Reemergence of rapidity-scale uncertainty in soft-collinear effective theory

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Re-emergence of rapidity scale uncertainty in SCET

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The artificial separation of a full-theory mode into distinct collinear and soft modes in SCET leads to divergent integrals over rapidity, which are not present in the full theory. Rapidity divergence introduces an additional scale into the problem, giving rise to its own renormalization group with respect to this new scale. Two contradicting claims exist in the literature concerning rapidity scale uncertainty. One camp has shown that the results of perturbative calculations depend on the precise choice of rapidity scale. The other has derived an all-order factorization formula with no dependence on rapidity scale, by using a form of analytic regulator to regulate rapidity divergences. We deliver a simple resolution to this controversy by deriving an alternative form of the all-order factorization formula with an analytic regulator that, despite being formally rapidity scale independent, reveals how rapidity scale dependence arises when it is truncated at a finite order in perturbation theory. With our results, one can continue to take advantage of the technical ease and simplicity of the analytic regulator approach while correctly taking into account rapidity scale dependence. As an application, we update our earlier study of WW production with jet-veto by including rapidity scale uncertainty. While the central values of the predictions are unchanged, the scale uncertainties are increased and consistency between the NLL and NNLL calculations are improved.

I. INTRODUCTION

The paradigm and framework of effective field theory (EFT) is essential in our conceptual understanding of quantum field theory as well as in practical calculations. In an EFT, by definition, we have removed (i.e., integrated out) the modes of the full theory that are necessarily highly off-shell with the given initial and final states, so that the Lagrangian only contains the modes that can be on-shell and may produce singularities in loop and/or phase-space integrals. Also fundamental in EFT is to have small parameters besides coupling constants and assign to each field in the theory a definite power-counting rules in terms of those parameters, so that we have manifestly well-controlled expansions at the Lagrangian level. Often, defining such power-counting rules requires classifying the EFT modes into smaller groups of modes and introduce a separate field for each group.

Soft collinear effective theory (SCET) [1–5] [6, 7] is an excellent example of EFT with such further mode separations. In a typical application of SCET to collider physics, we have *collinear modes,* i.e., the energetic modes nearly parallel to one beam direction, and *anti-collinear modes,* i.e., those nearly parallel to the other beam. It is also often convenient or necessary to introduce *soft modes,* which are between collinear and anti-collinear modes.[1] To introduce our notation and nomenclature, let us quickly review the power-counting rules for those modes. Let n and $\bar n$ be the 4-vectors $(1,0,0,1)$ and $(1,0,0,-1)$, respectively. Any 4-vector v can then be decomposed uniquely as $v^\mu = v_+\, n^\mu/2 + v_-\, \bar n^\mu/2 + v_\perp^\mu$, where $v_+ \equiv \bar n\cdot v$ and $v_- \equiv n\cdot v$, and $n\cdot v_\perp = \bar n\cdot v_\perp = 0$. Then, a collinear momentum p, an anti-collinear momentum $\bar p$, and a soft momentum $p_{\rm s}$ are characterized by the following scaling behavior in powers of a small power-counting parameter $\lambda \ll 1$:

$$\begin{aligned} p &= (p_+, p_-, p_\perp) \sim (1, \lambda^2, \lambda) M\,, \\ \bar p &= (\bar p_+, \bar p_-, \bar p_\perp) \sim (\lambda^2, 1, \lambda) M\,, \\ p_{\rm s} &= (p_{\rm s+}, p_{\rm s-}, p_{\rm s\perp}) \sim (\lambda, \lambda, \lambda) M\,, \end{aligned} \tag{1}$$

where M is the characteristic high energy scale of the process in question. Perturbative computation in SCET is a dual expansion in λ and the coupling constant $\alpha_{\rm s}$. All the three modes have the same level of virtuality, $p^2 \sim \bar p^2 \sim p_{\rm s}^2 \sim \lambda^2 M^2$, but their rapidities η (along the beam axis) are vastly different, i.e., $\mathrm{e}^\eta \sim \lambda^{-1}$, $\sim \lambda$, and ~ 1, respectively. As noted earlier, the EFT should describe those three groups of modes by three separate fields $\phi(x)$, $\bar\phi(x)$, and $\phi_{\rm s}(x)$ (even if they all originally belong to the same field $\Phi(x)$ in the full theory) so that we can assign a definite power of λ to each field, and thereby each term, in the Lagrangian.

Such separation of modes, however, can lead to additional divergences in loop and/or phase-space integrals [8–10]. Just like we get ultraviolet (UV) divergences by ignoring the boundary (or cutoff) between the highly off-shell modes we have integrated out and the nearly on-shell modes we have kept, we get *rapidity divergences* by neglecting the boundaries between collinear, soft, and anti-collinear modes. Then, just like UV divergence leads to a renormalization group (RG) with respect to a scale parameter μ, rapidity divergence leads to its own RG with respect to a different scale parameter ν [11–13]. As usual, the μ RG equations (RGEs) can be used to resum large logarithms arising from a (Lorentz invariant) hierarchy of scales in the problem. Similarly, the ν-RGEs can be used to resum large logarithms from a (frame-dependent) hierarchy of rapidity scales (as in Eq. (1)).

[1] We restrict our discussion to $\mathrm{SCET_{II}}$-type observables, for which collinear, anti-collinear and soft modes suffice. Other observables may require different modes such as ultra-soft modes.

A natural question, then, is whether there are additional scale uncertainties associated with the precise choice of initial and final values of ν in solving the ν-RGEs, analogous to the familiar scale uncertainties from the choice of μ. At first sight, it might appear that the ν-RG should not constitute an *independent* source of uncertainties. Firstly, both μ- and ν-RGEs resum the same large logarithms of the form $\log(1/\lambda)$, because the theory actually contains only one large ratio of scales, $1/\lambda$, as can be seen in the scaling laws (1). Since there is only one kind of logarithms being resummed, one might argue that there should only be one source of scale uncertainties. Secondly, recall that the usual μ scale dependence arises because α_s depends on μ and we do not know its exact μ dependence.[2] That is why physical quantities like cross-sections depend on μ even though we demanded that they should not. In contrast, being a Lorentz invariant quantity, α_s cannot depend on ν, because ν is a rapidity cutoff, i.e., a proxy for the frame-dependent boundaries separating collinear, soft, and anti-collinear modes. Then, the ν-independence requirement for physical quantities would just render the perturbation series to be trivially independent of ν order-by-order in α_s, leaving us no additional scale uncertainties. Indeed, in Refs. [14–17], all-order factorization formulae were derived that are explicitly ν independent, based on a form of analytic regulator [18] for cutting off rapidity divergences. However, in a seminal paper on this subject [13], a thorough, explicit analysis demonstrates that the ν-RG constitutes an independent source of scale uncertainties, where ν is varied independently of μ. A recent work [19] also shows that rapidity scale dependence can have significant numerical impact.

In this paper, we provide a simple resolution to this puzzle. We will show exactly how ν dependence is concealed in the seemingly ν-independent factorization formula in Ref. [17], by deriving an alternative all-order factorization formula. Our formula is equivalent to that of Ref. [17] if we were to compute to all orders in perturbation theory. They are not equivalent when truncated at a finite order, however, and our formula vividly shows how it acquires ν dependence at a finite order and allows us to quantify the scale uncertainty associated with the precise choice of ν. Our understanding also captures how ν dependence arises in spite of α_s being ν independent, as well as how it constitutes an independent source of scale uncertainties even though there is only one kind of logarithms being resummed.

For the derivation of our factorization formula, we will employ a form of analytic regulator to regulate rapidity divergences that is similar to, but different from, what has used by Refs. [17, 18]. Our analytic regulator shares the same great advantage that there is no need to perform 'zero-bin subtraction' [20] to avoid double-counting modes in the overlapping regions, as the integrals in the overlapping regions become scaleless and thus vanish once consistently multipole-expanded [21]. This type of analytic regulator is also shown to preserve fundamental properties such as gauge invariance [18]. The difference between our and their forms of analytic regulator is that our regulator treats the collinear and anti-collinear sectors in a symmetric manner by introducing two independent rapidity RG scales, while the regulator of Refs. [17, 18] treats the two sectors asymmetrically but needs only one rapidity scale. As a consistency check, we will show in Appendix A that our factorization formula can also be obtained from the regulator of Refs. [17, 18]. Therefore, we can continue to take advantage of the enormous technical simplicity and ease of the analytic regulator approach for practical calculations while correctly taking into account rapidity scale uncertainties at the same time.

