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Impact of Axions on Confinement in Three and Two Dimensions

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Abstract

In this paper we discuss well-known three- and two-dimensional models with confinement, namely, the Polyakov compact electrodynamics in 3D and two-dimensional $CP(N-1)$ sigma model, and reveal changes in the confining regimes of these model upon adding the axion field.

In both cases the addition of axion has a drastic impact. In the $CP(N-1)$ model the axion-induced deconfinement was known previously, but we discuss a new feature not considered in the previous publication.

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1 Introduction

In this paper, we discuss well-known three- and two-dimensional models with linear confinement, namely the Polyakov compact electrodynamics in 3D and two-dimensional $CP(N-1)$ sigma model, and reveal changes the confining regimes undergo if one adds the axion field. Historically the first was the Polyakov model [1]. A confining string in this model appears due to the fact that a mass is generated for the dual photon by the instanton-monopole contribution. As a result, domain walls (more exactly, domain lines) appear. They play the role of the confining strings. The dual photon-axion mixing drastically changes the domain line composition resulting in a “leakage” of a part of the electric flux of the probe charges into the Coulomb regime. We also show that the same phenomenon takes place in pure Yang-Mills theory on $R^3 \times S_L^1$ with small circumference of S_L^1 , provided an appropriate deformation is added. It is important to stress that this phenomenon apply only to the case of a small circle and can not be extended to the infinite circle limit, i.e., Yang-Mills on R^4 . Axion does not affect the confining mechanism in four dimensions.

As was discussed in [2], in 2D (nonsupersymmetric) $CP(N-1)$ model axions, being added in a conventional way, result in deconfinement. Here we show that formerly stable mesons start decaying into “quark-antiquark” pairs with exponentially suppressed probabilities at large N ,

$$w \sim \exp\left(-cf_a^{3/2}N^{1/4}\right),$$

where f_a is the axion constant (dimensionless in two dimensions), and c is a numerical coefficient. Thus, at $N = \infty$ the theory is still confining, but at $1 \ll N < \infty$ a “weak” deconfinement occurs. At $N \sim 2$ one can expect full-blown deconfinement, much in the same way as it occurs in supersymmetric model [3].

2 Polyakov’s Confinement in 2+1 Dimensions

We start from a brief review of Polyakov’s compact electrodynamics in three dimensions, and outline confinement mechanism in this model. Then, we show that inclusion of axion completely destroys linear confinement of electric charges. Polyakov’s model of color confinement [1] was historically the first gauge model in which linear confinement of probe electric charges was analytically established in $2+1$ dimensions.

2.1 Preliminaries

To make QED compact, Polyakov suggested to embed it in the Georgi–Glashow model in $1+2$ dimensions [1]. Conventional ’t Hooft–Polyakov monopoles [4, 5] have to be reinterpreted as instantons in the Euclidean version of the model (we will refer to them as to monopole-instantons). To begin with, we will briefly outline the Polyakov mechanism limiting ourselves to the $SU(2)$ case.

The Lagrangian of the Georgi–Glashow model [6] in 2+1 dimensions includes gauge fields and a real scalar field, both in the adjoint representation of $SU(2)$. The Lagrangian of the model is obtained from Yang-Mills in four dimensions by reducing to 3D (see Section 3),

$$\mathcal{L} = \frac{1}{4g_{3D}^2} G_{\mu\nu}^a G_{\mu\nu}^a + \frac{1}{2} (\nabla_\mu \chi^a) (\nabla_\mu \chi^a) - \lambda (\chi^a \chi^a - v^2)^2, \quad (1)$$

where g_{3D} is the 3D coupling constant and $\mu, \nu = 1, 2, 3$; the covariant derivative in the adjoint acts as

$$\nabla_\mu \chi^a = \partial_\mu \chi^a + \varepsilon^{abc} A_\mu^b \chi^c, \quad (2)$$

and the Euclidean metric is $g_{\mu\nu} = \text{diag}\{+1, +1, +1\}$. It is understood that $\lambda \rightarrow 0$, thus the last term is just a shorthand for the boundary condition of the χ field,

$$(\chi^a \chi^a)_{\text{vac}} = v^2, \quad (3)$$

where v is a real positive parameter. One can always choose the gauge in such a way that

$$\chi^{1,2} = 0, \quad \chi^3 = v. \quad (4)$$

Then, the third component of A_μ (i.e. A_μ^3) remains massless. At distances larger than $1/m_W$ the field A_μ^3 acts as a *bona fide* photon. At the same time, the $A_\mu^\pm = \frac{1}{\sqrt{2}g_{3D}} (A_\mu^1 \mp A_\mu^2)$ components become W -bosons; they acquire a mass $m_W = g_{3D}v$. This is why the model is referred to as compact electrodynamics.

The classical equations of motion which follow from Eq. (1) (second order differential equations) can be replaced by first-order equations,

$$-\frac{1}{2g_{3D}} \varepsilon_{\mu\nu\rho} G_{\nu\rho}^a = \pm \nabla_\mu \chi^a. \quad (5)$$

The monopole-instanton action is

$$S_{\text{inst}} = 4\pi \frac{v}{g_{3D}} \equiv 4\pi \frac{m_W}{g_{3D}^2}. \quad (6)$$

The monopole-instanton in the model at hand has four collective coordinates: three translational and one phase corresponding to the unbroken $U(1)$ subgroup of $SU(2)$. After integrating over the $U(1)$ collective coordinate, we obtain the instanton measure in the form

$$d\mu_{\text{inst}} = \text{const} \times m_W^3 d^3x_0 \exp(-S_{\text{inst}}). \quad (7)$$

The validity of the quasiclassical approximation demands that $v \gg g_{3D}$ and hence, $S_{\text{inst}} \gg 1$. As a result, the instanton measure carries an exponential suppression.

