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Abstract

We derive expressions for the neutrino mixing parameters that result from complex perturbations on (1) the Majorana neutrino mass matrix (in the basis of charged lepton mass eigenstates) and on (2) the charged lepton mass matrix, for arbitrary initial (unperturbed) mixing matrices. In the first case, we find that the phases of the elements of the perturbation matrix, and the initial values of the Dirac and Majorana phases, strongly impact the leading order corrections to the neutrino mixing parameters and phases. For experimentally compatible scenarios wherein the initial neutrino mass matrix has $\mu-\tau$ symmetry, we find that the Dirac phase can take any value under small perturbations. Similarly, in the second case, perturbations to the charged lepton mass matrix can generate large corrections to the mixing angles and phases of the PMNS matrix. As an illustration of our generalized procedure, we apply it to a situation in which nonstandard scalar and nonstandard vector interactions simultaneously affect neutrino oscillations.

1 Introduction

After decades of neutrino oscillation experiments, the mixing pattern in the lepton sector has been well established [1]. There are one small and two large mixing angles, and two mass squared differences that differ by a factor of 30 in the neutrino sector. Numerous neutrino mixing scenarios have been proposed in the literature to explain such a non-trivial mixing pattern; for a recent review, see Ref. [2]. The most attractive scenarios are those with mixing patterns motivated by simple symmetries, such as tri-bimaximal mixing [3], bimaximal mixing [4], and golden ratio mixing [5]. All three mixing scenarios have $\theta_{13} = 0$, $\theta_{23} = 45°$, and are a subset of the more general $\mu - \tau$ symmetry [6, 7]. However, recent measurements of θ_{13} by short-baseline reactor experiments Daya Bay [11], RENO [12], Double Chooz [13], and long-baseline accelerator experiments T2K [14], MINOS [15] strongly disfavor $\theta_{13} = 0$. Therefore models with simple symmetries need additional features to explain the observed neutrino mixing pattern.

A modified approach to explain the data is to treat the simple mixing scenarios as the underlying model and add perturbations to accommodate the discrepancy between theoretical predictions and experimental data. In Ref. [7], we took $\mu - \tau$ symmetry (in the charged lepton basis), as the underlying model, and added real perturbations to the Majorana neutrino mass matrices to explain the data. We found that small perturbations can cause large corrections to θ_{12}, and the experimental data can be explained by most $\mu - \tau$ symmetric mixing scenarios with perturbations of similar magnitude.

After the discovery that the mixing angle θ_{13} is relatively large, many scenarios without $\mu - \tau$ symmetry have been proposed [16]. Motivated by this, and by the fact that real perturbations have no effect on the Dirac and Majorana phases, in this paper we consider arbitrary initial mixing for the underlying model and generalize the perturbation results to the complex space. Under the assumption that the charged lepton mass matrix is diagonal, we derive analytic formulae for the leading order (LO) corrections to the three mixing angles and the Dirac and Majorana phases. We find that the phases of the elements of the perturbation matrix, and the initial values of the Dirac and Majorana phases, strongly impact the LO corrections to the neutrino mixing parameters. We also perform a numerical study of complex perturbations on initial neutrino mass matrices with $\mu - \tau$ symmetry. We find that the Dirac phase can take any value under small perturbations for experimentally compatible scenarios.

Since the mixings in both the charged lepton sector and neutrino sector contribute to the observed PMNS matrix, we explore the case in which the charged lepton mass matrix is not diagonal, and consider small complex perturbations to the charged lepton mass matrix as well. Since the initial mixing matrix in the charged lepton sector is unconstrained, small perturbations in the charged lepton sector could have large effects on the 1-2 mixing in the charged lepton sector, which lead to large changes in the mixing angles and phases in the PMNS matrix.

In addition, as an application of our generalized perturbation procedure, we study neutrino oscillations with matter effects from both nonstandard scalar and nonstandard vector interactions. Nonstandard scalar interactions add small perturbations to the neutrino mass matrix, which yield corrections to the vacuum mixing angles and mass-squared differences. By using our formalism, we demonstrate how expressions for neutrino oscillation probabilities that simultaneously depend on nonstandard scalar and nonstandard vector interactions, can be obtained.

This paper is organized as follows. In Section 2, we work in the diagonal charged lepton basis and derive analytic formulae for the mixing parameters that result from real and complex perturbations. In Section 3, we perform a numerical analysis of complex perturbations to neutrino mass matrices with $\mu - \tau$ symmetry. In Section 4, we calculate corrections to the three mixing angles and phases in the PMNS matrix from perturbations in the charged lepton sector. In Section 5, we apply our perturbation results to study neutrino oscillations with both nonstandard scalar and nonstandard vector interactions. We summarize our results in Section 6.

2 Perturbations on the neutrino mass matrix

Our goal in this section is to obtain the LO corrections to all the physical parameters under small perturbations on the initial Majorana neutrino mass matrix assuming the charged lepton mass matrix to be diagonal. The final (resultant) mass matrix can be written as the sum of an initial matrix M_0 and a perturbation matrix E, i.e.,

$$M = M_0 + E = U_0^* \overline{M_0} U_0^\dagger + \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{pmatrix}, \tag{1}$$

where $\overline{M_0} = \mathrm{diag}(m_1^0, m_2^0, m_3^0)$, and U_0 is the initial mixing matrix.

The final mass matrix can also be written as

$$M = U^* \overline{M} U^\dagger, \tag{2}$$

where U and $\overline{M}$ have the same form as U_0 and $\overline{M_0}$. From neutrino oscillation experiments we know that m_1 and m_2 are nearly degenerate, so here we assume $|\epsilon_{ij}|\,, |\delta m_{21}^0| \ll |\delta m_{31}^0|$.

2.1 Real case

For simplicity, we consider the real case first. The mixing matrix U_0 can be written as

$$U_0 = R_{23}^0 R_{13}^0 R_{12}^0\,, \tag{3}$$

where R_{ij}^0 is the rotation matrix in the $i-j$ plane with a rotation angle θ_{ij}^0. Then Eq. (1) can be rewritten as

$$\begin{aligned} M &= m_1^0 \mathrm{I} + R_{23}^0 R_{13}^0 R_{12}^0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^0 & 0 \\ 0 & 0 & \delta m_{31}^0 \end{pmatrix} (R_{12}^0)^T (R_{13}^0)^T (R_{23}^0)^T + E \\ &= m_1^0 \mathrm{I} + R_{23}^0 R_{13}^0 \left[R_{12}^0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta m_{21}^0 & 0 \\ 0 & 0 & \delta m_{31}^0 \end{pmatrix} (R_{12}^0)^T + E' \right] (R_{13}^0)^T (R_{23}^0)^T\,, \end{aligned} \tag{4}$$

where $E' = (R^0_{13})^T (R^0_{23})^T E R^0_{23} R^0_{13}$, $\delta m^0_{ji} = m^0_j - m^0_i$, and I is the 3×3 identity matrix. We employ the following notation:

$$\begin{aligned}
&\epsilon_1 = \epsilon_{11}\,, \quad \epsilon_2 = \epsilon_{12} c^0_{23} - \epsilon_{13} s^0_{23}\,, \quad \epsilon_3 = \epsilon_{12} s^0_{23} + \epsilon_{13} c^0_{23}\,, \\
&\epsilon_4 = \epsilon_{22} (c^0_{23})^2 + \epsilon_{33} (s^0_{23})^2 - \epsilon_{23} s^0_{2\times 23}\,, \\
&\epsilon_5 = \epsilon_{23} c^0_{2\times 23} + \frac{1}{2}(\epsilon_{22} - \epsilon_{33}) s^0_{2\times 23}\,, \\
&\epsilon_6 = \epsilon_{22} (s^0_{23})^2 + \epsilon_{33} (c^0_{23})^2 + \epsilon_{23} s^0_{2\times 23}\,,
\end{aligned} \tag{5}$$

where c^0_{ij}, s^0_{ij}, $c^0_{2\times ij}$ and $s^0_{2\times ij}$ denote $\cos\theta^0_{ij}$, $\sin\theta^0_{ij}$, $\cos(2\theta^0_{ij})$ and $\sin(2\theta^0_{ij})$, respectively. Then $\epsilon'_{ij} \equiv (E')_{ij}$ can be written explicitly as

$$\begin{aligned}
&\epsilon'_{11} = \epsilon_1 (c^0_{13})^2 + \epsilon_6 (s^0_{13})^2 - \epsilon_3 s^0_{2\times 13}\,, \quad \epsilon'_{12} = \epsilon_2 c^0_{13} - \epsilon_5 s^0_{13}\,, \\
&\epsilon'_{13} = \epsilon_3 c^0_{2\times 13} + \frac{1}{2}(\epsilon_1 - \epsilon_6) s^0_{2\times 13}\,, \quad \epsilon'_{22} = \epsilon_4\,, \\
&\epsilon'_{23} = \epsilon_2 s^0_{13} + \epsilon_5 c^0_{13}\,, \quad \epsilon'_{33} = \epsilon_1 (s^0_{13})^2 + \epsilon_6 (c^0_{13})^2 + \epsilon_3 s^0_{2\times 13}\,.
\end{aligned} \tag{6}$$