As an application of our factorization formula and method of estimating rapidity scale uncertainty, we will update our earlier work of the WW production with jet-veto at the LHC [22], which used the formulation of Ref. [17] and thereby lacked scale uncertainties from ν dependence as well as suffered from another accidental cancellation that led to an underestimation of the total scale uncertainties. We will see that, while the central values of the predictions remain completely unaltered, the scale uncertainties increase and the consistency between the next-to-leading logarithm (NLL) and next-to-next-to-leading logarithm (NNLL) calculations improves.

II. THE SET-UP, INGREDIENTS, AND GOAL

Even though the conceptual points we will be making are quite general, for definiteness we discuss a SCET formalism that is applicable to the resummation of jet-veto logarithms for the production of a color-neutral object [23], such as a higgs boson [17, 23, 24] [21] [25–27], W^+W^- [22, 28] or other di-bosons [29], at a hadron collider. Jet veto rejects an event if it contains a jet whose transverse momentum is larger than a prescribed jet-veto scale $p_T^{\rm veto}$.[3] Using M to denote the invariant mass of the color-neutral system, and keeping only the relevant features explicit, we can express the differential jet-veto cross-section with respect to M as[4]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}M} \sim H(\mu)\, Z_s(\mu,\nu,\bar\nu)\, B(\mu,\nu)\, \bar B(\mu,\bar\nu)\,, \qquad (2)$$

where μ is the usual RG scale associated with UV divergence, while both ν and $\bar\nu$ are rapidity RG scales. For

[2] In practice, there is also scale dependence from parton distribution functions (PDFs), but that is beside the point. One could talk about QED instead of QCD.

[3] We consider the simplest case where $p_T^{\rm veto}$ is constant, but it could in principle be chosen to depend on other variables in the process, such as the rapidity of the jet [30].

[4] If the color-neutral state is a higgs boson, it would be a total jet-veto cross-section with M fixed to the higgs boson mass.

generality, we have introduced two rapidity scales because in principle the location of collinear-vs-soft boundary is independent of that of soft-vs-anti-collinear boundary. We now define and explain each of H, $Z_{\rm s}$, B and $\bar{B}$ below.

The *hard function* $H(\mu)$ is the squared amplitude for the hard parton collision with the center-of-momentum energy M that produces the color-neutral system in question. Being identical to the corresponding squared amplitude in the full theory, the hard function has no rapidity divergences and is hence independent of ν and $\bar{\nu}$. The scale μ has the usual interpretation as the scale or boundary beyond which the modes would be regarded as necessarily highly off-shell. So, μ is a proxy for the level of virtuality of the degrees of freedom in the EFT, which suggests $\mu^2 \sim p^2 \sim \bar{p}^2 \sim p_{\rm s}^2 \sim \lambda^2 M^2$. Further noting that $p_\perp \sim \bar{p}_\perp \sim p_{{\rm s}\perp} \sim \lambda M \sim p_{\rm T}^{\rm veto}$ in the presence of jet-veto, we expect $\mu \sim p_{\rm T}^{\rm veto}$. Later we will see how this choice is indeed forced upon us by the formalism. The job of μ-RGEs is thus to take us from the high scale $\mu \sim M$ (the *hard scale*) where the SCET is matched onto the full theory, down to the low scale $\mu \sim p_{\rm T}^{\rm veto}$ (the *factorization scale*) where the cross-section (2) is evaluated, resumming some of the logarithms of the form $\log(M/p_{\rm T}^{\rm veto})$. (The remaining logarithms of the form $\log(M/p_{\rm T}^{\rm veto})$ will be resummed by rapidity RGEs, as we discuss below.)

The *beam functions* $B(\mu,\nu)$ and $\bar{B}(\mu,\bar{\nu})$ [31, 32] are respectively the PDFs for the collinear and anti-collinear partons *with the jet-veto.* That is, if we schematically write a quark PDF as

$$\phi \sim \sum_X \langle N|\overline{\psi}|X\rangle\langle X|\psi|N\rangle\,, \tag{3}$$

where ψ and N are respectively the collinear quark field and the nucleon in question, and the summation goes over all possible hadronic states X, then the beam function for the same quark and nucleon is given by

$$B \sim \sum_X{}' \langle N|\overline{\psi}|X\rangle\langle X|\psi|N\rangle\,, \tag{4}$$

which is the same as before down to every little (implicit) detail except that the summation $\sum'$ only goes over X that satisfies the jet-veto condition. The collinear beam function $B(\mu,\nu)$ suffers from rapidity divergence when we send k_+ (the + component of integration variables) toward zero and thus 'invade' the territory of soft modes. The regulator we introduce in Section III regulates this divergence and trades it for a scale ν. This regulator does not introduce any other new scale, so we expect the logarithms from k_+ integration to have the form $\log(\nu/\xi P_+)$, because the only physical scale in the + direction is ξP_+ (where ξ is the parton momentum fraction and $P_+/2$ the proton momentum in the collinear beam). This suggests that the good scale choice that minimizes the logarithms should be $\nu \sim \xi P_+$. Similarly, the anti-collinear beam function $\bar{B}(\mu,\bar{\nu})$ has rapidity divergence as $k_- \to 0$, which is regulated in favor of a scale $\bar{\nu}$. Our regulator by design treats the collinear and anti-collinear sectors symmetrically, so we expect $\bar{\nu} \sim \bar{\xi} P_-$ (where $\bar{\xi}$ and $P_-/2$ are the parton momentum fraction and proton momentum in the anti-collinear beam). Integrations over the $\perp$ component are also divergent, which we regulate by dimensional regularization acting on the $d-2$ transverse dimensions with the scale μ. As we already discussed, the $\perp$ component controls the degree of virtuality of the partons, so we expect the μ-dependent logarithms to have the form $\log(\mu/p_{\rm T}^{\rm veto})$ in the presence of jet-veto, suggesting the choice $\mu \sim p_{\rm T}^{\rm veto}$.

Finally, $Z_{\rm s}(\mu,\nu,\bar{\nu})$ is called the *soft function*, which is a matrix element of a SCET operator consisting only of soft modes, evaluated between states containing only soft modes. Rapidity divergences arise when we send k_+ or k_- to infinity and invade the land of collinear or anti-collinear modes. The regulator trades the k_+ and k_- divergences for ν and $\bar{\nu}$, respectively. Alternatively, in our context, the expression (2) tells us that $Z_{\rm s}$ can also be regarded as a renormalization constant that absorbs the rapidity divergences and ν- and $\bar{\nu}$-dependences of the product of operators $B\bar{B}$. In the former interpretation, the soft modes are 'integrated in' in the SCET and we can directly calculate the soft function diagrammatically. In the latter, the soft modes are not in the theory and $Z_{\rm s}$ is determined by matching $Z_{\rm s}B\bar{B}$ onto the full theory at some appropriate scale. Either way, what we actually do in practice are essentially identical, while both interpretations can be useful conceptually. So we will discuss the two viewpoints in parallel below. Most importantly, whether it is viewed as a soft function or renormalization constant, $Z_{\rm s}$ should not know anything about ξP_+, $\bar{\xi} P_-$, nor M. Thus, the degree of virtuality imposed by jet-veto, $\mu \sim p_{\rm T}^{\rm veto}$, is the only physical scale entering $Z_{\rm s}$, so the good scale choice must be given by $\nu \sim \bar{\nu} \sim \mu$ to avoid large $\log(\nu/\mu)$ and $\log(\bar{\nu}/\mu)$.

We can now state our goal. For B and $\bar{B}$, we have high scales $\nu \sim \xi P_+$ and $\bar{\nu} \sim \bar{\xi} P_-$, which are both of order M. For $Z_{\rm s}$, we have low scales $\nu \sim \bar{\nu} \sim \mu$, which is of order $p_{\rm T}^{\rm veto}$. The job of rapidity RGEs is then to take us from the high scales to the low scales, resumming the logarithms of the form $\log(M/p_{\rm T}^{\rm veto})$, the same form as what the μ-RGEs resum. As we alluded in Section I, this is not surprising as there is actually only one large ratio of scales, $1/\lambda$, in the theory. This is also reflected to the correlation between the rapidity scales and the virtuality scale, $\nu \sim \bar{\nu} \sim \mu$. So, our problem is to analyze how imperfect this correlation can be at a finite order in perturbation theory and to quantify the associated uncertainty.

