Consistent type IIB reductions to maximal 5D supergravity

Arnaud Baguet, Olaf Hohm, and Henning Samtleben

Phys. Rev. D **92**, 065004 — Published 8 September 2015

DOI: 10.1103/PhysRevD.92.065004

MIT-CTP/4670

Consistent Type IIB Reductions to Maximal 5D Supergravity

Arnaud Baguet[1], Olaf Hohm[2] and Henning Samtleben[1]

[1] *Université de Lyon, Laboratoire de Physique, UMR 5672, CNRS*
École Normale Supérieure de Lyon
46, allée d'Italie, F-69364 Lyon cedex 07, France
arnaud.baguet@ens-lyon.fr, henning.samtleben@ens-lyon.fr

[2] *Center for Theoretical Physics*
Massachusetts Institute of Technology
Cambridge, MA 02139, USA
ohohm@mit.edu

Abstract

We use exceptional field theory as a tool to work out the full non-linear reduction ansaetze for the $\mathrm{AdS}_5 \times S^5$ compactification of IIB supergravity and its non-compact counterparts in which the sphere S^5 is replaced by the inhomogeneous hyperboloidal space $H^{p,q}$. The resulting theories are the maximal 5D supergravities with gauge groups $\mathrm{SO}(p,q)$. They are consistent truncations in the sense that every solution of 5D supergravity lifts to a solution of IIB supergravity. In particular, every stationary point and every holographic RG flow of the scalar potentials for the compact and non-compact 5D gaugings directly lift to solutions of IIB supergravity.

1 Introduction

It is a notoriously difficult problem to establish the consistency of Kaluza-Klein truncations. Consistency requires that any solution of the lower-dimensional theory can be lifted to a solution of the original higher-dimensional theory [1]. While this condition is trivially satisfied for torus compactifications, the compactification on curved manifolds is generically inconsistent except for very specific geometries and matter content of the theories. Even in the case of maximally symmetric spherical geometries, consistency only holds for a few very special cases [2] and even then the proof is often surprisingly laborious. An example for a Kaluza-Klein truncation for which a complete proof of consistency was out of reach until recently is that of type IIB supergravity on $\mathrm{AdS}_5 \times \mathrm{S}^5$, which is believed to have a consistent truncation to the maximal SO(6) gauged supergravity in five dimensions constructed in [3–5]. In general not even the form of the non-linear Kaluza-Klein reduction ansatz for the higher-dimensional fields is explicitly known, in which case it is not even known how to perform the Kaluza-Klein reduction in principle. If the reduction ansatz is known it remains the task to show that the internal coordinate dependence of the higher-dimensional field equations factors out such that these equations consistently reduce to those of the lower-dimensional theory. Despite these complications, consistency proofs have been obtained over the years for various special cases. The maximal eleven-dimensional supergravity admits consistent Kaluza-Klein truncations on $\mathrm{AdS}_4 \times \mathrm{S}^7$ [6] and $\mathrm{AdS}_7 \times \mathrm{S}^4$ [7]. Subsectors of truncations of type IIB to five dimensions have been shown to be consistent in [8–10, 13–17]. More recently, a consistent truncation of massive type IIA supergravity on S^6 has been found [18].

In this paper we will present the explicit and complete reduction formulas for a large class of truncations of type IIB supergravity to maximal five-dimensional gauged supergravity, by working out the details of the general construction of [19]. This includes the famous reduction on $\mathrm{AdS}_5 \times \mathrm{S}^5$ to the maximal $D = 5$ SO(6) gauged supergravity of [5], but also reductions to non-compact gaugings, corresponding to truncations with non-compact (hyperboloidal) internal manifolds. Consistency of the latter has first been conjectured in [20] and more recently been discussed in [21, 22]. The crucial new ingredient that makes our construction feasible is the recently constructed 'exceptional field theory' (EFT) [23–26] and its associated extended geometry, see [27–30], and [31–34] for the closely related double field theory. Within this framework, the complicated geometric IIB reductions can very conveniently be formulated as Scherk-Schwarz reductions on an exceptional space-time.

In order to illustrate this point, it is useful to compare it with the toy example of an S^2 compactification of the D-dimensional Einstein-Maxwell theory, whose volume form provides the source for the U(1) field strength. With a particular dilaton coupling, this theory not only permits a vacuum solution with S^2 as the compact space but also a consistent Kaluza-Klein truncation around this vacuum to a $(D-2)$-dimensional theory [2]. The required dilaton couplings are precisely those that follow from embedding the original theory as the S^1 reduction of pure gravity in $D+1$ dimensions. While the consistency of this reduction can be shown by a direct computation, a far more elegant proof relies on this geometric origin. As shown in [35], from the point of view of $(D+1)$ dimensional Einstein gravity, the original S^2 reduction takes the form of a Scherk-Schwarz (or DeWitt) reduction on a three-dimensional SO(3) group manifold

via the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. For Scherk-Schwarz reductions, however, consistency is guaranteed from symmetry arguments [36], which then implies the consistency of the S^2 reduction of the Einstein-Maxwell theory. In this sense, the consistency of the S^2 reduction hinges on the fact that the original theory is secretly a 'geometric' theory in higher dimensions (namely pure Einstein gravity).

Similarly, in exceptional field theory maximal supergravity is reformulated on an extended higher-dimensional space that renders the theory covariant w.r.t. the exceptional U-duality groups in the series $\mathrm{E}_{d(d)}$, $2 \leqslant d \leqslant 8$. In this case, the higher-dimensional theory is not simply Einstein gravity, but EFT is subject to a covariant constraint that implies that only a subspace of the extended space is physical. Solving the constraint accordingly one obtains either type IIB or eleven-dimensional supergravity. Importantly, the gauge symmetries of EFT are governed by 'generalized Lie derivatives' that unify the usual diffeomorphism and tensor gauge transformations of supergravity into generalized diffeomorphisms of the extended space. Specifically, for the $\mathrm{E}_{6(6)}$ EFT that will be employed in this paper the generalized Lie derivative for vector fields V^M, W^M, $M, N = 1, \ldots, 27$, in the fundamental representation $\mathbf{\bar{27}}$ reads [28,37]

$$\big(\mathbb{L}_V W\big)^M \;\equiv\; V^N \partial_N W^M - W^N \partial_N V^M + 10\, d^{MNP}\, d_{KLP}\, \partial_N V^K\, W^L \,, \tag{1.1}$$

where d^{MNK} is a (symmetric) invariant tensor of $\mathrm{E}_{6(6)}$. Here the first two terms represent the standard Lie bracket or derivative on the extended 27-dimensional space, while the new term encodes the non-trivial modification of the diffeomorphism algebra.

It was shown in [19] how sphere compactifications of the original supergravities and their non-compact cousins can be realized in EFT through generalized Scherk-Schwarz compactifications, which are governed by $\mathrm{E}_{d(d)}$ valued 'twist' matrices. In terms of the duality covariant fields of EFT the reduction formulas take the form of a simple Scherk-Schwarz ansatz (see (2.1) below), proving the consistency of the corresponding Kaluza-Klein truncation. Although this settles the issue of consistency it may nevertheless be useful to have the explicit reduction formulas in terms of the *conventional* supergravity fields. This requires the dictionary for identifying the original supergravity fields in the EFT formulation. In this paper we work out the explicit reduction formulas for the complete set of type IIB supergravity fields, using the general embedding of type IIB supergravity into the $\mathrm{E}_{6(6)}$ EFT given in [38]. In particular, this includes all components of the IIB self-dual four form. Results for the scalar sector in the compact case have appeared in [11, 12, 39, 40]. The components of the twist matrix give rise to various conventional tensors, including for instance the Killing vectors in the case of S^5 but also various higher Killing-type tensors. We analyze the identities satisfied by these tensors by decomposing the Lie derivatives (1.1), which can be thought of as giving generalized Killing equations on the extended space. Various identities that appear miraculous from the point of view of standard geometry but are essential for consistency of the Kaluza-Klein ansatz are thereby explained in terms of the higher-dimensional $\mathrm{E}_{6(6)}$ covariant geometry of EFT.

This paper is not completely self-contained in that we assume some familiarity with the $\mathrm{E}_{6(6)}$ EFT of [24]. Our recent review [38], which also gives the complete embedding of type IIB, can serve as a preparatory article. In particular, we use the same conventions. The rest of this paper is organized as follows. In sec. 2 we briefly review the generalized Scherk-Schwarz ansatz and the consistency conditions for the $\mathrm{E}_{6(6)}$ EFT and give the twist matrices. The twist matrix

gives rise to a set of generalized vectors of the extended space satisfying an algebra of generalized Lie derivatives (1.1) akin to the algebra of Killing vector fields on a conventional manifold. In sec. 3 we analyze the various components of this equation and give the explicit solutions in terms of various Killing-type tensors. In sec. 4 we review the class of $D = 5$ gauged supergravities that will be embedded into type IIB. Finally, in sec. 5 we work out the complete Kaluza-Klein ansatz by using the general embedding of type IIB established in [38]. In particular, we show how to reconstruct the self-dual 4-form of type IIB from the EFT fields. Along the way, we show that the reduction ansatz reduces the ten-dimensional self-duality equations to the equations of motion of the $D = 5$ theory. While this is guaranteed by the general argument, its explicit realization requires an impressive interplay of Killing vector/tensor identities and the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset space structure of the five-dimensional scalar fields. In sec. 6 we summarize the final results, the full set of reduction formulas, and comment on the fermionic sector. Some technically involved computations are relegated to an appendix.

2 Generalized Scherk-Schwarz reduction

We begin by giving the generalized Scherk-Schwarz ansatz in terms of the variables of exceptional field theory. This ansatz is governed by a group-valued twist matrix $U \in \mathrm{E}_{6(6)}$ and a scale factor ρ, both of which depend only on the internal coordinates Y. For the bosonic EFT fields, the general reduction ansatz reads [19]

$$\begin{aligned}
\mathcal{M}_{MN}(x,Y) &= U_M{}^{\underline{K}}(Y)\, U_N{}^{\underline{L}}(Y)\, M_{\underline{K}\underline{L}}(x)\,, \\
g_{\mu\nu}(x,Y) &= \rho^{-2}(Y)\, \mathrm{g}_{\mu\nu}(x)\,, \\
\mathcal{A}_\mu{}^M(x,Y) &= \rho^{-1}(Y) A_\mu{}^{\underline{N}}(x) (U^{-1})_{\underline{N}}{}^M(Y)\,, \\
\mathcal{B}_{\mu\nu\,M}(x,Y) &= \rho^{-2}(Y) U_M{}^{\underline{N}}(Y)\, B_{\mu\nu\,\underline{N}}(x)\,.
\end{aligned} \tag{2.1}$$

Here, indices M, N label the fundamental representation **27** of $\mathrm{E}_{6(6)}$, and the four lines refer to the internal metric, external metric, vector fields and two-forms, respectively, see [24] for details. In order for the ansatz (2.1) to be consistent, U and ρ need to factor out homogeneously of all covariant expressions defining the action and equations of motion. This is the case provided the following two consistency equations ('twist equations') are satisfied:

$$\begin{aligned}
\partial_N (U^{-1})_{\underline{K}}{}^N - 4\,(U^{-1})_{\underline{K}}{}^N\, \rho^{-1}\partial_N \rho &= 3\,\rho\,\vartheta_{\underline{K}}\,, \\
\big[(U^{-1})_{\underline{M}}{}^K (U^{-1})_{\underline{N}}{}^L \partial_K U_L{}^{\underline{P}}\big]_{\mathbf{351}} &= \frac{1}{5}\,\rho\,\Theta_{\underline{M}}{}^{\boldsymbol{\alpha}} t_{\boldsymbol{\alpha}\underline{N}}{}^{\underline{P}}\,.
\end{aligned} \tag{2.2}$$

Here the *constant* tensors are $\vartheta_{\underline{K}}$, which defines the embedding tensor of 'trombone' gaugings, and $\Theta_{\underline{M}}{}^{\boldsymbol{\alpha}}$, which defines the embedding tensor of conventional gaugings.