2.2 Compact electrodynamics

The mechanism we are interested in is applicable at distances $\gg m_W^{-1}$. Then the presence of the W -bosons in the spectrum of the model is irrelevant, and one can focus on “massless” fields (the meaning of the quotation marks will become clear shortly). There are two such fields: the photon and oscillation quanta of χ^3 ,

$$\chi^3 = v + \beta. \quad (8)$$

In what follows, we will omit the isospace index 3 to ease the notation. We will endow the β field with a mass m_β such that $m_W \gg m_\beta \gg m_\varphi$ (i.e. $\lambda \neq 0$, albeit small), see Eq. (17). Then it plays no role and can be ignored in what follows. In three dimensions, the photon field has only one physical (transverse) polarization. This means that the photon field must have a dual description in terms of one scalar field φ of the angular type [1].

In the absence of source (probe) charges, one can always use the so-called first order formalism. Consider $F_{\mu\nu}$ to be an independent variable and implement

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (9)$$

via introducing a field φ with the action

$$\Delta\mathcal{S}_\varphi = \int d^3x (\partial_\mu \varphi) \epsilon_{\mu\alpha\beta} F_{\alpha\beta}. \quad (10)$$

Now, varying with respect to the photon field we arrive at

$$F_{\mu\nu} - \epsilon_{\mu\nu\alpha} \partial_\alpha \varphi - \epsilon_{\mu\nu\alpha} a \partial_\alpha \beta = 0; \quad (11)$$

see Eq. (20) which explains the occurrence of the last term in the right-hand side, cf. (41). As was mentioned, the β field is sufficiently heavy and can be ignored in the low-energy limit of compact electrodynamics. Therefore, one can ignore the last term in (11) (much in the same way as in (41)). Introducing a static electric charge as a source term J , we conclude that

$$\begin{aligned} F_{\mu\nu} &\propto \epsilon_{\mu\nu\alpha} \partial_\alpha \varphi, \\ J_\mu &= \partial_\nu F_{\mu\nu} = \partial_\nu \epsilon_{\mu\nu\alpha} \partial_\alpha \varphi. \end{aligned} \quad (12)$$

The latter equality is only possible because of singularities (vortices) in the angular field φ . For the standard minimal vortex,

$$F_{\mu\nu} = \frac{g_{3D}^2}{2\pi} Q \epsilon_{\mu\nu\rho} (\partial_\rho \varphi). \quad (13)$$

The normalization of the φ field is chosen in such a way that the values $\varphi = 0, \pm 2\pi, \pm 4\pi, \dots$ are identified, i.e., φ is defined on S^1 , the circle with unit radius. Then, given the coefficient

in (13), the minimal probe electric charge ($Q = 1/2$ in the model at hand) creates a minimal single winding vortex of the φ field. The original energy functional reduces to

$$\mathcal{E} = \frac{1}{2g_{3D}^2} \int d^2x \left(\vec{E}^2 + B^2 \right) = \frac{g_{3D}^2}{32\pi^2} \int d^2x \left[\left(\vec{\nabla}\varphi \right)^2 + \dot{\varphi}^2 \right]. \quad (14)$$

At this level, the dual photon field φ is massless. Instanton-induced interaction generates a potential for the φ field, however,

$$\mathcal{L}_{\text{inst}} = \frac{1}{2} \mu^3 \exp(\pm i \varphi), \quad \mu^3 \sim m_W^3 \exp(-S_{\text{inst}}), \quad (15)$$

where \pm refers to monopole-instanton (anti-instanton). Assembling the monopole-instanton and anti-instanton contributions, we arrive at the following effective Lagrangian for the field φ :

$$\begin{aligned} \mathcal{L}_{\text{dual}} &= \frac{\kappa^2}{2} (\partial_\mu \varphi) (\partial_\mu \varphi) + \mu^3 \cos \varphi, \\ \kappa &= \frac{g_{3D}}{4\pi}. \end{aligned} \quad (16)$$

This is the Lagrangian of the *sine-Gordon* model. The dual photon mass is readily calculable from (16),

$$m_\varphi = \mu^{3/2} \kappa^{-1}, \quad (17)$$

which is exponentially small. The potential in (16) is 2π periodic, as expected.

2.3 Domain line as a confining string

The 2π periodicity of $\mathcal{L}_{\text{dual}}$ and the mass generation in (17) results in the existence of domain lines of the type [7]

$$\varphi = 2 \left[\arcsin \tanh(m_\varphi y) + \frac{\pi}{2} \right], \quad (18)$$

interpolating between $\varphi_{\text{vac}} = 0$ at $y = -\infty$ and $\varphi_{\text{vac}} = 2\pi$ at $y = \infty$, where y is one of two coordinates in the two-dimensional $\{x, y\}$ plane. The transverse size of the domain line is obviously $\sim m_\varphi^{-1}$, while its tension is

$$T = 8\mu^{3/2} \kappa = 8m_\varphi \kappa^2. \quad (19)$$

Note that this tension is much larger than m_φ^2 .

The above domain line is in fact a string that ensures linear confinement of the probe electric charges in compact electrodynamics. Indeed, the necessary conditions for the topological defect to be a string are: (i) the defect is a one-dimensional object; (ii) while traveling away from the defect in the transverse direction, at large distances, we should find ourselves in one and the same vacuum no matter in which direction we go. The first requirement is obviously satisfied for a long domain line. The second requirement is also satisfied since for the compact field φ we have physically the same vacuum on both sides of the domain line.

For the linear regime to set up, the distance between the probe charges L must be $L \gg m_\varphi^{-1}$. The tension of this string is given in (19).

2.4 Axion's impact

If we introduce in (1) an appropriately normalized vacuum angle θ ,

$$\Delta S_\theta = \frac{\theta}{16\pi^2} \epsilon_{\mu\nu\alpha} G_{\mu\nu}^a (\nabla_\alpha \chi^a), \quad (20)$$

then the instanton Lagrangian (15) takes the form

$$\mathcal{L}_{\text{inst}} = \frac{1}{2} \mu^3 \exp [\pm i (\varphi + \theta)]. \quad (21)$$

It is obvious that the θ angle can be absorbed in φ and completely disappears from the physics in compact electrodynamics. This observation was first made by Polyakov in the 1970s.

