rm ISCO}/M$	$\mathcal{D}/M$	$M\omega_\varphi\vert_{\rm ISCO}$	$M\omega_\theta\vert_{\rm ISCO}$
a_0	2.000000	6.000000	0	0.06804138	0.06804138
b_0	0	-3.265986	0	0.05092593	0.04166667
c_0	-0.2500000	-0.2962963	0	0.03717075	0.02488551
d_0	0	-0.1436429	0	0.02797068	0.01521776
e_0	-0.07812500	-0.08957762	0	0.02176680	0.009504151
f_0	0	-0.06362468	0	0.01744794	0.005997155
a_1	0	0	0.5000000	0	0
b_1	0	0	0	0	0
c_1	0	0	-0.1250000	0	0
d_1	0	0	0	0	0
e_1	0	0	-0.06250000	0	0
f_1	0	0	0	0	0
a_2	-0.07656250	-0.1047904	0.1520833	0.001563316	0.001563316
b_2	0	-0.1201586	0	0.003733697	0.003267414
c_2	-0.005273438	0.01442503	-0.06562500	0.003913488	0.002948313
d_2	0	0.03108340	0	0.003160772	0.002041599
e_2	0.0007317631	0.02534504	-0.01282676	0.002252579	0.001211371
f_2	0	0.03275054	0	0.001261845	0.0005010288
a_3	-0.05482722	-0.05057329	0.09658358	0.0007597788	0.0007597788
b_3	0	-0.06564501	0	0.001933365	0.001708284
c_3	0.003374719	0.02695641	-0.06503722	0.001759287	0.001360011
d_3	0	0.03707053	0	0.0009886223	0.0006954496
e_3	-0.003268537	0.008889195	0.006444151	0.0005021134	0.0004182817
f_3	0	0.01125766	0	0.0001173107	0.0002713766
a_4	-0.05139096	-0.03780985	0.08178788	0.0005969829	0.0005969829
b_4	0	-0.05431288	0	0.001640950	0.001457307
c_4	0.007351713	0.03876245	-0.06717236	0.001340086	0.001049576
d_4	0	0.05336676	0	0.0003245650	0.0002579998
e_4	-0.01308955	-0.0002670172	0.02912691	-0.00008270490	0.0001598391
f_4	0	0.00005147144	0	-0.0002712206	0.0002346154
a_5	-0.05266569	-0.03321722	0.07886477	0.0005272127	0.0005272127
b_5	0	-0.04890924	0	0.001475785	0.001313209
c_5	0.01223578	0.05011869	-0.07508136	0.0009815575	0.0007719423
d_5	0	0.06775197	0	-0.0003132155	-0.0001796837
e_5	-0.02668707	-0.01513785	0.05662227	-0.0005664351	-0.00003872048
f_5	0	-0.02079280	0	-0.0003557189	0.0003967294
a_6	-0.05753945	-0.03250101	0.08245910	0.0005165825	0.0005165825
b_6	0	-0.04824270	0	0.001458090	0.001298530
c_6	0.01812429	0.06363000	-0.08788829	0.0007567590	0.0005928172
d_6	0	0.08594476	0	-0.0008980528	-0.0005936325
e_6	-0.04690661	-0.03465520	0.09290842	-0.0009679387	-0.0001794779
f_6	0	-0.04866939	0	-0.0002465314	0.0007018829
a_7	-0.06565095	-0.03416370	0.09098999	0.0005425110	0.0005425110
b_7	0	-0.05069519	0	0.001533246	0.001365759
c_7	0.02592898	0.08041305	-0.1064184	0.0005835871	0.0004504112
d_7	0	0.1086434	0	-0.001525555	-0.001044296
e_7	-0.07619661	-0.06177540	0.1427116	-0.001328310	-0.0002654697
f_7	0	-0.08762124	0	0.0001137773	0.001223565

	$r_{\rm ergo}/M$	$M^2 R_{\rm intr}$
l_0	2.000000	0.5000000
m_0	0	0
n_0	-0.06640625	-0.05859375
p_0	-0.2500000	-0.3750000
q_0	-0.04687500	-0.2343750
u_0	0.03515625	0.1054688
l_1	0	0
m_1	0	0
n_1	0	0
p_1	0	0
q_1	0	0
u_1	0	0
l_2	-0.07656250	0.03828125
m_2	0.02239583	-0.02182674
n_2	-0.02390784	-0.01729976
p_2	-0.02766927	-0.1164568
q_2	0.003249614	-0.02236320
u_2	0.02138998	0.1302572
l_3	-0.05482722	0.02741361
m_3	0.02660141	-0.02038983
n_3	-0.02726616	0.001655395
p_3	-0.02322669	-0.08694771
q_3	0.006734437	0.01982431
u_3	0.01726319	0.1306188
l_4	-0.05139096	0.02789366
m_4	0.03391131	-0.02622298
n_4	-0.04523654	0.01648181
p_4	-0.02655960	-0.1007545
q_4	0.01024171	0.05758840
u_4	0.02190527	0.1906275
l_5	-0.05266569	0.02948113
m_5	0.04318465	-0.03204816
n_5	-0.06780117	0.03771938
p_5	-0.03094887	-0.1122069
q_5	0.01536541	0.1033644
u_5	0.02574869	0.2495574
l_6	-0.05753945	0.03296015
m_6	0.05574249	-0.04017469
n_6	-0.1009268	0.06678447
p_6	-0.03761820	-0.1310279
q_6	0.02256511	0.1634876
u_6	0.03145506	0.3323836
l_7	-0.06565095	0.03820390
m_7	0.07278610	-0.05103945
n_7	-0.1481975	0.1073399
p_7	-0.04685712	-0.1565866
q_7	0.03308656	0.2435265
u_7	0.03891437	0.4418871

Table II. Numerical values of the coefficients of the expressions (35),(40),(49), (52)-(53) (left panel), and of the ergosphere and intrinsic curvature radius (39)], (38) (right panel).

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