In order to obtain the final mixing matrix that diagonalizes M, we use a procedure that is similar to that in Ref. [17]. We first put zeros in the 2-3 and 1-3 entries of the matrix in the square bracket of Eq. (4) by using rotations $R_{23}(\delta'_{23})$ and $R_{13}(\delta'_{13})$, respectively. To LO in $\mathcal{O}(|\epsilon_{ij}|/|\delta m^0_{31}|)$, we have

$$\delta'_{23} \approx \frac{\epsilon'_{23}}{\delta m^0_{31}}\,, \quad \delta'_{13} \approx \frac{\epsilon'_{13}}{\delta m^0_{31}}\,, \tag{7}$$

and the LO correction to m^0_3 is

$$\delta m_3 = \epsilon'_{33}\,. \tag{8}$$

Note that since $|\epsilon_{ij}| \ll |\delta m^0_{31}|$, after the two rotations in the 2-3 and 1-3 planes, the matrix in the square bracket of Eq. (4) becomes block-diagonal and the 1-2 submatrix remains unchanged to leading order. Hence we can rewrite Eq. (4) as

$$M = m^0_1 \mathrm{I} + V \begin{pmatrix} M' & 0 \\ 0 & \delta m^0_{31} \end{pmatrix} V^T + O(|\epsilon_{ij}|^2/\delta m^0_{31})\,, \tag{9}$$

where $V = R^0_{23} R^0_{13} R_{23}(\delta'_{23}) R_{13}(\delta'_{13}) R^0_{12}$, and

$$M' = \begin{pmatrix} \epsilon'_{11}(c^0_{12})^2 + \epsilon'_{22}(s^0_{12})^2 - \epsilon'_{12} s^0_{2\times 12} & \epsilon'_{12} c^0_{2\times 12} + \frac{1}{2}(\epsilon'_{11} - \epsilon'_{22}) s^0_{2\times 12} \\ \epsilon'_{12} c^0_{2\times 12} + \frac{1}{2}(\epsilon'_{11} - \epsilon'_{22}) s^0_{2\times 12} & \epsilon'_{11}(s^0_{12})^2 + \epsilon'_{22}(c^0_{12})^2 + \epsilon'_{12} s^0_{2\times 12} + \delta m^0_{21} \end{pmatrix}\,, \tag{10}$$

which can be diagonalized by the rotation $R_{12}(\xi')$ with

$$\xi' = \frac{1}{2} \arctan \frac{2\epsilon'_{12} c^0_{2\times 12} - (\epsilon'_{22} - \epsilon'_{11}) s^0_{2\times 12}}{(\epsilon'_{22} - \epsilon'_{11}) c^0_{2\times 12} + 2\epsilon'_{12} s^0_{2\times 12} + \delta m^0_{21}} . \tag{11}$$

The corrections to m_1 and m_2 can be written as

$$\delta m_i = \frac{\epsilon'_{11} + \epsilon'_{22}}{2} \pm \frac{1}{2} \left[\delta m^0_{21} - \sqrt{\Delta} \right] , \tag{12}$$

where $\Delta = (\delta m^0_{21})^2 + 4(\epsilon'_{12})^2 + (\epsilon'_{22} - \epsilon'_{11})^2 + 2\delta m^0_{21} \left[2\epsilon'_{12} s^0_{2\times 12} + (\epsilon'_{22} - \epsilon'_{11}) c^0_{2\times 12} \right]$, and the plus (minus) sign is for $i = 1$ (2). The final mass matrix is diagonalized by the following mixing matrix

$$U = R^0_{23} R^0_{13} R_{23}(\delta'_{23}) R_{13}(\delta'_{13}) R^0_{12} R_{12}(\xi') . \tag{13}$$

By comparing it to the standard parametrization, we find the LO corrections to the three mixing angles to be

$$\begin{aligned}
\delta\theta_{13} &= \delta'_{13} = \frac{\epsilon'_{13}}{\delta m^0_{31}} , \\
\delta\theta_{23} &= \frac{\delta'_{23}}{c^0_{13}} = \frac{\epsilon'_{23}}{c^0_{13} \delta m^0_{31}} , \\
\delta\theta_{12} &= \xi' = \frac{1}{2} \arctan \frac{2\epsilon'_{12} c^0_{2\times 12} - (\epsilon'_{22} - \epsilon'_{11}) s^0_{2\times 12}}{(\epsilon'_{22} - \epsilon'_{11}) c^0_{2\times 12} + 2\epsilon'_{12} s^0_{2\times 12} + \delta m^0_{21}} ,
\end{aligned} \tag{14}$$

where we have ignored the next-to-leading order correction to θ_{12}, which is $\mathcal{O}(|\epsilon_{ij}|/|\delta m^0_{31}|)$. For $\theta^0_{13} = 0$ and $\theta^0_{23} = \pi/4$, it is easy to verify that the corrections in Eq. (14) yield the results of Ref. [7] for the LO corrections,[1] which were obtained using degenerate perturbation theory. As noted in Ref. [7], the near degeneracy of m_1 and m_2 ($|\delta m^0_{21}| \ll |\delta m^0_{31}|$) implies that $\delta\theta_{12}$ can be large for small perturbations ($|\epsilon_{ij}| \ll |\delta m^0_{31}|$).

2.2 Complex case

For the complex case, the most general form for U_0 is

$$U_0 = R_{23}(\theta^0_{23}) U_{13}(\theta^0_{13}, \delta^0) R_{12}(\theta^0_{12}) P(\phi^0_2, \phi^0_3) , \tag{15}$$

[1]Also, for the next-to-leading order correction to θ_{12}, we obtain Eq. (14) of Ref. [7], except that δm^0_{21} in the denominator should be replaced by $\delta m^{(1)}_{21} \equiv (m^0_2 + \delta m_2) - (m^0_1 + \delta m_1)$.

where

$$R_{23}(\theta_{23}^0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23}^0 & s_{23}^0 \\ 0 & -s_{23}^0 & c_{23}^0 \end{pmatrix} , \quad U_{13}(\theta_{13}^0, \delta^0) = \begin{pmatrix} c_{13}^0 & 0 & e^{-i\delta^0} s_{13}^0 \\ 0 & 1 & 0 \\ -e^{i\delta^0} s_{13}^0 & 0 & c_{13}^0 \end{pmatrix} ,$$

$$R_{12}(\theta_{12}^0) = \begin{pmatrix} c_{12}^0 & s_{12}^0 & 0 \\ -s_{12}^0 & c_{12}^0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad P(\phi_2^0, \phi_3^0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_2^0/2} & 0 \\ 0 & 0 & e^{i\phi_3^0/2} \end{pmatrix} . \tag{16}$$

Because of the nonzero Majorana phases, in general, the mixing matrix would not remain unchanged if we subtract the identity matrix multiplied by a constant from the mass matrix. Hence we use a slightly different procedure to obtain the LO corrections for the complex case. We rewrite the final mass matrix as

$$M = U_0^* \overline{M_0} U_0^\dagger + E = U_0^* \left[\overline{M_0} + \tilde{E} \right] U_0^\dagger , \tag{17}$$

where $\tilde{E} = U_0^T E U_0$ can be explicitly written as

$$\tilde{E} = \begin{pmatrix} a & b e^{i\phi_2^0/2} & d e^{i\phi_3^0/2} \\ b e^{i\phi_2^0/2} & c e^{i\phi_2^0} & f e^{i(\phi_2^0+\phi_3^0)/2} \\ d e^{i\phi_3^0/2} & f e^{i(\phi_2^0+\phi_3^0)/2} & g e^{i\phi_3^0} \end{pmatrix} , \tag{18}$$

with

$$\begin{aligned}
a &= \epsilon_4 (s_{12}^0)^2 + [\epsilon_1 (c_{13}^0)^2 - \epsilon_3 s_{2\times13}^0 e^{i\delta^0} + \epsilon_6 (s_{13}^0)^2 e^{2i\delta^0}](c_{12}^0)^2 + (\epsilon_5 s_{13}^0 e^{i\delta^0} - \epsilon_2 c_{13}^0) s_{2\times12}^0 , \\
b &= \epsilon_2 c_{13}^0 c_{2\times12}^0 + [\epsilon_1 (c_{13}^0)^2 - \epsilon_4 + \epsilon_6 (s_{13}^0)^2 e^{2i\delta^0}] c_{12}^0 s_{12}^0 - [\epsilon_3 s_{2\times12}^0 s_{13}^0 c_{13}^0 + \epsilon_5 c_{2\times12}^0 s_{13}^0] e^{i\delta^0} , \\
c &= \epsilon_4 (c_{12}^0)^2 + [\epsilon_1 (c_{13}^0)^2 - \epsilon_3 s_{2\times13}^0 e^{i\delta^0} + \epsilon_6 (s_{13}^0)^2 e^{2i\delta^0}](s_{12}^0)^2 - (\epsilon_5 s_{13}^0 e^{i\delta^0} - \epsilon_2 c_{13}^0) s_{2\times12}^0 , \\
d &= (\epsilon_1 c_{12}^0 c_{13}^0 - \epsilon_2 s_{12}^0) s_{13}^0 e^{-i\delta^0} - \epsilon_6 c_{12}^0 c_{13}^0 s_{13}^0 e^{i\delta^0} + \epsilon_3 c_{12}^0 c_{2\times13}^0 - \epsilon_5 c_{13}^0 s_{12}^0 , \\
f &= (\epsilon_1 s_{12}^0 c_{13}^0 + \epsilon_2 c_{12}^0) s_{13}^0 e^{-i\delta^0} - \epsilon_6 s_{12}^0 c_{13}^0 s_{13}^0 e^{i\delta^0} + \epsilon_3 s_{12}^0 c_{2\times13}^0 + \epsilon_5 c_{13}^0 c_{12}^0 , \\
g &= \epsilon_6 (c_{13}^0)^2 + \epsilon_3 s_{2\times13}^0 e^{-i\delta^0} + \epsilon_1 (s_{13}^0)^2 e^{-2i\delta^0} ,
\end{aligned} \tag{19}$$