III. (RAPIDITY) RG EQUATIONS

In this section, we will derive an all-order form of factorization formula to be used in Eq. (2), that is, the precise form of "$Z_{\rm s}(\mu,\nu,\bar{\nu})\,B(\mu,\nu)\,\bar{B}(\mu,\bar{\nu})$" there. The reader who wishes to skip the derivation and read about

the discussion of rapidity scale uncertainty may directly move on to Section IV.

To regulate rapidity divergences, we employ the following form of analytic regulator, where for each phase-space integral[5] with momentum k we insert

$$\left(\frac{\nu}{k_+}\right)^{\alpha}\theta(k_+-k_-)+\left(\frac{\bar\nu}{k_-}\right)^{\bar\alpha}\theta(k_--k_+)\,. \tag{5}$$

For a collinear k and at the leading order in λ (i.e., $\mathcal{O}(\lambda^0)$), we should neglect k_- in comparison with k_+ inside $\theta(k_+-k_-)$, which renders the step function trivial. Thus, this regulator in practice simply amounts to inserting $(\nu/k_+)^\alpha$ in the phase-space integration for each collinear particle, and similarly just inserting $(\bar\nu/k_-)^{\bar\alpha}$ for each anti-collinear particle. This symmetric treatment of the collinear and anti-collinear sectors then implies that B and $\bar B$ are identical up to the trivial relabelling $\nu\leftrightarrow\bar\nu$, $\alpha\leftrightarrow\bar\alpha$, etc.

As all regulators do, the regulator (5) introduces artificial new scales into the problem, ν and $\bar\nu$, which are proxies for the boundaries separating collinear modes from soft modes, and anti-collinear modes from soft modes, respectively. The requirement that physical observables should be independent of ν and $\bar\nu$ leads to rapidity RGEs. We also introduce separate jet-veto scales $p_{\rm T}^{\rm veto}$ and $\bar p_{\rm T}^{\rm veto}$ for the collinear and anti-collinear sectors, respectively, as they can in principle be chosen independently and it has proven useful to do so [17].[6]

Now, as we take the $\alpha\to0$ and $\bar\alpha\to0$ limits, $1/\alpha$ and $1/\bar\alpha$ poles appear. Just like a $1/\epsilon$ pole in dimensional regularization comes with $\log\mu$, the $1/\alpha$ and $1/\bar\alpha$ poles are respectively accompanied by

$$L\equiv\log\frac{\nu}{\xi P_+}\,,\quad \bar L\equiv\log\frac{\bar\nu}{\bar\xi P_-}\,. \tag{6}$$

Here, $\xi P_+/2$ and $\bar\xi P_-/2$ are respectively the energies of the collinear and anti-collinear partons that enter the hard function H, while $P_+/2$ and $P_-/2$ are the energies of their respective parent hadrons. These logarithms are the only sources of ν and $\bar\nu$ dependences. The form of L can be understood by observing that the only physical scale entering in k_+ integration in the collinear sector is ξP_+. The form of $\bar L$ then follows from symmetry.

The beam functions also depend on the scale μ of dimensional regularization, which regulates divergence from $k_\perp$ integration. Then, since the jet-veto scale is the only physical scale in the $\perp$ direction in the EFT (as the hard physics of order M has been already integrated out), the only dependence on μ besides through $\alpha_{\rm s}(\mu)$ is via the following logarithms:

$$L_\perp\equiv\log\frac{\mu^2}{(p_{\rm T}^{\rm veto})^2},\quad \bar L_\perp\equiv\log\frac{\mu^2}{(\bar p_{\rm T}^{\rm veto})^2}. \tag{7}$$

A. The μ-RG equations

Without loss of generality, the μ-RGEs can be written as

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\log B(\mu,\nu)&=\sum_{p=0}^{\infty}f^{(p)}(\xi,\alpha_{\rm s},L_\perp)\,L^p\,,\\ \mu\frac{\partial}{\partial\mu}\log \bar B(\mu,\bar\nu)&=\sum_{p=0}^{\infty}f^{(p)}(\bar\xi,\alpha_{\rm s},\bar L_\perp)\,\bar L^p\,,\end{aligned} \tag{8}$$

where the μ dependences are in $\alpha_{\rm s}$, $L_\perp$, and $\bar L_\perp$, while the ν and $\bar\nu$ dependences are in L and $\bar L$, respectively. The functions $f^{(p)}$ in the $\bar B$ equation are the same functions as those in the B equation, owing to the symmetry of analytic regulator (5).

The equations (8) are actually too general and their forms can be constrained significantly. Physical observables of our interest, such as the WW jet-veto cross-section, depend only on the product $Z_{\rm s}B\bar B$, as in Eq. (2). Being a renormalization constant, $Z_{\rm s}$ cannot depend on dynamical variables such as ξ, $\bar\xi$, P_+, or P_-. Alternatively, being the soft function, i.e., a matrix element of soft fields between soft states, it cannot depend on the momenta of the (anti-)collinear partons nor of the colliding hadrons. We therefore require that the sum of the right-hand sides of the μ-RGEs (8), i.e.,

$$A\equiv\sum_{p=0}^{\infty}f^{(p)}(\xi,\alpha_{\rm s},L_\perp)\,L^p+\sum_{p=0}^{\infty}f^{(p)}(\bar\xi,\alpha_{\rm s},\bar L_\perp)\,\bar L^p\,, \tag{9}$$

should be independent of ξ, $\bar\xi$, P_+, and P_- at any values of μ, ν, and $\bar\nu$. These four variables, however, are not all independent of each other but are constrained as $\xi P_+\bar\xi P_-=M^2$ because we are considering the quantity of the form (2), which is a function of M. So, we instead consider

$$\widetilde A\equiv A+\ell\left(\frac{\xi P_+\bar\xi P_-}{M^2}-1\right), \tag{10}$$

where ℓ is a Lagrange multiplier varied independently of ξ, $\bar\xi$, P_+, and P_-, so that the condition $\partial\widetilde A/\partial\ell=0$ gives us the constraint back. Then, the requirement $\partial\widetilde A/\partial P_+=0$ implies

$$f^{(1)}=\ell\,,\quad f^{(2)}=f^{(3)}=\cdots=0\,. \tag{11}$$

The first relation here means that $f^{(1)}(\xi,\alpha_{\rm s},L_\perp)$ is actually independent of ξ, i.e., $f^{(1)}(\xi,\alpha_{\rm s},L_\perp)=f^{(1)}(\alpha_{\rm s},L_\perp)$, because the Lagrange multiplier ℓ is by definition independent of the remaining independent variables, ξ, $\bar\xi$, P_+,

[5] As shown in Ref. [18], rapidity divergences only occur in phase space integrals.

[6] However, we still assume $\bar p_{\rm T}^{\rm veto}\sim p_{\rm T}^{\rm veto}$ to avoid introducing another hierarchy of scales into the theory. We will see later that the formalism would indeed break down if $\bar p_{\rm T}^{\rm veto}$ is very different from $p_{\rm T}^{\rm veto}$.

and P_-. However, ℓ may still depend on other parameters and variables. Most generally, it may depend on $\alpha_{\rm s}$. It cannot actually depend on $L_\perp$ nor $\bar{L}_\perp$ for the following reason. Our symmetric treatment of the collinear and anti-collinear sectors implies that if ℓ depended on $L_\perp$, it should also depend on $\bar{L}_\perp$ symmetrically, i.e., $\ell(\alpha_{\rm s}, L_\perp, \bar{L}_\perp) = \ell(\alpha_{\rm s}, \bar{L}_\perp, L_\perp)$. However, the above condition $f^{(1)}(\alpha_{\rm s}, L_\perp) = \ell$ tells us that ℓ does not depend on $\bar{L}_\perp$, so by symmetry it does not depend on $L_\perp$ either. We thus conclude that $f^{(1)}(\alpha_{\rm s}, L_\perp) = f^{(1)}(\alpha_{\rm s})$. Then, applying this and the conditions (11) to the requirement $\partial\widetilde{A}/\partial\xi = 0$, we get

$$\frac{\partial}{\partial\xi} f^{(0)}(\xi, \alpha_{\rm s}, L_\perp) = 0\,. \tag{12}$$

We thus have narrowed down the μ-RGEs (8) to

$$\begin{aligned} \mu\frac{\partial}{\partial\mu}\log B(\mu,\nu) &= f^{(1)}(\alpha_{\rm s})\,L + f^{(0)}(\alpha_{\rm s}, L_\perp)\,,\\ \mu\frac{\partial}{\partial\mu}\log \bar{B}(\mu,\bar{\nu}) &= f^{(1)}(\alpha_{\rm s})\,\bar{L} + f^{(0)}(\alpha_{\rm s}, \bar{L}_\perp)\,. \end{aligned} \tag{13}$$