For the subsequent analysis it is convenient to reformulate these consistency conditions by rescaling the twist matrix by ρ,

$$\widehat{U}^{-1} \equiv \rho^{-1}\, U^{-1}\,. \tag{2.3}$$

This rescaling is such that $\widehat{U}^{-1}$ can be viewed as a generalized vector of the same density weight as the gauge parameters. Accordingly, one can define generalized Lie derivatives w.r.t. this

vector. The consistency conditions can then be brought into the compact form

$$\mathbb{L}_{\widehat{U}^{-1}_{\underline{M}}}\,\widehat{U}^{-1}_{\underline{N}} \;\equiv\; -X_{\underline{MN}}{}^{\underline{K}}\,\widehat{U}^{-1}_{\underline{K}}\;, \tag{2.4}$$

where $X_{\underline{MN}}{}^{\underline{K}}$ are constants related to the $D=5$ embedding tensor by

$$X_{\underline{MN}}{}^{\underline{K}} \;=\; \left(\Theta_{\underline{M}}{}^{\boldsymbol{\alpha}} + \frac{9}{2}\,\vartheta_{\underline{L}}(t^{\boldsymbol{\alpha}})_{\underline{M}}{}^{\underline{L}}\right)(t_{\boldsymbol{\alpha}})_{\underline{N}}{}^{\underline{K}} - \delta_{\underline{N}}{}^{\underline{K}}\vartheta_{\underline{M}}\;. \tag{2.5}$$

This implies in particular that the first equation in (2.2) can be written as

$$\mathbb{L}_{\widehat{U}^{-1}_{\underline{M}}}\,\rho \;=\; -\vartheta_{\underline{M}}\,\rho\;. \tag{2.6}$$

In [19], the consistency equations (2.2) were solved for the sphere and hyperboloid compactifications, with gauge groups $\mathrm{SO}(p,6-p)$ and $\mathrm{CSO}(p,q,6-p-q)$, explicitly in terms of SL(6) group-valued twist matrices. Specifically, with the fundamental representation of $\mathrm{E}_{6(6)}$ decomposing as

$$\{Y^M\} \;\longrightarrow\; \{Y^{ab}, Y_{a\alpha}\}\;, \tag{2.7}$$

into $(15,1)\oplus(6',2)$ under $\mathrm{SL}(6)\times\mathrm{SL}(2)$, we single out one of the fundamental SL(6) indices $a\to(0,i)$ to define the SL(6) matrix $U_a{}^b$ as

$$\begin{aligned}
U_0{}^0 &\equiv (1-v)^{-5/6}\,(1+uK(u,v))\;,\\
U_0{}^i &\equiv -\eta_{ij}y^j\,(1-v)^{-1/3}\,K(u,v)\;,\\
U_i{}^0 &\equiv -\eta_{ij}y^j\,(1-v)^{-1/3}\;,\\
U_i{}^j &\equiv (1-v)^{1/6}\,\delta^{ij}\;,
\end{aligned} \tag{2.8}$$

with the combinations

$$u \;\equiv\; y^i\delta_{ij}y^j\;,\qquad v \;\equiv\; y^i\eta_{ij}y^j\;. \tag{2.9}$$

Here η_{ij} is the metric

$$\eta_{ij} = \mathrm{diag}\,(\,\underbrace{1,\ldots,1}_{p-1},\underbrace{-1,\ldots,-1}_{6-p}\,)\;, \tag{2.10}$$

and we define similarly the $\mathrm{SO}(p,6-p)$ invariant metric η_{ab} with signature $(p,6-p)$. Note that in (2.9) we use two different metrics, one Euclidean, the other pseudo-Euclidean. The function $K(u,v)$ is the solution of the differential equation

$$2(1-v)\,(u\,\partial_v K + v\,\partial_u K) \;=\; ((7-2p)(1-v)-u)\,K - 1\;, \tag{2.11}$$

which can be solved analytically. For instance, for $p=6$, i.e., for gauge group SO(6) relevant for the S^5 compactification, the solution reads

$$p=6\;:\qquad K(u) \;=\; \frac{1}{2}\,u^{-3}\,\left(u(u-3)+\sqrt{u(1-u)}\,\left(3\arcsin\sqrt{u}+c_0\right)\right)\;, \tag{2.12}$$

with constant c_0. We refer to [19] for other explicit forms. The inverse twist matrix is given by

$$\begin{aligned}
(U^{-1})_0{}^0 &= (1-v)^{5/6} \,, \\
(U^{-1})_0{}^i &= \eta_{ij} y^j \, (1-v)^{1/3} \, K(u,v) \,, \\
(U^{-1})_i{}^0 &= \eta_{ij} y^j \, (1-v)^{1/3} \,, \\
(U^{-1})_i{}^j &= (1-v)^{-1/6} \left(\delta^{ij} + \eta_{ik} \eta_{jl} \, y^k y^l \, K(u,v) \right) \,.
\end{aligned} \tag{2.13}$$

Finally, the density factor ρ is given by

$$\rho = (1-v)^{1/6} \,. \tag{2.14}$$

Upon embedding the SL(6) twist matrix (2.8) into $\mathrm{E}_{6(6)}$, one may verify that it satisfies the consistency equations (2.2) with an embedding tensor that describes the gauge group $\mathrm{SO}(p,q)$, where the physical coordinates are embedded into the EFT coordinates via (2.7) according to

$$y^i = Y^{[0i]} \,. \tag{2.15}$$

With the above form of the generalized Scherk-Schwarz ansatz and the explicit form of the twist matrix and the scale factor we have given the complete embedding of the corresponding sphere and hyperboloid compactifications into the $\mathrm{E}_{6(6)}$ EFT. It is instructive, however, to clarify this embedding by analyzing it in terms of more conventional geometric objects. Therefore, in the next section we will analyze the consistency conditions (2.4) under the appropriate decomposition (that embeds, for instance, the standard algebra of Killing vector fields on a sphere) and thereby reconstruct the above solution in a more conventional language. In particular, this will clarify the geometric significance of the function K, which is related to the four-form whose exterior derivative defines the volume form on the five-sphere.

3 Untangling the twist equations

3.1 General analysis

We now return to the 'twist equations' (2.4) and decompose them w.r.t. the subgroup appropriate for the type IIB solution of the section constraint, i.e.

$$\begin{aligned}
\mathrm{E}_{6(6)} &\longrightarrow \mathrm{GL}(5) \times \mathrm{SL}(2) \,, \\
\mathbf{27} &\longrightarrow (\mathbf{5},\mathbf{1}) \oplus (\mathbf{5}',\mathbf{2}) \oplus (\mathbf{10},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \,.
\end{aligned} \tag{3.1}$$

Accordingly, the fundamental index on the generalized vector $\widehat{U}^{-1}$ decomposes as

$$(\widehat{U}^{-1})_{\underline{M}}{}^M = \left\{ \mathcal{K}_{\underline{M}}{}^m \,,\, \mathcal{R}_{\underline{M}\, m\alpha} \,,\, \mathcal{Z}_{\underline{M}\, mnk} \,,\, \mathcal{S}_{\underline{M}\, n_1 \ldots n_5 \alpha} \right\} \,, \tag{3.2}$$

in terms of GL(5) indices $m,n = 1,\ldots,5$ and SL(2) indices $\alpha,\beta = 1,2$. In order to give the decomposition of the twist equations (2.4) in terms of these objects we use the definition

(1.1) of the generalized Lie derivative and the decomposition of the d-symbol (3.28) in [38]. A straightforward computation, largely analogous to those in, e.g., sec. 3.3 of [38], then yields

$$-X_{\underline{MN}}{}^{\underline{K}}\mathcal{K}_{\underline{K}}{}^{m} = \mathcal{L}_{\mathcal{K}_{\underline{M}}}\mathcal{K}_{\underline{N}}{}^{m} , \tag{3.3}$$

$$-X_{\underline{MN}}{}^{\underline{K}}\mathcal{R}_{\underline{K}\,m\alpha} = \mathcal{L}_{\mathcal{K}_{\underline{M}}}\mathcal{R}_{\underline{N}\,m\alpha} - \mathcal{L}_{\mathcal{K}_{\underline{N}}}\mathcal{R}_{\underline{M}\,m\alpha} + \partial_m\left(\mathcal{K}_{\underline{N}}{}^{n}\mathcal{R}_{\underline{M}\,n\alpha}\right) , \tag{3.4}$$

$$\begin{aligned}-X_{\underline{MN}}{}^{\underline{K}}\mathcal{Z}_{\underline{K}\,kmn} &= \mathcal{L}_{\mathcal{K}_{\underline{M}}}\mathcal{Z}_{\underline{N}\,kmn} - \mathcal{L}_{\mathcal{K}_{\underline{N}}}\mathcal{Z}_{\underline{M}\,kmn} + 3\,\partial_{[k}\left(\mathcal{K}_{\underline{N}}{}^{l}\mathcal{Z}_{\underline{M}\,mn]l}\right)\\ &\quad + 3\sqrt{2}\,\varepsilon^{\alpha\beta}\,\partial_{[k}\mathcal{R}_{\underline{M}\,m|\alpha|}\mathcal{R}_{\underline{N}\,n]\beta} ,\end{aligned} \tag{3.5}$$

$$\begin{aligned}-X_{\underline{MN}}{}^{\underline{K}}\mathcal{S}_{\underline{K}\,n_1\ldots n_5\alpha} &= \mathcal{L}_{\mathcal{K}_{\underline{M}}}\mathcal{S}_{\underline{N}\,n_1\ldots n_5\alpha}\\ &\quad + 20\sqrt{2}\left(\mathcal{Z}_{\underline{N}\,[n_1n_2n_3}\partial_{n_4}\mathcal{R}_{\underline{M}\,n_5]\alpha} - \partial_{[n_1}\mathcal{Z}_{\underline{M}\,n_2n_3n_4}\mathcal{R}_{\underline{N}\,n_5]\alpha}\right) .\end{aligned} \tag{3.6}$$

We will now successively analyze these equations. We split the index as $\underline{M} \to \{A, u\}$, where A, B denote the 'gauge group directions' and u, v the remaining ones, and assume that the only non-vanishing entries of $X_{\underline{MN}}{}^{\underline{K}}$ are

$$X_{AB}{}^{C} = -f_{AB}{}^{C} , \qquad X_{Au}{}^{v} = (D_A)_u{}^{v} , \tag{3.7}$$

given in terms of structure constants and representation matrices of the underlying Lie algebra of the gauge group, c.f. [41]. Let us emphasize that $X_{\underline{MN}}{}^{\underline{K}}$ is not assumed to be antisymmetric. In particular, for this ansatz we have, e.g., $X_{uA}{}^{v} = 0$. Let us also stress that this ansatz is not the most general, but it is sufficient for the purposes in this paper.

The first equation (3.3), specialized to external indices (A, B), implies that the vector fields $\mathcal{K}_A$ satisfy the Lie bracket algebra

$$\left[\mathcal{K}_A, \mathcal{K}_B\right]^m \equiv \mathcal{L}_{\mathcal{K}_A}\mathcal{K}_B{}^{m} = f_{AB}{}^{C}\,\mathcal{K}_C{}^{m} . \tag{3.8}$$

In view of standard Kaluza-Klein compactifications it is natural to interpret these vector fields as the Killing vectors of some internal geometry. We now *define* a metric w.r.t. which the $\mathcal{K}_A$ are indeed Killing vectors by setting for the inverse metric

$$\tilde{G}^{mn} \equiv \mathcal{K}_A{}^{m}\,\mathcal{K}_B{}^{n}\,\eta^{AB} , \tag{3.9}$$

with the Cartan-Killing metric $\eta_{AB} \equiv f_{AC}{}^{D}f_{BD}{}^{C}$. The internal metric $\tilde{G}_{mn}$ exists provided the Cartan-Killing metric is invertible and that there are sufficiently many vectors fields $\mathcal{K}_A{}^{m}$ to make $\tilde{G}^{mn}$ invertible. This assumption, which we will make throughout the following discussion, is satisfied in the examples below. Since by (3.8) the $\mathcal{K}_A$ transform under themselves according to the adjoint group action, under which the Cartan-Killing metric is invariant, it follows that the vectors are indeed Killing:

$$\mathcal{L}_{\mathcal{K}_A}\tilde{G}_{mn} \equiv \nabla_m\mathcal{K}_{An} + \nabla_n\mathcal{K}_{Am} = 0 , \tag{3.10}$$

where here and in the following ∇_m denotes the covariant derivative w.r.t. the metric (3.9), which is used to raise and lower indices. The other non-trivial components of (3.3), with external indices (A, u), (u, A) and (u, v), imply that the remaining vector fields $\mathcal{K}_u{}^{m}$ satisfy

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{K}_u{}^{m} = -(D_A)_u{}^{v}\,\mathcal{K}_v{}^{m} = 0 , \qquad \mathcal{L}_{\mathcal{K}_u}\mathcal{K}_v{}^{m} \equiv \left[\mathcal{K}_u, \mathcal{K}_v\right]^m = 0 . \tag{3.11}$$

For non-vanishing $\mathcal{K}_u$ the first equation can only be satisfied if the representation encoded by the $(D_A)_u{}^v$ includes the trivial (singlet) representation. In the following we will analyze the remaining equations under the assumption that the representation does not contain a trivial part, which then requires

$$\mathcal{K}_u{}^m \;=\; 0\,. \tag{3.12}$$

We next consider the second equation (3.4), specialized to external indices (A,u) and (u,A) to obtain

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{R}_{u\,m\alpha} \;=\; -(D_A)_u{}^v\,\mathcal{R}_{v\,m\alpha} \;=\; \partial_m\left(\mathcal{K}_A{}^n\mathcal{R}_{u\,n\alpha}\right)\,. \tag{3.13}$$

Writing out the Lie derivative on the left-hand side we obtain in particular

$$\mathcal{K}_A{}^n\left(\partial_m\mathcal{R}_{u\,n\alpha}-\partial_n\mathcal{R}_{u\,m\alpha}\right) \;=\; 0\,. \tag{3.14}$$

With the above assumption that the metric (3.9) is invertible it follows that the curl of $\mathcal{R}$ is zero. Hence we can write it in terms of a gradient,

$$\mathcal{R}_{u\,m\alpha} \;\equiv\; \partial_m\mathcal{Y}_{u\,\alpha}\,. \tag{3.15}$$

As we still have to solve the first equation of (3.13), we must demand that the function $\mathcal{Y}$ transforms under the Killing vectors in the representation D_A,