Similar to the real case, we apply a unitary matrix U_δ to $N \equiv \overline{M_0} + \tilde{E}$ such that there are zeros in the 2-3 and 1-3 entries of the matrix $U_\delta^T N U_\delta$. Since $|\epsilon_{ij}| \ll |\delta m_{31}^0|$, to LO in

$\mathcal{O}(|\epsilon_{ij}|/|\delta m^0_{31}|)$, U_δ can be written as

$$U_\delta = \begin{pmatrix} 1 & 0 & \delta_{13} \\ 0 & 1 & \delta_{23} \\ -\delta^*_{13} & -\delta^*_{23} & 1 \end{pmatrix} \,, \tag{20}$$

where

$$\delta_{13} \approx \frac{|d| e^{-i\phi_{13}}}{|m^0_3 - m^0_1 e^{-2i\phi_{13}}|} \,, \quad \delta_{23} \approx \frac{|f| e^{-i\phi_{23}}}{|m^0_3 - m^0_1 e^{-2i\phi_{23}}|} \,, \tag{21}$$

with $\tan\phi_{13} = \frac{m^0_3+m^0_1}{m^0_3-m^0_1}\tan\left[\arg(d) + \phi^0_3/2\right]$ and $\tan\phi_{23} = \frac{m^0_3+m^0_1}{m^0_3-m^0_1}\tan\left[\arg(f) + \frac{\phi^0_2+\phi^0_3}{2}\right]$. After block-diagonalization, the LO correction to m_3 is

$$\delta m_3 = \left| m^0_3 + g e^{i\phi^0_3} \right| - m^0_3 \,. \tag{22}$$

Note that the 1-2 submatrix of N remains unchanged to leading order after the block-diagonalization. Using the procedure described in Appendix A, we diagonalize this submatrix using the unitary matrix

$$U_{12}(\xi,\phi) = \begin{pmatrix} c_\xi & s_\xi e^{-i\phi} & 0 \\ -s_\xi e^{i\phi} & c_\xi & 0 \\ 0 & 0 & 1 \end{pmatrix} \,, \tag{23}$$

where

$$\phi = \arctan\frac{|a+m^0_1|\sin(\phi_a-\phi_b) - |ce^{i\phi^0_2}+m^0_2|\sin(\phi_c-\phi_b)}{|a+m^0_1|\cos(\phi_a-\phi_b) + |ce^{i\phi^0_2}+m^0_2|\cos(\phi_c-\phi_b)} \,, \tag{24}$$

$$\xi = \frac{1}{2}\arctan\frac{2|b|}{|ce^{i\phi^0_2}+m^0_2|\cos(\phi_c+\phi-\phi_b) - |a+m^0_1|\cos(\phi_a-\phi-\phi_b)} \,, \tag{25}$$

with $\phi_a = \arg(a+m^0_1)$, $\phi_b = \arg(b) + \phi^0_2/2$ and $\phi_c = \arg(ce^{i\phi^0_2}+m^0_2)$. In addition, we obtain the LO corrections to m_1 and m_2 as

$$\begin{aligned} \delta m_1 &= \left|(a+m^0_1)c^2_\xi + (ce^{i\phi^0_2}+m^0_2)s^2_\xi e^{2i\phi} - 2bs_\xi c_\xi e^{i\phi}\right| - m^0_1 \,, \\ \delta m_2 &= \left|(a+m^0_1)s^2_\xi e^{-2i\phi} + (ce^{i\phi^0_2}+m^0_2)c^2_\xi + 2bs_\xi c_\xi e^{-i\phi}\right| - m^0_2 \,. \end{aligned} \tag{26}$$

The final mixing matrix that diagonalizes M and makes the diagonal elements real and nonnegative can be written as

$$U = U_0 U_\delta U_{12}(\xi, \phi) P \,, \tag{27}$$

where $P = \mathrm{diag}(e^{i\omega_1/2}, e^{i\omega_2/2}, e^{i\omega_3/2})$, and

$$\begin{aligned}
\omega_1 &= -\arg\left[(a+m_1^0)c_\xi^2 + (ce^{i\phi_2^0}+m_2^0)s_\xi^2 e^{2i\phi} - 2bs_\xi c_\xi e^{i\phi}\right] , \\
\omega_2 &= -\arg\left[(a+m_1^0)s_\xi^2 e^{-2i\phi} + (ce^{i\phi_2^0}+m_2^0)c_\xi^2 + 2bs_\xi c_\xi e^{-i\phi}\right] , \\
\omega_3 &= -\arg\left(m_3^0 + ge^{i\phi_3^0}\right) .
\end{aligned} \tag{28}$$

As shown in Appendix B, the right-multiplication of $U_{12}(\xi, \phi)$ does not change θ_{13} and θ_{23}. Hence, the LO corrections to θ_{13} and θ_{23} come from the right-multiplication of U_δ. Since δ_{13} and δ_{23} are suppressed by a factor of $|\epsilon_{ij}|/|\delta m_{31}^0|$, while ξ and ϕ are not, the LO corrections to θ_{12} and the Dirac phases come from the right-multiplication of $U_{12}(\xi, \phi)$, and the LO corrections to the Majorana phases come from both $U_{12}(\xi, \phi)$ and P.

By comparing U to the standard parametrization, we obtain the LO corrections to the three mixing angles:

$$\delta\theta_{13} = \frac{|d| c_{12}^0 \cos(\delta^0 - \frac{\phi_3^0}{2} - \phi_{13})}{|m_3^0 - m_1^0 e^{-2i\phi_{13}}|} + \frac{|f| s_{12}^0 \cos(\delta^0 + \frac{\phi_2^0 - \phi_3^0}{2} - \phi_{23})}{|m_3^0 - m_1^0 e^{-2i\phi_{23}}|} \tag{29}$$

$$\delta\theta_{23} = -\frac{|d| s_{12}^0 \cos(\frac{\phi_3^0}{2} + \phi_{13})}{|m_3^0 - m_1^0 e^{-2i\phi_{13}}|} + \frac{|f| c_{12}^0 \cos(\frac{\phi_2^0 - \phi_3^0}{2} - \phi_{23})}{|m_3^0 - m_1^0 e^{-2i\phi_{23}}|} \tag{30}$$

$$\delta\theta_{12} = \arcsin\sqrt{\sin^2(\theta_{12}^0 + \xi) - \sin(2\theta_{12}^0)\sin(2\xi)\sin^2\frac{\phi_2^0 + 2\phi}{4}} - \theta_{12}^0 \,, \tag{31}$$

where t_{ij}^0 denotes $\tan\theta_{ij}^0$. The LO corrections to the three phases can be written as

$$\Delta\delta = \alpha - \beta \,, \tag{32}$$

$$\Delta\phi_2 = -2(\alpha + \beta) + \omega_2 - \omega_1 \,, \tag{33}$$

$$\Delta\phi_3 = -2\beta + \omega_3 - \omega_1 \,. \tag{34}$$

where

$$\alpha = -\arctan\frac{\tan\theta_{12}^0 \tan\xi \sin(\phi_2^0/2+\phi)}{1-\tan\theta_{12}^0\tan\xi\cos(\phi_2^0/2+\phi)}\,, \tag{35}$$

and

$$\beta = \arctan\frac{\tan\xi \sin(\phi_2^0/2+\phi)}{\tan\theta_{12}^0+\tan\xi\cos(\phi_2^0/2+\phi)}\,. \tag{36}$$

From Eq. (31), we see that $\delta\theta_{12}$ varies from $-\xi$ to $+\xi$ depending on the initial Majorana phase ϕ_2^0 and the perturbation phase ϕ. Since ξ and ϕ depend only on the ratios of linear combinations of ϵ_{ij}'s and δm_{21}^0, large corrections to θ_{12} and the Dirac and Majorana phases are possible even for small perturbations. However, corrections can be small in special cases, e.g., if ϕ_2^0 is close to 180° for the inverted hierarchy, ϕ approaches 90° and ξ is suppressed by a factor of $|\epsilon_{ij}|/(m_2^0+m_1^0)$, so that the corrections to θ_{12} and the Dirac and Majorana phases are also small.