Next, since the hard function $H(\mu)$ is independent of ν and $\bar{\nu}$, the ν and $\bar{\nu}$ dependences in the product $B\bar{B}$ should be completely cancelled by $Z_{\rm s}$ in the product $Z_{\rm s}B\bar{B}$. So, we must have

$$\mu\frac{\partial}{\partial\mu}\log Z_{\rm s}(\mu,\nu,\bar{\nu}) = -f^{(1)}(\alpha_{\rm s})\,L_{\rm s} + F^{(0)}(\alpha_{\rm s}, L_\perp, \bar{L}_\perp)\,, \tag{14}$$

where

$$L_{\rm s} \equiv \log\frac{\nu\bar{\nu}}{\mu^2}\,. \tag{15}$$

We can determine $f^{(1)}$ and a combination of $f^{(0)}$ and $F^{(0)}$ from the requirement that the μ dependence in the product $Z_{\rm s}B\bar{B}$ must be cancelled when $Z_{\rm s}B\bar{B}$ is multiplied by the hard function H, i.e., $\mu\,\partial\log(Z_{\rm s}B\bar{B}H)/\partial\mu = 0$. The μ-RGE of the hard function can be parametrized as

$$\mu\frac{\rm d}{{\rm d}\mu}\log H(\mu) = 2\Gamma L_M + 4\gamma\,, \tag{16}$$

where $L_M \equiv \log\bigl(M^2/\mu^2\bigr)$ with the coefficients Γ (the cusp anomalous dimension) and γ (the anomalous dimension) that can be calculated order-by-order in $\alpha_{\rm s}$ using perturbation theory. To identify $f^{(1)}$, we demand that the derivatives $\mu\,\partial/\partial\mu$ and $M\,\partial/\partial M$ commute when acting on $\log(Z_{\rm s}B\bar{B}H)$. This identifies the M dependence in L_M with that in $L + \bar{L}$ from Eq. (13), i.e.,[7]

$$f^{(1)}(\alpha_{\rm s}) = 2\Gamma\,. \tag{17}$$

[7] Mathematically, a μ-independent integration constant should be included in the right-hand side of the relation (17). However, such integration constant is absent in perturbation theory as all terms in RGEs contain $\alpha_{\rm s}$, which does depend on μ.

Cancelling the remaining μ dependence in $\log(Z_{\rm s}B\bar{B}H)$, we then get

$$f^{(0)}(\alpha_{\rm s}, L_\perp) + f^{(0)}(\alpha_{\rm s}, \bar{L}_\perp) + F^{(0)}(\alpha_{\rm s}, L_\perp, \bar{L}_\perp) = -4\gamma\,. \tag{18}$$

B. The ν-RG equations

Physical quantities must be independent of the renormalization scheme. In particular, the solution of RGEs must be independent of the shape of the path in the μ-ν-$\bar{\nu}$ space and can only depend on the endpoints of the path. This path independence requires that

$$\begin{aligned} \mu\frac{\partial}{\partial\mu}\nu\frac{\partial}{\partial\nu}\log B(\mu,\nu) &= \nu\frac{\partial}{\partial\nu}\mu\frac{\partial}{\partial\mu}\log B(\mu,\nu)\\ &= f^{(1)}(\alpha_{\rm s})\,, \end{aligned} \tag{19}$$

where the second equality has used Eq. (13). Thus, $\mu\,\partial/\partial\mu$ acting on $\nu\,\partial\log B/\partial\nu$ gives us a ν-independent function, $f^{(1)}(\alpha_{\rm s})$. So, if there are any ν-dependent terms in $\nu\,\partial\log B/\partial\nu$, they must be *in*dependent of μ so that they get annihilated by $\mu\,\partial/\partial\mu$. However, since the perturbation series is an expansion in $\alpha_{\rm s}$, every term does depend on μ via $\alpha_{\rm s}$. Therefore, there cannot be any ν-dependent terms in $\nu\,\partial\log B/\partial\nu$. We thus arrive at the general form of rapidity RGEs:

$$\begin{aligned} \nu\frac{\partial}{\partial\nu}\log B(\mu,\nu) &= g(\mu)\,,\\ \bar{\nu}\frac{\partial}{\partial\bar{\nu}}\log \bar{B}(\mu,\bar{\nu}) &= \bar{g}(\mu)\,, \end{aligned} \tag{20}$$

where

$$g(\mu) \equiv \sum_{p=0}^{\infty} g^{(p)}(\alpha_{\rm s})\,L_\perp^p\,, \quad \bar{g}(\mu) \equiv \sum_{p=0}^{\infty} g^{(p)}(\alpha_{\rm s})\,\bar{L}_\perp^p\,. \tag{21}$$

(Note that these are expansions in $L_\perp$ and $\bar{L}_\perp$ unlike the μ-RGEs (8), which were expansions in L or $\bar{L}$.) We have written $g^{(p)}$ as a function of only $\alpha_{\rm s}$, as it turns out that $g^{(p)}$ is independent of ξ (or $\bar{\xi}$) as we will see shortly. We again require that the product $Z_{\rm s}B\bar{B}$ to be independent of ν and $\bar{\nu}$, which implies that the rapidity RGEs for $Z_{\rm s}$ must be given by

$$\begin{aligned} \nu\frac{\partial}{\partial\nu}\log Z_{\rm s}(\mu,\nu,\bar{\nu}) &= -g(\mu)\,,\\ \bar{\nu}\frac{\partial}{\partial\bar{\nu}}\log Z_{\rm s}(\mu,\nu,\bar{\nu}) &= -\bar{g}(\mu)\,. \end{aligned} \tag{22}$$

The path independence for $Z_{\rm s}$ does not provide us with any new constraints. First, the $Z_{\rm s}$ rapidity RGEs (22) trivially lead to $[\nu\,\partial/\partial\nu,\, \bar{\nu}\,\partial/\partial\bar{\nu}]\log Z_{\rm s} = 0$. Also, the path independence condition (19) for B and the similar condition for $\bar{B}$ readily imply that $[\mu\,\partial/\partial\mu,\, \nu\,\partial/\partial\nu]\log Z_{\rm s} = [\mu\,\partial/\partial\mu,\, \bar{\nu}\,\partial/\partial\bar{\nu}]\log Z_{\rm s} = 0$.

C. The all-order factorization formula and the choice of scales

We are now ready to solve the rapidity RGEs (20) and (22) for an arbitrary but fixed μ. Solving them is trivial as the right-hand sides of all the ν-RGEs are independent of ν and $\bar{\nu}$. We solve for B starting from an 'initial' point $\nu = \nu_{\rm B}$ to an arbitrary final ν, $\bar{B}$ from $\bar{\nu}_{\rm B}$ to $\bar{\nu}$, and $Z_{\rm S}$ from $(\nu_{\rm S}, \bar{\nu}_{\rm S})$ to $(\nu, \bar{\nu})$. Then, combining all the solutions, we get

$$\begin{aligned} &Z_{\rm S}(\mu,\nu,\bar{\nu})\, B(\mu,\nu)\, \bar{B}(\mu,\bar{\nu}) \\ &= \left(\frac{\nu_{\rm S}}{\nu_{\rm B}}\right)^{g(\mu)} \left(\frac{\bar{\nu}_{\rm S}}{\bar{\nu}_{\rm B}}\right)^{\bar{g}(\mu)} \hat{Z}_{\rm S} \hat{B} \hat{\bar{B}}(\mu) \end{aligned} \tag{23}$$

with

$$\hat{Z}_{\rm S} \hat{B} \hat{\bar{B}}(\mu) \equiv Z_{\rm S}(\mu, \nu_{\rm S}, \bar{\nu}_{\rm S})\, B(\mu, \nu_{\rm B})\, \bar{B}(\mu, \bar{\nu}_{\rm B})\,. \tag{24}$$

To determine what value of μ we should choose when we evaluate $Z_{\rm S}(\mu,\nu,\bar{\nu})\, B(\mu,\nu)\, \bar{B}(\mu,\bar{\nu})$ in the factorization formula (23), let us examine more closely the path independence condition (19). Applying the relation (17) to it, we can rewrite it as a condition for $g(\mu)$:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g(\mu) = f^{(1)}(\alpha_{\rm s}) = 2\Gamma\,. \tag{25}$$