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{Y}_{u\,\alpha} \;=\; -(D_A)_u{}^v\,\mathcal{Y}_{v\,\alpha}\,, \tag{3.16}$$

for then (3.13) follows with the covariant relation (3.15). Finally, specializing (3.4) to external indices (A,B), we obtain

$$f_{AB}{}^C\mathcal{R}_{C\,m\alpha} \;=\; \mathcal{L}_{\mathcal{K}_A}\mathcal{R}_{B\,m\alpha}-\mathcal{L}_{\mathcal{K}_B}\mathcal{R}_{A\,m\alpha}+\partial_m\big(\mathcal{K}_B{}^n\mathcal{R}_{A\,n\alpha}\big)\,. \tag{3.17}$$

This equation is solved by $\mathcal{R}_{A\,m\alpha}=0$, and the latter indeed holds for the SL(6) valued twist matrix to be discussed below. In addition, we will find that for these twist matrices also the components $\mathcal{Z}_u$ and $\mathcal{S}_A$ are zero, and therefore in the following we analyze the equations for this special case,

$$\mathcal{R}_{A\,m\alpha} \;=\; \mathcal{Z}_{u\,mnk} \;=\; \mathcal{S}_{A\,n_1\ldots n_5\,\alpha} \;=\; 0\,. \tag{3.18}$$

Let us now turn to the third equation (3.5), which will constrain the $\mathcal{Z}$ tensor. Specializing to external indices (A,B), we obtain

$$f_{AB}{}^C\,\mathcal{Z}_{C\,kmn} \;=\; \mathcal{L}_{\mathcal{K}_A}\mathcal{Z}_{B\,kmn}-\mathcal{L}_{\mathcal{K}_B}\mathcal{Z}_{A\,kmn}+3\,\partial_{[k}\big(\mathcal{K}_B{}^l\mathcal{Z}_{A\,mn]l}\big)\,, \tag{3.19}$$

where we used (3.18). Writing out the second Lie derivative on the right-hand side, this can be reorganized as

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{Z}_{B\,kmn}-4\,\mathcal{K}_B{}^p\,\partial_{[p}\mathcal{Z}_{A\,kmn]} \;=\; f_{AB}{}^C\,\mathcal{Z}_{C\,kmn}\,. \tag{3.20}$$

In order to solve this equation we make the following ansatz

$$\mathcal{Z}_{A\,klm} \;\equiv\; -\frac{1}{4}\sqrt{2}\,\mathcal{K}_{A\,klm}-\frac{1}{8}\sqrt{2}\,\mathcal{K}_A{}^p\,\tilde{C}_{pklm}\,, \tag{3.21}$$

in terms of a four-form $\tilde{C}$, where we chose the normalization for later convenience, and we defined the Killing tensor

$$\mathcal{K}_{A\,klm} \;\equiv\; \frac{1}{2}\,\tilde{\omega}_{klmpq}\,\mathcal{K}_A{}^{pq}\,, \qquad \mathcal{K}_{A\,mn} \;\equiv\; 2\,\nabla_{[m}\mathcal{K}_{A\,n]}\,, \tag{3.22}$$

with the volume form $\tilde{\omega}_{klmpq} \equiv |\tilde{G}|^{1/2}\,\varepsilon_{klmpq}$. We recall that all internal indices are raised and lowered with $\tilde{G}_{mn}$ defined in (3.9).

It remains to determine $\tilde{C}_{pklm}$ from the above system of equations. In order to simplify the result of inserting (3.21) into (3.20) we can use that the Killing tensor term transforms 'covariantly' under the Lie derivative,

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{K}_{B\,mnk} \;=\; f_{AB}{}^{C}\,\mathcal{K}_{C\,mnk}\,, \tag{3.23}$$

which follows from the corresponding property (3.8) of the Killing vectors. For the second term on the left-hand side of (3.20), however, we have to compute,

$$\begin{aligned}\mathcal{K}_B{}^p\nabla_{[p}\mathcal{K}_{A\,kmn]} \;&=\; \mathcal{K}_B{}^p\nabla_{[p}\big(\tfrac{1}{2}\,\tilde{\omega}_{kmn]lq}\,\mathcal{K}_A{}^{lq}\big) \;=\; \mathcal{K}_B{}^p\,\tilde{\omega}_{lq[kmn}\nabla_{p]}\nabla^{[l}\mathcal{K}_A{}^{q]} \\ &=\; -\tfrac{1}{2}\,\mathcal{K}_B{}^p\,\tilde{\omega}_{kmnpl}\,\nabla_q\nabla^{[l}\mathcal{K}_A{}^{q]} \;=\; \tfrac{1}{2}\,\mathcal{K}_B{}^p\,\tilde{\omega}_{kmnpl}\,\nabla_q\nabla^{q}\mathcal{K}_A{}^{l}\,.\end{aligned} \tag{3.24}$$

Here we used the $D=5$ Schouten identity $\tilde{\omega}_{[lqkmn}\nabla_{p]} \equiv 0$ and that the Killing tensor written as $\mathcal{K}_{A\,mn} = 2\,\nabla_m\mathcal{K}_{A\,n}$ is automatically antisymmetric as a consequence of the Killing equations (3.10). Using the latter fact again, the last expression simplifies as follows

$$\nabla_q\nabla^q\mathcal{K}_A{}^{l} \;=\; -\nabla_q\nabla^l\mathcal{K}_A{}^{q} \;=\; -\big[\nabla_q,\nabla^l\big]\,\mathcal{K}_A{}^{q} \;=\; -\,\tilde{\mathcal{R}}^{lp}\,\mathcal{K}_{A\,p}\,. \tag{3.25}$$

We will see momentarily that (3.20) can be solved analytically by the above ansatz (3.21) if the metric $\tilde{G}$ is Einstein. We thus assume this to be the case, so that the Ricci tensor reads $\tilde{\mathcal{R}}_{mn} = \lambda\,\tilde{G}_{mn}$, for some constant λ. Using this in (3.25) and inserting back into (3.24) we obtain

$$\mathcal{K}_B{}^p\nabla_{[p}\mathcal{K}_{A\,kmn]} \;=\; \frac{\lambda}{2}\,\tilde{\omega}_{kmnpl}\,\mathcal{K}_A{}^{p}\mathcal{K}_B{}^{l}\,. \tag{3.26}$$

Next, insertion of the second term in (3.21) into (3.20) yields the contribution

$$\mathcal{L}_{\mathcal{K}_A}\big(\mathcal{K}_B{}^p\,\tilde{C}_{pkmn}\big) + 4\,\mathcal{K}_B{}^p\,\partial_{[p}\big(\mathcal{K}_A{}^q\,\tilde{C}_{kmn]q}\big) \;=\; f_{AB}{}^{C}\mathcal{K}_C{}^p\,\tilde{C}_{pkmn} + 5\,\mathcal{K}_A{}^p\mathcal{K}_B{}^q\,\partial_{[p}\tilde{C}_{qkmn]}\,. \tag{3.27}$$

Here we used (3.8) and combined the terms from $\mathcal{L}_{\mathcal{K}_A}\tilde{C}_{pkmn}$ with those from the second term on the left-hand side. Employing now (3.26) and (3.27) we find that insertion of (3.21) into (3.20) yields

$$0 \;=\; \mathcal{K}_A{}^p\mathcal{K}_B{}^q\big(5\,\partial_{[p}\tilde{C}_{qkmn]} - 4\,\lambda\,\tilde{\omega}_{pqkmn}\big)\,. \tag{3.28}$$

Thus, we have determined $\tilde{C}$, up to closed terms, to be

$$5\,\partial_{[p}\tilde{C}_{qkmn]} \;=\; 4\lambda\,|\tilde{G}|^{1/2}\varepsilon_{kmnpq}\,, \tag{3.29}$$

which can be integrated to solve for $\tilde{C}_{klmn}$, since in five coordinates the integrability condition is trivially satisfied. In total we have proved that the (A,B) component of the third equation (3.5) of the system is solved by (3.21). We also note that the remaining components of (3.5) are identically satisfied under the assumption (3.18). (For the (u,v) component this requires using

that the exterior derivative of $\mathcal{R}_{u\,m\alpha}$ vanishes by (3.15).) For the subsequent analysis it will be important to determine how $\tilde{C}$ transforms under the Killing vectors. To this end we recall that in the definition (3.21) $\tilde{C}$ is the only 'non-covariant' contribution, which therefore accounts for the second term on the left-hand side of the defining equation (3.20). From this we read off

$$\mathcal{L}_{\mathcal{K}_A}\tilde{C}_{mnkl} \;=\; -16\sqrt{2}\,\partial_{[m}\mathcal{Z}_{A\,nkl]}\;. \tag{3.30}$$

Finally, we turn to the last equation (3.6), which determines $\mathcal{S}_u$. Under the assumptions (3.12), (3.18), the (u,v) and (u,A) components trivialize, while the (A,u) component implies

$$\mathcal{L}_{\mathcal{K}_A}\mathcal{S}_{u\,n_1\ldots n_5\alpha} \;=\; -(D_A)_u{}^v\mathcal{S}_{v\,n_1\ldots n_5\alpha} + 20\sqrt{2}\,\partial_{[n_1}\mathcal{Z}_{A\,n_2n_3n_4}\,\mathcal{R}_{|u|n_5]\alpha}\;. \tag{3.31}$$

We will now show that this equation is solved by

$$\mathcal{S}_{u\,n_1\ldots n_5\alpha} \;=\; a\,\tilde{\omega}_{n_1\ldots n_5}\,\mathcal{Y}_{u\alpha} - \frac{5}{4}\,\tilde{C}_{[n_1\ldots n_4}\,\partial_{n_5]}\mathcal{Y}_{u\alpha}\;, \tag{3.32}$$

in terms of the volume form of $\tilde{G}_{mn}$, the function defined in (3.15) and the four-form defined via (3.29). Here, a is an arbitrary coefficient, while we set the second coefficient to the value that is implied by the following analysis. We first note that $\mathcal{L}_{\mathcal{K}_A}\tilde{\omega}_{n_1\ldots n_5} = 0$, which follows from the invariance under the Killing vectors of the metric $\tilde{G}$ defining $\tilde{\omega}$. Second, we recall (3.16), which states that the function $\mathcal{Y}_u$ transforms 'covariantly' under $\mathcal{L}_{\mathcal{K}_A}$ (i.e., w.r.t. the representation matrices D_A). Thus, all terms in (3.32) transform covariantly, except for the four-form $\tilde{C}$, whose 'anomalous' transformation must therefore account for the second term in $\mathcal{L}_{\mathcal{K}_A}\mathcal{S}_u$ on the right-hand side of (3.31). Using the anomalous transformations of $\tilde{C}$ given in (3.30), it then follows that (3.32) solves (3.31) for arbitrary coefficient a. This concludes our general discussion of the system of equations (3.3)–(3.6).

3.2 Explicit tensors

We now return to the explicit twist matrices and read off the tensors whose general structure we discussed in the previous subsection. To this end we have to split the $\mathrm{E}_{6(6)}$ indices further in order to make contact with the twist matrices given in (2.8), (2.13). As it turns out, for these twist matrices the split of indices $V_{\underline{M}} \equiv (V_A, V_u)$ discussed before (3.7), coincides with the split $27 = 15 + 12$ of (2.7)

$$V_{\underline{M}} \;\equiv\; (V_A, V_u) \;\equiv\; (V_{[ab]}, V^{a\alpha})\;, \qquad a,b \;=\; 0,\ldots,5\;, \quad \alpha,\beta \;=\; 1,2\;. \tag{3.33}$$

In several explicit formulas we will have to split $[ab]$ further,

$$[ab] \;\equiv\; ([0i],[ij])\;, \qquad i,j = 1\ldots,5\;. \tag{3.34}$$

Similarly, we perform the same index split for the fundamental index M under $\mathrm{E}_{6(6)} \to \mathrm{SL}(6)$ (and then further to $\mathrm{GL}(5)\times\mathrm{SL}(2)$ according to (3.1)), thus giving up in the following the distinction between bare and underlined indices. Let us note that we employ the convention

$$V^{0i} \;\equiv\; \tfrac{1}{\sqrt{2}}V^i\;, \tag{3.35}$$

in agreement with the summation conventions of ref. [24]. In order to read off the various tensors from the twist matrices let us first canonically embed the SL(6) matrix $U_a{}^b$ into $\mathrm{E}_{6(6)}$. Under the above index split we have

$$U_M{}^{\underline{N}} \;=\; \begin{pmatrix} U_{[ab]}{}^{[cd]} & U_{[ab]}{}^{c\alpha} \\ U^{a\alpha,[cd]} & U^{a\alpha,}{}_{b\beta} \end{pmatrix} \;=\; \begin{pmatrix} U_{[a}{}^{c}\, U_{b]}{}^{d} & 0 \\ 0 & \delta^{\alpha}{}_{\beta}\,(U^{-1})_{b}{}^{a} \end{pmatrix} . \tag{3.36}$$

With this embedding, and recalling the convention (3.35), we can identify the Killing vector fields with components of the twist matrices as follows,

$$\mathcal{K}_{[ab]}{}^{m} \;\equiv\; \sqrt{2}\,(\widehat{U}^{-1})_{ab}{}^{m0} , \tag{3.37}$$

which yields

$$\mathcal{K}_{[0i]}{}^{m}(y) \;=\; -\frac{1}{2}\sqrt{2}\,(1-v)^{1/2}\,\delta_i^m , \qquad \mathcal{K}_{[ij]}{}^{m}(y) \;=\; \sqrt{2}\,\delta^m_{[i}\,\eta_{j]k}y^k . \tag{3.38}$$

