Probing CP violation systematically in differential distributions

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Probing CP violation systematically in differential distributions

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We revisit the topic of *triple-product* asymmetries which probe CP violation through differential distributions. We construct distributions with well-defined discrete symmetry properties and characterize the asymmetries formed upon them. It is stressed that the simplest asymmetries may not be optimal. We explore systematic generalizations having limited reliance on the process dynamics and phase-space parametrization. They exploit larger fractions of the information contained in differential distributions and may lead to increased sensitivities to CP violation. Our detailed treatment of the case of spinless four-body decays paves the way for further experimental studies.

I. INTRODUCTION

The fully differential rates of some multibody meson decays are being more and more accurately measured. In the search for new sources of CP violation, such processes present several advantages. They often feature a rich variety of interfering contributions from which differences in CP-violating—*weak*—phases could manifest themselves. In addition, the multiplication of measurable independent four-vectors permits the construction of so-called *triple-product* observables. Those have a couple of interesting characteristics. Unlike total rate asymmetries between CP-conjugate processes, their sensitivity to small differences in CP-violating phases is not conditioned by the presence of CP-conserving—*strong* or *unitary*—phase differences. They can also be measured using *untagged* samples in which CP-conjugate processes need not be distinguished, provided their fractions are equal.

In this paper, we explore the variety of possible *triple-product* observables. The ever-increasing amount of data collected allows finer details of the differential distributions for which they are proxies to become measurable. We stress that the most common asymmetries may not be the most sensitive ones, due to cancellations in phase-space integrals. As much as possible, we would like to abstract our treatment from the particular dynamics of the studied process. In many multibody decays, only phenomenological descriptions of various degrees of accuracy are achieved. They may not capture all the fine details of interfering contributions which could reveal CP violation. A systematic procedure that is less likely to miss unpredicted forms of CP violation is therefore desirable. Although we will mostly focus, for concreteness, on four-body meson decays involving spinless particles, our discussion has a wider range of application.

A. Differential CP violation

Let us consider two transitions of amplitudes $\mathcal{M}(\{\lambda_i, p_i\})$ and $\bar{\mathcal{M}}(\{\lambda_{\bar{\imath}}, p_{\bar{\imath}}\})$. They involve an equal number particles respectively labeled by i and $\bar{\imath}$, with helicities $\lambda_{i,\bar{\imath}}$ and four-momenta $p_{i,\bar{\imath}}$. We would like to perform a comparison of those two amplitudes phase-space point by phase-space point so we take $\lambda_{\bar{\imath}} = \lambda_i$ as well as $p_{\bar{\imath}} = p_i$.

If these two processes are CP conjugate of each other, with $\bar{\imath} = \mathrm{CP}[i]$, CP violation at any phase-space point takes the form of a difference between the squared moduli of

$$\mathcal{M}(\{\lambda_i, p_i\}) \quad \text{and} \quad \bar{\mathcal{M}}(\{\lambda_i, \bar{p}_i\})$$

where $\bar{p} \equiv \mathrm{P}[p]$ is the parity conjugate of the momentum p. Testing CP conservation phase-space point by phase-space point thus implies a comparison of the differential rates of two processes involving CP-conjugate particles of identical helicities but opposite three-momenta.

It reveals useful to define an operator, called *motion reversal* and denoted here by $\hat{\mathrm{T}}$, that reverts both momentum and spin three-vectors [1, 2]. Its action on helicities and momenta is thus identical to that of CP and it can be viewed as the unitary component of the antiunitary time-reversal operator T. It is therefore sometimes called *naive T*. In general, the amplitudes above can then be decomposed into two pieces that are respectively $\hat{\mathrm{T}}$-even and $\hat{\mathrm{T}}$-odd [3]:

$$\begin{aligned}\mathcal{M}(\{\lambda_i, p_i\}) &= \mathcal{M}_e(\{\lambda_i, p_i\}) + \mathcal{M}_o(\{\lambda_i, p_i\}),\\ \bar{\mathcal{M}}(\{\lambda_i, \bar{p}_i\}) &= \bar{\mathcal{M}}_e(\{\lambda_i, \bar{p}_i\}) + \bar{\mathcal{M}}_o(\{\lambda_i, \bar{p}_i\})\\ &= \bar{\mathcal{M}}_e(\{\lambda_i, p_i\}) - \bar{\mathcal{M}}_o(\{\lambda_i, p_i\}).\end{aligned}$$

Those two terms can receive several contributions whose absorptive parts [4, 5] take the form of CP-even phases δ. One can then write

$$\begin{aligned}\mathcal{M}_e(\{\lambda_i, p_i\}) &= a_e^j\ e^{i(\delta_e^j + \varphi_e^j)},\\ \bar{\mathcal{M}}_e(\{\lambda_i, p_i\}) &= a_e^j\ e^{i(\delta_e^j - \varphi_e^j)},\\ \mathcal{M}_o(\{\lambda_i, p_i\}) &= a_o^k\ e^{i(\delta_o^k + [\varphi_o^k + \pi/2])},\\ \bar{\mathcal{M}}_o(\{\lambda_i, p_i\}) &= a_o^k\ e^{i(\delta_o^k - [\varphi_o^k + \pi/2])},\end{aligned} \tag{1}$$

with implicit summation over the j, k indices, and real $a_{e,o}^{j,k}$, $\delta_{e,o}^{j,k}$, $\varphi_{e,o}^{j,k}$ functions of the helicities and momenta

$\{\lambda_i, p_i\}$. The above conventions imply that all CP violation is encoded in the CP-odd phases $\varphi_{e,o}^{j,k}$. When they vanish,

$$\bar{\mathcal{M}}_e(\{\lambda_i, p_i\}) = +\mathcal{M}_e(\{\lambda_i, p_i\}),$$
$$\bar{\mathcal{M}}_o(\{\lambda_i, p_i\}) = -\mathcal{M}_o(\{\lambda_i, p_i\}),$$

so that the CP-conjugate rates are identical, phase-space point by phase-space point. As the physical amplitude is defined up to an overall phase, a departure from zero for differences in these $\varphi_{e,o}^{j,k}$ is what we are after.

B. CP violation without CP-even phases

The $\hat{\mathrm{T}}$-transformed differential rates are obviously accessible experimentally since the measured momenta can be artificially reversed. For processes involving only scalars in their initial and final states, $\hat{\mathrm{T}}$ is actually equivalent to parity conjugation P. The measured differential rates of any pair of CP-conjugate processes can therefore be decomposed into four pieces of definite $\hat{\mathrm{T}}$ and CP transformation properties:

$$\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-}^{\text{even}}_{\text{odd}}}^{\hat{\mathrm{T}}\text{-}^{\text{even}}_{\text{odd}}} \equiv \frac{\mathbb{I}\pm\hat{\mathrm{T}}}{2}\,\frac{\mathbb{I}\pm\mathrm{CP}}{2}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi} \tag{2}$$

with the shorthand $\Phi \equiv \{\lambda_i, p_i\}$.

For simplicity, let us assume there are respectively two and one contribution(s) to the $\hat{\mathrm{T}}$-even and $\hat{\mathrm{T}}$-odd parts of the amplitude in the process under scrutiny:

$$\mathcal{M}(\{\lambda_i, p_i\}) = a_e^1 e^{i(\delta_a^1+\varphi_a^1)} + a_e^2\, e^{i(\delta_a^2+\varphi_a^2)} + i a_o^1\ e^{i(\delta_o^1+\varphi_o^1)},$$
$$\bar{\mathcal{M}}(\{\lambda_i, \bar{p}_i\}) = a_e^1 e^{i(\delta_a^1-\varphi_a^1)} + a_e^2\, e^{i(\delta_a^2-\varphi_a^2)} + i a_o^1\ e^{i(\delta_o^1-\varphi_o^1)}.$$

All functions of the phase space are evaluated at $\{\lambda_i, p_i\}$. Note the convention of Eq. (1) causes the appearance of a factor of i in front of the $\hat{\mathrm{T}}$-odd term. Up to a flux factor, the squared modulus of this expression and of its CP conjugate provides us with the differential rates which can be decomposed as prescribed in Eq. (2):

$$\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-even}}^{\hat{\mathrm{T}}\text{-even}} \propto\ a_e^1\, a_e^1 + a_e^2\, a_e^2 + a_o^1\, a_o^1 + 2\, a_e^1\, a_e^2\ \cos(\delta_e^1-\delta_e^2)\cos(\varphi_e^1-\varphi_e^2),$$

$$\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-even}}^{\hat{\mathrm{T}}\text{-odd}} \propto\ 2\, a_e^1\, a_o^1\ \sin(\delta_e^1-\delta_o^1)\ \cos(\varphi_e^1-\varphi_o^1) + 2\, a_e^2\, a_o^1\ \sin(\delta_e^2-\delta_o^1)\ \cos(\varphi_e^2-\varphi_o^2),$$

$$\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-even}} \propto -2\, a_e^1\, a_e^2\ \sin(\delta_e^1-\delta_e^2)\ \sin(\varphi_e^1-\varphi_e^2),$$

$$\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-odd}} \propto\ 2\, a_e^1\, a_o^1\ \cos(\delta_e^1-\delta_o^1)\sin(\varphi_e^1-\varphi_o^1) + 2\, a_e^2\, a_o^1\ \cos(\delta_e^2-\delta_o^1)\sin(\varphi_e^2-\varphi_o^1).$$

The last two expression above vanish in the CP limit. There are thus two distinct kinds of CP-violating differential rates [6]: the presence of the $\hat{\mathrm{T}}$-even one requires non-vanishing differences in CP-even phases δ while the $\hat{\mathrm{T}}$-odd–CP-odd does not. This can be understood as, in the absence of absorptive part to the amplitude, $\hat{\mathrm{T}}$ is equivalent to T so that CPT conservation imposes any CP-odd quantity to be also $\hat{\mathrm{T}}$ odd [7].