Note that the corrections in the complex case are strongly dependent on the phases of ϵ_{ij}, and the initial values of the Dirac and Majorana phases. If we take ϵ_{ij}'s to be real, and set $\delta^0=\phi_2^0=\phi_3^0=0$ in Eqs. (29), (30) and (31), we recover Eq. (14).

3 Perturbations to $\mu-\tau$ symmetry

As an illustration of our analytic results, we study perturbations on initial neutrino mass matrices with $\mu-\tau$ symmetry. There are four classes of mixing with $\mu-\tau$ symmetry [7]: (a) $\theta_{23}^0=45°,\theta_{13}^0=0$; (b) $\theta_{23}^0=45°,\theta_{12}^0=0$; (c) $\theta_{23}^0=45°,\theta_{12}^0=90°$; (d) $\theta_{23}^0=45°,\delta^0=\pm 90°$. In Ref. [8], it was shown that the initial class (a) can be perturbed to class (d) for a specific model. Here we reproduce the results of Ref. [8] by applying our general perturbation formulae. The complex neutrino mass matrix of Ref. [8] can be written (in our phase convention) as

$$M = m_0\begin{pmatrix} 1+2\delta & 0 & 0 \\ 0 & \delta & -(1+\delta) \\ 0 & -(1+\delta) & \delta \end{pmatrix} + m_0\begin{pmatrix} 2\delta' & \delta'' & -\delta''^* \\ \delta'' & 0 & 0 \\ -\delta''^* & 0 & 0 \end{pmatrix}, \tag{37}$$

where m_0 is a common mass parameter, δ, δ' are real and $|\delta'|$, $|\delta''| \ll |\delta|$. We treat the first term on the right-hand side of Eq. (37) as the initial mass matrix and the second term as the perturbation. The initial mass matrix has class (a) $\mu - \tau$ symmetry. In the standard parametrization, we have $\theta_{23}^0 = \frac{\pi}{4}$, $\theta_{12}^0 = \theta_{13}^0 = \phi_2^0 = 0$ and $\phi_3^0 = \pi$. The three initial masses are $m_1^0 = m_2^0 = m_0(1+2\delta)$, and $m_3^0 = m_0$. In this case, Eq. (19) is greatly simplified:

$$a = 2m_0\delta' \, , \qquad b = \sqrt{2}m_0 \mathrm{Re}\,(\delta'') \, , \qquad c = f = g = 0 \, , \qquad d = i\sqrt{2}m_0 \mathrm{Im}\,(\delta'') \, . \tag{38}$$

From Eqs. (21), (24) and (25), we find

$$\delta_{23} = 0 \, , \qquad \delta_{13} \approx \frac{\mathrm{Im}\,(\delta'')}{\sqrt{2}\delta} \, , \qquad \phi = 0 \, , \qquad \xi = \frac{1}{2} \arctan \frac{\sqrt{2}\mathrm{Re}\,(\delta'')}{-\delta'} \, . \tag{39}$$

Then the final mixing matrix can be written as

$$\begin{aligned} U &= R_{23}(\frac{\pi}{4})P(0,\pi)R_{13}(\delta_{13})R_{12}(\xi) \\ &= R_{23}(\frac{\pi}{4})U_{13}(\delta_{13},\frac{\pi}{2})R_{12}(\xi)P(0,\pi) \, . \end{aligned} \tag{40}$$

Hence, the final mixing angles and the Dirac phase are

$$\theta_{23} = \frac{\pi}{4} \, , \qquad \theta_{12} = \frac{1}{2} \arctan \frac{\sqrt{2}\mathrm{Re}\,(\delta'')}{-\delta'} \, , \qquad \theta_{13} = \frac{\mathrm{Im}\,(\delta'')}{\sqrt{2}\delta} \, , \qquad \delta = \frac{\pi}{2} \, , \tag{41}$$

as in Ref. [8]. Note that the initial class (a) is perturbed to class (d), and that the large change of the Dirac phase δ coincides with the deviation of θ_{13} from 0.

The general form of the neutrino mass matrix with class (d) $\mu - \tau$ symmetry and its associated generalized CP symmetry has been recognized in Ref. [9], and deviations from it were discussed in Ref. [10]. It has been shown in Ref. [9] that the general forms of the neutrino mass matrices with class (a) and (d) $\mu - \tau$ symmetry are (in our phase convention)

$$M_a = \begin{pmatrix} x & y & -y \\ y & z & -w \\ -y & -w & z \end{pmatrix} \, , \quad \text{and} \quad M_d = \begin{pmatrix} u & r & -r^* \\ r & s & -v \\ -r^* & -v & s^* \end{pmatrix} \, , \tag{42}$$

respectively. Here x, y, z, w, r, s are complex and u, v are real. Hence, any perturbation matrix of the form

$$E = \begin{pmatrix} \mathrm{Re}(\epsilon_{11}) - i\mathrm{Im}(x) & \epsilon_{12} & -\epsilon_{12}^* + 2i\mathrm{Im}(y) \\ \epsilon_{12} & \epsilon_{22} & \mathrm{Re}(\epsilon_{23}) + i\mathrm{Im}(w) \\ -\epsilon_{12}^* + 2i\mathrm{Im}(y) & \mathrm{Re}(\epsilon_{23}) + i\mathrm{Im}(w) & \epsilon_{22}^* - 2i\mathrm{Im}(z) \end{pmatrix} \, , \tag{43}$$

Table 1: Best-fit values and 2σ ranges of the oscillation parameters [18], with $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$.

Parameter	$\theta_{12}(^\circ)$	$\theta_{13}(^\circ)$	$\theta_{23}(^\circ)$	$\delta m^2(10^{-5}\text{eV}^2)$	$\lvert\Delta m^2\rvert(10^{-3}\text{eV}^2)$
Normal hierarchy	$33.7^{+2.1}_{-2.1}$	$8.80^{+0.73}_{-0.77}$	$41.4^{+6.6}_{-2.6}$	$7.54^{+0.46}_{-0.39}$	$2.43^{+0.12}_{-0.13}$
Inverted hierarchy	$33.7^{+2.1}_{-2.1}$	$8.91^{+0.70}_{-0.82}$	$42.4^{+9.5}_{-3.2}$	$7.54^{+0.46}_{-0.39}$	$2.38^{+0.12}_{-0.13}$

perturbs the initial mass matrix with class (a) $\mu - \tau$ symmetry to class (d) $\mu - \tau$ symmetry.

We now perform a numerical search to find perturbations that fit the experimental data for initial neutrino mass matrices with $\mu - \tau$ symmetry. We select class (d) and scan θ_{12}^0 and θ_{13}^0 over the range $[0, 90^\circ]$. Since the initial mass matrices of classes (a), (b) and (c) do not depend on δ^0, the perturbation results of class (d) will cover the other classes, e.g., the perturbation results for bimaximal mixing would be the same as that of class (d) with $\theta_{13}^0 = 0$ and $\theta_{12}^0 = 45^\circ$. Since we work in the basis in which the charged lepton mass matrix is diagonal, the mixing matrix in the neutrino sector is the same as the observed PMNS matrix. We also choose $m_1 = 0$ for the normal hierarchy (or $m_3 = 0$ for the inverted hierarchy), so the best-fit values from the global fit in Table 1 define the other two final masses and the three final mixing angles.

We characterize the size of the perturbation as the root-mean-square (RMS) value of the perturbations,

$$\epsilon_{\text{RMS}} = \sqrt{\frac{\text{Tr}[E^\dagger E]}{9}} = \sqrt{\frac{\sum_{i,j=1}^{3} |\epsilon_{ij}|^2}{9}}\,, \tag{44}$$

where i and j sum over neutrino flavors. ϵ_{RMS} is determined by the three initial masses, two initial Majorana phases, two final Majorana phases and one final Dirac phase.