Satisfying this condition order-by-order in $L_\perp$ in the expansion (21), we get the following nontrivial recursion relations among the coefficient functions $g^{(p)}(\alpha_{\rm s})$:

$$\begin{aligned} 2g^{(1)} + \beta g'^{(0)} &= f^{(1)} = 2\Gamma\,, \\ 2pg^{(p)} + \beta g'^{(p-1)} &= 0 \qquad (p \geq 2)\,, \end{aligned} \tag{26}$$

where $g'^{(p)} \equiv \mathrm{d}g^{(p)}/\mathrm{d}\alpha_{\rm s}$ and $\beta \equiv \mu\, \mathrm{d}\alpha_{\rm s}/\mathrm{d}\mu$. Let us parametrize $g^{(0)}$, Γ, and β as

$$\begin{aligned} g^{(0)} &= a_{\rm s}\, d_1 + a_{\rm s}^2\, d_2 + \cdots\,, \\ \Gamma &= a_{\rm s}(\Gamma_0 + a_{\rm s}\Gamma_1 + \cdots)\,, \\ \beta &= -2\alpha_{\rm s} a_{\rm s}(\beta_0 + a_{\rm s}\beta_1 + \cdots)\,, \end{aligned} \tag{27}$$

where $a_{\rm s} \equiv \alpha_{\rm s}/(4\pi)$, and all the coefficients, $d_{1,2,\cdots}$, $\Gamma_{0,1,\cdots}$, $\beta_{0,1,\cdots}$, can be calculated in perturbation theory. Since none of these coefficients depend on ξ, the functions $g^{(p)}$ are indeed independent of ξ for all p as we alluded earlier. Combined with the parametrization (27), the recursion relations (26) give

$$\begin{aligned} g^{(1)} &= a_{\rm s}\Gamma_0 + a_{\rm s}^2(\Gamma_1 + \beta_0 d_1) + \cdots\,, \\ g^{(2)} &= \qquad\quad a_{\rm s}^2 \frac{\Gamma_0\beta_0}{2} \qquad\quad + \cdots\,. \end{aligned} \tag{28}$$

Note that $g^{(p)} = \mathcal{O}(\alpha_{\rm s}^p)$ for all $p \geq 1$. This means that in order for us to be able to reliably truncate the expansions (21) at any finite order in $\alpha_{\rm s}$, the logarithms $L_\perp$ and $\bar{L}_\perp$ must be small, i.e., $L_\perp, \bar{L}_\perp \ll \alpha_{\rm s}^{-1}$. This is possible only if we choose

$$\mu \sim p_{\rm T}^{\rm veto} \sim \bar{p}_{\rm T}^{\rm veto}\,, \tag{29}$$

There are two things to note here. Firstly, the right choice of the factorization scale at which we should evaluate the factorization formula (23) is on the order of the jet-veto scale $p_{\rm T}^{\rm veto}$, rather than the hard scale M. Secondly, our formalism would not be applicable to the case in which $p_{\rm T}^{\rm veto}$ and $\bar{p}_{\rm T}^{\rm veto}$ are hierarchically different. In such case, there would remain large logarithms of the form $\log(p_{\rm T}^{\rm veto}/\bar{p}_{\rm T}^{\rm veto})$ that are not resummed by our formalism.

The remaining scales $\nu_{\rm B}$, $\bar{\nu}_{\rm B}$, $\nu_{\rm S}$, and $\bar{\nu}_{\rm S}$ should be chosen to minimize L, $\bar{L}$, and $L_{\rm S}$. These logarithms vanish if

$$\nu_{\rm B} = \xi P_+\,, \quad \bar{\nu}_{\rm B} = \bar{\xi} P_-\,, \quad \nu_{\rm S} = \bar{\nu}_{\rm S} = \mu\,. \tag{30}$$

The factorization formula (23) then becomes

$$\begin{aligned} &Z_{\rm S}(\mu,\nu,\bar{\nu})\, B(\mu,\nu)\, \bar{B}(\mu,\bar{\nu}) \\ &= \left(\frac{\mu}{\xi P_+}\right)^{g(\mu)} \left(\frac{\mu}{\bar{\xi} P_-}\right)^{\bar{g}(\mu)} \hat{Z}_{\rm S} \hat{B} \hat{\bar{B}}(\mu)\,. \end{aligned} \tag{31}$$

However, since our formalism is valid only for $\bar{p}_{\rm T}^{\rm veto} \sim p_{\rm T}^{\rm veto}$, there is no practical benefit to be gained by choosing the two jet-veto scales differently in experimental analyses. So, hereafter, we will mostly focus on the special case,

$$\bar{p}_{\rm T}^{\rm veto} = p_{\rm T}^{\rm veto}\,, \quad \bar{L}_\perp = L_\perp\,, \quad \bar{g} = g\,. \tag{32}$$

The factorization formula (31) now takes a very simple form:

$$Z_{\rm S}(\mu,\nu,\bar{\nu})\, B(\mu,\nu)\, \bar{B}(\mu,\bar{\nu}) = \left(\frac{\mu^2}{M^2}\right)^{g(\mu)} \hat{Z}_{\rm S} \hat{B} \hat{\bar{B}}(\mu)\,. \tag{33}$$

Before closing this section, we would like to remind the reader that the only properties of the analytic regulator (5) that were actually used in the derivation of the formula (33) are the forms of the logarithms (6) and (7). Therefore, any other regulator that only produces those logarithms will lead to exactly the same form of factorization formula as Eq. (33). It is actually even more robust than that as we will see in Appendix A where a regulator that gives different forms of logarithms still arrives at essentially the same factorization formula, although it seems difficult to pin down exactly how robust it is. Finally, let us also remark that it should be straightforward to repeat our analysis for objects other than the jet-veto beam and soft functions, such as the transverse momentum dependent parton distribution functions (TMD-PDFs) for small $p_{\rm T}$ resummation.

IV. MAIN RESULTS AND DISCUSSIONS

Let us now examine what happens to the factorization formula (31) or (33) if we deviate from the scale choice (30). We are now ready to show that, while such

deviations cancel out order-by-order in α_s, the cancellation is incomplete if the series is truncated in a way consistent with the EFT power-counting rules, rendering rapidity RG to be an independent source of scale uncertainty.

A. Finite-order truncation and additional scale uncertainties from rapidity RG.

If we could calculate $Z_S B\bar{B}$ exactly or to all orders in perturbation theory, the scale choice (30) would be unique and unambiguous. This is a corollary of the fact that the right-hand sides of ν-RGEs (20) and (22) are independent of ν and $\bar{\nu}$. For example, one might object to the choice (30) that the vanishing of L_S only implies $\nu_S\bar{\nu}_S = \mu^2$, so we could instead choose $\nu_S = r\mu$ and $\bar{\nu}_S = r^{-1}\mu$ with some r. Glancing back at Eq. (23), we see that this (apparently) alternative choice would amount to multiplying the prefactor $(\mu/\xi P_+)^g\,(\mu/\bar{\xi}P_-)^{\bar{g}}$ in the factorization formula (31) by a factor of $r^{g-\bar{g}}$, while changing $\hat{Z}_S$ to $Z_S(\mu, r\mu, r^{-1}\mu)$ at the same time. But the ν-RGE (22) tells us that $Z_S(\mu, r\mu, r^{-1}\mu) = r^{-g+\bar{g}}\hat{Z}_S$, so r cancels out. In fact, any one of ν_B, $\bar{\nu}_B$, ν_S and $\bar{\nu}_S$ can be varied independently, e.g., $\nu_B = \xi P_+ \to r\,\xi P_+$, without producing any net effect. This 'r invariance' is exact, and since α_s is independent of ν and $\bar{\nu}$, it is exact order-by-order in α_s.