On the other hand, the $\hat{\mathrm{T}}$-odd–CP-even piece of the differential rate could be used to isolate relatively small differences in CP-even phases δ, in the absence of CP-odd phase φ. It can thus serve to better understand final-state interactions.

C. Untagged samples

Another remarkable characteristic of the $\hat{\mathrm{T}}$-odd-CP-odd part of the differential rate is that it can be measured with samples which contain an equal number of events from CP-conjugated processes. It can also be evaluated in the decay of self-conjugate states like the Z and h bosons, or any Majorana fermion. This can be understood by rewriting the $\hat{\mathrm{T}}$-odd–CP-odd differential rate defined in Eq. (2) as

$$\frac{\mathbb{I}-\hat{\mathrm{T}}}{2}\left(\frac{\mathbb{I}+\mathrm{CP}\hat{\mathrm{T}}}{2}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right),$$

using the fact that $\hat{\mathrm{T}}$ is an involution: $\hat{\mathrm{T}}^2 = \mathbb{I}$. It only involves $\mathrm{d}(\Gamma+\bar{\Gamma})/\mathrm{d}\Phi$ evaluated at the phase-space point $\{\lambda_i, p_i\}$ and at its $\hat{\mathrm{T}}$ conjugate $\{\lambda_i, \bar{p}_i\}$.

Other discrete symmetry operators can be introduced. In particular, let us denote a permutation of the external particles as $\mathrm{E}\{i_1, i_2, \ldots, i_n\} = \{\mathrm{E}[i_1], \mathrm{E}[i_2], \ldots, \mathrm{E}[i_n]\}$. For transitions involving a self-conjugate subset of external particles, there is an especially relevant permutation E^* that takes each particle in the subset to its CP conjugate. For example, $\mathrm{E}^*\{K^+, K^-, \pi^+, \pi^-\} = \{K^-, K^+, \pi^-, \pi^+\}$.

A part of the differential rate that is odd under a permutation E can also be used to test CP conservation with samples containing an equal number of events from CP-conjugate processes:

$$\frac{\mathbb{I}-\mathrm{E}}{2}\left(\frac{\mathbb{I}+\mathrm{CP}\,\mathrm{E}}{2}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right).$$

However, resorting to such samples is only desirable when a subset of the particles involved is self conjugate. Experimentally, the tagging that discriminates between the CP-conjugate processes then comes with an efficiency cost. Importantly, without tagging, what is then actually measured is

$$\frac{\mathbb{I}+\mathrm{CP}\hat{\mathrm{T}}\,\mathrm{E}^*}{2}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}.$$

In an untagged sample, one can therefore measure two CP-odd differential rates that are either $\hat{\mathrm{T}}$-odd–E^*-even or $\hat{\mathrm{T}}$-even–E^*-odd:

$$\frac{\mathbb{I}\pm\hat{\mathrm{T}}}{2}\;\frac{\mathbb{I}\mp\mathrm{E}^*}{2}\;\frac{\mathbb{I}-\mathrm{CP}}{2}\;\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi} = \frac{\mathbb{I}\pm\hat{\mathrm{T}}}{2}\;\frac{\mathbb{I}\mp\mathrm{E}^*}{2}\left(\frac{\mathbb{I}+\mathrm{CP}\hat{\mathrm{T}}\,\mathrm{E}^*}{2}\;\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right).$$

Some asymmetries of either kind were measured by the LHCb Collaboration in its study of the $B_s^0 \to K^+K^-\pi^+\pi^-$ decay with an untagged sample [8] (see discussion in Section II.H).

On the contrary, the differential rates of identical $\hat{\mathrm{T}}$ and E^* parities are CP even in an untagged sample:

$$\frac{\mathbb{I}\pm\hat{\mathrm{T}}}{2}\;\frac{\mathbb{I}\pm\mathrm{E}^*}{2}\left(\frac{\mathbb{I}+\mathrm{CP}\hat{\mathrm{T}}\,\mathrm{E}^*}{2}\;\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right) = \frac{\mathbb{I}\pm\hat{\mathrm{T}}}{2}\;\frac{\mathbb{I}\pm\mathrm{E}^*}{2}\;\frac{\mathbb{I}+\mathrm{CP}}{2}\;\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}.$$

As in the tagged sample case, they provide an handle on the CP-even phases.

D. Integrated observables

No phase-space integration or spin averaging is in principle required to test for the existence of CP-violating phases. Such procedures are only applied because of practical constraints like finite statistics. The total rate asymmetry is constructed upon the $\hat{\mathrm{T}}$-even–CP-odd differential rate

$$\int \mathrm{d}\Phi\;\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-even}}. \tag{3}$$

A second family of observables can be obtained from integrals of its $\hat{\mathrm{T}}$-odd–CP-odd homologue

$$\int \mathrm{d}\Phi\;f(\Phi)\;\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-odd}} \tag{4}$$

with some $\hat{\mathrm{T}}$-odd function $f(\Phi)$ without which the phase-space integral would vanish. Similarly, any $\hat{\mathrm{T}}$-even function $g(\Phi)$ could be inserted in the $\hat{\mathrm{T}}$-even–CP-odd integral to construct observables sharing the properties of the total rate asymmetry.

As a product of a $\hat{\mathrm{T}}$-odd kinematic function with a $\hat{\mathrm{T}}$-odd–CP-even differential rate, the observables of Eq. (4) are $\hat{\mathrm{T}}$-even and CP-odd but have not definite T transformation properties.

E. $\hat{\mathrm{T}}$-oddity and triple products

There are two tensors available to construct Lorentz invariants from spin and momenta four-vectors. The metric $g_{\mu\nu}$ leads to $\hat{\mathrm{T}}$-even contractions like invariant masses, and the completely antisymmetric $\epsilon_{\mu\nu\rho\sigma}$ produces $\hat{\mathrm{T}}$-odd combinations of four-vectors.

Dot products and antisymmetric contractions of four-momenta and Pauli-Lubański spin vectors (respectively denoted by p and w) have definite P parities. The P-even combinations are:

$$p_1\cdot p_2,\quad w_1\cdot w_2,$$
$$\epsilon_{\mu\nu\rho\sigma}\;p_1^\mu p_2^\nu p_3^\sigma w_4^\rho,\quad \text{and}\quad \epsilon_{\mu\nu\rho\sigma}\;p_1^\mu w_2^\nu w_3^\sigma w_4^\rho,$$

while the P-odd ones are:

$$p_1\cdot w_2,$$
$$\epsilon_{\mu\nu\rho\sigma}\;p_1^\mu p_2^\nu p_3^\sigma p_4^\rho,\quad \epsilon_{\mu\nu\rho\sigma}\;p_1^\mu p_2^\nu w_3^\sigma w_4^\rho,\quad \epsilon_{\mu\nu\rho\sigma}\;w_1^\mu w_2^\nu w_3^\sigma w_4^\rho.$$

The sensitivities to discrete symmetry violation of observables having definite P and $\hat{\mathrm{T}}$ transformation properties, in the presence or absence of absorptive parts in the amplitude, are listed on p. 519 of Ref. [6].

The completely antisymmetric Lorentz structure can originate directly from Lagrangian couplings like $i\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$, or arise in the presence of chiral fermions, since $\gamma^5 = \frac{i}{4!}\,\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$. Because it is completely antisymmetric, however, a necessary condition for the presence of a $\hat{\mathrm{T}}$-odd part $\mathcal{M}_o$ in an amplitude is the availability of four independent and distinguishable four-vectors. In a process involving scalars or particles of unmeasured spins, at least five external momenta are therefore required.

In a reference frame where $a^\mu = (a^0, \mathbf{0})$, the completely antisymmetric combination of four four-vectors $\epsilon_{\mu\nu\rho\sigma}\;a^\mu\,b^\nu\,c^\rho\,d^\sigma$ reduces to a $a^0\;\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{d})$ scalar *triple product* (for $\epsilon_{0123}\equiv+1$). The observables constructed from the $\hat{\mathrm{T}}$-odd parts of the differential rate are therefore customarily called *triple-product* asymmetries. A significant amount of effort, both theoretical and experimental has been devoted to their study. A *triple-product* asymmetry has been measured in $K_L\to\pi^+\pi^-e^+e^-$ [9] and applications are also found in heavy meson [10–26] [27–31], baryons [32, 33], top [34], Z [35], Higgs [36–39], and beyond-the-standard-model [40] physics.