It is straightforward to verify that these vectors satisfy the Lie bracket algebra (3.8). Specifically,

$$\big[\mathcal{K}_{ab},\mathcal{K}_{cd}\big]^m \;=\; -\sqrt{2}f_{ab\,,cd}{}^{ef}\mathcal{K}_{ef}{}^{m} , \qquad f_{ab\,,cd}{}^{ef} \;\equiv\; 2\,\delta_{[a}{}^{[e}\eta_{b][c}\delta_{d]}{}^{f]} , \tag{3.39}$$

with the $\mathrm{SO}(p,6-p)$ metric η_{ab}. The Killing tensors defined in (3.22) are then found to be

$$\begin{aligned} \mathcal{K}_{[0i]mnk} \;&=\; -\sqrt{2}\,\varepsilon_{mnkij}\,y^j , \\ \mathcal{K}_{[ij]mnk} \;&=\; -\sqrt{2}(1-v)^{-\frac{1}{2}}\,\varepsilon_{mnkpq}\big(\,\delta_i{}^p\,\delta_j{}^q - 2\,\delta_{[i}{}^p\eta_{j]l}\,y^q y^l\,\big) . \end{aligned} \tag{3.40}$$

We can now define the metric $\tilde{G}$ as in (3.9) w.r.t. which these vectors are Killing, using the Cartan-Killing form $\eta^{ab,cd} = \eta^{a[c}\eta^{d]b}$. This yields for the metric and its inverse

$$\begin{aligned} \tilde{G}_{mn} \;&=\; \eta_{mn} + (1-v)^{-1}\eta_{mp}\eta_{nq}y^p y^q , \\ \tilde{G}^{mn} \;&=\; \eta^{mn} - y^m y^n . \end{aligned} \tag{3.41}$$

One may verify that this metric describes the homogeneous space $\mathrm{SO}(p,q)/\mathrm{SO}(p-1,q)$ with

$$\tilde{\mathcal{R}}_{mn} \;=\; 4\,\tilde{G}_{mn} , \tag{3.42}$$

determining the constant above, $\lambda = 4$. The associated volume form is given by

$$\tilde{\omega}_{mnklp} \;=\; (1-v)^{-\frac{1}{2}}\,\varepsilon_{mnklp} . \tag{3.43}$$

Next we give the function defining $\mathcal{R}$ in (3.15) w.r.t. the above index split,

$$\mathcal{R}_{u\,m\alpha} \;=\; \mathcal{R}^{a\beta}{}_{m\alpha} \;=\; \partial_m \mathcal{Y}^{a\beta}{}_{\alpha} , \tag{3.44}$$

for which we read off from the twist matrix

$$\mathcal{Y}^{a\beta}{}_{\alpha} \;=\; \mathcal{Y}^a\delta^{\beta}_{\alpha} \qquad \text{with} \qquad \mathcal{Y}^a(y) \;\equiv\; \begin{cases} (1-v)^{1/2} & a=0 \\ y^i & a=i \end{cases} . \tag{3.45}$$

In agreement with (3.16) this transforms in the fundamental representation of the algebra of Killing vector fields (3.38). Specifically,

$$\mathcal{L}_{\mathcal{K}_{[ab]}}\mathcal{Y}^c \;=\; \mathcal{K}_{[ab]}{}^m\,\partial_m\mathcal{Y}^c \;=\; \sqrt{2}\,\delta^c{}_{[a}\,\mathcal{Y}_{b]}\;, \tag{3.46}$$

where $\mathcal{Y}_a$ is obtained from $\mathcal{Y}^a$ by means of η_{ab}. Let us also emphasize that the $\mathcal{Y}_a$ can be viewed as 'fundamental harmonics', satisfying

$$\Box\mathcal{Y}^a \;=\; -5\,\mathcal{Y}^a\;, \tag{3.47}$$

in that all higher harmonics can then be constructed from them. For instance, the Killing vectors themselves can be written as

$$\mathcal{K}_{[ab]m} \;=\; \sqrt{2}\big(\partial_m\mathcal{Y}_{[a}\big)\,\mathcal{Y}_{b]}\;. \tag{3.48}$$

Next we compute the four-form $\tilde{C}_{mnkl}$ by integrating (3.29). An explicit solution can be written in terms of the function K from (2.11) as

$$\tilde{C}_{mnkl} \;=\; \lambda\,(1-v)^{-1/2}\,\varepsilon_{mnklq}\,\big(K\delta^{qr}\eta_{rs}+\delta^q_s\big)y^s\;, \tag{3.49}$$

whose exterior derivative is indeed proportional to the volume form (3.43) for the metric $\tilde{G}_{mn}$. Together with the Killing vectors and tensors defined above, the $\mathcal{Z}$ tensor is now uniquely determined according to (3.21). Moreover, it is related to the twist matrix according to

$$\mathcal{Z}_{[ab]mnk} \;=\; \frac{1}{2}\,\varepsilon_{mnkpq}\,\big(\widehat{U}^{-1}\big)_{[ab]}{}^{[pq]} \;=\; \frac{1}{2}\,\varepsilon_{mnkpq}\,\rho^{-1}\,(U^{-1})_{[a}{}^p\,(U^{-1})_{b]}{}^q\;, \tag{3.50}$$

which agrees with (3.21) for $\lambda = 4$.

Finally, let us turn to the tensor $\mathcal{S}_u$ whose general form is given in (3.32). Under the above index split it is convenient to write this tensor as

$$\mathcal{S}_{u\,n_1\ldots n_5\,\beta} \;\equiv\; \mathcal{S}^{a\alpha}{}_{n_1\ldots n_5\,\beta} \;\equiv\; \mathcal{S}^a\,\varepsilon_{n_1\ldots n_5}\,\delta^\alpha{}_\beta\;, \tag{3.51}$$

which is read off from the twist matrix as

$$\mathcal{S}^{a\alpha}{}_{n_1\ldots n_5\,\beta} \;=\; \varepsilon_{n_1\ldots n_5}(\widehat{U}^{-1})^{a\alpha}{}_{0\beta} \;=\; \varepsilon_{n_1\ldots n_5}\,\rho^{-1}\,\delta^\alpha{}_\beta\,U_0{}^a\;, \tag{3.52}$$

leading with (2.8) to

$$\mathcal{S}^a \;=\; \begin{cases} (1-v)^{-1}\,(1+uK) & a=0 \\ -\eta_{ij}y^j\,(1-v)^{-1/2}\,K & a=i \end{cases}\;. \tag{3.53}$$

One may verify that this agrees with (3.32) for

$$a \;=\; 1\;, \qquad \lambda \;=\; 4\;. \tag{3.54}$$

3.3 Useful identities

In this final paragraph we collect various identities satisfied by the above Killing-type tensors. These will be useful in the following sections when explicitly verifying the consistency of the Kaluza-Klein truncations. We find

$$\mathcal{K}^{[ab]}{}_{mn}\mathcal{K}_{[cd]}{}^{n} = -\sqrt{2}\, f_{cd,ef}{}^{ab}\mathcal{K}^{[ef]}{}_{m} + 2\,\partial_m\left(\delta_{[c}{}^{[a}\mathcal{Y}^{b]}\mathcal{Y}_{d]}\right) , \quad (3.55)$$

$$\mathcal{K}^{[ab]}{}_{n}\mathcal{K}_{[cd]}{}^{n} = 2\,\delta_{[c}{}^{[a}\,\mathcal{Y}^{b]}\mathcal{Y}_{d]} , \quad (3.56)$$

$$\mathcal{K}_{[ab]}{}^{k}\mathcal{Z}_{[cd]\,kmn} + \mathcal{K}_{[cd]}{}^{k}\mathcal{Z}_{[ab]\,kmn} = -\frac{1}{8}\,\varepsilon_{abcdef}\,\mathcal{K}^{[ef]}{}_{mn} , \quad (3.57)$$

$$\mathcal{K}^{[ab]}{}_{mn}\mathcal{K}_{[cd]}{}^{m}\mathcal{K}_{[ef]}{}^{n} = 4\sqrt{2}\,\delta_{[c}{}^{[a}\,\mathcal{Y}_{d]}\mathcal{Y}_{[e}\,\delta_{f]}{}^{b]} , \quad (3.58)$$

$$\mathcal{K}_{[cd]}{}^{m}\mathcal{K}_{[ab]}{}^{n}\mathcal{K}_{[ef]}{}^{l}\,\partial_l\mathcal{K}^{[ab]}{}_{mn} = -8\,\eta_{e[c}\,\mathcal{Y}_{d]}\mathcal{Y}_f + 8\,\eta_{f[c}\,\mathcal{Y}_{d]}\mathcal{Y}_e , \quad (3.59)$$

which can be verified using the explicit tensors determined above.

4 The $D = 5$ supergravity

The $D = 5$ gauged theory with gauge group $\mathrm{SO}(p, q)$ was originally constructed in [3–5]. For our purpose, the most convenient description is its covariant form found in the context of general gaugings [41] to which we refer for details.[1] In the covariant formulation, the $D = 5$ gauged theory features 27 propagating vector fields $A_\mu{}^M$ and up to 27 topological tensor fields $B_{\mu\nu\,M}$. The choice of gauge group and the precise number of tensor fields involved is specified by the choice of an embedding tensor $Z^{MN} = Z^{[MN]}$ in the **351** representation of $\mathrm{E}_{6(6)}$. E.g. the full non-abelian vector field strengths are given by

$$F_{\mu\nu}{}^M \;=\; 2\,\partial_{[\mu}A_{\nu]}{}^M + \sqrt{2}\,X_{KL}{}^M A_{[\mu}{}^K A_{\nu]}{}^L - 2\sqrt{2}\,Z^{MN}B_{\mu\nu\,N}\;, \tag{4.2}$$

with the tensor $X_{KL}{}^M$ carrying the gauge group structure constants and defined in terms of the embedding tensor Z^{MN} as

$$X_{MN}{}^P \;=\; d_{MNQ}Z^{PQ} + 10\,d_{MQS}d_{NRT}d^{PQR}Z^{ST}\;. \tag{4.3}$$

The $\mathrm{SO}(p, q)$ gaugings preserve the global SL(2) subgroup of the symmetry group $\mathrm{E}_{6(6)}$ of the ungauged theory, more specifically the centralizer of its subgroup SL(6) . Accordingly, the vector fields in the **27** of $\mathrm{E}_{6(6)}$ can be split as

$$A_\mu{}^M \;\longrightarrow\; \left\{A_\mu{}^{ab}, A_{\mu\,a\alpha}\right\}\;, \qquad a, b = 0, \ldots, 5\;, \quad \alpha = 1, 2\;, \tag{4.4}$$

into 15 SL(2) singlets and 6 SL(2) doublets, c.f. (3.33). The 27 two-forms $B_{\mu\nu\,M}$ split accordingly, with only the 6 SL(2) doublets $B_{\mu\nu}{}^{a\alpha}$ entering the supergravity Lagrangian. In the basis (4.4), the only non-vanishing components of the embedding tensor Z^{MN} are

$$Z_{a\alpha,b\beta} \;\equiv\; -\frac{1}{2}\sqrt{5}\,\varepsilon_{\alpha\beta}\eta_{ab}\;, \tag{4.5}$$

where the normalization has been chosen such as to match the later expressions. With (4.3), we thus obtain[2]

$$X_{MN}{}^K \;:\; \begin{cases} X_{ab,cd}{}^{ef} = f_{ab,cd}{}^{ef} \\ X_{ab}{}^{c\alpha}{}_{d\beta} = -\delta_{[a}{}^c\eta_{b]d}\,\delta^\alpha_\beta \end{cases}\;, \tag{4.7}$$

with the $\mathrm{SO}(p, 6-p)$ structure constants $f_{ab,cd}{}^{ef}$ from (3.39).

The form of the field strength (4.2) is the generic structure of a covariant field strength in gauged supergravity, with non-abelian Yang-Mills part and a Stückelberg type coupling to the

[1] To be precise, and to facilitate the embedding of this theory into EFT, we choose the normalization of [24] for vector and tensor fields which differs from [41] as

$$\mathcal{A}_\mu{}^M{}_{[1312.0614]} = \frac{1}{\sqrt{2}}A_\mu{}^M{}_{[\mathrm{hep-th}/0412173]}\;, \quad \mathcal{B}_{\mu\nu\,M[1312.0614]} = -\frac{1}{4}B_{\mu\nu\,M[\mathrm{hep-th}/0412173]}\;, \tag{4.1}$$

together with a rescaling of the associated symmetry parameters. Moreover, we have set the coupling constant of [41] to $g = 1$.