The initial Dirac phase in class (d) is fixed to be $\pm 90^\circ$. To evaluate the change in the Dirac phase due to the perturbations, we fix $\theta_{12}^0 = 45^\circ$ and scan over δ and θ_{13}^0 to find the minimum RMS value of the perturbation, $\epsilon_{\text{RMS}}^{\text{min}}$, that results in the best-fit parameters. The results for $\delta^0 = 90^\circ$ are shown in Fig. 1. The results for $\delta^0 = -90^\circ$ (or 270°) are symmetric to that of $\delta^0 = 90^\circ$ with $\delta \to 360^\circ - \delta$. From Fig. 1, we see that for $\theta_{13}^0 \leq 20^\circ$ it is possible for the final Dirac phase to have any value under small perturbations ($\epsilon_{\text{RMS}}^{min} \lesssim 10$ meV), i.e.,

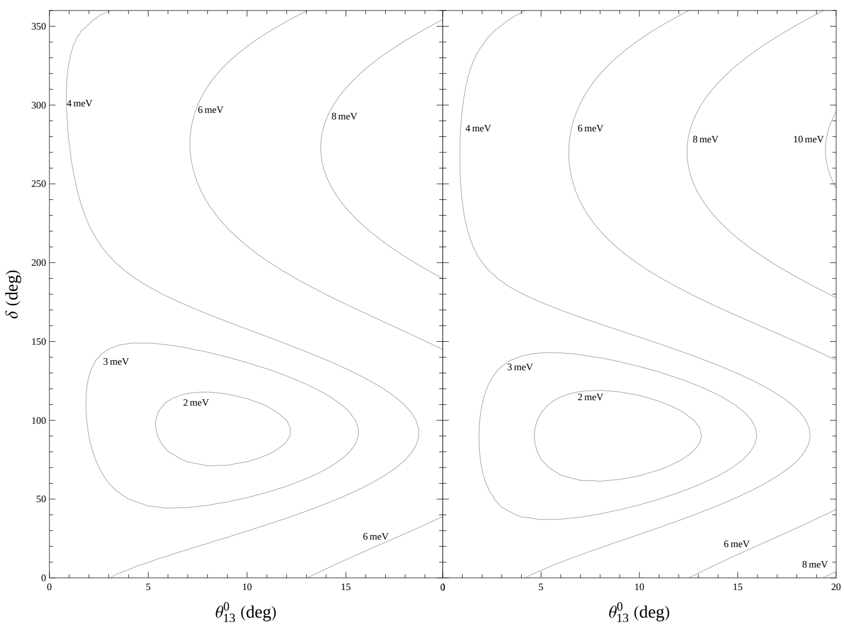

Figure 1: Iso-$\epsilon_{\rm RMS}^{\rm min}$ contours in the (θ_{13}^0, δ) plane that give the best-fit parameters for $\mu - \tau$ symmetry with $\theta_{23}^0 = \theta_{12}^0 = 45°, \delta^0 = 90°$. The left panel is for the normal hierarchy with $m_1 = 0$ and the right panel is for the inverted hierarchy with $m_3 = 0$.

the correction to the Dirac phase can be large for small perturbations.

4 Perturbations in the charged lepton sector

In the basis in which the charged lepton mass matrix is not diagonal, the observed PMNS mixing matrix is

$$U_{PMNS} = U_l^\dagger U_\nu \,, \tag{45}$$

where U_l and U_ν are the mixing matrices in the charged lepton and neutrino sector, respectively. For an arbitrary charged lepton mass matrix M_l, we have

$$(M_l)^\dagger M_l = U_l \overline{M_l}^2 (U_l)^\dagger \,, \tag{46}$$

where $\overline{M_l} = \text{diag}\,(m_e, m_\mu, m_\tau)$.

Suppose the charged lepton mass matrix is also the result of small perturbations to an initial mass matrix, i.e., $M_l = M_l^0 + E_l$, where $(E_l)_{ij} \equiv \epsilon_{ij}^l$ and $|\epsilon_{ij}^l| \ll m_\tau$. If the initial mixing matrix in the charged lepton sector is U_l^0, i.e.,

$$(M_l^0)^\dagger M_l^0 = U_l^0 (\overline{M_l^0})^2 (U_l^0)^\dagger \,, \tag{47}$$

then to LO, we get

$$\begin{aligned}(M_l)^\dagger M_l &\approx U_l^0 (\overline{M_l^0})^2 (U_l^0)^\dagger + (M_l^0)^\dagger E_l + E^\dagger M_l^0 \\ &= U_l^0 \left[\overline{M_l^0}^2 + N^l \right] (U_l^0)^\dagger \,, \end{aligned} \tag{48}$$

where $N^l = (U_l^0)^\dagger \left[(M_l^0)^\dagger E_l + E^\dagger M_l^0 \right] U_l^0$. Note that since U_l^0 is unconstrained, the size of each element of the N^l matrix could be of order $m_\tau |\epsilon_{ij}^l|$.

If $(\overline{M_l^0})^2 + N^l$ is diagonalized by a unitary matrix U_δ^l, i.e.,

$$\overline{M_l^0}^2 + N^l = \begin{pmatrix} (m_e^0)^2 + N_{11}^l & N_{12}^l & N_{13}^l \\ (N_{11}^l)^* & (m_\mu^0)^2 + N_{22}^l & N_{23}^l \\ (N_{13}^l)^* & (N_{23}^l)^* & (m_\tau^0)^2 + N_{33}^l \end{pmatrix} = U_\delta^l \begin{pmatrix} m_e^2 & 0 & 0 \\ 0 & m_\mu^2 & 0 \\ 0 & 0 & m_\tau^2 \end{pmatrix} (U_\delta^l)^\dagger \,, \tag{49}$$

then the PMNS matrix can be written as

$$U_{PMNS} = (U_l^0 U_\delta^l)^\dagger U_\nu^0 = (U_\delta^l)^\dagger U_0 \,, \tag{50}$$

where $U_0 = (U_l^0)^\dagger U_\nu^0$ has the most general form of Eq. (15). Since $N_{ij}^l \sim m_\tau |\epsilon_{ij}^l|$, the 2-3 and 1-3 mixing angles in U_δ^l are suppressed by a factor of $N_{ij}^l / m_\tau^2 \sim |\epsilon_{ij}^l| / m_\tau$. To LO in $\mathcal{O}(|\epsilon_{ij}^l| / m_\tau)$, U_δ^l can be parametrized as

$$U_\delta^l = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta_{23}^l e^{-i\phi_{23}^l} \\ 0 & -\delta_{23}^l e^{i\phi_{23}^l} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \delta_{13}^l e^{-i\phi_{13}^l} \\ 0 & 1 & 0 \\ -\delta_{13}^l e^{i\phi_{13}^l} & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_{12}^l & \sin\theta_{12}^l e^{-i\phi_{12}^l} & 0 \\ -\sin\theta_{12}^l e^{i\phi_{12}^l} & \cos\theta_{12}^l & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{51}$$

where

$$\begin{aligned} \delta_{13}^l &\approx \frac{|N_{13}^l|}{m_\tau^2} \,, \quad \delta_{23}^l \approx \frac{|N_{23}^l|}{m_\tau^2} \,, \\ \theta_{12}^l &\approx \frac{1}{2} \arctan \frac{2|N_{12}^l|}{m_\mu^2 + N_{22}^l - N_{11}^l} \,, \end{aligned} \tag{52}$$

and $\phi_{ij} \approx -\arg N^l_{ij}$.

If θ^l_{12} is also very small, the LO corrections to the three mixing angles in the PMNS matrix are

$$\begin{aligned}
\delta\theta_{13} &= -\theta^l_{12} s^0_{23} \cos(\delta^0 - \phi^l_{12}) - \delta^l_{13} c^0_{23} \cos(\delta^0 - \phi^l_{13})\,, \\
\delta\theta_{23} &= -\delta^l_{23} \cos\phi^l_{23} - \delta^l_{13} s^0_{23} t^0_{13} \cos(\delta^0 - \phi^l_{13}) + \theta^l_{12} c^0_{23} t^0_{13} \cos(\delta^0 - \phi^l_{12})\,, \\
\delta\theta_{12} &= \frac{1}{c^0_{13}} (\delta^l_{13} s^0_{23} \cos\phi^l_{13} - \theta^l_{12} c^0_{23} \cos\phi^l_{12})\,.
\end{aligned} \tag{53}$$

However, in general, since $N^l_{ij} \sim m_\tau |\epsilon^l_{ij}|$, and if $|\epsilon^l_{ij}| \sim m^2_\mu / m_\tau = 6$ MeV, θ^l_{12} could be very large, which will give large corrections to the mixing angles in the PMNS matrix. Thus the situation in the charged sector is similar to that in the neutrino sector: the near-degeneracy of m_e and m_μ (on the scale of m_τ) can lead to large corrections in 1-2 space.

For large θ^l_{12}, the analytical expressions for the corrections to the mixing angles in the PMNS matrix are cumbersome. Here as an illustration, we consider the very simple scenario in which

$$U^l_\delta = \begin{pmatrix} \cos\theta^l_{12} & \sin\theta^l_{12} & 0 \\ -\sin\theta^l_{12} & \cos\theta^l_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{54}$$

Then from Eq. (50), the final mixing angles in the PMNS matrix are given by

$$c_{13} c_{23} = c^0_{13} c^0_{23}\,, \tag{55}$$

$$s^2_{13} = (s^0_{13})^2 (c^l_{12})^2 + (s^0_{23})^2 (c^0_{13})^2 (s^l_{12})^2 - 2 s^0_{13} c^0_{13} s^0_{23} s^l_{12} c^l_{12} \cos\delta^0\,, \tag{56}$$

$$\begin{aligned}
c^2_{13} s^2_{12} = \big[&(c^l_{12} c^0_{13} s^0_{12} - s^l_{12} c^0_{12} c^0_{23})^2 + (s^l_{12})^2 (s^0_{12})^2 (s^0_{13})^2 (s^0_{23})^2 \\
&+ 2 s^l_{12} s^0_{12} s^0_{13} s^0_{23} (c^l_{12} c^0_{13} s^0_{12} - s^l_{12} c^0_{12} c^0_{23}) \cos\delta^0 \big]\,,
\end{aligned} \tag{57}$$

where c^l_{12} denotes $\cos\theta^l_{12}$, and s^l_{12} denotes $\sin\theta^l_{12}$. As an example, if θ^l_{12} is the Cabibbo angle and the initial PMNS matrix has bimaximal symmetry ($\theta^0_{12} = 45°, \theta^0_{13} = 0$), then the resulting θ_{12} and θ_{13} are consistent with the observed values to within 2σ.