F. Asymmetries

The simplest *up-down triple-product* asymmetries are based on the sign of one of the constructible *triple product* (see Eq. (4))

$$f(\Phi) = \mathrm{sign}\{\,\epsilon_{\mu\nu\rho\sigma}\;a^\mu b^\nu c^\rho d^\sigma\,\}. \tag{5}$$

The usual quantities defined in the literature

$$\overleftrightarrow{A}_{\hat{\mathrm{T}}} \equiv \frac{\int\mathrm{d}\Phi\;f(\Phi)\;\left[\;\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-even}}^{\hat{\mathrm{T}}\text{-odd}} \overleftrightarrow{\pm} \left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-odd}}\;\right]}{\int\mathrm{d}\Phi\;\left[\;\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-even}}^{\hat{\mathrm{T}}\text{-even}} \overleftrightarrow{\pm} \left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-even}}\;\right]}$$

are ratios of integrated $\hat{\mathrm{T}}$-odd and $\hat{\mathrm{T}}$-even differential rates and have no definite CP transformation properties. The converse could only be argued when differences of

CP-even phases are proven vanishing. In the notations of Section I.B,

$$A_{\hat{\mathrm{T}}} \propto 2\, a_e^j\, a_o^k\, \sin\left[(\delta_e^j - \delta_e^k) + (\varphi_e^j - \varphi_o^k)\right]$$

then actually becomes a probe for small differences in CP-odd phases φ. On the contrary,

$$\mathcal{A}_{\hat{\mathrm{T}}}^{\mathrm{CP}} \equiv \frac{1}{2}(A_{\hat{\mathrm{T}}} - \bar{A}_{\hat{\mathrm{T}}}),$$

is always CP-odd. $\bar{A}_{\mathrm{P}}$ is occasionally defined as the CP$\hat{\mathrm{T}}$-conjugate of $A_{\hat{\mathrm{T}}}$ and has then a sign opposite to $\bar{A}_{\hat{\mathrm{T}}} \equiv \mathrm{CP}[A_{\hat{\mathrm{T}}}]$ defined here. With this alternative convention, $\mathcal{A}_{\hat{\mathrm{T}}}^{\mathrm{CP}}$ becomes a sum. Other asymmetries were for instance listed in Ref. [26]. Instead of $\mathcal{A}_{\hat{\mathrm{T}}}^{\mathrm{CP}}$, one may consider

$$\widetilde{\mathcal{A}}_{\hat{\mathrm{T}}}^{\mathrm{CP}} \equiv \frac{\int \mathrm{d}\Phi\ f(\Phi)\ \left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-odd}}}{\int \mathrm{d}\Phi\ \left.\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\right|_{\text{CP-even}}^{\hat{\mathrm{T}}\text{-even}}}.$$

This choice corresponds to the more common one when the total rate asymmetry of Eq. (3) vanishes. Using $\widetilde{\mathcal{A}}_{\hat{\mathrm{T}}}^{\mathrm{CP}}$, the $\hat{\mathrm{T}}$-even–CP-odd and $\hat{\mathrm{T}}$-odd–CP-odd families of observables can be kept independent. Uncertainties in the relative abundance of the two CP-conjugate initial states can however make the use of $\mathcal{A}_{\hat{\mathrm{T}}}^{\mathrm{CP}}$ experimentally preferable.

G. Dilutions and $f(\Phi)$ sets

The 'sign' function used in Eq. (5) is not the only possible weight function $f(\Phi)$ that could be used in phase-space integrals of $\hat{\mathrm{T}}$-odd differential rates. This choice confines, experimentally, to counting events in regions of phase space. Moreover, the adjunction of any $\hat{\mathrm{T}}$-even factor in the 'sign' argument besides the antisymmetric contraction $\epsilon_{\mu\nu\rho\sigma}\ a^\mu b^\nu c^\rho d^\sigma$ would obviously yield other potentially interesting observables. Using a basis of $\hat{\mathrm{T}}$-odd functions on Φ, it is also possible to decompose the $\hat{\mathrm{T}}$-odd–CP-odd differential rate in moments (see Refs. [41–43] about the method of moments). As in Ref. [28], a binning of the phase space could also be defined and a chi-squared test carried out to assess local departures from zero in the $\hat{\mathrm{T}}$-odd–CP-odd piece of the differential rate. This would correspond to choosing, for the $f(\Phi)$s, a set of characteristic functions that evaluate to 1 in one bin and vanish elsewhere. At least three categories of $f(\Phi)$ functions can thus be used to describe the $\hat{\mathrm{T}}$-odd–CP-odd piece of the differential decay rate:

– 'sign' functions defining a signed partition of the phase space,
– a $\hat{\mathrm{T}}$-odd basis on Φ providing a decomposition in moments,
– characteristic functions defining a phase-space binning.

To avoid dilutions in the integral of Eq. (4), the functions chosen should ideally change sign wherever the $\hat{\mathrm{T}}$-odd–CP-odd piece of the differential decay rate itself changes sign. The bins' boundaries should also be placed there.

The question of what set of $f(\Phi)$ functions would yield the best sensitivity to CP violation is non trivial and depends on the process at hand. Actually, when the form of the differential decay rate is known with confidence, one may rely on an unbinned likelihood fit to the data for extracting CP-violating parameters. Such *amplitude analyses* have notably been carried out for several B meson decays: *e.g.*, for $B_s^0 \to K^+K^-K^+K^-$, dominated by a $\phi\phi$ intermediate state [29], or for $B^0 \to K^+K^-K^+\pi^-$, dominated by a ϕK^{*0} resonant intermediate state [27, 31].

Trustworthy parametrizations also make it possible to determine the asymmetries relevant in the study of the CP-odd phases that might appear in perturbative processes like $h \to \ell^+\ell^-\ \ell'^+\ell'^-$, or $e^+e^- \to h\ \ell^+\ell^-$ [39, 44–48]. Observables of optimal statistical significance can then also be determined [49].

In the hadronic decays of heavy mesons however, the parametrization provided by a resonance model is only phenomenological and, although it may capture accurately enough the main features of the studied process, new sources of CP violation may only be observable in finer details. Using tests of CP violation that have a limited reliance upon the process dynamics and its parametrization is therefore desirable.

II. SPINLESS FOUR-BODY DECAYS

Four-body decays involving only spinless particles are simple examples of processes in which four independent four-vectors can be measured. In these cases, $\hat{\mathrm{T}}$ is equivalent to P and there is actually one single independent antisymmetric $\epsilon_{\mu\nu\rho\sigma}$ contraction which involves the external particles' four-momenta. All $\hat{\mathrm{T}}$-odd functions of the phase space are built upon it.

In the following, we will focus on this simple case and investigate how to define appropriate signed partitions (or binnings) of the phase space. For concreteness, we will often refer to the specific $D^0 \to K^+K^-\pi^+\pi^-$ decay. Its differential rate, as well as the one of the corresponding CP-conjugate process, has recently been measured with an impressive accuracy by the LHCb Collaboration [28]. Note we will only consider time-integrated quantities while the LHCb Collaboration also recorded the time dependence of the decay rate.

A. Phase-space parametrization

Hadronic multibody decays often receive contributions of various topologies. The ones so far measured in $D^0 \to K^+K^-\pi^+\pi^-$ are displayed in Table I. A given resonance structure would be most appropriately described with a parametrization of the phase space that includes

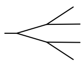
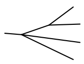
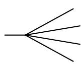
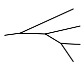
	Intermediate states	Br $\times 10^4$
	$(\phi\rho^0)_S, \quad \phi \to K^+K^-, \quad \rho^0 \to \pi^+\pi^-$	9.3 ± 1.2
	D	0.83 ± 0.23
	$(K^{*0}\overline{K}^{*0})_S, \quad K^{*0} \to K^\pm\pi^\mp$	1.48 ± 0.30
	$\phi(\pi^+\pi^-)_S, \quad \phi \to K^+K^-$	2.50 ± 0.33
	$(K^-\pi^+)_P(K^+\pi^-)_S$	2.6 ± 0.5
	$K_1^+K^-, \quad K_1^+ \to K^{*0}\pi^+$	1.8 ± 0.5
	$K_1^-K^+, \quad K_1^- \to \overline{K}^{*0}\pi^-$	0.22 ± 0.12
	$K_1^+K^-, \quad K_1^+ \to \rho^0K^+$	1.14 ± 0.26
	$K_1^-K^+, \quad K_1^- \to \rho^0K^-$	1.46 ± 0.25
	$K^*(1410)^+K^-, \ K^*(1410)^+ \to K^{*0}\pi^+$	1.02 ± 0.26
	$K^*(1410)^-K^+, \ K^*(1410)^- \to \overline{K}^{*0}\pi^-$	1.14 ± 0.25

TABLE I. The different measured contributions [50] to $\mathrm{Br}(D^0 \to K^+K^-\pi^+\pi^-) = (24.3 \pm 1.2) \times 10^{-4}$ as listed by the Particle Data Group [51], the corresponding decay topologies—or resonance structures—, and branching fractions. The S, P, and D indices indicate the partial waves in which the particle pairs are produced.

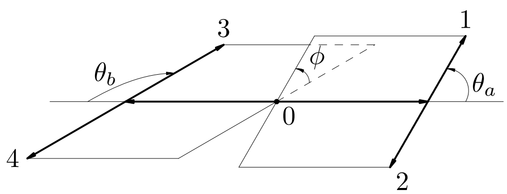


FIG. 1. Parametrization of the phase space of a $0 \to 1\,2\,3\,4$ four-body decay privileging the $(1\,2)$ and $(3\,4)$ subsystems. The momenta of the final-state particle pairs are pictured in their joint rest frames.

the invariant masses in which resonances occur. Such a description would likely be the most sensitive to the interferences between the several partial-wave contributions to that topology. On the contrary, the effects of the resonances occurring in other invariant masses would be diluted. Therefore, at this point already, the parametrization of the four-body phase space challenges our aim at a description independent of the process dynamics.

One may first consider the partial wave decomposition of the dominant resonant intermediate state. Let us here focus on the $0 \to a\,b \to (1\,2)\,(3\,4)$ topology found in the $\phi\rho^0$ resonant contribution that accounts for about 40% of the $D^0 \to K^+K^-\pi^+\pi^-$ branching fraction. Repeating the analysis that follows for different parametrizations would be required to obtain better sensitivities to other decay topologies. A $(1\,4)\,(2\,3)$ pairing would for instance allow to better probe CP violation involving a $K^{*0}\overline{K}^{*0}$ resonant intermediate state in the $D^0 \to K^+K^-\pi^+\pi^-$ decay.