[2] The totally symmetric cubic d-symbol of $\mathrm{E}_{6(6)}$ in the $\mathrm{SL}(6) \times \mathrm{SL}(2)$ basis (4.4) is given by

$$d^{MNK} \;:\; d^{ab}{}_{c\alpha,d\beta} = \frac{1}{\sqrt{5}}\,\delta^{ab}_{cd}\,\varepsilon_{\alpha\beta}\;, \quad d^{ab,cd,ef} = \frac{1}{\sqrt{80}}\,\varepsilon^{abcdef}\;. \tag{4.6}$$

two-forms. In the present case, we can make use of the tensor gauge symmetry which acts by shift $\delta A_{\mu\, a\alpha} = \Xi_{\mu\, a\alpha}$ on the vector fields, to eliminate all components $A_{\mu\, a\alpha}$ from the Lagrangian and field equations. This is the gauge we are going to impose in the following, which brings the theory in the form of [5].[3] As a result, the covariant object (4.2) splits into components carrying the SO(p,q) Yang-Mills field strength, and the two-forms $B_{\mu\nu}{}^{a\alpha}$, respectively,

$$F_{\mu\nu}{}^{M} \;=\; \begin{cases} F_{\mu\nu}{}^{ab} \;\equiv\; 2\,\partial_{[\mu}A_{\nu]}{}^{ab} + \sqrt{2}\, f_{cd,ef}{}^{ab}\, A_{\mu}{}^{cd} A_{\nu}{}^{ef} \\ F_{\mu\nu\, a\alpha} \;\equiv\; \sqrt{10}\,\varepsilon_{\alpha\beta}\eta_{ab}\, B_{\mu\nu}{}^{b\beta} \end{cases} . \tag{4.9}$$

In particular, fixing of the tensor gauge symmetry implies that the two-forms $B_{\mu\nu}{}^{a\alpha}$ turn into topologically massive fields, preserving the correct counting of degrees of freedom, [42]. The Lagrangian and field equations are still conveniently expressed in terms of the combined object $F_{\mu\nu}{}^{M}$. E.g. the first order duality equation between vector and tensor fields is given by

$$3\, D_{[\mu}B_{\nu\rho]}{}^{a\alpha} \;=\; \frac{1}{2\sqrt{10}}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau} M^{a\alpha}{}_{N}\, F^{\sigma\tau\, N} , \tag{4.10}$$

which upon expanding around the scalar origin and with (4.9) yields the first order topologically massive field equation for the two-form tensors. The full bosonic Lagrangian reads

$$\begin{aligned} \mathcal{L} \;=\;& \sqrt{|\mathrm{g}|}\, R - \frac{1}{4}\sqrt{|\mathrm{g}|}\, M_{MN}\, F_{\mu\nu}{}^{M} F^{\mu\nu\, N} + \frac{1}{24}\sqrt{|\mathrm{g}|}\, D_{\mu}M_{MN} D^{\mu}M^{MN} \\ & + \varepsilon^{\mu\nu\rho\sigma\tau}\left(\frac{5}{4}\,\varepsilon_{\alpha\beta}\,\eta_{ab}\, B_{\mu\nu}{}^{a\alpha} D_{\rho}B_{\sigma\tau}{}^{b\beta} + \frac{1}{24}\sqrt{2}\,\varepsilon_{abcdef}\, A_{\mu}{}^{ab}\,\partial_{\nu}A_{\rho}{}^{cd}\,\partial_{\sigma}A_{\tau}{}^{ef}\right) \\ & + \frac{1}{16}\,\varepsilon^{\mu\nu\rho\sigma\tau}\varepsilon_{abcdef}\, f_{gh,ij}{}^{ab}\, A_{\mu}{}^{cd} A_{\nu}{}^{gh} A_{\rho}{}^{ij}\left(\partial_{\sigma}A_{\tau}{}^{ef} + \tfrac{1}{5}\sqrt{2}\, f_{kl,mn}{}^{ef} A_{\sigma}{}^{kl} A_{\tau}{}^{mn}\right) \\ & - \sqrt{|\mathrm{g}|}\, V(M_{MN}) . \end{aligned} \tag{4.11}$$

Here, the 42 scalar fields parameterize the coset space $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ via the symmetric $\mathrm{E}_{6(6)}$ matrix M_{MN} which can be decomposed in the basis (4.4) as

$$M_{MN} \;=\; \begin{pmatrix} M_{ab,cd} & M_{ab}{}^{c\gamma} \\ \mathcal{M}^{a\alpha}{}_{bc} & M^{a\alpha,c\gamma} \end{pmatrix} , \tag{4.12}$$

with the SO$(p,6-p)$ covariant derivatives defined according to

$$D_{\mu}X^{a} \;\equiv\; \partial_{\mu}X^{a} + \sqrt{2}\, A_{\mu}{}^{ab}\,\eta_{bd}\, X^{d} , \tag{4.13}$$

and similarly on the different blocks of (4.12). The scalar potential V in (4.11) is given by the following contraction of the generalized structure constants (4.7) with the scalar matrix (4.12)

$$V(M_{MN}) \;=\; \frac{1}{30}\, M^{MN} X_{MP}{}^{Q}\left(5\, X_{NQ}{}^{P} + X_{NR}{}^{S}\, M^{PR} M_{QS}\right) . \tag{4.14}$$

[3] To be precise: this holds with a rescaling of p-forms according to

$$\mathcal{A}_{\mu}{}^{ab}{}_{[1312.0614]} = -\sqrt{2}\, A_{\mu}{}^{ab}{}_{\mathrm{GRW}} , \qquad \sqrt{5}\,\mathcal{B}_{\mu\nu}{}^{a\alpha}{}_{[1312.0614]} = B_{\mu\nu}{}^{a\alpha}{}_{\mathrm{GRW}} , \tag{4.8}$$

and with their coupling constant set to $g_{\mathrm{GRW}} = 2$.

For later use, let us explicitly state the vector field equations obtained from (4.11) which take the form

$$\begin{aligned} 0 &= \varepsilon_{\mu\nu\rho\sigma\tau}\left(\eta_{c[a}\,D^{\tau}M_{b]d,N}M^{N,cd}+\sqrt{2}\,D_{\lambda}\left(F^{\tau\lambda\,N}M_{N,ab}\right)\right) \\ &\quad+\frac{3}{2}\,\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{cd}F_{\rho\sigma]}{}^{ef}+60\,\varepsilon_{\alpha\beta}\,\eta_{ac}\eta_{bd}\,B_{[\mu\nu}{}^{c\alpha}B_{\rho\sigma]}{}^{d\beta}\,. \end{aligned} \tag{4.15}$$

We will also need part of the scalar field equations that are obtained by varying in (4.11) the scalar matrix (4.12) with an SL(6) generator $X_a{}^b$

$$\begin{aligned} 0 &= \frac{1}{4}\,D^{\mu}(M^{ad\,K}D_{\mu}M_{K\,bd})-\frac{1}{2}\,M_{bc\,N}\,F_{\mu\nu}{}^{ac}F^{\mu\nu\,N}+\frac{1}{4}\,\sqrt{10}\,\eta_{bc}\,\varepsilon_{\alpha\beta}\,M^{a\alpha}{}_{N}\,B_{\mu\nu}{}^{c\beta}F^{\mu\nu\,N} \\ &\quad+\left(2\,M^{ae,fc}+\frac{4}{15}\,M^{de,h(a}M^{c)j,fg}M_{dg,hj}+\frac{1}{15}\,M^{de,h(a}M^{c)\beta,f\alpha}M_{d\alpha,h\beta}\right)\eta_{bc}\eta_{ef} \\ &\quad-\frac{2}{15}\left(M^{de,k(a}M^{c)\alpha}{}_{dg}M_{k\alpha}{}^{fg}+M^{de,h(a}M^{c)g}{}_{d\alpha}M^{f\alpha}{}_{hg}\right)\eta_{ef}\eta_{bc}\;-\;[\text{trace}]_b{}^a\,. \end{aligned} \tag{4.16}$$

5 The IIB reduction ansatz

In terms of the $\mathrm{E}_{6(6)}$ EFT fields, the reduction ansatz is given by the simple factorization (2.1) with the twist matrix U given by (2.13). In order to translate this into the original IIB theory, we may first decompose the EFT fields under (3.1), according to the IIB solution of the section constraint, and collect the expressions for the various components. We do this separately for EFT vectors, two-forms, metric, and scalars, and subsequently derive the expressions for three- and four-forms from the IIB self-duality equations, as outlined in the general case in [38]. In a second step, we can then recombine the various EFT components into the original IIB fields, upon applying the explicit dictionary [24, 38] from IIB into EFT.

In particular, the explicit expression for the full IIB metric allows one to determine the background metric, i.e. the IIB metric at the point where all $D = 5$ scalar fields are set to zero. This metric may or may not extend to a solution of the IIB field equations, depending on whether the scalar potential of the $D = 5$ theory has a stationary point at its origin. It is known [5] that this is the case for the $D = 5$ theories with gauge group SO(6) and SO(3, 3), with AdS and dS vacuum, respectively. Accordingly, the internal manifolds S^5 and $H^{3,3}$ extend to solutions of the full IIB field equations, with the external geometry given by AdS_5 or dS_5, respectively.

5.1 IIB supergravity

Let us briefly review our conventions for the $D = 10$ IIB supergravity [43–45]. The IIB field equations can be most compactly obtained from the pseudo-action

$$\begin{aligned} S &= \int d^{10}\hat{x}\,\sqrt{|G|}\,\Big(\hat{R}+\frac{1}{4}\partial_{\hat{\mu}}m_{\alpha\beta}\partial^{\hat{\mu}}m^{\alpha\beta}-\frac{1}{12}\hat{F}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3}{}^{\alpha}\hat{F}^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\,\beta}m_{\alpha\beta} \\ &\qquad\qquad-\frac{1}{30}\hat{F}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5}\hat{F}^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5}\Big) \\ &\quad-\frac{1}{864}\int d^{10}\hat{x}\,\varepsilon_{\alpha\beta}\,\varepsilon^{\hat{\mu}_1\ldots\hat{\mu}_{10}}C_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4}\hat{F}_{\hat{\mu}_6\hat{\mu}_7\hat{\mu}_8}{}^{\alpha}\hat{F}_{\hat{\mu}_8\hat{\mu}_9\hat{\mu}_{10}}{}^{\beta}\,. \end{aligned} \tag{5.1}$$

Here, $D = 10$ coordinates are denoted by $x^{\hat{\mu}}$, and the action carries the field strengths

$$\begin{aligned}
\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}}{}^{\alpha} &\equiv 3\,\partial_{[\hat{\mu}}\hat{C}_{\hat{\nu}\hat{\rho}]}{}^{\alpha}\,, \\
\hat{F}_{\hat{\mu}_1\ldots\hat{\mu}_5} &\equiv 5\,\partial_{[\hat{\mu}_1}\hat{C}_{\hat{\mu}_2\ldots\hat{\mu}_5]} - \frac{5}{4}\,\varepsilon_{\alpha\beta}\,\hat{C}_{[\hat{\mu}_1\hat{\mu}_2}{}^{\alpha}\hat{F}_{\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5]}{}^{\beta}\,,
\end{aligned} \tag{5.2}$$

of two- and four-form gauge potential. After variation, the field equations derived from (5.1) have to be supplemented with the standard self-duality equations for the 5-form field strength

$$\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}} = \frac{1}{5!}\sqrt{|G|}\,\varepsilon_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5}\,\hat{F}^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5}\,. \tag{5.3}$$

Finally, the symmetric SL(2) matrix $m_{\alpha\beta}$ parametrizes the coset space SL(2)/SO(2) and carries dilaton and axion. In the notation of [44] it is parametrized by a complex scalar B as

$$m_{\alpha\beta} \equiv (1-BB^*)^{-1}\begin{pmatrix} (1-B)(1-B^*) & i(B-B^*) \\ i(B-B^*) & (1+B)(1+B^*) \end{pmatrix}. \tag{5.4}$$