However, the whole point of using an EFT is to resum the large logarithm $\log(1/\lambda)$. So, in the EFT power counting, $\alpha_s\log(1/\lambda)$ is by definition parametrically regarded as $\mathcal{O}(1)$. Thus, the EFT perturbation series is *not* a literal power series in α_s but an effective power series in α_s *with* $\log(1/\lambda)$ *counted as* $\mathcal{O}(\alpha_s^{-1})$. When the factorization formula (33) is truncated at a finite order according to this EFT power counting, the r invariance is no longer exact. Let us say we are aiming at an $\mathcal{O}(\alpha_s^k)$ accuracy with the EFT power counting (i.e., the so-called N^{k+1}LL accuracy). Then, the right-hand side of factorization formula (33) should be truncated as

$$\left(\frac{\mu^2}{M^2}\right)^{h_1\alpha_s+h_2\alpha_s^2+\cdots+h_{k+1}\alpha_s^{k+1}} \left[\hat{Z}_S\hat{B}\hat{\bar{B}}(\mu)\right]_{\text{computed to }\mathcal{O}(\alpha_s^k)}, \tag{34}$$

where h_n are α_s-independent coefficients defined via the expansion $g(\mu) = \sum_{n=1}^{\infty} h_n\,\alpha_s^n$. Recalling the definition (24) and the scale choices (29) and (30), we see that the object $\left[\hat{Z}_S\hat{B}\hat{\bar{B}}(\mu)\right]$ has no logarithms of ν or $\bar{\nu}$ at all, nor any large logarithms of μ. That is why the object $\left[\hat{Z}_S\hat{B}\hat{\bar{B}}(\mu)\right]$ is calculated above to literally $\mathcal{O}(\alpha_s^k)$, with no need to take into account the EFT power counting. In contrast, the exponent of μ^2/M^2 is computed above to $\mathcal{O}(\alpha_s^{k+1})$, because $\log(M^2/\mu^2) \sim \log(M^2/(p_T^{\text{veto}})^2) \sim \log(1/\lambda) \sim \mathcal{O}(\alpha_s^{-1})$ and the EFT power counting must be applied.

Let us now examine how $Z_S B\bar{B}$ changes from the 'central value' (34) if we deviate from the scale choice (30). Again, looking back at Eq. (23) (with $\bar{g} = g$) and using the ν-RGEs (20) and (22), we see that the prefactor $(\mu^2/M^2)^{h_1\alpha_s+h_2\alpha_s^2+\cdots+h_{k+1}\alpha_s^{k+1}}$ will get multiplied by $r^{h_1\alpha_s+h_2\alpha_s^2+\cdots+h_{k+1}\alpha_s^{k+1}}$. For the product of matrix elements $\left[\hat{Z}_S\hat{B}\hat{\bar{B}}(\mu)\right]_{\text{to }\mathcal{O}(\alpha_s^k)}$, there are two possibilities depending on how the r dependence is calculated, which agree to $\mathcal{O}(\alpha_s^k)$ but differ at $\mathcal{O}(\alpha_s^{k+1})$ and hence leading to different estimates of rapidity scale uncertainty of $\mathcal{O}(\alpha_s^{k+1})$. One way is to start with an $\mathcal{O}(\alpha_s^k)$-truncated "initial" $\hat{Z}_S\hat{B}\hat{\bar{B}}$ evaluated at the scale (30) and then evolve it to a different scale choice by using the $\mathcal{O}(\alpha_s^k)$-truncated ν-RGEs. The other way is to directly evaluate $\hat{Z}_S\hat{B}\hat{\bar{B}}$ at $\mathcal{O}(\alpha_s^k)$ at a point away from the choice (30). In the first method, the product of matrix element $\left[\hat{Z}_S\hat{B}\hat{\bar{B}}(\mu)\right]_{\text{to }\mathcal{O}(\alpha_s^k)}$ would be simply multiplied by $r^{-h_1\alpha_s-h_2\alpha_s^2-\cdots-h_k\alpha_s^k}$. Then, combined with the factor of $r^{h_1\alpha_s+h_2\alpha_s^2+\cdots+h_{k+1}\alpha_s^{k+1}}$ coming from $(\mu^2/M^2)^{h_1\alpha_s+h_2\alpha_s^2+\cdots+h_{k+1}\alpha_s^{k+1}}$, we would be left with a simple net r dependence, $r^{h_{k+1}\alpha_s^{k+1}}$. This is not optimal for the purpose of estimating rapidity scale uncertainty, however, because it is only sensitive to a single coefficient h_{k+1}, which may be accidentally small or large. On the other hand, the second method leads to

$$\begin{aligned} &Z_S(\mu,\nu,\bar{\nu})\,B(\mu,\nu)\,\bar{B}(\mu,\bar{\nu}) \\ &= \left(\frac{\mu^2}{M^2}\right)^{h_1\alpha_s+\cdots+h_{k+1}\alpha_s^{k+1}} r^{h_1\alpha_s+h_2\alpha_s^2+\cdots+h_{k+1}\alpha_s^{k+1}} \\ &\quad\times\left[r^{-h_1\alpha_s-h_2\alpha_s^2-\cdots-h_k\alpha_s^k}\,\hat{Z}_S\hat{B}\hat{\bar{B}}(\mu)\right]_{\text{to }\mathcal{O}(\alpha_s^k)}, \end{aligned} \tag{35}$$

which is more robust as it involves not only all of h_1 through h_{k+1} but also the whole perturbative series of $\hat{Z}_S\hat{B}\hat{\bar{B}}(\mu)$ to $\mathcal{O}(\alpha_s^k)$. Hence, we expect that varying r by an $\mathcal{O}(1)$ factor in the expression (35) should give us a plausible estimate of rapidity scale uncertainty of $\mathcal{O}(\alpha_s^{k+1})$. We therefore propose the formula (35) as our N^{k+1}LL truncation of the factorization formula (33) with rapidity scale uncertainty appropriately taken into account.

B. Comparison with the literature

We can now discuss exactly what is missing in the all-order factorization formula of Ref. [17], which exhibits no rapidity scale dependence. They used an analytic regulator different from Eq. (5), which treats the collinear and anti-collinear sectors differently, but as we will show in Appendix A, it nonetheless leads to a factorization formula essentially identical to Eq. (33):

$$Z_S(\mu)\,B(\mu,\nu)\,\bar{B}(\mu,\nu) = \left(\frac{\mu^2}{M^2}\right)^{g(\mu)} \hat{Z}_S\hat{B}\hat{\bar{B}}(\mu), \tag{36}$$

where $\hat{Z}_S\hat{B}\hat{\bar{B}}(\mu) \equiv Z_S(\mu)\,B(\mu,\nu_B)\,\bar{B}(\mu,\bar{\nu}_B)$ and we have taken $\bar{p}_T^{\text{veto}} = p_T^{\text{veto}}$ as we did for Eq. (33). Then, repeat-

ing the analysis of Section IV A, we arrive at a correct $\mathrm{N}^{k+1}\mathrm{LL}$ expression

$$\begin{aligned} &Z_{\mathrm{S}}(\mu)\, B(\mu,\nu)\, \bar{B}(\mu,\nu) \\ &= r^{h_{k+1}\alpha_{\mathrm{s}}^{k+1}} \left(\frac{\mu^2}{M^2}\right)^{h_1\alpha_{\mathrm{s}}+\cdots+h_{k+1}\alpha_{\mathrm{s}}^{k+1}} \Big[\hat{Z}_{\mathrm{S}}\hat{B}\hat{\bar{B}}(\mu)\Big]_{\text{to } \mathcal{O}(\alpha_{\mathrm{s}}^k)}, \end{aligned} \tag{37}$$

which is essentially the same as Eq. (35).

To obtain the form of factorization formula of Ref. [17] from Eq. (36), we first set $r = 1$, since Ref. [17] has made the scale choice $\nu_{\mathrm{B}} = \xi P_+$ and $\bar{\nu}_{\mathrm{B}} = \mu^2/(\bar{\xi}P_-)$ (see Appendix A) implicitly and did not consider variations from it. We also rewrite μ^2/M^2 as $\mu^2/M^2 = [(p_{\mathrm{T}}^{\mathrm{veto}})^2/M^2]\,\mathrm{e}^{L_\perp}$. Then, instead of the formula (37), we would get

$$\begin{aligned} &Z_{\mathrm{S}}(\mu)\, B(\mu,\nu)\, \bar{B}(\mu,\nu) \\ &= \left(\frac{(p_{\mathrm{T}}^{\mathrm{veto}})^2}{M^2}\right)^{h_1\alpha_{\mathrm{s}}+\cdots+h_{k+1}\alpha_{\mathrm{s}}^{k+1}} \Big[Z'_{\mathrm{S}}B'\bar{B}'(\mu)\Big]_{\text{to } \mathcal{O}(\alpha_{\mathrm{s}}^k)}, \end{aligned} \tag{38}$$

where $Z'_{\mathrm{S}}B'\bar{B}'(\mu) \equiv \mathrm{e}^{gL_\perp}\hat{Z}_{\mathrm{S}}\hat{B}\hat{\bar{B}}(\mu)$. This is the form presented in Ref. [17]. Since there is no r dependence, there is no rapidity scale uncertainty to talk about. One might argue that the lack of the r-dependent prefactor $r^{h_{k+1}\alpha_{\mathrm{s}}^{k+1}}$ could be (retrospectively) justified by saying that it is $\mathcal{O}(\alpha_{\mathrm{s}}^{k+1})$ for $r = \mathcal{O}(1)$. However, in our analysis of Section IV A, the factor $r^{h_{k+1}\alpha_{\mathrm{s}}^{k+1}}$ is a deviation in the prefactor from the central value $(\mu^2/M^2)^{h_1\alpha_{\mathrm{s}}+h_2\alpha_{\mathrm{s}}^2+\cdots+h_{k+1}\alpha_{\mathrm{s}}^{k+1}}$, which is an $\mathcal{O}(\alpha_{\mathrm{s}}^k)$ quantity in the EFT power counting. This "fundamental ambiguity in the exponentiation of the rapidity logarithms" was first pointed out in Ref. [13]. This ambiguity itself should be regarded as part of rapidity scale uncertainty, and our form (37) (or (35)) captures this ambiguity.