The standard Cabibbo-Maksymowicz parametrization [52] of the phase space Φ can be adopted to describe a four-body decay of $0 \to a\,b \to (1\,2)\,(3\,4)$ topology. It is based on two invariant masses m_a^2 and m_b^2 (which become constants in the narrow width approximation) and three angles (see Fig. 1). In the a and b subsystems' rest frames, the orientations of the final-state particles' momenta are respectively characterized by θ_a and θ_b, comprised in the $[0, \pi]$ interval. The relative orientation of the planes formed by the two pairs of momenta is measured by $\phi \in [-\pi, \pi]$. Note that θ_a and θ_b are $\hat{\mathrm{T}}$-even while ϕ is $\hat{\mathrm{T}}$-odd (and P-odd). The whole ϕ dependence of a differential distribution of definite $\hat{\mathrm{T}}$ transformation properties can thus be obtained from the $[0, \pi]$ interval. The angle ϕ also determines the sign of the *triple product*:

$$\epsilon_{\mu\nu\rho\sigma}\, p_1^\mu\, p_2^\nu\, p_3^\rho\, p_4^\sigma = \frac{1}{8} m_a m_b \sqrt{\lambda(m_0^2, m_a^2, m_b^2)}\ \mathrm{s}\theta_a\ \mathrm{s}\theta_b\ \mathrm{s}\phi\,.$$

where $\lambda(x^2, y^2, z^2) \equiv (x+y+z)(x+y-z)(x-y+z)(x-y-z)$ is the usual Källén function, and $m_{1,2,3,4}$ have been neglected. We will occasionally use shorthands like $\mathrm{c}\phi \equiv \cos\phi$, $\mathrm{s}^2\theta \equiv \sin^2\theta$, $\mathrm{s}2\theta \equiv \sin(2\theta)$.

B. Differential decay rates

For a decay to four spinless particles forming two intermediates states of angular momentum j_a, j_b, the amplitude can be expressed in terms of spherical harmonics. With a spinless initial state, the two intermediate states have equal helicities λ. We can therefore write

$$\mathcal{M} = 4\pi \sum_{j_a, j_b, \lambda} A_\lambda^{j_a, j_b}(m_a^2, m_b^2)\ Y_{j_a}^\lambda(\theta_a, \phi)\ Y_{j_b}^\lambda(\theta_b, 0)^*$$

with $|\lambda| \le \min(j_a, j_b)$, and partial-wave amplitudes $A_\lambda^{j_a, j_b}$ of mass-dimension -1. General expression for n-body phase spaces and arbitrary spins can be derived from Refs. [53–55]. Our normalization is chosen such that the squared amplitude integrated over the $\theta_{a,b}$ and ϕ angles takes the form

$$\int \frac{\mathrm{d}\,\mathrm{c}\theta_a}{2} \frac{\mathrm{d}\,\mathrm{c}\theta_b}{2} \frac{\mathrm{d}\phi}{2\pi}\ |\mathcal{M}|^2 = \sum_{j_a, j_b, \lambda} |A_\lambda^{j_a, j_b}(m_a^2, m_b^2)|^2.$$

In the $j_a = 1 = j_b$ case relevant for the $\phi\rho^0$ intermediate state of $D^0 \to K^+K^-\pi^+\pi^-$, one can define the linear polarization amplitudes

$$A_0 \equiv A_0^{1,1}, \qquad A_{\parallel,\perp} \equiv \frac{1}{\sqrt{2}}\left(A_{+1}^{1,1} \pm A_{-1}^{1,1}\right),$$

where, for conciseness, we omitted the $m_{a,b}^2$ dependences. The amplitude then writes

$$\frac{1}{3}\mathcal{M} = A_0\, \mathrm{c}\theta_a\ \mathrm{c}\theta_b + \frac{A_\parallel}{\sqrt{2}} \mathrm{s}\theta_a\ \mathrm{s}\theta_b\ \mathrm{c}\phi - i\frac{A_\perp}{\sqrt{2}} \mathrm{s}\theta_a\ \mathrm{s}\theta_b\ \mathrm{s}\phi$$

where the last term is $\hat{\mathrm{T}}$ odd because of its $\mathrm{s}\phi$ dependence. Its factor of i respects the phase conventions of

Eq. (1). Denoting by $\bar{A}_{0,\parallel,\perp}$ the linear polarization amplitudes of the CP-conjugate process, the corresponding differential decay rate can be decomposed, as described before, in four pieces of definite $\hat{\mathrm{T}}$ and CP transformation properties:

$$\begin{aligned}
\frac{2m_0}{9}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\bigg|_{\text{CP-even}}^{\hat{\mathrm{T}}\text{-even}} &= \frac{|A_0|^2+|\bar{A}_0|^2}{2}\;\mathrm{c}^2\theta_a\;\mathrm{c}^2\theta_b\\
&+\frac{|A_\parallel|^2+|\bar{A}_\parallel|^2}{4}\;\mathrm{s}^2\theta_a\;\mathrm{s}^2\theta_b\;\mathrm{c}^2\phi\\
&+\frac{|A_\perp|^2+|\bar{A}_\perp|^2}{4}\;\mathrm{s}^2\theta_a\;\mathrm{s}^2\theta_b\;\mathrm{s}^2\phi\\
&+\frac{\mathrm{Re}\{A_0A_\parallel^*+\bar{A}_0\bar{A}_\parallel^*\}}{4\sqrt{2}}\;\mathrm{s}2\theta_a\;\mathrm{s}2\theta_b\;\mathrm{c}\phi\,,\\
\frac{2m_0}{9}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\bigg|_{\text{CP-even}}^{\hat{\mathrm{T}}\text{-odd}} &= \frac{\mathrm{Im}\{A_\perp A_0^*+\bar{A}_\perp\bar{A}_0^*\}}{4\sqrt{2}}\;\mathrm{s}2\theta_a\;\mathrm{s}2\theta_b\;\mathrm{s}\phi\\
&+\frac{\mathrm{Im}\{A_\perp A_\parallel^*+\bar{A}_\perp\bar{A}_\parallel^*\}}{4}\;\mathrm{s}^2\theta_a\;\mathrm{s}^2\theta_b\;\mathrm{s}2\phi\,,\\
\frac{2m_0}{9}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\bigg|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-even}} &= \frac{|A_0|^2-|\bar{A}_0|^2}{2}\;\mathrm{c}^2\theta_a\;\mathrm{c}^2\theta_b\\
&+\frac{|A_\parallel|^2-|\bar{A}_\parallel|^2}{4}\;\mathrm{s}^2\theta_a\;\mathrm{s}^2\theta_b\;\mathrm{c}2\phi\\
&+\frac{|A_\perp|^2-|\bar{A}_\perp|^2}{4}\;\mathrm{s}^2\theta_a\;\mathrm{s}^2\theta_b\;\mathrm{s}2\phi\\
&+\frac{\mathrm{Re}\{A_0A_\parallel^*-\bar{A}_0\bar{A}_\parallel^*\}}{4\sqrt{2}}\;\mathrm{s}2\theta_a\;\mathrm{s}2\theta_b\;\mathrm{c}\phi\,,\\
\frac{2m_0}{9}\,\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\bigg|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-odd}} &= \frac{\mathrm{Im}\{A_\perp A_0^*-\bar{A}_\perp\bar{A}_0^*\}}{4\sqrt{2}}\;\mathrm{s}2\theta_a\;\mathrm{s}2\theta_b\;\mathrm{s}\phi\\
&+\frac{\mathrm{Im}\{A_\perp A_\parallel^*-\bar{A}_\perp\bar{A}_\parallel^*\}}{4}\;\mathrm{s}^2\theta_a\;\mathrm{s}^2\theta_b\;\mathrm{s}2\phi\,,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{d}\Phi \equiv\; &\frac{\sqrt{\lambda(m_0^2,m_a^2,m_b^2)}}{8\pi m_0^2}\,\frac{\sqrt{\lambda(m_a^2,m_1^2,m_2^2)}}{8\pi m_a^2}\\
&\frac{\sqrt{\lambda(m_b^2,m_3^2,m_4^2)}}{8\pi m_b^2}\;\frac{\mathrm{d}\,\mathrm{c}\theta_a}{2}\;\frac{\mathrm{d}\,\mathrm{c}\theta_b}{2}\;\frac{\mathrm{d}\phi}{2\pi}\;\frac{\mathrm{d}m_a^2}{2\pi}\;\frac{\mathrm{d}m_b^2}{2\pi}.
\end{aligned}$$

The linear polarization amplitudes may receive different contributions each having a CP-even phase δ_X^i and a CP-odd phase φ_X^i:

$$A_X(m_a^2,m_b^2) \equiv \sum_i a_X^i(m_a^2,m_b^2)\;e^{i[\delta_X^i(m_a^2,m_b^2)+\varphi_X^i]}$$

for real-valued a_X^i, δ_X^i, φ_X^i and $X = 0$, $\parallel$, or $\perp$. The corresponding CP-conjugate quantities are then given by $\bar{A}_X \equiv \sum_i a_X^i\; e^{i[\delta_X^i-\varphi_X^i]}$, so that

$$\begin{aligned}
&\mathrm{Re}\{A_XA_Y^*\pm\bar{A}_X\bar{A}_Y^*\}/2\\
&\qquad=\pm\sum_{i,j}a_X^ia_Y^j\;{}^{\cos}_{\sin}(\delta_X^i-\delta_Y^j)\;{}^{\cos}_{\sin}(\varphi_X^i-\varphi_Y^j),\\
&\mathrm{Im}\{A_XA_Y^*\pm\bar{A}_X\bar{A}_Y^*\}/2\\
&\qquad=+\sum_{i,j}a_X^ia_Y^j\;{}^{\sin}_{\cos}(\delta_X^i-\delta_Y^j)\;{}^{\cos}_{\sin}(\varphi_X^i-\varphi_Y^j).
\end{aligned}$$

In this specific example, again, the different pieces of the partial rate exhibit the sensitivities to the CP-even and -odd phases described earlier.