There are eight parameters in Eqs. (55), (56) and (57). We use the best-fit values in Table 1 for the normal hierarchy to fix θ_{12}, θ_{13} and θ_{23}. Then for given values of θ^0_{13} and δ^0,

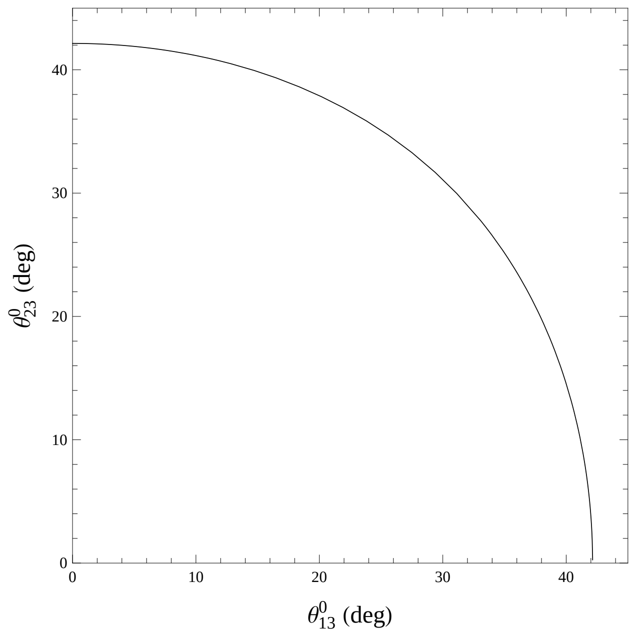

Figure 2: Dependence of θ_{23}^0 on θ_{13}^0 for small perturbations in the charged lepton sector when U_δ^l is given by Eq. (54), and the three mixing angles in the PMNS matrix are fixed by the best-fit values of the global fit in Table 1 for the normal hierarchy.

the other three unknown parameters θ_{23}^0, θ_{12}^l and θ_{12}^0 are determined by the three equations. First, we obtain θ_{23}^0 from Eq. (55) for a given θ_{13}^0. Note that the constraints on θ_{23}^0 and θ_{13}^0 are symmetric for fixed θ_{13} and θ_{23}, which can be seen from Fig. 2. Then we scan θ_{12}^l from $[-90°, 90°]$ to find solutions to Eq. (56) for a given δ^0. For each solution of θ_{12}^l, we obtain θ_{12}^0 from Eq. (57) by scanning θ_{12}^0 from $[-90°, 90°]$. Note that we only scan the first and fourth quadrants of θ_{12}^l (θ_{12}^0) because Eq. (56) (Eq. 57) is only sensitive to the relative sign between the cosine and sine of θ_{12}^l (θ_{12}^0). Once we obtain θ_{23}^0, θ_{12}^l and θ_{12}^0 for given values of θ_{13}^0 and δ^0, the resulting PMNS matrix is completely determined (except for the diagonal Majorana phase matrix) from Eqs. (50), (54) and (15). By comparing the PMNS matrix with the standard parametrization, the resulting Dirac phase δ is also obtained for given values of θ_{13}^0 and δ^0. The dependence of θ_{12}^l, θ_{12}^0 and δ on δ^0 for different values of θ_{13}^0 is shown in Fig. 3. From Figs. 2 and 3 we see that the initial mixing angles and the initial Dirac phase can be very different from their observed values in the PMNS matrix due to small perturbations in the charged lepton sector.

Generally, perturbations in both the charged lepton and neutrino sector will be present.

In this case, one must first use the procedure described in this section to find the corrections to the initial PMNS matrix from perturbations in the charged lepton sector alone, then use the new PMNS matrix to rotate to the basis in which the final charged lepton mass matrix is diagonal, and ultimately use the procedure described in Section 2 to find the final corrections to the parameters in the PMNS matrix from perturbations in the neutrino sector.

5 Neutrino oscillations with nonstandard interactions

We now apply our generalized perturbation procedure to a phenomenological study of neutrino oscillations that are affected by nonstandard scalar and nonstandard vector interactions simultaneously.

As ν_e propagate in matter, they scatter on electrons via the V-A interaction. This is described by the MSW potential [19], which is added to the vacuum oscillation Hamiltonian:

$$H = \frac{1}{2E_\nu} U \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} \sqrt{2} G_F N_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{58}$$

where G_F is the Fermi constant, N_e is the electron number density in the medium, and U and m_i are the mixing matrix and eigenmasses in vacuum, respectively.

New physics beyond the Standard Model can be probed by studying model-independent nonstandard interactions in neutrino oscillation experiments; for a recent review see Ref. [20]. Most studies of nonstandard interactions are focused on the vector interaction, which can be described by effective four-fermion operators of the form,

$$\mathcal{L}_{\rm V} = \frac{G_F}{\sqrt{2}} \epsilon^V_{\alpha\beta} \left[\bar{\nu}_\alpha \gamma^\rho (1-\gamma^5) \nu_\beta\right] \left[\bar{f} \gamma_\rho (1 \pm \gamma^5) f\right] + \text{h.c.} , \tag{59}$$

where $f = u, d, e$ is a charged fermion field, and $\epsilon^V_{\alpha\beta}$ are dimensionless parameters that denote the strength of the deviation from the standard interactions. Similar to the MSW term, the matter effect due to the nonstandard vector interaction modifies the oscillation Hamiltonian by additional potential terms, $\sqrt{2} G_F N_f \epsilon^V_{\alpha\beta}$.

In addition, consider nonstandard scalar interactions, which may arise from a Lagrangian

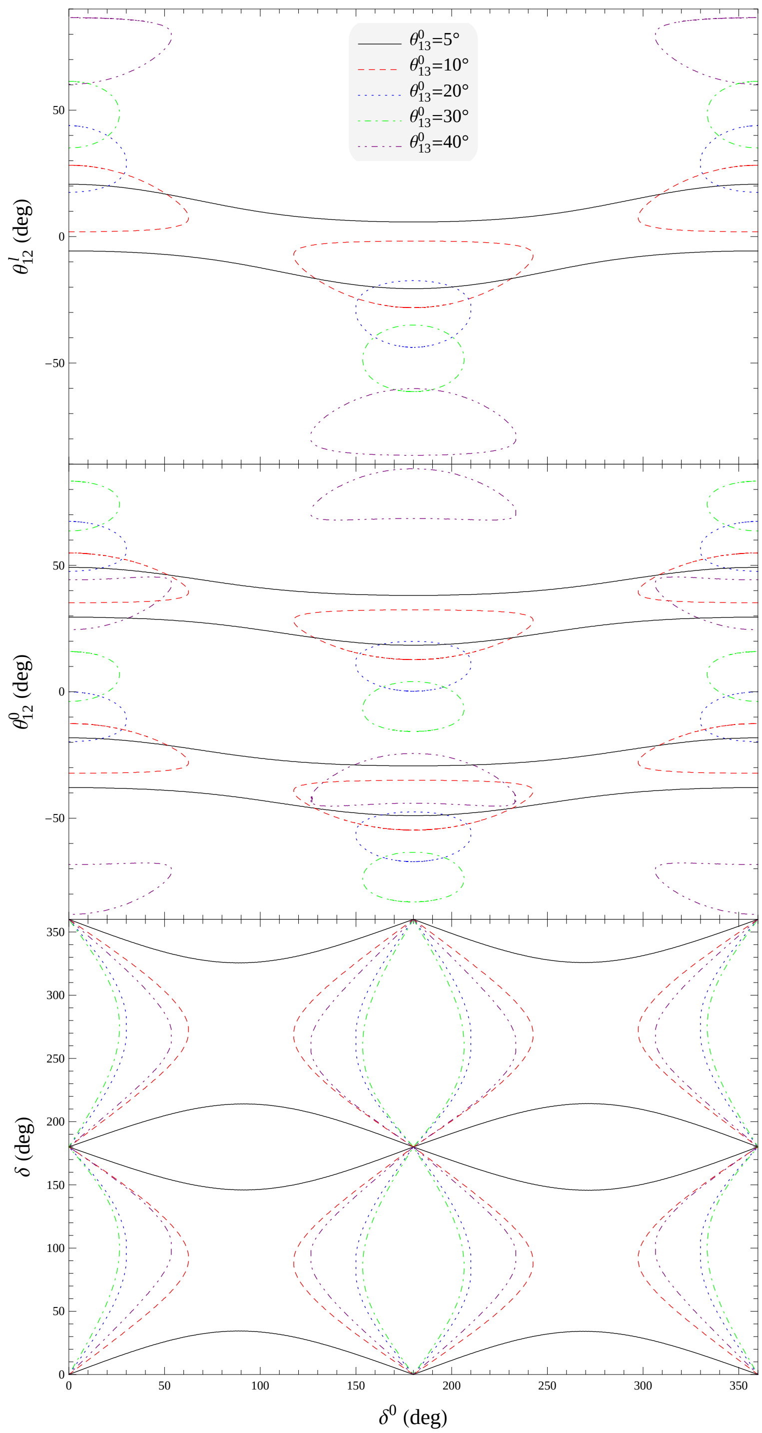

Figure 3: Dependence of θ_{12}^l (top), θ_{12}^0 (middle) and δ (bottom) on δ^0 for different values of θ_{13}^0 for small perturbations in the charged lepton sector when U_δ^l is given by Eq. (54), and the three mixing angles in the PMNS matrix are fixed by the best-fit values of the global fit in Table 1 for the normal hierarchy.