As a first step for the reduction ansatz, we perform the $5+5$ Kaluza-Klein decomposition of coordinates $\{x^{\hat{\mu}}\} = \{x^{\mu}, y^m\}$ and fields, starting from the ten-dimensional vielbein

$$E_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} (\det\phi)^{-1/3}\,e_{\mu}{}^{\underline{a}} & A_{\mu}{}^{m}\phi_m{}^{\underline{\alpha}} \\ 0 & \phi_m{}^{\underline{\alpha}} \end{pmatrix}, \tag{5.5}$$

but keeping the dependence on all 10 coordinates. Decomposition of the p-forms in standard Kaluza-Klein manner then involves the projector $P_{\mu}{}^{\hat{\nu}} = E_{\mu}{}^{\underline{a}}E_{\underline{a}}{}^{\hat{\nu}}$ together with a further redefinition of fields due to the Chern-Simons contribution in (5.2), see [38] for details. This leads to the components

$$\begin{aligned}
C_{mn}{}^{\alpha} &\equiv \hat{C}_{mn}{}^{\alpha}\,, \\
C_{\mu\,m}{}^{\alpha} &\equiv \hat{C}_{\mu m}{}^{\alpha} - A_{\mu}{}^{p}\hat{C}_{pm}{}^{\alpha}\,, \\
C_{\mu\nu}{}^{\alpha} &\equiv \hat{C}_{\mu\nu}{}^{\alpha} - 2A_{[\mu}{}^{p}\hat{C}_{|p|\nu]}{}^{\alpha} + A_{\mu}{}^{p}A_{\nu}{}^{q}\hat{C}_{pq}{}^{\alpha}\,, \\
C_{mnkl} &\equiv \hat{C}_{mnkl}\,, \\
C_{\mu\,nkl} &\equiv \hat{C}_{\mu nkl} - A_{\mu}{}^{p}\hat{C}_{pnkl} - \tfrac{3}{8}\varepsilon_{\alpha\beta}\,C_{\mu\,[n}{}^{\alpha}C_{kl]}{}^{\beta}\,, \\
C_{\mu\nu\,kl} &\equiv \hat{C}_{\mu\nu kl} - 2A_{[\mu}{}^{p}\hat{C}_{|p|\nu]kl} + A_{\mu}{}^{p}A_{\nu}{}^{q}\hat{C}_{pqkl} - \tfrac{1}{8}\,\varepsilon_{\alpha\beta}\,C_{\mu\nu}{}^{\alpha}C_{kl}{}^{\beta}\,, \\
C_{\mu\nu\rho\,m} &\equiv \hat{C}_{\mu\nu\rho\,m} - 3A_{[\mu}{}^{p}\hat{C}_{|p|\nu\rho]\,m} + 3A_{[\mu}{}^{p}A_{\nu}{}^{q}\hat{C}_{|pq|\rho]\,m} - A_{\mu}{}^{p}A_{\nu}{}^{q}A_{\rho}{}^{r}\hat{C}_{pqrm} \\
&\quad - \tfrac{3}{8}\varepsilon_{\alpha\beta}\,C_{[\mu\nu}{}^{\alpha}C_{\rho]\,m}{}^{\beta}\,, \\
C_{\mu\nu\rho\sigma} &\equiv \hat{C}_{\mu\nu\rho\sigma} - 4A_{[\mu}{}^{p}\hat{C}_{|p|\nu\rho\sigma]} + 6A_{[\mu}{}^{p}A_{\nu}{}^{q}\hat{C}_{|pq|\rho\sigma]} - 4A_{[\mu}{}^{p}A_{\nu}{}^{q}A_{\rho}{}^{r}\hat{C}_{|pqr|\sigma]} \\
&\quad + A_{\mu}{}^{p}A_{\nu}{}^{q}A_{\rho}{}^{r}A_{\sigma}{}^{s}\hat{C}_{pqrs}\,.
\end{aligned} \tag{5.6}$$

in terms of which the reduction ansatz is most naturally given in the following.

5.2 Vector and two-form fields

Breaking the 27 EFT vector fields according to (3.1) into

$$\{\mathcal{A}_{\mu}{}^{m}, \mathcal{A}_{\mu\,m\alpha}, \mathcal{A}_{\mu\,kmn}, \mathcal{A}_{\mu\,\alpha}\}\,, \tag{5.7}$$

we read off the reduction ansatz from (2.1), (3.2), which in particular gives rise to

$$\begin{aligned} \mathcal{A}_\mu{}^m(x,y) &= \mathcal{K}_{[ab]}{}^m(y)\, A_\mu^{ab}(x)\ , \\ \mathcal{A}_{\mu\, kmn}(x,y) &= \mathcal{Z}_{[ab]\, kmn}(y)\, A_\mu^{ab}(x)\ . \end{aligned} \tag{5.8}$$

The Kaluza-Klein vector field $\mathcal{A}_\mu{}^m = A_\mu{}^m$ thus reduces in the standard way with the 15 Killing vectors $\mathcal{K}_{[ab]}{}^m(y)$ whose algebra defines the gauge group of the $D = 5$ theory. Note, however, that these extend to Killing vectors of the internal space-time metric only in case of the compact gauge group SO(6). In the general case, as discussed above, the $\mathcal{K}_{[ab]}{}^m(y)$ are the Killing vector fields of an auxiliary homogeneous Lorentzian metric (3.9), compare also [20–22]. The vector field components $\mathcal{A}_{\mu\, kmn}$ are expressed in terms of the same 15 $D = 5$ vector fields. Their internal coordinate dependence is not exclusively carried by Killing vectors and tensors, but exhibits via the tensor $\mathcal{Z}_{[ab]\, kmn}(y)$ an inhomogeneous term carrying the four-form $\tilde{C}_{mnkl}$ according to (3.21).[4] This is similar to reduction formulas for the dual vector fields in the S^7 reduction of $D = 11$ supergravity [46], which, however, in the present case already show up among the fundamental vectors.

For the remaining vector field components, the ansatz (2.1), (3.2), at first yields the reduction formulas

$$\begin{aligned} \mathcal{A}_{\mu\, m\alpha}(x,y) &= \mathcal{R}^{a\beta}{}_{m\alpha}(y)\, A_{\mu a\beta}(x) \;=\; \partial_m \mathcal{Y}^a(y)\, A_{\mu\, a\alpha}(x)\ , \\ \mathcal{A}_{\mu\, \alpha}(x,y) &= \mathcal{S}^a(y)\, A_{\mu\, a\alpha}(x) \\ &= \tfrac{1}{5!}|\tilde{G}|^{1/2} \left(5!\, \mathcal{Y}^a(y) - \frac{5}{4}\, \tilde{\omega}^{klmnp} \tilde{C}_{klmn}\, \partial_p \mathcal{Y}^a(y) \right) A_{\mu\, a\alpha}(x)\ , \end{aligned} \tag{5.9}$$

in terms of the 12 vector fields $A_{\mu\, a\alpha}$ in $D = 5$ and the tensors defined in (3.32) and (3.44). However, as discussed in the previous section, for the SO(p,q) gauged theories, a natural gauge fixing of the two-form tensor gauge transformations allows to eliminate these vector fields in exchange for giving topological mass to the two-forms. As a result, the final reduction ansatz reduces to

$$\mathcal{A}_{\mu\, m\alpha} \;=\; 0 \;=\; \mathcal{A}_{\mu\, \alpha}\ . \tag{5.10}$$

For the two-forms, upon breaking them into GL(5) components

$$\{ \mathcal{B}_{\mu\nu}{}^\alpha, \mathcal{B}_{\mu\nu\, mn}, \mathcal{B}_{\mu\nu}{}^{m\alpha}, \mathcal{B}_{\mu\nu\, m} \}\ , \tag{5.11}$$

similar reasoning via (2.1) and evaluation of the twist matrix $\rho^{-2}\, U_M{}^{\underline{N}}$ gives the following ansatz for the SL(2) doublets

$$\begin{aligned} \mathcal{B}_{\mu\nu}{}^\alpha(x,y) &= \mathcal{Y}_a(y)\, B_{\mu\nu}{}^{a\alpha}(x)\ , \\ \mathcal{B}_{\mu\nu}{}^{m\alpha}(x,y) &= \mathcal{Z}_a{}^m(y)\, B_{\mu\nu}{}^{a\alpha}(x)\ , \end{aligned} \tag{5.12}$$

[4] This seems to differ from the ansatz derived in [40]. The precise comparison should take into account that the A_μ, $B_{\mu\nu}$ are non-gauge-invariant vector potentials. In the present discussion, the inhomogeneous term in $\mathcal{Z}_{[ab]\, kmn}(y)$ played a crucial role in the verification of the proper algebraic relations.

in terms of the 12 topologically massive two-form fields of the $D = 5$ theory. Here, $\mathcal{Z}_a{}^m(y)$ is the vector density, given by[4]

$$\mathcal{Z}_a{}^m \;=\; |\tilde{G}|^{1/2}\left(\tilde{G}^{mn}\partial_n\mathcal{Y}_a + \frac{1}{96}\,\tilde{\omega}^{mklpq}\tilde{C}_{klpq}\mathcal{Y}_a\right)\,, \tag{5.13}$$

in terms of the Lorentzian metric $\tilde{G}_{mn}$, vector field $\mathcal{Y}_a$, and four-form $\tilde{C}_{klmn}$. As is obvious from their index structure, the fields $\mathcal{B}_{\mu\nu}{}^{m\alpha}$ contribute to the dual six-form doublet of the IIB theory, but not to the original IIB fields. Accordingly, for matching the EFT Lagrangian to the IIB dynamics, these fields are integrated out from the theory [24, 38]. For the IIB embedding of $D = 5$ supergravity, we will thus only need the first line of (5.12).

For the remaining two-form fields, the reduction ansatz (2.1) yields the explicit expressions

$$\begin{aligned}\mathcal{B}_{\mu\nu\,m}(x,y) &= \mathcal{Z}^{[ab]}{}_m(y)\,B_{\mu\nu\,ab}(x)\,,\\ \mathcal{B}_{\mu\nu\,mn}(x,y) &= -\frac{1}{4}\sqrt{2}\,\mathcal{K}^{[ab]}{}_{mn}(y)\,B_{\mu\nu\,ab}(x)\,,\end{aligned} \tag{5.14}$$

with the Killing tensor $\mathcal{K}^{[ab]}{}_{mn} = 2\,\partial_{[m}\mathcal{K}^{[ab]}{}_{n]}$, and the tensor density $\mathcal{Z}^{[ab]}{}_m$ given by

$$\mathcal{Z}^{[ab]}{}_m \;=\; |\tilde{G}|^{1/2}\left(\mathcal{K}^{[ab]}{}_m + \frac{1}{192}\,\tilde{\omega}^{klnpq}\mathcal{K}^{[ab]}{}_{mk}\,\tilde{C}_{lnpq}\right)\,. \tag{5.15}$$

Here, the 15 $D = 5$ two-forms $B_{\mu\nu\,ab}$ are in fact absent in the $\mathrm{SO}(p,q)$ supergravities, described in the previous section. In principle, they may be introduced on-shell, employing the formulation of these theories given in [41, 47], however, subject to an additional (three-form tensor) gauge freedom, which subsequently allows one to set them to zero. Hence, in the following we adopt $B_{\mu\nu\,ab}(x) = 0$, such that (5.14) reduces to

$$\mathcal{B}_{\mu\nu\,m} \;=\; 0 \;=\; \mathcal{B}_{\mu\nu\,mn}\,. \tag{5.16}$$

Within EFT, consistency of this choice with the reduction ansatz (5.14) can be understood by the fact that the fields $\mathcal{B}_{\mu\nu\,m}$ (related to the IIB dual graviton) do not even enter the EFT Lagrangian, while the fields $\mathcal{B}_{\mu\nu\,mn}$ enter subject to gauge freedom

$$\delta\mathcal{B}_{\mu\nu\,mn} \;=\; 2\,\partial_{[m}\Lambda_{n]\,\mu\nu}\,, \tag{5.17}$$

(descending from tensor gauge transformations of the IIB four-form potential), which allows us to explicitly gauge the reduction ansatz (5.14) to zero.

Combining the reduction formulas for the EFT fields with the explicit dictionary given in sec. 5.2 of [38], we can use the results of this section to give the explicit expressions for the different components (5.6) of the type IIB form fields. This gives the following reduction formulae

$$C_{\mu\nu}{}^{\alpha}(x,y) \;=\; \sqrt{10}\,\mathcal{Y}_a(y)\,B_{\mu\nu}{}^{a\alpha}(x)\,, \tag{5.18}$$

$$\begin{aligned}C_{\mu m}{}^{\alpha}(x,y) &= 0\,,\\ C_{\mu\nu\,mn}(x,y) &= \frac{\sqrt{2}}{4}\,\mathcal{K}_{[ab]}{}^{k}(y)\mathcal{Z}_{[cd]\,kmn}(y)\,A_{[\mu}{}^{ab}(x)A_{\nu]}{}^{cd}(x)\,,\\ C_{\mu kmn}(x,y) &= \frac{\sqrt{2}}{4}\,\mathcal{Z}_{[ab]\,kmn}(y)\,A_{\mu}{}^{ab}(x)\,,\end{aligned} \tag{5.19}$$

for two- and four form gauge potential in the basis after standard Kaluza-Klein decomposition. In the next subsection, we collect the expressions for the scalar components $C_{mn}{}^{\alpha}$ and C_{klmn}, and in subsection 5.5 we derive the reduction formulas for the last missing components $C_{\mu\nu\rho\,m}$, and $C_{\mu\nu\rho\sigma}$ of the four-form.

Let us finally note that with the reduction formulas given in this section, also the non-abelian EFT field strengths of the vector fields factorize canonically, as can be explicitly verified with the identities given in (3.8), (3.19). Explicitly, we find

$$\begin{aligned}
\mathcal{F}_{\mu\nu}{}^{m} &\equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]}{}^{m} - \mathcal{A}_{\mu}{}^{n}\partial_{n}\mathcal{A}_{\nu}{}^{m} + \mathcal{A}_{\nu}{}^{n}\partial_{n}\mathcal{A}_{\mu}{}^{m} \\
&= \mathcal{K}_{[ab]}{}^{m}(y)\left(2\,\partial_{[\mu}A_{\nu]}{}^{ab}(x) + \sqrt{2}\,f_{cd,ef}{}^{ab}A_{\mu}{}^{cd}A_{\nu}{}^{ef}(x)\right) \\
&= \mathcal{K}_{[ab]}{}^{m}(y)\,F_{\mu\nu}{}^{ab}(x)\;, \\
\mathcal{F}_{\mu\nu\,kmn} &\equiv 2\,\partial_{[\mu}\mathcal{A}_{\nu]\,kmn} - 2\,\mathcal{A}_{[\mu}{}^{l}\partial_{l}\mathcal{A}_{\nu]\,kmn} - 3\,\partial_{[k}\mathcal{A}_{[\mu}{}^{l}\mathcal{A}_{\nu]\,mn]l} + 3\,\mathcal{A}_{[\mu}{}^{l}\partial_{[k}\mathcal{A}_{\nu]\,mn]l} \\
&= \mathcal{Z}_{[ab]\,kmn}(y)\,F_{\mu\nu}{}^{ab}(x)\;,
\end{aligned} \tag{5.20}$$

in terms of the non-abelian $\mathrm{SO}(p,q)$ field strength $F_{\mu\nu}{}^{ab}(x)$ from (4.9).