This statement does not extend to the axion field, however, because adding the axion field ($\theta \rightarrow \theta + a$) introduces an extra dynamical degree of freedom. The Lagrangian (16) now takes the form²

$$\mathcal{L}_{\text{eff}} = \frac{\kappa^2}{2} (\partial_\mu \varphi) (\partial_\mu \varphi) + \frac{f_a^2}{2} (\partial_\mu a) (\partial_\mu a) + \mu^3 \cos(\varphi + a), \quad (22)$$

where a is the axion field and f_a is the axion constant. The axion field is compact too, $a = 0, \pm 2\pi, \pm 4\pi, \dots$ are identified. It is crucial that only one linear combination of φ and a acquires a mass, the orthogonal combination,

$$A \equiv \varphi - x a, \quad x = \frac{f_a^2}{\kappa^2} \quad (23)$$

stays massless. The parameter x measures the strength of the axion interaction with the dual photon. In the limit $x \rightarrow \infty$ the axion decouples and we return to the pure Polyakov model.³ Diagonalization transforms the Lagrangian (22) into

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \frac{f_\Phi^2}{2} (\partial_\mu \Phi) (\partial_\mu \Phi) + \frac{f_A^2}{2} (\partial_\mu A) (\partial_\mu A) + \mu^3 \cos \Phi, \\ \Phi &= \varphi + a, \quad f_\Phi^2 = \kappa^2 \frac{x}{1+x}, \quad f_A^2 = f_a^2 \frac{1}{x(1+x)}. \end{aligned} \quad (24)$$

A domain line can be built only out of the Φ field; the A field cannot be excited inside the domain boundary strip because it is massless. There is no solution for A other than $A=$

²It is assumed here that for given θ there are no other vacua entangled in the θ evolution. At strong coupling this need not be the case. For instance, in QCD with N_f light flavors the θ dependence appears as $f(\theta/N_f)$, where the function f is 2π periodic. This is due to the fact that N_f vacuum states are entangled in the θ evolution. At $\theta = \pi, 3\pi, 5\pi$ one jumps from one vacuum to another. The assumption of a single vacuum involved in the θ evolution is not important for our statement.

³Exponentially small values of x (i.e. exponentially small f_a) must be excluded from our consideration since we need to maintain the axion- φ mass much smaller than the masses m_W and m_β .

constant (which can be put to zero) for all y and x . The domain line solution is obtained from (18) by the substitution

$$m_\varphi \rightarrow m_\Phi = \mu^{3/2} f_\Phi^{-1} = m_\varphi \sqrt{\frac{1+x}{x}}. \quad (25)$$

Its tension (i.e., the tension of the Φ string) is

$$T_\Phi = 8\mu^{3/2} f_\Phi = T_\varphi \sqrt{\frac{x}{1+x}}. \quad (26)$$

Since across the domain line $\Delta\Phi = 2\pi$ and A is not excited, and using

$$\varphi = \frac{x\Phi + A}{1+x}, \quad (27)$$

we conclude that

$$\delta\varphi = 2\pi \frac{x}{1+x}. \quad (28)$$

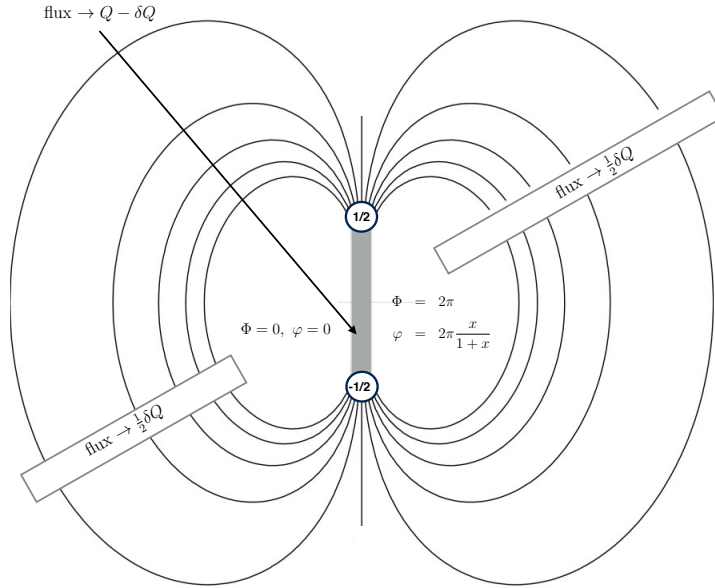


Figure 1: Electromagnetic dipole in three-dimensional compact electrodynamics. A part of the flux goes through the string (domain line), while the remaining flux is dispersed in a Coulomb-like manner.

Next, we observe that it is only the φ field (or the photon F_{0j} , $j = 1, 2$) which interacts with the static probe charge. The minimal electric $U(1)$ charge $\frac{1}{2}$ is represented in the dual

language by the φ vortex with 2π winding. Since in the presence of axion in the model the winding of the φ component of the vortex must be smaller than 2π the endpoint of the domain line will support the electric charge

$$Q - \delta Q \equiv \frac{1}{2} - \frac{1}{2} \frac{1}{1+x}. \quad (29)$$

The remainder of the electric field flux, corresponding to $\delta Q = \frac{1}{2} \frac{1}{1+x}$, is not squeezed inside the domain line (string), but rather spreads out in a Coulomb-like manner (typical of a long electric dipole), as shown in Fig. 4. If $x \gg 1$ we return back to the Polyakov 3D string. If $x \ll 1$, the string dissolves.

3 Four-dimensional Yang-Mills on a cylinder

In the previous sections, we considered the three-dimensional Polyakov model (with the additional term (20)). In this section, we show that the model studied in the previous section can be obtained starting from a four-dimensional theory.

To this end, we start from Yang-Mills theory on $R^3 \times S_L^1$, assuming that the circumference of the circle, L , is small, and then analyze the resulting theory in the low-energy limit. Over the past few year, there has been a significant interest in studying QCD in cylindrical geometry. It was proposed that one can study pure Yang-Mills (with no fermions) on $R^3 \times S^1$ by adding double trace deformations [8] (more on this below). The deformed theory preserves the center symmetry, exhibits the confinement phenomenon, and can serve as a theoretical laboratory to understand various phenomena in gauge theories. In this regard, it will also be instructive to add an axion to deformed Yang-Mills on $R^3 \times S_L^1$ and study various limiting cases of this model.