C. Beyond the most common observables

Interestingly, with a single resonant intermediate state having $j_a = 1 = j_b$, the total rate asymmetry based on the integral of Eq. (3) vanishes when the A_0 coefficient receives contributions of identical phases, or one single contribution. (The terms involving other linear polarization amplitudes vanish upon phase-space integration.) In such a case, only could a differential rate study provide information about CP violation.

Without assumption about the presence of identical phases, the most common *up-down* integrated asymmetry based on the sign of the *triple product*

$$\int\mathrm{d}\Phi\;\;\mathrm{sign}\{\,\mathrm{s}\phi\,\}\;\;\frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\bigg|_{\text{CP-odd}}^{\hat{\mathrm{T}}\text{-odd}}$$

also vanishes in this simple case. This illustrates—in an extreme way—that phase-space integration may result in losses of sensitivity to CP-violating phases. A non-trivial phase-space dependent $\hat{\mathrm{T}}$-even factor in the $\hat{\mathrm{T}}$-odd–CP-odd differential rate can make it change sign where the *triple product* does not.

Such dilutions can obviously be overcome when a trustworthy parametrization of the differential rate is known. Taking seriously the simplified parametrization of the D^0 decay presented above, the bare examination of the differential rates indicates that more information about CP-odd and -even phases is contained in the piece-wise integrals of Table II upon which asymmetries could be constructed.

However, as already stressed, the parametrization of heavy mesons' hadronic decays is only phenomenological and may miss some fine interference details that have the potential of revealing new sources of CP violation. We would therefore wish to adopt a more systematic approach that does not rely on strong theoretical assumptions about the process dynamics.

D. A first look at the data

This point can be made more concrete using the recent experimental study of the $D^0 \to K^+K^-\pi^+\pi^-$ decay. The LHCb Collaboration displays in Ref. [28] the measured $m_{\pi^+\pi^-}$, $m_{K^+K^-}$, $\cos\theta_\pi$, $\cos\theta_K$ and ϕ distributions for both D^0 and $\bar{D}^0$ as well as $\sin\phi > 0$ and $\sin\phi < 0$. This allows for the marginalized differential

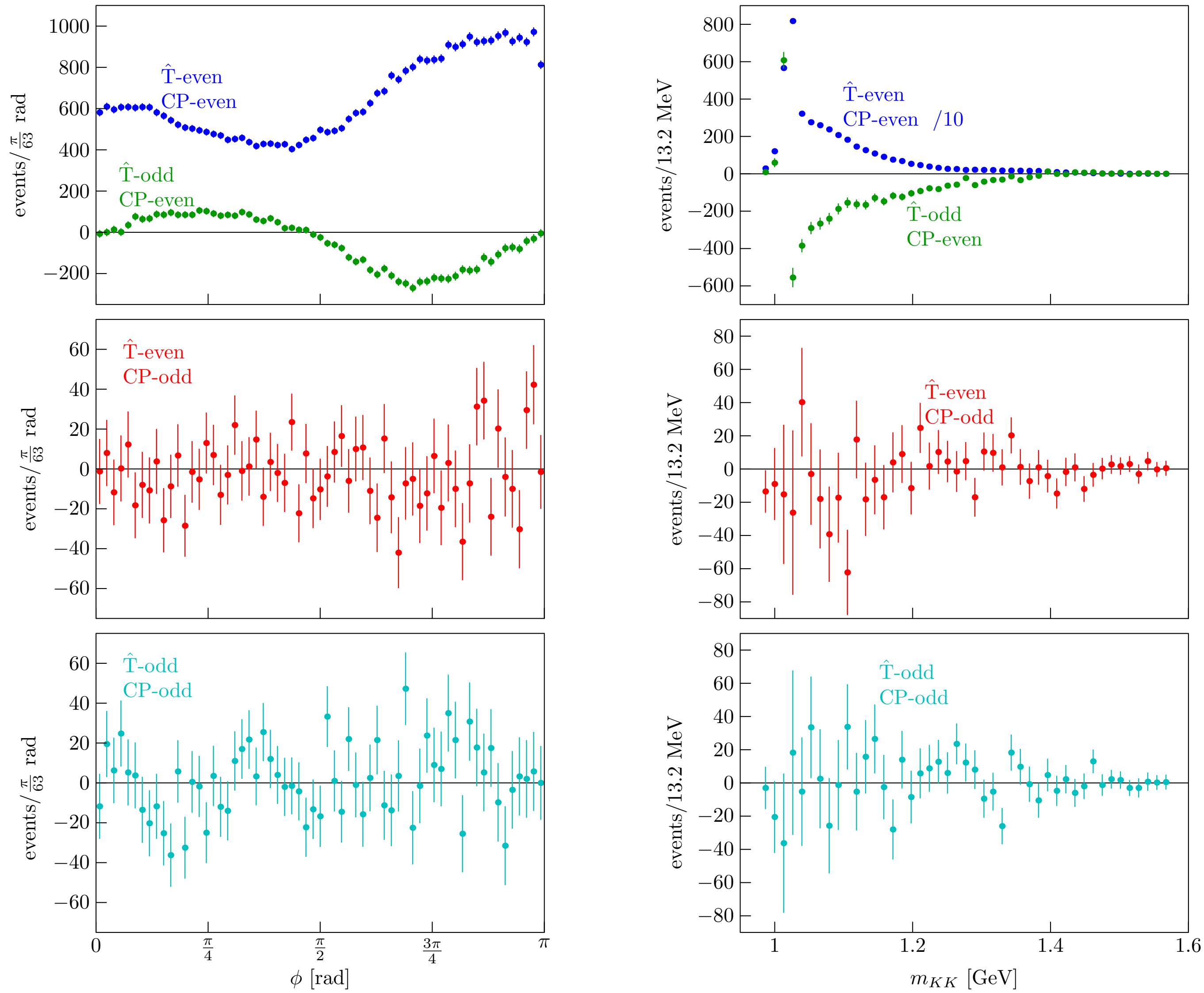


FIG. 2. Decomposition of the measured $D \to K^+K^-\pi^+\pi^-$ differential rate into components of definite $\hat{\mathrm{T}}$ and CP transformation properties, projected onto the ϕ angle and m_{KK} invariant mass. The uncertainties on the LHCb data points in Fig. 3(e-f) and Fig. 2(c-d) of Ref. [28] have been assumed equal to $\sqrt{N}+8$ and uncorrelated.

distributions of definite $\hat{\mathrm{T}}$ and CP properties to be derived. The left panel of Fig. 2 for instance shows those differential distributions projected onto the ϕ angle (*i.e.*, marginalized over the four other phase-space variables). The respective $\mathrm{c}^2\phi$ and $\mathrm{s}2\phi$ dependences of the $\hat{\mathrm{T}}$-even–CP-even and $\hat{\mathrm{T}}$-odd–CP-even differential rates expected from a dominant $\phi\rho^0$ contribution are clearly visible while the $\hat{\mathrm{T}}$-even–CP-odd and $\hat{\mathrm{T}}$-odd–CP-odd distributions are roughly compatible with zero.

An oscillatory pattern can however be distinguished in the $\hat{\mathrm{T}}$-odd–CP-odd differential rate. The $A_n \equiv \int \mathrm{d}\Phi\ \mathrm{sign}\{\sin n\phi\}\ \frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\Big|^{\hat{\mathrm{T}}\text{-odd}}_{\text{CP-odd}}$ asymmetries (see Table III) notably point at the presence of a sizable $\sin 8\phi$ contribution: the A_8 departure from zero is of about 2.6 standard deviations (2.0 standard deviations for A_2 and A_{13}). If genuine, this rapid oscillatory behavior would indicate the presence of a CP-violating phase difference but would not have contributed to the asymmetries that could be expected from a simple $\phi\rho^0$ parametrization. Whether any resonance model considered as providing a fair description of that process would have included a contribution oscillating so rapidly is also unclear.

E. Even more angular asymmetries

Clearly, one way in which the presence of CP-violating phases could be probed without relying on a full description of the dynamics of the process studied would be to evaluate systematically a wider range of *triple-product*

$$\int d\Phi \ \mathrm{sign}\{\, c\theta_a \ c\theta_b \ c\phi \,\} \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-even}}_{\text{CP-even}} = + \frac{2\sqrt{2}}{\pi} \int \frac{dm_a^2}{2\pi} \frac{dm_b^2}{2\pi} \ \mathcal{N} \sum_{i,j} a_0^i \, a_\parallel^j \ \cos(\delta_0^i - \delta_\parallel^j) \ \cos(\varphi_0^i - \varphi_\parallel^j)$$

$$\int d\Phi \ \mathrm{sign}\{\, c\theta_a \ c\theta_b \ s\phi \,\} \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-odd}}_{\text{CP-even}} = + \frac{2\sqrt{2}}{\pi} \int \frac{dm_a^2}{2\pi} \frac{dm_b^2}{2\pi} \ \mathcal{N} \sum_{i,j} a_\perp^i \, a_0^j \ \sin(\delta_\perp^i - \delta_0^j) \ \cos(\varphi_\perp^i - \varphi_0^j)$$

$$\int d\Phi \ \mathrm{sign}\{\, c\theta_a \ c\theta_b \ c\phi \,\} \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-even}}_{\text{CP-odd}} = - \frac{2\sqrt{2}}{\pi} \int \frac{dm_a^2}{2\pi} \frac{dm_b^2}{2\pi} \ \mathcal{N} \sum_{i,j} a_0^i \, a_\parallel^j \ \sin(\delta_0^i - \delta_\parallel^j) \ \sin(\varphi_0^i - \varphi_\parallel^j)$$