of the form

$$\mathcal{L}_S = \lambda_\nu^{\alpha\beta}\bar{\nu}_\alpha\nu_\beta\phi + \lambda_f\bar{f}f\phi\,, \tag{60}$$

where ϕ is a new scalar field, and $\lambda_\nu^{\alpha\beta}$ and λ_f are dimensionless coupling constants for neutrinos and charged fermions, respectively. In a mean field approximation, the nonstandard scalar interaction will shift elements of the neutrino mass matrix by [21],

$$\epsilon_{\alpha\beta} \approx \frac{\lambda_\nu^{\alpha\beta}}{m_\phi^2}\lambda_f N_f\,, \tag{61}$$

where m_ϕ is the mass of the scalar field, N_f is the number density of the charged fermion f, which is assumed to be nonrelativistic.

Tests of the inverse square law of the gravitational force put stringent m_ϕ-dependent limits on the coupling of a new scalar field to the nucleon field [22]. For m_ϕ in the range, 10^{-6} eV to 10^{-10} eV, the current experimental upper limit of λ_N varies from 10^{-21} to 10^{-22} [23]. Since

$$\epsilon_{\alpha\beta} \simeq 0.46 \text{ meV } \left(\frac{\lambda_\nu}{10^{-4}}\right)\left(\frac{\lambda_N}{10^{-21}}\right)\left(\frac{N_f}{N_A/\text{cm}^3}\right)\left(\frac{10^{-6}\text{ eV}}{m_\phi}\right)^2\,, \tag{62}$$

and $N_f \sim 1N_A/\text{cm}^3 \sim 10^{10}$ eV^3 on earth and $N_f \sim 100N_A/\text{cm}^3 \sim 10^{12}$ eV^3 in the solar core where most solar neutrinos are produced, in these environments, a λ_ν of order 10^{-3} gives a mass shift of order 1 meV for $m_\phi = 10^{-6}$ eV. Such $\epsilon_{\alpha\beta}$ values are possible for much smaller values of λ_ν when $m_\phi < 10^{-6}$ eV.

In the presence of both nonstandard scalar and nonstandard vector interactions, the effective Hamiltonian for neutrino oscillations can be written as

$$H_{eff} = \frac{1}{2E_\nu}M_{eff}^\dagger M_{eff} + \sqrt{2}G_F N_e\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + \sqrt{2}G_F N_f\begin{pmatrix}\epsilon_{ee}^V & \epsilon_{e\mu}^V & \epsilon_{e\tau}^V\\ \epsilon_{e\mu}^{V*} & \epsilon_{\mu\mu}^V & \epsilon_{\mu\tau}^V\\ \epsilon_{e\tau}^{V*} & \epsilon_{\mu\tau}^{V*} & \epsilon_{\tau\tau}^V\end{pmatrix}\,, \tag{63}$$

where the effective mass matrix has the form

$$M_{eff} = U^*\begin{pmatrix}m_1 & 0 & 0\\ 0 & m_2 & 0\\ 0 & 0 & m_3\end{pmatrix}U^\dagger + \begin{pmatrix}\epsilon_{11} & \epsilon_{12} & \epsilon_{13}\\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23}\\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33}\end{pmatrix}\,. \tag{64}$$

We apply our generalized perturbation procedure to the study of both nonstandard scalar and nonstandard vector interactions. By incorporating the the corrections to the vacuum oscillation parameters (arising from the scalar interaction) into the nonstandard vector interaction formulae, we immediately obtain new formulae for oscillation probabilities with both nonstandard scalar and nonstandard vector interactions. Taking the oscillations of ν_μ in long baseline experiments as an example, the result for the ν_μ survival probability is [24],

$$\begin{aligned}
P_{\mu\mu} \simeq 1 &- s^2_{2\times 23}\left[\sin^2\frac{\Delta m^2_{31}L}{4E}\right] \\
&- |\epsilon^V_{\mu\tau}|\cos\phi^V_{\mu\tau}s_{2\times 23}\left[s^2_{2\times 23}(\sqrt{2}G_F N_e L)\sin\frac{\Delta m^2_{31}L}{2E} + 4c^2_{2\times 23}\frac{2\sqrt{2}G_F N_e E}{\Delta m^2_{31}}\sin^2\frac{\Delta m^2_{31}L}{4E}\right] \\
&+ (|\epsilon^V_{\mu\mu}| - |\epsilon^V_{\tau\tau}|)s^2_{2\times 23}c_{2\times 23}\left[\frac{\sqrt{2}G_F N_e L}{2}\sin\frac{\Delta m^2_{31}L}{2E} - 2\frac{2\sqrt{2}G_F N_e E}{\Delta m^2_{31}}\sin^2\frac{\Delta m^2_{31}L}{4E}\right], \quad (65)
\end{aligned}$$

where $\Delta m^2_{31} = m^2_3 - m^2_1$, $s_{2\times ij} = \sin 2\theta_{ij}$, $c_{2\times ij} = \cos 2\theta_{ij}$, and $\phi^V_{\mu\tau} = \arg\epsilon^V_{\mu\tau}$. After replacing $\Delta m^2_{31} \to \Delta m^2_{31} + 2(m_3\delta m_3 - m_1\delta m_1)$ and $\theta_{23} \to \theta_{23} + \delta\theta_{23}$, where the shifts in m_i and θ_{23} can be easily obtained from our perturbation results in Section 2, the new formula for both nonstandard scalar and nonstandard vector interactions is as follows:

$$\begin{aligned}
P_{\mu\mu} \simeq 1 &- s^2_{2\times 23}\left[\sin^2\frac{\Delta m^2_{31}L}{4E}\right] \\
&-2\delta\theta_{23}\sin 4\theta_{23}\sin^2\frac{\Delta m^2_{31}L}{4E} - \frac{(m_3\delta m_3 - m_1\delta m_1)L}{2E}s^2_{2\times 23}\sin\frac{\Delta m^2_{31}L}{2E} \\
&- |\epsilon^V_{\mu\tau}|\cos\phi^V_{\mu\tau}s_{2\times 23}\left[s^2_{2\times 23}(\sqrt{2}G_F N_e L)\sin\frac{\Delta m^2_{31}L}{2E} + 4c^2_{2\times 23}\frac{2\sqrt{2}G_F N_e E}{\Delta m^2_{31}}\sin^2\frac{\Delta m^2_{31}L}{4E}\right] \\
&+ (|\epsilon^V_{\mu\mu}| - |\epsilon^V_{\tau\tau}|)s^2_{2\times 23}c_{2\times 23}\left[\frac{\sqrt{2}G_F N_e L}{2}\sin\frac{\Delta m^2_{31}L}{2E} - 2\frac{2\sqrt{2}G_F N_e E}{\Delta m^2_{31}}\sin^2\frac{\Delta m^2_{31}L}{4E}\right]. \quad (66)
\end{aligned}$$

We see that cancellations between the nonstandard scalar and vector terms are possible, a study of which is beyond the scope of this paper.

6 Summary

We introduced a generalized procedure to study complex perturbations on Majorana neutrino mass matrices. In the charged lepton basis, we derived analytic formulae for the corrections to the three mixing angles, and the Dirac and Majorana phases for arbitrary initial mixing. Since m_1 and m_2 are nearly degenerate, the corrections to θ_{12} and the Dirac and Majorana phases could be very large. We performed a numerical analysis on the mass matrices with

$\mu-\tau$ symmetry to illustrate our analytical results, and found that the final Dirac phase can take any value under small perturbations.

We also studied the scenario in which the charged lepton mass matrix is not diagonal, and considered perturbations on the charged lepton mass matrix. We found that small perturbations in the charged lepton sector give small mixing in the 1-3 and 2-3 sectors, but the mixing in the 1-2 sector could be potentially large due to the near degeneracy of m_e and m_μ (on the scale of m_τ), which could lead to large corrections to all three mixing angles in the PMNS matrix.