5.3 EFT scalar fields and metric

Similar to the discussion of the form fields, the reduction of the EFT scalars can be read off from (2.1) upon proper parametrization of the matrix $\mathcal{M}_{MN}$. We recall from [24,38] that $\mathcal{M}_{MN}$ is a real symmetric $\mathrm{E}_{6(6)}$ matrix parametrized by the 42 scalar fields

$$\{G_{mn}, C_{mn}{}^{\alpha}, C_{klmn}, m_{\alpha\beta}\}\;, \tag{5.21}$$

where $C_{mn}{}^{\alpha} = C_{[mn]}{}^{\alpha}$, and $C_{klmn} = C_{[klmn]}$ are fully antisymmetric in their internal indices, $G_{mn} = G_{(mn)}$ is the symmetric 5×5 matrix, representing the internal part of the IIB metric, and $m_{\alpha\beta} = m_{(\alpha\beta)}$ is the unimodular symmetric 2×2 matrix parametrizing the coset space $\mathrm{SL}(2)/\mathrm{SO}(2)$ carrying the IIB dilaton and axion. Decomposing the matrix $\mathcal{M}_{MN}$ into blocks according to the basis (5.7)

$$\mathcal{M}_{KM} = \begin{pmatrix} \mathcal{M}_{k,m} & \mathcal{M}_{k}{}^{m\beta} & \mathcal{M}_{k,mn} & \mathcal{M}_{k}{}^{\beta} \\ \mathcal{M}^{k\alpha}{}_{m} & \mathcal{M}^{k\alpha,m\beta} & \mathcal{M}^{k\alpha}{}_{mn} & \mathcal{M}^{k\alpha,\beta} \\ \mathcal{M}_{kl,m} & \mathcal{M}_{kl}{}^{m\beta} & \mathcal{M}_{kl,mn} & \mathcal{M}_{kl}{}^{\beta} \\ \mathcal{M}^{\alpha}{}_{m} & \mathcal{M}^{\alpha,m\beta} & \mathcal{M}^{\alpha}{}_{mn} & \mathcal{M}^{\alpha,\beta} \end{pmatrix}, \tag{5.22}$$

the scalar fields (5.21) can be read off from the various components of $\mathcal{M}_{MN}$ and its inverse $\mathcal{M}^{MN}$. We refer to [38] for the explicit formulas and collect the final result

$$\begin{aligned}
G^{mn} &= (\det G)^{1/3}\,\mathcal{M}^{m,n}\;, \\
m^{\alpha\beta} &= (\det G)^{2/3}\mathcal{M}^{\alpha,\beta}\;, \\
C_{mn}{}^{\alpha} &= \sqrt{2}\,\varepsilon^{\alpha\beta}(\det G)^{2/3}\,m_{\beta\gamma}\,\mathcal{M}^{\gamma}{}_{mn} \;=\; -\,\varepsilon^{\alpha\beta}(\det G)^{1/3}\,G_{nk}\,\mathcal{M}_{m\beta}{}^{k}\;, \\
C_{klmn} &= 2\,(\det G)^{2/3}\,\varepsilon_{klmnp}\,m_{\alpha\beta}\,\mathcal{M}^{\alpha,p\beta} \;=\; -\,\sqrt{2}\,(\det G)^{1/3}\varepsilon_{klmnp}G_{qr}\,\mathcal{M}^{pq,r}\;,
\end{aligned} \tag{5.23}$$

where G^{mn} and $m^{\alpha\beta}$ denote the inverse matrices of G_{mn} and $m_{\alpha\beta}$ from (5.21). The last two lines represent examples how the $C_{mn}{}^{\alpha}$ and C_{klmn} can be obtained in different but equivalent ways either from components of $\mathcal{M}_{MN}$ or $\mathcal{M}^{MN}$. This of course does not come as a surprise but is a simple consequence of the fact that the 27×27 matrix $\mathcal{M}_{MN}$ representing the 42-dimensional coset space $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ is subject to a large number of non-linear identities.

With (5.23), the reduction formulas for the EFT scalars are immediately derived from (2.1). For the IIB metric and dilaton/axion, this gives rise to the expressions

$$\begin{aligned} G^{mn}(x,y) &= \Delta^{2/3}(x,y)\,\mathcal{K}_{[ab]}{}^{m}(y)\mathcal{K}_{[cd]}{}^{n}(y)\,M^{ab,cd}(x)\,, \\ m^{\alpha\beta}(x,y) &= \Delta^{4/3}(x,y)\,\mathcal{Y}_a(y)\mathcal{Y}_b(y)\,M^{a\alpha,b\beta}(x)\,, \end{aligned} \tag{5.24}$$

with the function $\Delta(x,y)$ defined by

$$\Delta(x,y) \;\equiv\; \rho^3(y)\,(\det G)^{1/2} \;=\; (1-v)^{1/2}\,(\det G)^{1/2}\,, \tag{5.25}$$

and the 42 five-dimensional scalar fields parameterizing the symmetric $\mathrm{E}_{6(6)}$ matrix M_{MN} decomposed into an $\mathrm{SL}(6) \times \mathrm{SL}(2)$ basis as (4.12).

Similarly, the reduction formula for the internal components of the two-form $C_{mn}{}^{\alpha}$ is read off as

$$\begin{aligned} C_{mn}{}^{\alpha}(x,y) &= -\varepsilon^{\alpha\beta}\Delta^{2/3}(x,y)\,G_{nk}(x,y)\,\partial_m\mathcal{Y}^c(y)\,\mathcal{K}_{[ab]}{}^{k}(y)\,M^{ab}{}_{c\beta}(x) \\ &= -\frac{1}{2}\,\varepsilon^{\alpha\beta}\Delta^{4/3}(x,y)\,m_{\beta\gamma}(x,y)\,\mathcal{Y}_c(y)\,\mathcal{K}^{[ab]}{}_{mn}(y)\,M_{ab}{}^{c\gamma}(x)\,, \end{aligned} \tag{5.26}$$

featuring the inverse matrices of (5.24), with the two alternative expressions corresponding to using the different equivalent expressions in (5.23). To explicitly show the second equality in (5.26) requires rather non-trivial quadratic identities among the components (4.12) of an $\mathrm{E}_{6(6)}$ matrix, together with non-trivial identities among the Killing vectors and tensors. In contrast, this identity simply follows on general grounds from the equivalence of the two expressions in (5.23), i.e., it follows from the group property of $\mathcal{M}_{MN}$ and the twist matrix $U_M{}^{\underline{N}}$. Let us also stress, that throughout all indices on the Killing vectors $\mathcal{K}_{[ab]}{}^{m}$ and tensors are raised and lowered with the Lorentzian x-independent metric $\tilde{G}_{mn}(y)$ from (3.9), not with the space-time metric $G_{mn}(x,y)$.

Eventually, the same reasoning gives the reduction formula for C_{mnkl}

$$C_{klmn}(x,y) \;=\; 2\,\varepsilon_{klmnp}\,\Delta^{4/3}(x,y)\,m_{\alpha\beta}(x,y)\,\mathcal{Y}_a(y)\,\mathcal{Z}_b{}^{p}(y)\,M^{a\alpha,b\beta}(x)\,, \tag{5.27}$$

with $\mathcal{Z}_b{}^{p}(y)$ from (5.13). Explicitly, this takes the form

$$\begin{aligned} C_{klmn}(x,y) &= \tilde{\omega}_{klmnp}\,\Delta^{4/3}(x,y)\,m_{\alpha\beta}(x,y)\,\tilde{G}^{pq}(y)\,\partial_q\left(\Delta^{-4/3}(x,y)\,m^{\alpha\beta}(x,y)\right) \\ &\quad + \tilde{C}_{klmn}(y)\,. \end{aligned} \tag{5.28}$$

On the other hand, using the last identity in (5.23) to express C_{klmn}, the reduction formula is read off as

$$\begin{aligned} C_{klmn}(x,y) &= 4\sqrt{2}\,\Delta^{2/3}(x,y)\,\mathcal{Z}_{[ab]\,[klm}(y)G_{n]r}(x,y)\,\mathcal{K}_{[cd]}{}^{r}(y)\,M^{ab,cd}(x) \\ &= \tilde{C}_{mnkl}(y) \;-2\,\Delta^{2/3}(x,y)\,\mathcal{K}_{[ab]}{}^{p}(y)\,\mathcal{K}_{[cd]\,[klm}(y)\,G_{n]p}(x,y)\,M^{ab,cd}(x)\,, \end{aligned} \tag{5.29}$$

where we have used the explicit expression (3.21) for $\mathcal{Z}_{[ab]\,klm}$. Again, the equivalence between (5.28) and (5.29) is far from obvious, but a consequence of the group property of $\mathcal{M}_{MN}$ and the twist matrix $U_M{}^{\underline{N}}$. For the case of the sphere S^5, several of these reduction formulas have appeared in the literature [11–13, 39, 40]. Here we find that they naturally generalize to the case of hyperboloids, inducing the $D = 5$ non-compact $\mathrm{SO}(p, q)$ gaugings.

Let us finally spell out the reduction ansatz for the five-dimensional metric which follows directly from (2.1) as

$$g_{\mu\nu}(x, Y) \;=\; \rho^{-2}(y)\, \mathrm{g}_{\mu\nu}(x) \;. \tag{5.30}$$

Putting this together with the parametrization of the IIB metric in terms of the EFT fields, and the reduction (5.8) of the Kaluza-Klein vector field, we arrive at the full expression for the IIB metric

$$\begin{aligned} ds^2 \;&=\; \Delta^{-2/3}(x, y)\, \mathrm{g}_{\mu\nu}(x)\, dx^\mu dx^\nu \\ &\quad + G_{mn}(x, y) \left(dy^m + \mathcal{K}_{[ab]}{}^m(y) A^{ab}_\mu(x) dx^\mu\right) \left(dy^n + \mathcal{K}_{[cd]}{}^n(y) A^{cd}_\nu(x) dx^\nu\right) \;, \end{aligned} \tag{5.31}$$

in standard Kaluza-Klein form [48], with G_{mn} given by the inverse of (5.24).

5.4 Background geometry

It is instructive to evaluate the above formulas at the particular point where all $D = 5$ fields vanish, i.e. in particular the scalar matrix M_{MN} reduces to the identity matrix

$$M_{MN}(x) \;=\; \delta_{MN} \;. \tag{5.32}$$

This determines the background geometry around which the generalized Scherk-Schwarz reduction ansatz captures the fluctuations. Depending on whether or not the scalar potential of $D = 5$ gauged supergravity has a stationary point at the origin — which is the case for the SO(6) and SO(3, 3) gaugings [5] — this background geometry will correspond to a solution of the IIB field equations.

With (5.32) and the vanishing of the Kaluza-Klein vector fields, the IIB metric (5.31) reduces to

$$\begin{aligned} ds^2 \;&=\; \overset{\circ}{G}_{\hat\mu\hat\nu}\, dX^{\hat\mu} dX^{\hat\nu} \\ &\equiv\; (1 + u - v)^{1/2}\, \overset{\circ}{g}_{\mu\nu}(x)\, dx^\mu dx^\nu + (1 + u - v)^{-1/2} \left(\delta_{mn} + \frac{\eta_{mi}\eta_{nj} y^i y^j}{1 - v}\right) dy^m dy^n \;, \end{aligned} \tag{5.33}$$

where we have used the relations

$$\begin{aligned} \delta^{ac}\delta^{bd}\, \mathcal{K}_{[ab]}{}^m(y) \mathcal{K}_{[cd]}{}^n(y) \;&=\; (1 + u - v)\, \delta^{mn} - \eta_{mi}\eta_{nj} y^i y^j \;, \\ \overset{\circ}{\Delta} \;&=\; (1 + u - v)^{-3/4} \;. \end{aligned} \tag{5.34}$$

The internal metric of (5.33) is conformally equivalent to the hyperboloid $H^{p,6-p}$ defined by the embedding of the surface

$$z_1^2 + \cdots + z_p^2 - z_{p+1}^2 - \cdots - z_6^2 \;\equiv\; 1 \;, \tag{5.35}$$

in $\mathbb{R}^6$. This is a Euclidean five-dimensional space with isometry group $\mathrm{SO}(p)\times\mathrm{SO}(6-p)$, inhomogeneous for $p=2,3,4$. Except for $p=6$, this metric differs from the homogeneous Lorentzian metric defined in (3.9) with respect to which the Killing vectors and tensors parametrizing the reduction ansatz are defined.