$$\int d\Phi \ \mathrm{sign}\{\, c\theta_a \ c\theta_b \ s\phi \,\} \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-odd}}_{\text{CP-odd}} = + \frac{2\sqrt{2}}{\pi} \int \frac{dm_a^2}{2\pi} \frac{dm_b^2}{2\pi} \ \mathcal{N} \sum_{i,j} a_\perp^i \, a_0^j \ \cos(\delta_\perp^i - \delta_0^j) \ \sin(\varphi_\perp^i - \varphi_0^j)$$

$$\int d\Phi \ \mathrm{sign}\{\, s2\phi \,\} \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-odd}}_{\text{CP-even}} = + \frac{4}{\pi} \int \frac{dm_a^2}{2\pi} \frac{dm_b^2}{2\pi} \ \mathcal{N} \sum_{i,j} a_\perp^i \, a_\parallel^j \ \sin(\delta_\perp^i - \delta_\parallel^j) \ \cos(\varphi_\perp^i - \varphi_\parallel^j)$$

$$\int d\Phi \ \mathrm{sign}\{\, s2\phi \,\} \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-even}}_{\text{CP-odd}} = - \frac{4}{\pi} \int \frac{dm_a^2}{2\pi} \frac{dm_b^2}{2\pi} \ \mathcal{N} \sum_{i,j} a_\perp^i \, a_\perp^j \ \sin(\delta_\perp^i - \delta_\perp^j) \ \sin(\varphi_\perp^i - \varphi_\perp^j)$$

$$\int d\Phi \ \mathrm{sign}\{\, s2\phi \,\} \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-odd}}_{\text{CP-odd}} = + \frac{4}{\pi} \int \frac{dm_a^2}{2\pi} \frac{dm_b^2}{2\pi} \ \mathcal{N} \sum_{i,j} a_\perp^i \, a_\parallel^j \ \cos(\delta_\perp^i - \delta_\parallel^j) \ \sin(\varphi_\perp^i - \varphi_\parallel^j)$$

$$\int d\Phi \ \mathrm{sign}\{\, c2\phi \,\} \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-even}}_{\text{CP-odd}} = - \frac{4}{\pi} \int \frac{dm_a^2}{2\pi} \frac{dm_b^2}{2\pi} \ \mathcal{N} \sum_{i,j} a_\parallel^i \, a_\parallel^j \ \sin(\delta_\parallel^i - \delta_\parallel^j) \ \sin(\varphi_\parallel^i - \varphi_\parallel^j)$$

$$\text{with } \mathcal{N} \equiv \frac{1}{2m_0} \frac{\sqrt{\lambda(m_0^2, m_a^2, m_b^2)}}{8\pi m_0^2} \frac{\sqrt{\lambda(m_a^2, m_1^2, m_2^2)}}{8\pi m_a^2} \frac{\sqrt{\lambda(m_b^2, m_3^2, m_4^2)}}{8\pi m_b^2}$$

TABLE II. Piecewise integrals from which information about the CP-conserving and CP-violating phases between different polarization amplitudes could be extracted, for a $0 \to (1\,2)(3\,4)$ decay involving spinless particles and proceeding through two intermediate vector resonances.

n	A_n	n	A_n	n	A_n
1	$+58 \pm 132$	6	$+164 \pm 132$	11	$+128 \pm 132$
2	-259 ± 132	7	$+101 \pm 132$	12	$+164 \pm 132$
3	-2 ± 132	8	$+337 \pm 132$	13	$+268 \pm 132$
4	-134 ± 132	9	-40 ± 132	14	-107 ± 132
5	-225 ± 132	10	$+41 \pm 132$		

TABLE III. $A_n \equiv \int d\Phi \ \ \mathrm{sign}\{\sin n\phi\} \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-odd}}_{\text{CP-odd}}$ asymmetries in the data collected by the LHCb Collaboration on the $D^0 \to K^+K^-\pi^+\pi^-$ decay. The uncertainties on the data points of Fig. 3(e-f) in Ref. [28] have been assumed equal to $\sqrt{N} + 8$ and uncorrelated.

asymmetries of the form

$$\int d\Phi \ \ \mathrm{sign}\{f_l(\, c\theta_a \,) \ f_m(\, c\theta_b \,) \ \sin n\phi\} \ \ \frac{d\Gamma}{d\Phi}\Big|^{\hat{T}\text{-odd}}_{\text{CP-odd}}$$

for all combinations of reasonably large integers l, m, and n. In the case of spinless final states forming two pairs of resonant intermediate states, the natural set of functions f are products of the various $c\theta$ dependences arising in

$$\begin{aligned}
f_0(\, c\theta \,) &= 1, \\
f_1(\, c\theta \,) &= c\theta\,, \\
f_2(\, c\theta \,) &= 3\,c^2\theta \, - 1, \\
f_3(\, c\theta \,) &= c\theta\,(3\,c^2\theta \, - 1), \\
f_4(\, c\theta \,) &= c\theta\,(5\,c^2\theta \, - 3), \\
f_5(\, c\theta \,) &= 5\,c^2\theta \, - 1, \\
f_6(\, c\theta \,) &= 5\,c^2\theta \, - 3, \\
f_7(\, c\theta \,) &= c\theta\,(5\,c^2\theta \, - 1), \\
f_8(\, c\theta \,) &= c\theta\,(3\,c^2\theta \, - 1)\,(5\,c^2\theta \, - 3), \\
f_9(\, c\theta \,) &= (3\,c^2\theta \, - 1)\,(5\,c^2\theta \, - 1), \\
f_{10}(\, c\theta \,) &= c\theta\,(5\,c^2\theta \, - 3)\,(5\,c^2\theta \, - 1), \\
&\ \ \dots
\end{aligned}$$

TABLE IV. Natural set of functions of the $\theta_{a,b}$ angles for the systematic construction of asymmetries in $0 \to (1\,2)(3\,4)$ decays involving spinless particles.

	a_X [GeV^{-1}]	δ_X	φ_X
$X = \ 0:$	1	1	0
$\parallel:$	2	0	0
$\perp:$	1	1	0.05

TABLE V. Parameters chosen in the toy simulation of the $D \to \phi\rho^0 \to (K^+K^-)(\pi^+\pi^-)$ process.

spherical harmonics:

$$\begin{array}{lll} l=1: & \mathrm{c}\theta\,, & \\ 2: & 3\,\mathrm{c}^2\theta\,-1, & \\ 3: & \mathrm{c}\theta\,(5\,\mathrm{c}^2\theta\,-3), & 5\,\mathrm{c}^2\theta\,-1, \\ 4: & 35\,\mathrm{c}^4\theta\,-30\,\mathrm{c}^2\theta\,+3, & \mathrm{c}\theta\,(7\,\mathrm{c}^2\theta\,-3),\;\; 7\,\mathrm{c}^2\theta\,-1, \\ \ldots & & \end{array}$$

The dependences upon $\mathrm{s}\theta$ have been dropped as they have no influence on the sign of the associated Legendre polynomials P_l^m. Here, the set of f functions could therefore be defined as in Table IV, keeping in mind that another choice would be needed for final states carrying spin.

F. Invariant mass dependence

Let us still focus on the parametrization of the phase space privileging the $0 \to (1\,2)(3\,4)$ type of topology. Upon phase-space integration, the $m^2_{a,b}$ invariant mass dependence of the decay rate could also lead to losses of sensitivity to CP-violating phases. This happens when it causes the $\hat{\mathrm{T}}$-odd–CP-odd piece of the differential decay rate to change sign. Guessing where this could happen is in general difficult. However, when resonances are clearly identified, one at least knows the real parts of the associated propagators

$$\mathrm{Re}\left\{\frac{1}{m_a^2 - M^2 + i\Gamma M}\right\} = \frac{m_a^2 - M^2}{(m_a^2 - M^2)^2 + \Gamma^2 M^2}$$

change sign at the resonances (and could possibly appear in interferences).

Once again, a glimpse at the LHCb data shows such a behavior actually occurs in the m_{KK} invariant mass spectrum, although in the $\hat{\mathrm{T}}$-odd–CP-even piece of the differential rate which is not directly relevant for the extraction of CP-violating phases (see right panel of Fig. 2).

Therefore, when constructing asymmetries systematically one may also wish to consider $\mathrm{sign}\{m_a^2 - M_i^2\}$ and $\mathrm{sign}\{m_b^2 - M_j^2\}$ as weight functions, for the known resonances appearing at $M^2_{i,j}$ in the $m^2_{a,b}$ invariant mass spectra.

G. Binned analyses

Instead of constructing asymmetries, one may rather adopt the approach of Ref. [28] and bin the phase space. Care must however be taken in the binning choice. Putting together in one bin, regions of the phase space in which the $\hat{\mathrm{T}}$-odd–CP-odd part of the differential rate changes sign would result in sensitivity losses.