V. AN APPLICATION: THE WW PRODUCTION WITH JET-VETO

Let us now apply the factorization formula (35) to the WW production with jet-veto at $k = 1$, i.e., at NNLL. For the exponent coefficients h_1 and h_2, we need $\mathcal{O}(\alpha_{\mathrm{s}}^2)$ calculations. Combining the parametrization (27) and solutions (28) gives

$$\begin{aligned} h_1 &= \frac{1}{4\pi}(d_1 + \Gamma_0 L_\perp)\,, \\ h_2 &= \frac{1}{(4\pi)^2}\left[d_2 + (\Gamma_1 + \beta_0 d_1)L_\perp + \frac{\Gamma_0\beta_0}{2}L_\perp^2\right]. \end{aligned} \tag{39}$$

For the WW production at the 7- and 8-TeV LHC runs, the $q\bar{q}$ initial states dominate, so the coefficients $d_{1,2}$ and $\Gamma_{0,1}$ must be found for the fundamental representation of SU(3). From explicit diagrammatic calculations, one finds $d_1 = 0$, while d_2 can be found in Ref. [21], where it is denoted by d_2^{veto}, while $\Gamma_{0,1}$ and β_0 can be found in Ref. [14]. The dependence of jet-veto on the jet-radius parameter R enters the calculation through d_2^{veto}, and this R dependence was first studied in Refs. [33–35] [21].

For $\big[\hat{Z}_{\mathrm{S}}\hat{B}\hat{\bar{B}}(\mu)\big]$, we need $\mathcal{O}(\alpha_{\mathrm{s}})$ calculations. From an explicit 1-loop calculation, one finds that $\hat{B}$ is given by

$$\hat{B} = \sum_{i=q,g}\int_\xi^1 \frac{\mathrm{d}z}{z}\hat{I}_{q\leftarrow i}(z, p_{\mathrm{T}}^{\mathrm{veto}}, \mu)\,\phi_i(\xi/z,\mu)\,, \tag{40}$$

where ϕ_i is the PDF of parton i for the proton and

$$\begin{aligned} &\hat{I}_{j\leftarrow i}(z, p_{\mathrm{T}}^{\mathrm{veto}}, \mu) \\ &= \left[1 - a_{\mathrm{s}}\left(\Gamma_0\frac{L_\perp^2}{4} + \gamma_0 L_\perp\right)\right]\delta(1-z)\,\delta_{ji} \\ &\quad - a_{\mathrm{s}}\left[\mathcal{P}_{j\leftarrow i}^{(1)}(z)\,\frac{L_\perp}{2} - \mathcal{R}_{j\leftarrow i}(z)\right], \end{aligned} \tag{41}$$

where $\mathcal{P}_{j\leftarrow i}^{(1)}$ and $\mathcal{R}_{j\leftarrow i}$ can be found in Appendix A of Ref. [28]. $\hat{\bar{B}}$ is given by the same formula except for the obvious replacement of "q" by "$\bar{q}$". Then, the whole $\hat{Z}_{\mathrm{S}}\hat{B}\hat{\bar{B}}$ can be determined by matching it onto the corresponding full QCD expression at $\mu = p_{\mathrm{T}}^{\mathrm{veto}}$ and evolving it to another μ by using the μ-RGEs (13) and (14). Alternatively, $\hat{Z}_{\mathrm{S}}$ can be directly calculated diagrammatically by regarding it as the soft function. Either way, one finds $\hat{Z}_{\mathrm{S}} = 1$.

We can now update the scale uncertainty estimates for the WW jet-veto cross-sections presented in Ref. [22]. Note that the central values of the predictions, defined by $r = 1$ and $\mu = p_{\mathrm{T}}^{\mathrm{veto}}$, are clearly unaffected. However, scale uncertainties will necessarily be modified. In particular, the scale uncertainties arising from varying r from 1/2 to 2 will be added in quadrature to those from varying μ from $p_{\mathrm{T}}^{\mathrm{veto}}/2$ to $2p_{\mathrm{T}}^{\mathrm{veto}}$, treating the two sources of uncertainties as independent. We will not include power corrections as they were shown to be very small in Ref. [36] for the WW production at the 7- or 8-TeV LHC run with $p_{\mathrm{T}}^{\mathrm{veto}} = 25$–$30$ GeV. For other processes and/or choices of parameters, power corrections can become important, and a formalism for smoothly transitioning from the resummation-dominated regime to the fixed-order regime is described in Ref. [37]. Including the uncertainties from non-perturbative effects [38] [28] is beyond the scope of this paper.

In Fig. 1, the NLL and NNLL WW jet-veto cross-sections are shown for the $\sqrt{s} = 8$ TeV LHC run with the anti-k_{T} jet algorithm [39] with the jet-radius parameter $R = 1$. For the NLL, h_2 was discarded and $\hat{B}$ and $\hat{\bar{B}}$ are set to the corresponding PDFs. The 'π^2 resummation' [40–42] is included in all the cases as in our earlier work [22].[8] The plot on the right is based on the naive factorization formula as used in Ref. [22], where we see

[8] For a thorough discussion of scale uncertainty associated with 'π^2 vs no π^2', see Ref. [36].

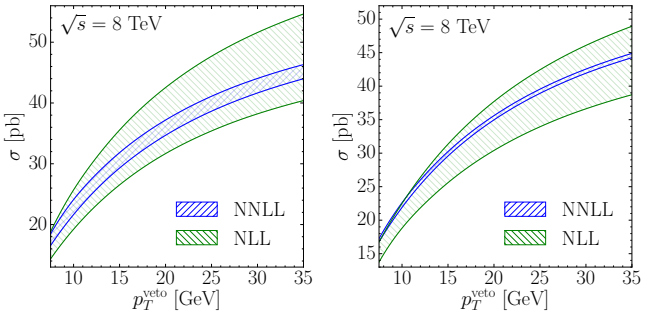

FIG. 1. The NLL and NNLL jet-veto cross-sections for WW production for the 8-TeV LHC run with $R = 1$. The scale-uncertainty bands in the left plot are obtained from the correct factorization formula (35), while in the right-hand plot, they are obtained following [22].

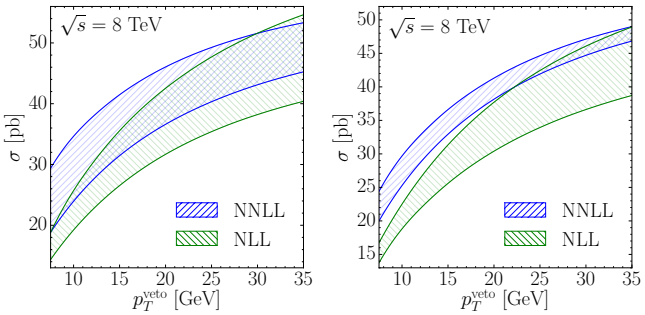

FIG. 2. Same as Fig. 1 but for $R = 0.4$.

that the NNLL uncertainty is ridiculously small ($\lesssim 1\%$), much smaller than the well-established NLO scale uncertainty of the *inclusive* WW cross-section, $\sim$ 3–4% [43]. On the other hand, the plot on the left is obtained from the correct factorization formula (35), which now exhibits scale uncertainties of reasonable size.

In Fig. 2, we have repeated the computations for $R = 0.4$, a typical value chosen in LHC experiments. Clearly, the plot on the left, which uses the correct factorization formula (35), displays a better convergence of the NLL and NNLL predictions than the plot on the right, which is the one presented in Ref. [22].