For definiteness the compactified direction will be aligned with the fourth axis x^4 which is taken to be a spatial direction. The basic distinction from the previous case is the occurrence of two types of monopole-instantons: the first one is the same as in three dimensional theory, while the second type of monopole-instanton is due to the nontrivial topology of S_L^1 , namely $\pi_1(S^1) = \mathbb{Z}$.

3.1 Theory and perturbative analysis

We consider $SU(2)$ Yang-Mills theory on $R^3 \times S_L^1$ along with an axion a :

$$S = \int_{R^3 \times S_L^1} d^4x \left[\frac{1}{4g^2} G_{mn}^a G_{mn}^a + \frac{F_a^2}{2} (\partial_m a)^2 - i \frac{a}{32\pi^2} G_{mn}^a \tilde{G}_{mn}^a \right], \quad (30)$$

where g and F_a are the four-dimensional gauge and axion constants, L is the S^1 circumference, and the Latin letters run over 1, 2, 3, 4. Equation (30) refers to Euclidean space.

Following [9], we introduce the axion field a via a heavy fermion Q in the fundamental representation coupled to a Higgs scalar X singlet under $SU(2)$,

$$\begin{aligned}\mathcal{L}_{Q+X} = & i\bar{Q}D_n\gamma_n Q + (\bar{Q}_L Q_R X + \text{H.c.}) \\ & + (\partial_n X)^2 - m_X^2 |X|^2 + \frac{\lambda}{2} |X|^4.\end{aligned}\tag{31}$$

The scalar field X has two degrees of freedom, its modulus and phase,

$$X = |X| \exp(i\alpha).$$

With a judicious choice of parameters the former will be very heavy and will determine the axion constant F_a while the latter will be promoted to the axion.

The vacuum expectation of X following from (31) is⁴

$$|X| = \frac{m_X}{\sqrt{\lambda}}.\tag{32}$$

By assuming $m_X \gg 1/L$ and λ small, we ensure that the fermion is very massive and can be ignored at energy scales much smaller than $1/L$, so it is irrelevant for what follows. However, the fermion loop produces the coupling of the axion field to the gauge bosons which is not suppressed by the fermion mass,

$$\Delta S = -\frac{i}{32\pi^2} a G_{mn} \tilde{G}_{mn}.\tag{33}$$

After dimensionally reducing the action (30) to R^3 , we obtain

$$\begin{aligned}S_{3D} = & L \int_{R^3} d^3x \left\{ \frac{1}{4g^2} G_{\mu\nu}^a G_{\mu\nu}^a + \frac{1}{2g^2} (\nabla_\mu \chi^a) (\nabla_\mu \chi^a) + \frac{F_a^2}{2} (\partial_\mu a)^2 \right. \\ & \left. - i \frac{a}{16\pi^2} \epsilon_{\mu\nu\rho} G_{\nu\rho}^a (\nabla_\mu \chi^a) + \mathcal{V}[\Omega] \right\},\end{aligned}\tag{34}$$

where χ^a is the component of the gauge field along the compact (fourth) direction which in 3D acts as a compact adjoint scalar. In fact, upon compactifying the theory on S_L^1 , one should sum up the tower of the Kaluza-Klein excitations of the gauge fields. This results in the Casimir potential given by the last term [10],

$$\mathcal{V}(\Omega) = -\frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \frac{|\text{Tr} \Omega^n|^2}{n^4},\tag{35}$$

where

$$\Omega = \exp \left[i \oint_{S_L^1} dx_4 \chi \right]$$

⁴For simplicity we take M_X to be real.

is the Polyakov line along S_L^1 . Without loss of generality, we can perform a global $SU(2)$ transformation to align χ along the τ_3 direction in the color space. Representing

$$\chi = \frac{v + \beta}{L} \frac{\tau_3}{2},$$

where v is the vacuum expectation value (VEV) and β is the field fluctuations, we find

$$\langle \Omega \rangle = \text{diag} (e^{iv/2}, e^{-iv/2}) . \quad (36)$$

The potential $\mathcal{V}(\Omega)$ is minimized at $v = 0$, and hence the center symmetry is maximally broken. $SU(2)$ gauge bosons are not Higgsed at $v = 0$. This prevents the Abelianization due to $SU(2) \rightarrow U(1)$, which is essential for our study of the theory using semi-classical methods.

In order to force the Abelianization, we add a deformation

$$\mathcal{V}_{\text{def}}[\Omega]$$

to the theory [8]. Such deformation can restore the center symmetry either fully or partially. In the special case when the potential is minimized at $\text{Tr}\Omega = 0$ (or at $v = \pi$), the center symmetry is exactly preserved. This can be achieved by adding a double trace deformation

$$\mathcal{V}_{\text{def}}^{\text{double trace}} = b |\text{Tr}\Omega|^2, \quad (37)$$

with some positive coefficient b . In this work we also consider the case

$$\text{Tr}\Omega \cong 0$$

(or $v \cong \pi$) which slightly shifts us away from the exact center-symmetric vacuum. As an example we will consider the following deformation:

$$\mathcal{V}_{\text{def}}[\Omega] = \frac{\tilde{b}}{16L^4} |\text{Tr}\Omega|^4 = \frac{\tilde{b}}{L^4} \cos^4\left(\frac{v}{2}\right), \quad (38)$$

where the numerical value of \tilde{b} is not crucial as long as it forces $v \cong \pi$. Just for definiteness we set

$$\tilde{b} = 1000.$$

Then the total potential $\mathcal{V} + \mathcal{V}_{\text{def}}$ has two minima at $v \cong 3.105$ and $v \cong 3.178$ which are close to π . Larger values of \tilde{b} will push v closer to π , while smaller values of \tilde{b} will pull v away from π .