In addition, we showed that using our generalized perturbation procedure, it is straightforward to study neutrino oscillations with both nonstandard scalar and nonstandard vector interactions.

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A Diagonalization of a 2×2 complex symmetric matrix

We show how to diagonalize a 2×2 complex symmetric matrix

$$M=\begin{pmatrix} a & b \\ b & c \end{pmatrix}=\begin{pmatrix} |a|e^{i\phi_a} & |b|e^{i\phi_b} \\ |b|e^{i\phi_b} & |c|e^{i\phi_c} \end{pmatrix}, \tag{67}$$

so that

$$U^T M U=\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \tag{68}$$

where m_1, m_2 are nonnegative real numbers, and U is an unitary matrix.

First, we diagonalize M with a unitary matrix V of the form

$$V=\begin{pmatrix} c_\xi & s_\xi e^{-i\phi} \\ -s_\xi e^{i\phi} & c_\xi \end{pmatrix}, \tag{69}$$

where c_ξ and s_ξ denotes $\cos\xi$ and $\sin\xi$, respectively:

$$V^T M V = \begin{pmatrix} ac_\xi^2 + cs_\xi^2 e^{2i\phi} - 2bs_\xi c_\xi e^{i\phi} & (ae^{-i\phi} - ce^{i\phi})s_\xi c_\xi + b(c_\xi^2 - s_\xi^2) \\ (ae^{-i\phi} - ce^{i\phi})s_\xi c_\xi + b(c_\xi^2 - s_\xi^2) & as_\xi^2 e^{-2i\phi} + cc_\xi^2 + 2bs_\xi c_\xi e^{-i\phi} \end{pmatrix} . \tag{70}$$

The diagonalization condition is

$$(ae^{-i\phi} - ce^{i\phi})s_\xi c_\xi + b(c_\xi^2 - s_\xi^2) = 0\,, \tag{71}$$

which implies the phase ϕ is

$$\phi = \arctan\frac{|a|\sin(\phi_a - \phi_b) - |c|\sin(\phi_c - \phi_b)}{|a|\cos(\phi_a - \phi_b) + |c|\cos(\phi_c - \phi_b)}\,, \tag{72}$$

and the mixing angle ξ is

$$\xi = \frac{1}{2}\arctan\frac{2|b|}{|c|\cos(\phi_c + \phi - \phi_b) - |a|\cos(\phi_a - \phi - \phi_b)}\,. \tag{73}$$

Also, the two eigenvalues can be written as

$$\begin{aligned} m_1 &= |ac_\xi^2 + cs_\xi^2 e^{2i\phi} - 2bs_\xi c_\xi e^{i\phi}|\,, \\ m_2 &= |as_\xi^2 e^{-2i\phi} + cc_\xi^2 + 2bs_\xi c_\xi e^{-i\phi}|\,. \end{aligned} \tag{74}$$

The final unitary matrix that diagonalizes M is

$$U = V \begin{pmatrix} e^{i\omega_1/2} & 0 \\ 0 & e^{i\omega_2/2} \end{pmatrix}\,, \tag{75}$$

where

$$\begin{aligned} \omega_1 &= -\arg\left(ac_\xi^2 + cs_\xi^2 e^{2i\phi} - 2bs_\xi c_\xi e^{i\phi}\right)\,, \\ \omega_2 &= -\arg\left(as_\xi^2 e^{-2i\phi} + cc_\xi^2 + 2bs_\xi c_\xi e^{-i\phi}\right)\,. \end{aligned} \tag{76}$$

B Right-multiplication

We calculate the change of mixing parameters when a general initial mixing matrix U_0 (see Eq. 15) is multiplied by the following unitary matrix from the right:

$$U_{12}(\xi,\phi) = \begin{pmatrix} c_\xi & s_\xi e^{-i\phi} & 0 \\ -s_\xi e^{i\phi} & c_\xi & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{77}$$

Multiplying U_0 on the right by U_{12} yields

$$
\begin{aligned}
U &= R_{23}(\theta_{23}^0)U_{13}(\theta_{13}^0,\delta^0)R_{12}(\theta_{12}^0)\begin{pmatrix}1 & 0 & 0\\ 0 & e^{i\phi_2^0/2} & 0\\ 0 & 0 & e^{i\phi_3^0/2}\end{pmatrix}U_{12}(\xi,\phi)\\
&= \begin{pmatrix} c_{13}^0\tilde{C}_{12} & c_{13}^0\tilde{S}_{12}^* e^{i\frac{\phi_2^0}{2}} & e^{-i(\delta^0-\frac{\phi_3^0}{2})}s_{13}^0\\ -c_{23}^0\tilde{S}_{12}-s_{23}^0s_{13}^0\tilde{C}_{12}e^{i\delta^0} & (c_{23}^0\tilde{C}_{12}^*-s_{23}^0s_{13}^0\tilde{S}_{12}^*e^{i\delta^0})e^{i\frac{\phi_2^0}{2}} & e^{i\frac{\phi_3^0}{2}}c_{13}^0s_{23}^0\\ s_{23}^0\tilde{S}_{12}-c_{23}^0s_{13}^0\tilde{C}_{12}e^{i\delta^0} & (-s_{23}^0\tilde{C}_{12}^*-c_{23}^0s_{13}^0\tilde{S}_{12}^*e^{i\delta^0})e^{i\frac{\phi_2^0}{2}} & e^{i\frac{\phi_3^0}{2}}c_{13}^0c_{23}^0\end{pmatrix}, \quad (78)
\end{aligned}
$$

where

$$\tilde{C}_{12} = c_{12}^0 c_\xi - s_{12}^0 s_\xi e^{i\frac{\phi_2^0+2\phi}{2}} \quad (79)$$

and

$$\tilde{S}_{12} = s_{12}^0 c_\xi + c_{12}^0 s_\xi e^{i\frac{\phi_2^0+2\phi}{2}} \quad (80)$$

are complex. Comparing U to the standard parametrization, we find that

$$\theta_{23} = \theta_{23}^0\,, \quad \theta_{13} = \theta_{13}^0\,, \quad (81)$$

and

$$\theta_{12} = \arcsin(|\tilde{S}_{12}|) = \arcsin\sqrt{\sin^2(\theta_{12}^0+\xi) - \sin(2\theta_{12}^0)\sin(2\xi)\sin^2\frac{\phi_2^0+2\phi}{4}}\,. \quad (82)$$

Note that after the right multiplication the phases of the resulting mixing matrix are not in the standard form. Defining

$$\alpha = \arg(\tilde{\mathrm{C}}_{12}) = -\arctan\frac{\tan\theta_{12}^0\tan\xi\sin(\phi_2^0/2+\phi)}{1-\tan\theta_{12}^0\tan\xi\cos(\phi_2^0/2+\phi)}\,, \quad (83)$$

$$\beta = \arg(\tilde{\mathrm{S}}_{12}) = \arctan\frac{\tan\xi\sin(\phi_2^0/2+\phi)}{\tan\theta_{12}^0+\tan\xi\cos(\phi_2^0/2+\phi)}\,, \quad (84)$$

we can write U as

$$U = \begin{pmatrix} c_{13}^0|\tilde{C}_{12}|e^{i\alpha} & c_{13}^0|\tilde{S}_{12}|e^{i(\frac{\phi_2^0}{2}-\beta)} & e^{-i(\delta^0-\frac{\phi_3^0}{2})}s_{13}^0\\ -c_{23}^0|\tilde{S}_{12}|e^{i\beta}-s_{23}^0s_{13}^0|\tilde{C}_{12}|e^{i(\delta^0+\alpha)} & (c_{23}^0|\tilde{C}_{12}|e^{-i\alpha}-s_{23}^0s_{13}^0|\tilde{S}_{12}|e^{i(\delta^0-\beta)})e^{i\frac{\phi_2^0}{2}} & e^{i\frac{\phi_3^0}{2}}c_{13}^0s_{23}^0\\ s_{23}^0|\tilde{S}_{12}|e^{i\beta}-c_{23}^0s_{13}^0|\tilde{C}_{12}|e^{i(\delta^0+\alpha)} & (-s_{23}^0|\tilde{C}_{12}|e^{-i\alpha}-c_{23}^0s_{13}^0|\tilde{S}_{12}|e^{i(\delta^0-\beta)})e^{i\frac{\phi_2^0}{2}} & e^{i\frac{\phi_3^0}{2}}c_{13}^0c_{23}^0\end{pmatrix}. \quad (85)$$

On removing the unphysical phases $\phi_e = \alpha$ and $\phi_\mu = \phi_\tau = \beta$ from the rows, the phases in the second and third columns match the form of the standard parametrization, with the Majorana phases shifted by

$$\Delta\phi_2 = -2(\alpha + \beta)\,, \tag{86}$$

$$\Delta\phi_3 = -2\beta\,, \tag{87}$$

and the Dirac phase shifted by

$$\Delta\delta = \alpha - \beta\,. \tag{88}$$

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