Comparing the two plots on the left, we see that the NNLL band in the $R = 0.4$ case is much broader than that in the $R = 1$ case. This broadening is mainly due to a larger value of h_2 in Eq. (35). The large value of h_2 for $R = 0.4$ comes from a large d_2 due to a $\log R$ multiplied by a large coefficient enhanced by factors of $C_{\rm A}$ and $T_{\rm F} n_{\rm f}$ (further multiplied by numerical factors of $\mathcal{O}(10)$) arising from gluon splitting.[9] Perhaps more importantly, note that a modest change in h_2 can lead to substantial change in the final numerical results as h_2 appears in the exponent of the ratio μ/M. Not having h_2, on the other hand, the NLL bands of the $R = 1$ and $R = 0.4$ cases have a similar width.

[9] For a detailed study of R dependence at $\mathcal{O}(\alpha_{\rm s}^3)$ and the resummation of $\log R$, see Refs. [44] and [45], respectively. Including the results of those studies into our analysis is beyond the scope of the present paper.

ACKNOWLEDGMENTS

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Appendix A: An analytic regulator with a collinear-anti-collinear asymmetry

Instead of the analytic regulator (5), we can also use a different analytic regulator where we use

$$\left(\frac{\nu}{k_+}\right)^\alpha \tag{A1}$$

for *both* the collinear and anti-collinear sectors [17, 18], which can be quite convenient for practical calculations due to its simplicity. While the asymmetric treatment of the collinear and anti-collinear sectors means we cannot obtain $\bar{B}$ from B by simply relabelling as $p_{\rm T}^{\rm veto} \to \bar{p}_{\rm T}^{\rm veto}$, etc., this asymmetric regulator has the simplicity that it only introduces one rapidity scale, ν. Since the action of the asymmetric regulator (A1) in the collinear sector is the same as the symmetric case (5), the form of L is unmodified from the form (6). On the other hand, in the anti-collinear sector, the presence of $(\nu/k_+)^\alpha$ instead of $(\bar{\nu}/k_-)^{\bar{\alpha}}$ implies that $\bar{L}$ should be changed to

$$\bar{L}' \equiv \log \frac{\nu \bar{\xi} P_-}{\mu^2} \,. \tag{A2}$$

This form can be understood from the fact that $k_+ = k_{\rm T}^2/k_-$ (with $k_{\rm T} \equiv |\vec{k}_\perp|$), where the $k_{\rm T}$ integration is cut off by dimensional regularization at the scale μ while the k_- integration picks up $\bar{\xi} P_-$, the only physical scale in the anti-collinear sector.

Then, repeating a similar analysis as we did with the symmetric regulator in Section III, we find that the μ-RGEs (13) and (14) are modified as

$$\begin{aligned}
\mu \frac{\partial}{\partial \mu} \log B(\mu, \nu) &= f^{(1)}(\alpha_{\rm s})\, L + f^{(0)}(\alpha_{\rm s}, L_\perp) \,, \\
\mu \frac{\partial}{\partial \mu} \log \bar{B}(\mu, \nu) &= -f^{(1)}(\alpha_{\rm s})\, \bar{L}' + \bar{f}^{(0)}(\alpha_{\rm s}, \bar{L}_\perp) \,, \\
\mu \frac{\partial}{\partial \mu} \log Z_{\rm s}(\mu, \nu) &= \widetilde{F}^{(0)}(\alpha_{\rm s}, L_\perp, \bar{L}_\perp) \,,
\end{aligned} \tag{A3}$$

where the B equation is the same as before as nothing has changed in the collinear sector, while the $\bar{B}$ and $Z_{\rm S}$ equations are changed. Due to the lack of symmetry, the function $\bar{f}^{(0)}$ is not a priori the same as $f^{(0)}$, and $\widetilde{F}_0$ has no reason to be the same as $F^{(0)}$ either. Also observe that the right-hand side of the $Z_{\rm S}$ equation above has no dependence on ν. The constraints (17) and (18) remain as before with the obvious replacements $f^{(0)}(\alpha_{\rm s}, \bar{L}_\perp) \to \bar{f}_0(\alpha_{\rm s}, \bar{L}_\perp)$ and $F^{(0)} \to \widetilde{F}^{(0)}$.

For rapidity RGEs, we find that the ν-RGEs (20) and (22) are modified as

$$\begin{aligned}
\nu\frac{\partial}{\partial\nu}\log B(\mu,\nu) &= \sum_{p=0}^{\infty} g^{(p)}(\alpha_{\rm s})\, L_\perp^p\,,\\
\bar{\nu}\frac{\partial}{\partial\nu}\log \bar{B}(\mu,\nu) &= -\sum_{p=0}^{\infty} g^{(p)}(\alpha_{\rm s})\, \bar{L}_\perp^p\,,\\
\nu\frac{\partial}{\partial\nu}\log Z_{\rm S}(\mu,\nu) &= -\sum_{p=0}^{\infty} g^{(p)}(\alpha_{\rm s})\, (L_\perp^p - \bar{L}_\perp^p)\,.
\end{aligned} \tag{A4}$$

Again, the B equation is unmodified. The negative sign in the $\bar{B}$ equation here is a consequence of the negative sign in the $\bar{B}$ equation in (A3) combined with a relation similar to Eq. (25) from path-independence requirement in the μ-ν space.

Solving Eq. (A4) for B from $\nu_{\rm B}$ to an arbitrary final ν, $\bar{B}$ from $\bar{\nu}_{\rm B}$ to ν, and $Z_{\rm S}$ from $\nu_{\rm S}$ to ν we find that the factorization formula (23) is modified as

$$\begin{aligned}
&Z_{\rm S}(\mu,\nu)\, B(\mu,\nu)\, \bar{B}(\mu,\nu)\\
&= \left(\frac{\nu_{\rm S}}{\nu_{\rm B}}\right)^{g(\mu)} \left(\frac{\bar{\nu}_{\rm B}}{\nu_{\rm S}}\right)^{\bar{g}(\mu)} \hat{Z}_{\rm S}\hat{B}\hat{\bar{B}}(\mu)
\end{aligned} \tag{A5}$$

with $\hat{Z}_{\rm S}\hat{B}\hat{\bar{B}}(\mu) \equiv Z_{\rm S}(\mu,\nu_{\rm S})\, B(\mu,\nu_{\rm B})\, \bar{B}(\mu,\bar{\nu}_{\rm B})$. We can again show '$r$ invariance' as we did in Section IV A, and the minimization of $\bar{L}'_\perp$ tells us that the choice for $\bar{\nu}_{\rm B}$ in Eq. (30) should be modified to

$$\bar{\nu}_{\rm B} = \frac{\mu^2}{\bar{\xi} P_-}\,, \tag{A6}$$

while $\nu_{\rm B}$ remains unchanged. For μ and $\nu_{\rm S}$, we again see $\nu_{\rm S} \sim \mu \sim p_{\rm T}^{\rm veto}$ by matching $\hat{Z}_{\rm S}\hat{B}\hat{\bar{B}}$ to the corresponding full QCD quantity, since $\hat{Z}_{\rm S}\hat{B}\hat{\bar{B}}$ (by definition) only depends on μ, $\nu_{\rm S}$, and $p_{\rm T}^{\rm veto}$, while the QCD counterpart has only μ and $p_{\rm T}^{\rm veto}$. Alternatively, if we integrate in the soft modes in the SCET and regard $Z_{\rm S}$ as the soft function, it is clear that $p_{\rm T}^{\rm veto}$ is the only physical scale in the problem and hence $\mu \sim \nu_{\rm S} \sim p_{\rm T}^{\rm veto}$. Hence, we choose $\nu_{\rm S} = \mu$ as in the symmetric regulator counterpart (30).

For the case of practical interest $\bar{p}_{\rm T}^{\rm veto} = p_{\rm T}^{\rm veto}$, we have $\bar{g} = g$, which makes $\nu_{\rm S}$ cancel out in Eq. (A5), and the ratio $\bar{\nu}_{\rm B}/\nu_{\rm B}$ becomes μ^2/M^2 as in the symmetric regulator case. Furthermore, from the ν-RGE for $Z_{\rm S}$ in Eq. (A4), we see that $Z_{\rm S}$ becomes ν independent when $\bar{g} = g$ (while it may still depend on μ). Therefore, the factorization formula (A5) reduces to Eq. (36), which is essentially identical to the symmetric regulator counterpart (33). The discussion of rapidity scale uncertainty in Section IV then goes through essentially unmodified and leads to the final factorization formula (37).

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