In summary, adding a suitable deformation will minimize the total potential at a non-zero expectation value of v , and thus, the $SU(2) \rightarrow U(1)$ breaking takes place. In both situations (an exact or nearly-exact center symmetry) it is guaranteed that the W -bosons with mass $\frac{v}{L}$ are heavy, provided we take the S_L^1 circle to be small, $L\Lambda_{\text{QCD}} \ll 1$, where Λ_{QCD} is the dynamical scale of the theory. This entails, in turn the freeze of the running

of the coupling constant g at a small value. As a result, we are able to perform reliable semi-classical calculations.

Ignoring the heavy W -bosons, the resulting 3D Abelian action takes the form

$$S_{U(1) \text{ 3D}} = L \int_{R^3} d^3x \left\{ \frac{1}{4g^2} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2g^2 L^2} (\partial_\mu \beta)^2 + \frac{F_a^2}{2} (\partial_\mu a)^2 \right. \\ \left. - i \frac{a}{16\pi^2 L} \epsilon_{\mu\nu\rho} F_{\nu\rho} \partial_\mu \beta + \mathcal{V}[\Omega] + \mathcal{V}_{\text{def}}[\Omega] \right\}, \quad (39)$$

where $F_{\mu\nu} = G_{\mu\nu}^{(3)}$ and the superscript (3) indicates the third direction in the color space.

Next, we obtain a dual description of the three dimensional photon by introducing an auxiliary term in the Lagrangian (cf. Sec. 2.2),

$$\Delta S_\phi = \frac{i}{8\pi} \int d^3x \epsilon_{\mu\nu\rho} \partial_\mu \phi F_{\nu\rho}. \quad (40)$$

Varying ΔS_ϕ with respect to ϕ , we obtain the Bianchi identity $\partial_\mu \epsilon_{\mu\nu\rho} F_{\nu\rho} = 0$. Also, varying $\Delta S_\phi + S_{U(1) \text{ 3D}}$ with respect to $F_{\mu\nu}$, we find

$$F_{\nu\rho} = -\frac{ig^2}{4\pi L} \left(\partial_\mu \phi - \frac{a}{2\pi} \partial_\mu \beta \right) \epsilon_{\mu\nu\rho}, \quad (41)$$

cf. Eq. (11). Substituting (41) in (39) and (40), we arrive at

$$\Delta S_\phi + S_{U(1) \text{ 3D}} = \int d^3x \left\{ \frac{1}{2g_{\text{3D}}^2 L^2} (\partial_\mu \beta)^2 + \frac{f_a^2}{2} (\partial a)^2 \right. \\ \left. + \frac{g_{\text{3D}}^2}{32\pi^2} \left(\partial_\mu \phi - \frac{a}{2\pi} \partial_\mu \beta \right)^2 + L\mathcal{V}[\Omega] + L\mathcal{V}_{\text{def}}[\Omega] \right\}, \quad (42)$$

where we defined the 3D coupling constant

$$g_{\text{3D}} \equiv g/\sqrt{L}$$

and the 3D axion constant

$$f_a \equiv \sqrt{L} F_a.$$

Since the potential $L\mathcal{V}[\Omega] + L\mathcal{V}_{\text{def}}[\Omega]$ is minimized at $v \cong \pi$, the field β acquires a mass $m_\beta \sim \frac{g}{L}$. Although m_β is parametrically smaller than the W -boson mass, $\frac{v}{L}$, it is still exponentially larger than the photon mass (which is acquired non-perturbatively). Ignoring the massive fields, we find that the perturbative infrared Lagrangian is given by

$$S_{\text{pert}} = \int d^3x \left[\frac{\kappa^2}{2} (\partial_\mu \phi)^2 + \frac{f_a^2}{2} (\partial a)^2 \right], \quad (43)$$

and $\kappa = g_{\text{3D}}/4\pi$, cf. (22).

3.2 Non-perturbative contribution

In the previous section, we considered the infrared effective description of appropriately deformed Yang-Mills theory on $R^3 \times S_L^1$ coupled to an axion. In this section, we take into account the instanton contribution to generate a derivative-free axion coupling.

Now, in addition to the 't-Hooft Polyakov monopole-instanton, we will have to deal with their Kaluza Klein excitations. In fact, there are an infinite number of (anti)monopole-instantons contributing to the partition function, thanks to the compact nature of S_L^1 . Fortunately enough, at weak coupling one has to take into account only the ones with the lowest action. Using the dual photon description, the effective vertices of the two main (anti)monopoles can be written as [11]

$$\begin{aligned}\mathcal{M} &= \rho^3 e^{-S_{\mathcal{M}}} e^{i(\phi + \frac{va}{2\pi})}, & \overline{\mathcal{M}} &= \rho^3 e^{-S_{\mathcal{M}}} e^{-i(\phi + \frac{va}{2\pi})} \\ \mathcal{K} &= \rho^3 e^{-S_{\mathcal{K}}} e^{-i(\phi - \frac{va}{2\pi})}, & \overline{\mathcal{K}} &= \rho^3 e^{-S_{\mathcal{K}}} e^{i(\phi - \frac{va}{2\pi})},\end{aligned}\tag{44}$$

where the bar denotes anti-monopole. The actions of the monopoles are

$$S_{\mathcal{M}} = \frac{4\pi}{g^2} v \quad \text{and} \quad S_{\mathcal{K}} = \frac{4\pi}{g^2} (2\pi - v),\tag{45}$$

and the pre-exponent

$$\rho^3 = \text{const} \times \frac{1}{L^3 g^4}.$$

\mathcal{M} is the conventional 't Hooft Polyakov monopole, while \mathcal{K} is the twisted, or lowest Kaluza-Klein, monopole. The full effective action takes the form

$$\begin{aligned}S_{\text{eff}} &= \int d^3x \left\{ \frac{\kappa^2}{2} (\partial_\mu \phi)^2 + \frac{f_a^2}{2} (\partial_\mu a)^2 \right. \\ &\quad \left. - 2\rho^3 e^{-S_{\mathcal{M}}} \cos\left(\phi + \frac{va}{2\pi}\right) - 2\rho^3 e^{-S_{\mathcal{K}}} \cos\left(\phi - \frac{va}{2\pi}\right) \right\}.\end{aligned}\tag{46}$$

In the center-symmetric vacuum $v = \pi$, both \mathcal{M} and \mathcal{K} have the same action. Correspondingly, the contribution from both monopoles have to be added with the same weight. Then the mixing term in the ϕ - a mass matrix vanishes.