These can be assessed using a toy simulation. We considered massless kaons and pions and generated event using MadGraph5 [56] with the following matrix elements

	$\sin n\phi$	$\sin n\phi\,(m^2_{KK}-m^2_\phi)$	$\sin n\phi\,(m^2_{\pi\pi}-m^2_\rho)$	$\sin n\phi\,\cos\theta_{KK}\cos\theta_{\pi\pi}$
$n=1:$	-2.0	0.86	1.2	-2.8
2 :	-4.0	0.88	3.4	0.40
3 :	0.20	0.15	-0.014	-1.4
4 :	0.30	-0.014	-0.52	-1.5
5 :	0.30	0.95	-1.2	-0.65
6 :	-0.40	0.20	1.0	0.057
7 :	-2.0	2.4	2.4	-0.042
8 :	-0.70	0.37	1.5	0.27
9 :	-0.60	-0.69	0.88	-0.75
10 :	0.60	-2.2	-0.78	1.0
11 :	-2.0	0.53	2.1	-0.15
12 :	-0.20	-0.092	0.55	-1.7
13 :	-1.0	0.30	1.2	0.67
14 :	0.20	1.7	0.30	-0.70

TABLE VI. Departure from zero expressed in standard deviations for a few asymmetries, computed with the simulated sample of $D \to \phi\rho^0 \to (K^+K^-)(\pi^+\pi^-)$ decays. Only statistical uncertainties are accounted for.

for the $D\rho\phi$, ϕKK, and $\rho\pi\pi$ interactions:

$$\begin{aligned} D\rho\phi:\ & \epsilon^\mu_\rho\ \epsilon^\nu_\phi\ p^\alpha_\rho\ p^\beta_\phi \Bigg(\mathcal{A}_0\ g_{\alpha\nu}g_{\beta\mu} \\ & \qquad + \mathcal{A}_\parallel \left\{ g_{\mu\nu}g_{\alpha\beta}\left[1 - \frac{p^2_\rho\, p^2_\phi}{(p_\rho\cdot p_\phi)^2}\right] - g_{\alpha\nu}g_{\beta\mu}\right\} \\ & \qquad + \mathcal{A}_\perp\ i\epsilon_{\mu\nu\alpha\beta}\Bigg), \\ \phi KK:\ & \epsilon^\mu_\phi\left(p_{K^+\mu} - p_{K^-\mu}\right), \\ \rho\pi\pi:\ & \epsilon^\mu_\rho\left(p_{\pi^+\mu} - p_{\pi^-\mu}\right). \end{aligned}$$

The linear polarization amplitude described earlier are then

$$A_0 = \frac{\mathcal{A}_0\ \lambda(m^2_D, m^2_{KK}, m^2_{\pi\pi})}{12(m^2_{KK} - m^2_\phi + im_\phi\Gamma_\phi)(m^2_{\pi\pi} - m^2_\rho + im_\rho\Gamma_\rho)},$$

$$A_\parallel = \frac{\mathcal{A}_\parallel\ \frac{m_{KK}m_{\pi\pi}}{m^2_D - m^2_{KK} - m^2_{\pi\pi}}\lambda(m^2_D, m^2_{KK}, m^2_{\pi\pi})}{6(m^2_{KK} - m^2_\phi + im_\phi\Gamma_\phi)(m^2_{\pi\pi} - m^2_\rho + im_\rho\Gamma_\rho)},$$

$$A_\perp = \frac{\mathcal{A}_\perp\ m_{KK}m_{\pi\pi}\sqrt{\lambda(m^2_D, m^2_{KK}, m^2_{\pi\pi})}}{6(m^2_{KK} - m^2_\phi + im_\phi\Gamma_\phi)(m^2_{\pi\pi} - m^2_\rho + im_\rho\Gamma_\rho)}.$$

where each of the $\mathcal{A}_{0,\parallel,\perp}$ were given both a CP-even and CP-odd phase: $\mathcal{A}_X = a_X\ e^{i(\delta_X + \varphi_X)}$ for $X = 0, \parallel, \perp$. These parameters were fixed as in Table V and 40 000 D^0 and $\bar{D}^0$ decays generated. The decomposition of the ϕ differential distribution obtained is displayed in the left panel of Fig. 3.

A larger magnitude for $a_\parallel$ than for $a_\perp$ causes the $\hat{\mathrm{T}}$-even–CP–even piece of the differential rate to have a dip at $\pi/2$. The non-vanishing difference in CP-conserving phases $\delta_\parallel - \delta_\perp$ sources the $\sin 2\phi$ dependence of the $\hat{\mathrm{T}}$-odd–CP-even contribution. No structure is generated in

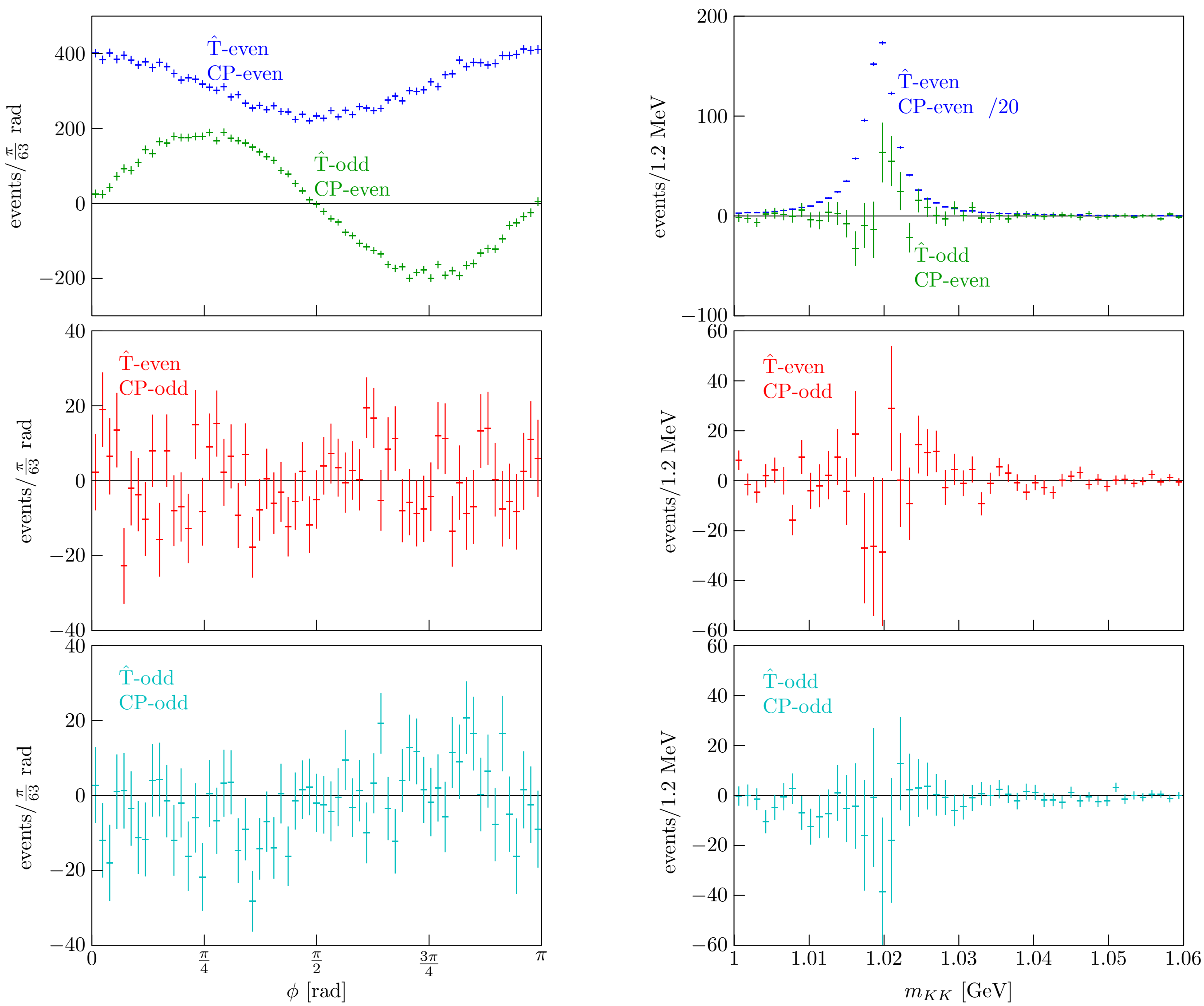


FIG. 3. Simulated $D \to \phi\rho^0 \to (K^+K^-)(\pi^+\pi^-)$ decay and partial rate decomposition in components of definite $\hat{\mathrm{T}}$ and CP transformation properties as in Eq. (2), projected onto the ϕ angle and m_{KK} invariant mass. Only statistical uncertainties are displayed.

the $\hat{\mathrm{T}}$-even–CP-odd distribution while a small difference in CP-violating phases $\varphi_\parallel - \varphi_\perp$ allows for a $\sin 2\phi$ dependence in the $\hat{\mathrm{T}}$-odd–CP-odd differential rate. Due to limited statistics, the latter is barely visible in Fig. 3. Additionally, a $\sin 2\theta_a \sin 2\theta_b \sin\phi$ dependence of each piece of the differential rate is washed out upon integration over the $\theta_{a,b}$ angles. One can also notice a sign change in the $\hat{\mathrm{T}}$-odd–CP-even differential rate projected on the m_{KK} variable at the $m_\phi = 1.02$ GeV resonance (see right panel of Fig. 3).

Computing

$$\int \mathrm{d}\Phi \;\; \mathrm{sign}\left\{ f_l(\,\mathrm{c}\theta_a\,)\; f_m(\,\mathrm{c}\theta_b\,)\;\sin n\phi \right.$$
$$\left. \prod_i (m_a^2 - M_i^2) \prod_j (m_b^2 - M_j^2) \right\} \; \frac{\mathrm{d}\Gamma}{\mathrm{d}\Phi}\bigg|^{\hat{\mathrm{T}}\text{-odd}}_{\text{CP-odd}} \qquad (6)$$

asymmetries as prescribed earlier, one observes the expected excesses for $(l, m; n) = (0, 0; 2)$ and $(1, 1; 1)$. They are of 4.0 and 2.8 standard deviations, respectively (see Table VI, only statistical uncertainties have been accounted for). Using additional $\mathrm{sign}\{m_{KK}^2 - m_\phi^2\}$ and $\mathrm{sign}\{m_{\pi\pi}^2 - m_\rho^2\}$ weight functions does not enhance the excesses' significance.