3.3 $S_{\mathcal{M}} = S_{\mathcal{K}}$

If $S_{\mathcal{M}} = S_{\mathcal{K}} \equiv \mathcal{S}$, then the potential in (46) takes the form

$$V = -4\rho^3 e^{-\mathcal{S}} \cos \phi \cos \frac{a}{2}.\tag{47}$$

It is obvious that the solution with pure ϕ domain wall presented in Sec. 2.3 and $a = 0$ goes through. If this solution is stable, then we can conclude that in this case axion's impact on confinement is absent.

The stability of the $a = 0$ solution can be checked by linearizing the equation for a near $a = 0$ and by determining the lowest energy eigenvalue. The equation is

$$\left(-d^2/dy^2 + \frac{\rho^3 e^{-\mathcal{S}}}{f_a^2} \cos \phi_0(y)\right) a = \varepsilon_a a \quad (48)$$

(see Fig. 2 for the potential), where y is the direction perpendicular to the wall line and $\phi_0(y)$ is the solution for the ϕ domain line discussed in Sec. 2.3. The lowest eigenvalue wavefunction must satisfy the boundary conditions $a(y = \pm\infty) = 0$.

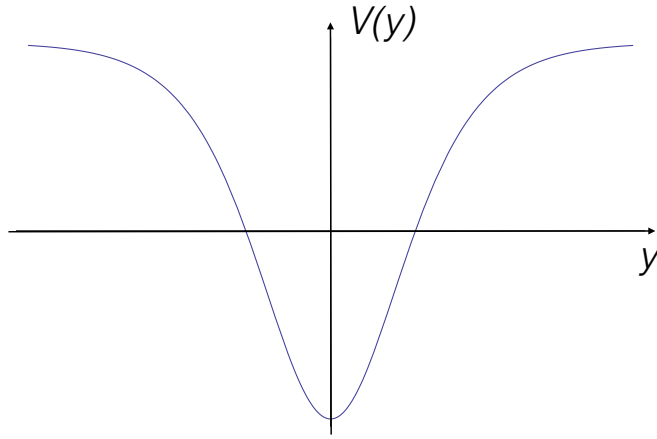


Figure 2: Potential in Eq. (48).

To calculate the lowest eigenvalue, it is convenient to pass to dimensionless variables,

$$\tilde{y} = m_\phi y, \quad \tilde{\varepsilon}_a = \frac{\varepsilon_a}{m_\phi^2}, \quad m_\phi^2 = 4 \frac{\rho^3 e^{-\mathcal{S}}}{\kappa^2}. \quad (49)$$

Then, (48) becomes

$$\left(-d^2/d\tilde{y}^2 + \frac{1}{4x} \cos \phi_0(\tilde{y})\right) a = \tilde{\varepsilon}_a a, \quad (50)$$

where x is defined in (23). At this point, it is important to emphasize that the free parameter x is unavoidable since it is responsible for the strength of the axion interaction. Numerical calculations yield that the lowest eigenvalue is positive for ⁵

$$x > \frac{1}{4},$$

⁵We remind the reader that in the limit $x \rightarrow \infty$ the axion decouples. Thus, the regime $x > 1/4$ is continuously connected to the limit of the axion decoupling.

e.g.,

$$\tilde{\varepsilon}_{a \text{ lowest}} \cong 0.023$$

at $x = 10$. The $a = 0$ solution is stable at least locally. At $x = 1/4$ one can solve Eq. (50) analytically. One finds that at $x = 1/4$ the lowest eigenvalue is exactly at 0 and the zero eigenmode is

$$a_0 = 2 \operatorname{sech} \tilde{y}. \quad (51)$$

For $x < \frac{1}{4}$, Eq. (50) yields negative eigenvalues, e.g., $\tilde{\varepsilon}_{a \text{ lowest}} \cong -0.715$ at $x = 0.1$, indicating the instability of the dual photon domain wall solution of Sec. 2.3 with regards to generation of the a field.

The emergence of negative eigenvalues in (50) for $x < \frac{1}{4}$ forces us to examine the problem from the opposite side: we set $\phi = 0$ and solve for a to find a purely axion domain wall solution,

$$a_0 = 4 \left[\arcsin \tanh(m_a y) + \frac{\pi}{2} \right], \quad m_a^2 = \frac{\rho^3 e^{-S}}{f_a^2}. \quad (52)$$

To check the stability of the solution (52), we linearize the equation of motion of ϕ near $\phi = 0$ in the background of (52) to find the eigenvalue equation

$$\left(-d^2/dy^2 + 4 \frac{\rho^3 e^{-S}}{\kappa^2} \cos\left(\frac{a_0(y)}{2}\right) \right) \phi = \varepsilon_\phi \phi. \quad (53)$$

Using the dimensionless variables

$$\tilde{y} = m_a y, \quad \tilde{\varepsilon}_\phi = \frac{\varepsilon_\phi}{m_a^2}$$

we obtain

$$\left(-d^2/d\tilde{y}^2 + 4x \cos\left(\frac{a_0(\tilde{y})}{2}\right) \right) \phi = \tilde{\varepsilon}_\phi \phi. \quad (54)$$

Equation (54) is identical to Eq. (50) upon the replacement $x \rightarrow 1/(16x)$. Thus, at $x < 1/4$ the axion domain line appears to make the $\phi = 0$ solution locally stable because the lowest eigenvalue $\tilde{\varepsilon}_\phi > 0$.

However, it is clear that beyond small perturbations the $\phi = 0$ solution is unstable. We know for sure that near the electric probe sources $\phi \neq 0$. This means that the lowest energy configuration has both components, a strongly modified ϕ wall significantly different from that of Eq. (2.3), and a correspondingly modified a wall. In this way confinement of charges will be maintained since the 2π periodicity in ϕ is preserved in the equations. This argument is confirmed by numerical analysis of the combined ϕ - a wall in the potential (47) at $x < 1/4$, as shown in Fig. 3. The ϕ wall is clearly seen, with 2π vortices at the endpoints.

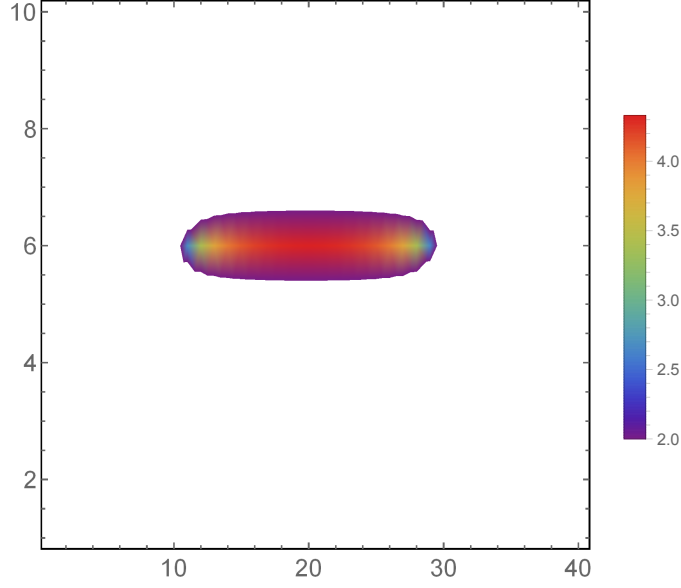


Figure 3: Numerical solution of the full equations of motion resulting from the action (46) in the case $S_{\mathcal{K}} = S_{\mathcal{M}}$. Our simulations in two dimensions are performed using the Gauss-Seidel relaxation method on a 40×40 grid with periodic boundary conditions and two probe charges inserted at (10,6) and (30,6). We plot the electric-field energy density $\frac{E^2}{2} = \frac{1}{2} [(\partial_1 \phi)^2 + (\partial_2 \phi)^2]$ for the parameters $\kappa = 1$, $2\rho^3 e^{-S_{\mathcal{K}}} = 2\rho^3 e^{-S_{\mathcal{M}}} = 1$, and $x = 0.005$. Even for such small values of x , we still can see the electric flux tube extending between the two probe charges. Similar simulations for $x = 0.005$ and $S_{\mathcal{K}} \neq S_{\mathcal{M}}$ show the dissolution of the electric flux tube.

3.4 $S_{\mathcal{M}} \neq S_{\mathcal{K}}$

To recover the situation in Sec. 2.4, we need to destroy the equality of \mathcal{M} and \mathcal{K} contributions. There are two ways to suppress one of the monopoles and hence to make the situation parallel with the case analyzed in Sec. (2.4). First, we can shift the value of v slightly from $v = \pi$. In this way we slightly depart from the exact center symmetry making $S_{\mathcal{K}} > S_{\mathcal{M}}$, or vice versa. Therefore, we can neglect either the \mathcal{K} or \mathcal{M} monopole contribution. This is a very good approximation in the small circle limit, $L\Lambda_{\text{QCD}} \ll 1$, where we have $g^2 \ll 1$.

The second option preserves the center symmetry, but introduces a 4-D massless fermion in the fundamental representation of $SU(2)$. According to the index theorem on $R^3 \times S^1$ [12], the fermion zero-mode will reside on one of the monopoles killing its contribution.

In both cases our action reduces to

$$S_{\text{eff}} = \int d^3x \left[\frac{\kappa^2}{2} (\partial_\mu \phi)^2 + \frac{f_a^2}{2} (\partial_\mu a)^2 + \mu^3 \cos \left(\phi \pm \frac{a}{2} \right) \right], \quad (55)$$

which is exactly the action analyzed in Sec. 2.4 after making a trivial shift $a/2 \rightarrow a$.

4 Axion in two-dimensional $CP(N-1)$ model

From the pioneering Witten paper [3], the large- N solution of the two-dimensional (non-supersymmetric) $CP(N-1)$ model was found. It was shown that the so-called n fields (N -plets in the gauged formulation, see below) are confined, only $n\bar{n}$ mesons appear in the spectrum of the model. Later it was realized (e.g. [2]) that introducing the axion field one dramatically changes the spectrum of the model: confinement is eliminated: $n\bar{n}$ mesons decay into their constituents (for a detailed discussion, see e.g., the review paper [13]). Here we will show that in fact, at large N , the above-mentioned mesons are very narrow, their decay rate is suppressed by $\exp(-cN^\kappa)$ where κ is a positive power, not necessarily integer. At $N=2$, however, one can expect that the asymptotic triplet states in the spectrum of the "axionless" model rapidly decay into the doublet states.

First, we briefly review the model and then explain why the decay rate is exponentially suppressed.

4.1 $CP(N-1)$ in the gauged formulation

The Lagrangian of $CP(N-1)$ model can be written as

$$\mathcal{L} = \frac{2}{g^2} [(\partial_\alpha + iA_\alpha) n_k^* (\partial_\alpha - iA_\alpha) n^k - \lambda (n_k^* n^k - 1)] , \quad (56)$$

where n^k is an N -component complex field,⁶ ($k = 1, 2, \dots, N$) subject to the constraint

$$n_k^* n^k = 1 . \quad (57)$$

Moreover, A_μ is an auxiliary gauge field which has no kinetic term in the bare Lagrangian.

The constraint (57) can be implemented by the Lagrange multiplier λ in (56). One could eliminate the field A_α in (57) by virtue of the equations of motion,

$$A_\alpha = -\frac{i}{2} n_k^* \overleftrightarrow{\partial}_\alpha n^k . \quad (58)$$

However, keeping in mind that a kinetic term for A_μ will be dynamically generated, we will not use (58).

Now, g^2 is a coupling constant; it is asymptotically free and defines a dynamical scale of the theory Λ ,

$$\Lambda^2 = M_{\text{uv}}^2 \exp\left(-\frac{8\pi}{Ng_0^2}\right) , \quad (59)$$

where M_{uv} is an ultraviolet cut-off and g_0^2 is the bare coupling.