The LHCb Collaboration partitioned the phase space in 32 bins (two bins per kinematic variable) and estimated the combined departure from zero using a chi-squared test [28]. The separation between the two bins of the ϕ variable was set at 1.99 rad (its domain is restricted to the $[0, \pi]$ interval here) and between -0.28 and $+0.28$ for $\cos\theta_{KK}$ and $\cos\theta_{\pi\pi}$.

In our simulated sample, a chi-squared test with only

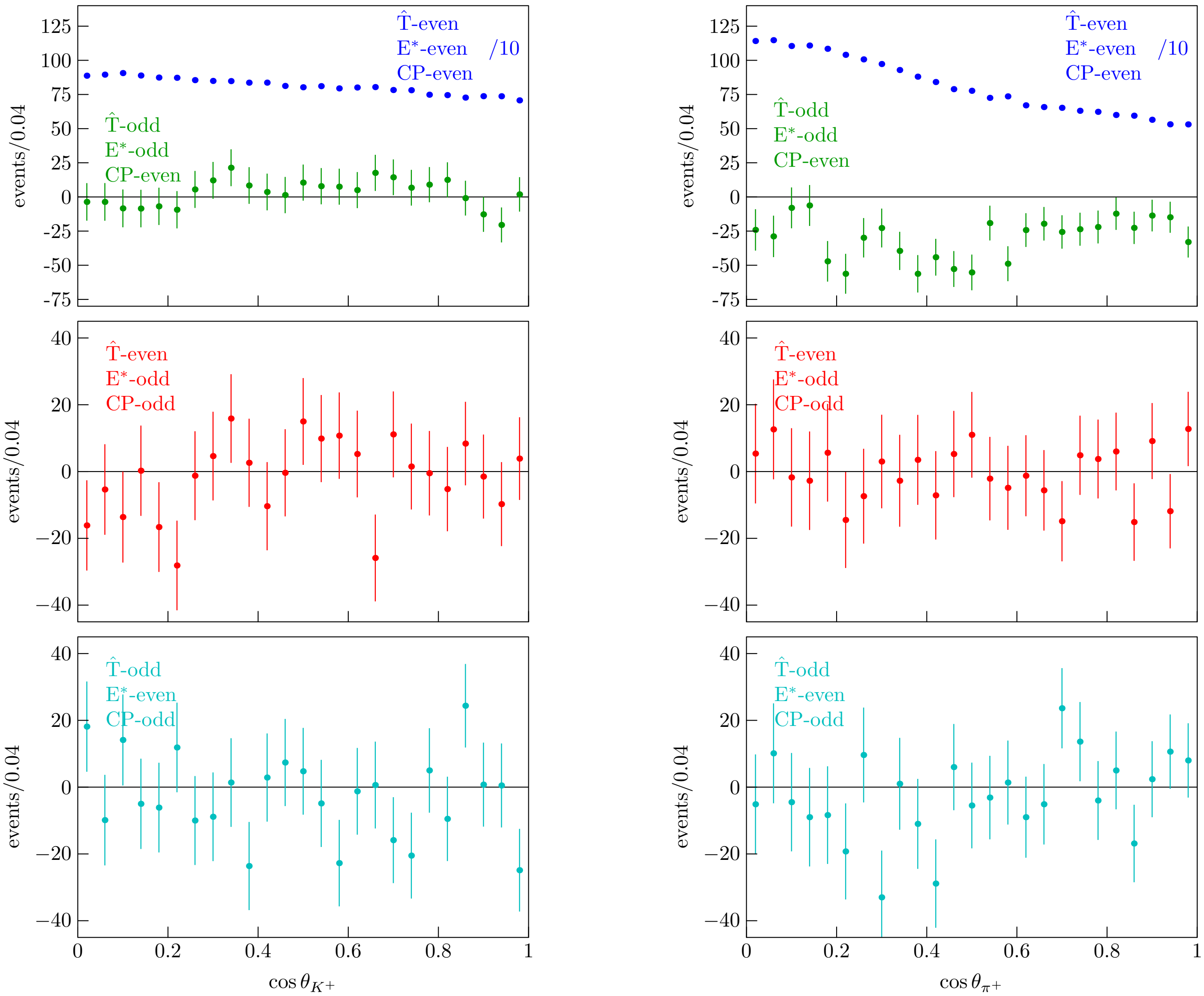


FIG. 4. The four components of the $D \to K^+K^-\pi^+\pi^-$ differential rate having definite $\hat{\mathrm{T}}$, E^*, and CP transformation properties that could have been measured with an untagged sample. The uncertainties on the LHCb data points in from Fig. 3(a-d) of Ref. [28] have been assumed equal to $\sqrt{N}+8$ and uncorrelated.

two bins of boundary located at $\pi/2$ in the in the ϕ variable gives a departure from zero for the $\hat{\mathrm{T}}$-odd–CP-odd differential rate equivalent to 3.9 Gaussian standard deviations. Once the bins' boundary is moved to 1.99 rad, this significance slightly diminishes to $3.7\,\sigma$. The loss of sensitivity is more significant for the only other binning relevant to this simplified simulation. With two bins in both the $\cos\theta_{KK}$ and $\cos\theta_{\pi\pi}$ directions, the $\hat{\mathrm{T}}$-odd–CP-odd differential rate departs from zero at the $2.8\,\sigma$ level when the bin boundaries are chosen at 0, and at the $1.1\,\sigma$ level only when they are respectively taken at the extreme values of -0.28 and $+0.28$.

The multiplication of unnecessary bins also leads to losses of sensitivity, in this scheme. With 8 bins having boundaries at $\pi/2$ in the ϕ angle and 0 in the $\cos\theta_{KK,\pi\pi}$ variables, one for instance obtains an overall departure from zero of 3.5 standard deviations.

H. Untagged D and $B \to K^+K^-\pi^+\pi^-$ samples

Although a tagging of the D^0 has been carried out by the LHCb Collaboration in this $D^0 \to K^+K^-\pi^+\pi^-$ decay, the self-conjugate final state could have motivated an untagged analysis. This is what was actually done in the study of the B_s^0 decay to the very same final state [8].

In both cases, the E^* permutation defined in Section I.C sends $\{K^+, K^-, \pi^+, \pi^-\}$ to $\{K^-, K^+, \pi^-, \pi^+\}$. In the parametrization of the phase space adopted thus far, it therefore acts trivially on the m_{KK}, $m_{\pi\pi}$ invariant masses, and on the ϕ azimuthal angle. The cosines of the polar angles in the K^+K^- and $\pi^+\pi^-$ subsystems undergo the following transformations:

$$\mathrm{E}^*[\cos\theta_{K^+}] = \cos\theta_{K^-} = -\cos\theta_{K^+},$$
$$\mathrm{E}^*[\cos\theta_{\pi^+}] = \cos\theta_{\pi^-} = -\cos\theta_{\pi^+}.$$

Practically, the untagged E^*-odd distributions can therefore be obtained by multiplying the weights of each recorded event by $\frac{1}{2}\,\mathrm{sign}\{\cos\theta_{K^+}\cos\theta_{\pi^+}\}$ and by considering the absolute values of both cosines as kinematic variables. The same procedure carried out with the variable ϕ yields the $\hat{\mathrm{T}}$-odd distributions. Using the LHCb measurement [28], we display in Fig. 4 the projection onto the $\cos\theta_{K^+}$ and $\cos\theta_{\pi^+}$ variables of the four differential rates that could have been measured with an untagged sample of $D \to K^+K^-\pi^+\pi^-$ events.

In its analysis of the $B_s^0 \to K^+K^-\pi^+\pi^-$ decay [8], the LHCb Collaboration used a parametrization of the phase space privileging resonances in the $K^+\pi^-$ and $K^-\pi^+$ invariant masses. A dominant $K^{*0}\overline{K}^{*0}$ intermediate state motivated this choice. In that parametrization, the permutation E^* exchanges the cosines of the polar angles defined in the two subsystem $c\theta_a$ and $c\theta_b$, as well as their respective invariant masses m_a and m_b. Various E^*-odd asymmetries can therefore be constructed by using weight functions $g(\Phi)$ (see Section I.D) proportional to either $c\theta_a - c\theta_b$, or $m_a - m_b$. The E^*-odd asymmetries measured in Ref. [8] were the ones possibly appearing for $K\pi$ subsystems forming partial waves of $j_{a,b} = 0$ and 1. The arguments presented here to motivate the systematic use of a wider range of $\hat{\mathrm{T}}$-odd–CP-odd asymmetries however also apply to $\hat{\mathrm{T}}$-even–E^*-odd–CP-odd ones.

III. CONCLUSIONS

CP violation in K and B decays has so far been observed mostly through time-independent and time-integrated rate asymmetries. As multibody decays are being measured with an ever increasing accuracy, it is desirable to devote more attention to their rich differential distributions.

Taking, as an illustrative example, the $D^0 \to K^+K^-\pi^+\pi^-$ decay whose differential distribution has recently been studied by the LHCb Collaboration [28], we propose to measure a large set of generalized *triple-product* asymmetries. Their choice is guided by the topology—or resonance structure—of the contribution under scrutiny, by the spin of the particles involved, and by the location of the known resonances. An illustration of the procedure and of the losses of sensitivity that may occur with a suboptimal partition of the phase-space is provided using a toy simulation. Such a procedure could obviously be applied to a wide range of other processes in which CP violation is searched for in differential distributions.

In charm decays, a signal of CP violation would clearly point at new physics. In B decays however, standard-model CP violation is expected to be visible in some cases. We did not investigate whether cleaner probes for physics beyond the standard model could be constructed from differential observables. Clearly, more theoretical studies in this direction would be necessary.

Our final point is to emphasize that more experimental studies are needed in order to devise observables optimized for specific processes. With the new data coming from LHCb and Belle II, such a task is timely.

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