In the absence of axion, the solution of the $CP(N-1)$ model at large N is determined by one loop and can be summarized as follows: the constraint (57) is dynamically eliminated

⁶Referred to as quarks or a soliton in Ref. [3].

so that all N fields n^k become independent degrees of freedom with the mass Λ . The photon field A_μ acquires a kinetic term

$$\mathcal{L}_{\gamma \text{ kin}} = -\frac{1}{4e^2} F_{\mu\nu}^2, \quad e^2 = \frac{12\pi\Lambda^2}{N}, \quad (60)$$

and also becomes “dynamical.” We use quotation marks here because in two dimensions the kinetic term (60) does not propagate any physical degrees of freedom; its effect reduces to an instantaneous Coulomb interaction,

$$V_{\text{Coul}} \sim \frac{\Lambda^2}{N} |z|. \quad (61)$$

Because of its linear growth we get linear confinement acting between the n , \bar{n} “quarks.”

4.2 Axion’s impact

Now we switch on the axion,

$$\mathcal{L}_a = \frac{1}{2} f_a^2 (\partial_\mu a)^2 + \frac{a}{2\pi} \varepsilon_{\alpha\gamma} \partial^\alpha A^\gamma, \quad (62)$$

where f_a is the axion constant. In two dimensions it is dimensionless. We will start from the limit $f_a \gg 1$, although this constraint is inessential.

Upon field rescaling, bringing kinetic terms to canonical normalization, one obtains

$$-\frac{1}{4} F_{\mu\nu}^2 + \frac{e}{2\pi f_a} a \varepsilon_{\alpha\gamma} \partial^\alpha A^\gamma + \frac{1}{2} (\partial_\mu a)^2. \quad (63)$$

After diagonalization, the photon becomes massive and a becomes its physical (propagating) degree of freedom. The mass is of order

$$m_\gamma \sim f_a^{-1} \Lambda N^{-1/2}. \quad (64)$$

Interaction between n fields is now mediated by massive quanta, hence, at distances larger than m_γ^{-1} the confining potential (61) is replaced by exponential fall off resulting in deconfinement at distances $\gg m_\gamma^{-1}$.

The decay rate w is determined by the Gamow mechanism. For nonrelativistic values of energy, $E \ll \Lambda$, we obtain

$$w \sim \exp \left[-2 \int dz \sqrt{\Lambda(V(z) - E)} \right] \sim \exp \left(-c f_a^{3/2} N^{1/4} \right), \quad (65)$$

where c is a numerical constant. As f_a and N decrease, the decay rate grows and becomes of order 1 at $f_a, N \sim 1$.

One can consider another mechanism of deconfinement. In the “axionless” $CP(N-1)$ model, there are $\sim N$ quasivacua split in energy, the splitting being of order of Λ^2/N (labeled

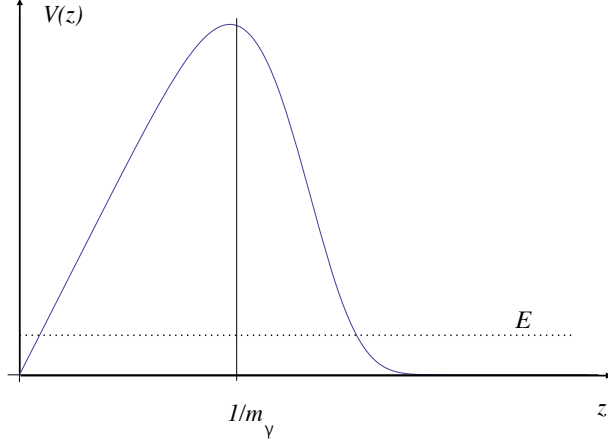


Figure 4: Coulomb interaction between n and \bar{n} cut off by the Yukawa exponent at distances larger than m_γ^{-1} .

by an integer k). Only the lower minimum is the true vacuum while all others are metastable excited states.⁷ At large N , the k dependence of the energy density on the quasivacua, as well as the θ dependence, is well-known

$$E_k(\theta) \sim N \Lambda^2 \left\{ 1 + \text{const} \left(\frac{2\pi k + \theta}{N} \right)^2 \right\}. \quad (66)$$

At $\theta = 0$, the genuine vacuum corresponds to $k = 0$, while the first excitation corresponds to $k = -1$. At $\theta = \pi$, these two vacua are degenerate, and at $\theta = 2\pi$ their roles interchange.

The energy split ensures kink confinement: kinks do not exist as asymptotic states — instead, they form kink-antikink mesons. The regions to the left of the kink and to the right of the antikink are the domains of the true vacuum (at $\theta = 0$ it corresponds to $k = 0$.) The region between the kink and antikink is an insertion of the adjacent quasivacuum with $k = -1$.

When we introduce the axion, the vacuum angle θ is replaced by a dynamical field, $a(t, z)$. In the regions to the left of the kink and to the right of the antikink $\langle a \rangle = 0$. If the region between the kink and antikink is large enough (this can happen e.g. if $m_a \sim \Lambda$), the axion field in this region adjusts itself in such way to minimize the energy,

$$\langle a \rangle = 0 \longrightarrow \langle a \rangle = 2\pi.$$

The intermediate false vacuum decays in the true vacuum, through the axion wall formation and restructuring of the n -field core in the middle. This probability can be estimated too,

$$\tilde{w} \sim \exp(-\tilde{c}N) \quad (67)$$

⁷In the large N limit the decay rate is exponentially small, $\sim \exp(-N)$.

and is smaller than (65) at large N .

5 Conclusions

We considered the impact of axions on confinement in two popular models. In the three dimensional Polyakov model the mixing between the dual photon and axion is crucial in changing the “string” (domain line) structure. This change leads to a “leakage” of a part of the electric flux to the Coulomb regime. At small f_a the domain line is built entirely from axions, and the electric flux disperses in the “bulk.”

In the two-dimensional $CP(N - 1)$ model deconfinement disappears at $N = \infty$ and f_a fixed. However, if we fix N and let f_a become small, we observe the full blown deconfinement.

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