Using that $\mathcal{Y}_a\mathcal{Y}_b\,\delta^{ab} = 1+u-v$, it follows from (5.24) that the IIB dilaton and axion are constant

$$\overset{\circ}{m}{}^{\alpha\beta} \;=\; \delta^{\alpha\beta}\,, \tag{5.36}$$

while the internal two-form (5.26) vanishes due to the fact that (5.32) does not break the $\mathrm{SL}(2)$. Eventually, the four-form C_{klmn} is most conveniently evaluated from (5.28) as

$$\begin{aligned}\overset{\circ}{C}_{klmn} &= \tilde{C}_{klmn} - \frac{8}{3}\,\tilde{\omega}_{klmnp}\,\tilde{G}^{pq}\,\overset{\circ}{\Delta}{}^{-1}\partial_q\overset{\circ}{\Delta}\\ &= 4\,\varepsilon_{klmnp}\,\eta^{pq}y^q\,(1-v)^{-1/2}\,\big(K(u,v)+(1+u-v)^{-1}\big)\,,\end{aligned} \tag{5.37}$$

which can also be confirmed from (5.29). In particular, its field strength is given by

$$5\,\partial_{[k}\overset{\circ}{C}_{lmnp]} \;=\; 8\,\varepsilon_{klmnp}\,\frac{p-4+(p-3)(u-v)}{(1-v)^{1/2}\,(1+u-v)^2}\,, \tag{5.38}$$

where we have used the differential equation (2.11) for the function $K(u,v)$. Again, it is only for $p=6$, that the background four-form potential $\overset{\circ}{C}_{klmn}$ coincides with the four-form $\tilde{C}_{klmn}$ that parametrizes the twist matrix $U_M{}^{\underline{N}}$.

With this ansatz, the type IIB field equations reduce to the Einstein equations, which in this normalization take the form

$$\overset{\circ}{R}_{mn} \;=\; \overset{\circ}{T}_{mn} \;\equiv\; \frac{25}{6}\,\partial_{[m}\overset{\circ}{C}_{klpq]}\partial_{[n}\overset{\circ}{C}_{rstu]}\,\overset{\circ}{G}{}^{kr}\overset{\circ}{G}{}^{ls}\overset{\circ}{G}{}^{pt}\overset{\circ}{G}{}^{qu}\,, \tag{5.39}$$

and similar for $\overset{\circ}{R}_{\mu\nu}$. With (5.33) and (5.38), the energy-momentum tensor takes a particularly simple form for $p=6$ and $p=3$:

$$\overset{\circ}{T}_{mn} \;=\; \begin{cases} 4\,\overset{\circ}{G}_{mn} & p=6\\ (1+u-v)^{-5/2}\,\overset{\circ}{G}_{mn} & p=3\end{cases}\,. \tag{5.40}$$

For the x-dependent background metric $\overset{\circ}{g}_{\mu\nu}(x)$ the most symmetric ansatz assumes an Einstein space (dS, AdS, or Minkowski)

$$R[\overset{\circ}{g}]_{\mu\nu} \;=\; k\,\overset{\circ}{g}_{\mu\nu}\,, \tag{5.41}$$

upon which the IIB Ricci tensor associated with (5.33) turns out to be blockwise proportional to the IIB metric for the same two cases $p=6$ and $p=3$

$$\begin{aligned}\overset{\circ}{R}_{mn} &= \begin{cases} 4\,\overset{\circ}{G}_{mn} & p=6\\ (1+u-v)^{-5/2}\,\overset{\circ}{G}_{mn} & p=3\end{cases}\,,\\ \overset{\circ}{R}_{\mu\nu} &= \begin{cases} k\,\overset{\circ}{G}_{\mu\nu} & p=6\\ -(1+u-v)^{-5/2}\,\big(1+(2-k)\,(1+u-v)^2\big)\,\overset{\circ}{G}_{\mu\nu} & p=3\end{cases}\,.\end{aligned} \tag{5.42}$$

Together it follows that (5.33), (5.37), (5.41) solve the IIB field equations for $p = 3, k = 2$ and $p = 6, k = -4$, c.f. [20]. The resulting backgrounds are $\mathrm{AdS}_5 \times S^5$ and $\mathrm{dS}_5 \times H^{3,3}$ and the induced $D = 5$ theories correspond to the SO(6) and the SO(3, 3) gaugings of [5], respectively. For $3 \neq p \neq 6$, the background geometry is not a solution to the IIB field equations. Let us stress, however, that also in these cases the reduction ansatz presented in the previous sections describes a consistent truncation of the IIB theory to an effectively $D = 5$ supergravity theory, just this theory does not have a simple ground state with all fields vanishing.

5.5 Reconstructing 3-form and 4-form

We have in the previous sections derived the reduction formulas for all EFT scalars, vectors, and two-forms. Upon using the explicit dictionary into the IIB fields [24, 38], this allows to reconstruct the major part of the original IIB fields. More precisely, among the components of the fundamental IIB fields only $\hat{C}_{\mu\nu\rho\, m}$ and $\hat{C}_{\mu\nu\rho\sigma}$ with three and four external legs of the IIB four-form potential remain undetermined from the previous analysis. These in turn can be reconstructed from the IIB self-duality equations, which are induced by the EFT dynamics. We refer to [38] for the details of the general procedure, which we work out in the following with the generalized Scherk-Schwarz reduction ansatz.

The starting point is the duality equation between EFT vectors and two-forms that follows from the Lagrangian

$$\partial_{[k}\left(\tilde{\mathcal{H}}_{|\mu\nu\rho|\, mn]} - \frac{1}{2}\, e\, \mathcal{M}_{mn],N}\, \mathcal{F}^{\sigma\tau\, N}\, \varepsilon_{\mu\nu\rho\sigma\tau}\right) \;=\; 0\,, \tag{5.43}$$

where $\mathcal{F}_{\mu\nu}{}^{N}$ is the non-abelian field strength associated with the vector fields $\mathcal{A}_{\mu}{}^{N}$, and $\tilde{\mathcal{H}}_{|\mu\nu\rho|\, mn}$ carries the field strength of the two-forms $\mathcal{B}_{\mu\nu\, mn}$. Taking into account the reduction ansatz (5.10), (5.16), it takes the explicit form

$$\begin{aligned}\tilde{\mathcal{H}}_{\mu\nu\rho\, mn} \;=\;& -\partial_{[\mu}\mathcal{A}_{\nu}{}^{k}\mathcal{A}_{\rho]\, kmn} - \mathcal{A}_{[\mu}{}^{k}\partial_{\nu}\mathcal{A}_{\rho]\, kmn} - \mathcal{F}_{[\mu\nu}{}^{k}\mathcal{A}_{\rho]\, kmn} - \mathcal{A}_{[\mu}{}^{k}\mathcal{F}_{\nu\rho]\, kmn} \\ & + 2\,\partial_{[m}\left(\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\mathcal{A}_{\rho]\, n]kl}\right)\,,\end{aligned} \tag{5.44}$$

in terms of the remaining vector fields and field strengths from (5.20). Since (5.43) is of the form of a vanishing curl, the equation can be integrated in the internal coordinates up to a curl $\partial_{[m}C_{n]\,\mu\nu\rho}$ related to the corresponding component of the IIB four-form, explicitly

$$\partial_{[m}C_{n]\,\mu\nu\rho} \;=\; \frac{1}{16}\sqrt{2}\, e\, \varepsilon_{\mu\nu\rho\sigma\tau}\, \mathcal{M}_{mn,N}\, \mathcal{F}^{\sigma\tau\, N} - \frac{1}{8}\sqrt{2}\, \tilde{\mathcal{H}}_{\mu\nu\rho\, mn}\,. \tag{5.45}$$

It is a useful consistency test of the present construction, that with the reduction ansatz described in the previous sections, the r.h.s. of this equation indeed takes the form of a curl in the internal variables. Let us verify this explicitly. Since the reduction ansatz is covariant, the first term reduces according to the form of its free indices $[mn]$, c.f. (5.14)

$$e\, \mathcal{M}_{mn,N}\, \mathcal{F}^{\sigma\tau\, N} \;=\; -\frac{1}{2}\,\sqrt{2}\,\partial_{[m}\mathcal{K}^{[ab]}{}_{n]}\left(\sqrt{|\mathrm{g}|}\, M_{ab,N}\, F^{\sigma\tau\, N}\right)\,, \tag{5.46}$$

which indeed takes the form of a curl. We recall that the $D = 5$ field strength $F_{\mu\nu}{}^{N}$ combines the 15 non-abelian field strengths $F_{\mu\nu}{}^{ab}$ and the 12 two-forms $B_{\mu\nu\, a\alpha}$ according to (4.9).

The reduction of the second term on the r.h.s. of (5.45) is less obvious, since $\tilde{\mathcal{H}}_{\mu\nu\rho\,mn}$ is not a manifestly covariant object, and we have computed it explicitly by combining its defining equation (5.44) with the reduction of the vector fields (5.8) and field strengths (5.20). With the identity (3.57) among the Killing vectors and tensors, the second term on the r.h.s. of (5.45) then reduces according to

$$\tilde{\mathcal{H}}_{\mu\nu\rho\,mn} \;=\; \frac{1}{8}\,\varepsilon_{abcdef}\,\mathcal{K}^{[ef]}{}_{mn}\,\Omega^{abcd}_{\mu\nu\rho} + 2\,\partial_{[m}\left(\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\mathcal{A}_{\rho]n]kl}\right)\,. \tag{5.47}$$

with the non-abelian $\mathrm{SO}(p,q)$ Chern-Simons form defined as

$$\Omega^{abcd}_{\mu\nu\rho} \;=\; \partial_{[\mu}A_{\nu}{}^{ab}A_{\rho]}{}^{cd} + F_{[\mu\nu}{}^{ab}A_{\rho]}{}^{cd}\,, \tag{5.48}$$

in terms of the $\mathrm{SO}(p,6-p)$ Yang-Mills field strength $F_{\mu\nu}{}^{ab}$. Again, (5.47) takes the form of a curl in the internal variables, such that equation (5.45) can be explicitly integrated to

$$\begin{aligned} C_{m\,\mu\nu\rho} \;=\; & -\frac{1}{32}\,\mathcal{K}^{[ab]}{}_{m}\left(2\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,M_{ab,N}F^{\sigma\tau\,N} + \sqrt{2}\,\varepsilon_{abcdef}\,\Omega^{cdef}_{\mu\nu\rho}\right) \\ & -\frac{1}{4}\sqrt{2}\,\mathcal{K}_{[ab]}{}^{k}\mathcal{K}_{[cd]}{}^{l}\,\mathcal{Z}_{[ef]\,mkl}\left(A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho]}{}^{ef}\right)\,. \end{aligned} \tag{5.49}$$

This yields the full reduction ansatz for the component $C_{m\,\mu\nu\rho}$. Obviously, $C_{m\,\mu\nu\rho}$ is determined by (5.45) only up to a gradient $\partial_m\Lambda_{\mu\nu\rho}$ in the internal variables, which corresponds to a gauge transformation of the IIB four-form. Choosing the reduction ansatz (5.49), we have thus made a particular choice for this gauge freedom.

In a similar way, the last missing component $C_{\mu\nu\rho\sigma}$ can be reconstructed by further manipulating the equations and comparing to the IIB self-duality equations [38]. Concretely, taking the external curl of (5.45) and using Bianchi identities and field equations on the r.h.s. yields a differential equation that can be integrated in the internal variable to

$$\begin{aligned} \frac{1}{24}\,e\varepsilon_{\mu\nu\rho\sigma\lambda}\,\varepsilon^{kpqrs}\,(\det G)^{-1}\,G_{nk}\,\widehat{\mathcal{D}}^{\lambda}c_{pqrs} \;=\; & 16\,D^{\mathrm{KK}}_{[\mu}C_{\nu\rho\sigma]\,n} - 30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^{\alpha}\partial_n\mathcal{B}_{\rho\sigma]}{}^{\beta} \\ & + 6\sqrt{2}\,\mathcal{F}_{[\mu\nu}{}^{k}\mathcal{A}_{\rho}{}^{l}\mathcal{A}_{\sigma]\,lkn} + 4\,\partial_n C_{\mu\nu\rho\sigma}\,, \end{aligned} \tag{5.50}$$

up to an external gradient $\partial_n C_{\mu\nu\rho\sigma}$ which carries the last missing component of the IIB four-form. Here, D^{KK}_{μ} denotes the Kaluza-Klein covariant derivative

$$D^{\mathrm{KK}}_{\mu}C_n \;\equiv\; \partial_\mu C_n - \mathcal{A}_{\mu}{}^{k}\partial_k C_n - \partial_n\mathcal{A}_{\mu}{}^{k}C_k\,, \qquad \text{etc.}\,, \tag{5.51}$$

and $\widehat{\mathcal{D}}_\mu c_{pqrs}$ is a particular combination of scalar covariant derivatives [38], which is most compactly defined via particular components of the scalar currents as

$$\mathcal{D}_\mu\mathcal{M}_{mn,N}\mathcal{M}^{N\,n} \;=\; -\frac{2\sqrt{2}}{4!}\,(\det G)^{-1}\,G_{mn}\,\varepsilon^{npqrs}\,\widehat{\mathcal{D}}_\mu c_{pqrs}\,, \tag{5.52}$$

where $\mathcal{D}_\mu$ refers to the full EFT derivative, covariant under generalized diffeomorphisms. Again, it is a useful consistency check of the construction that with the reduction ansatz developed so far, equation (5.50) indeed turns into a total gradient, from which we may read off the function $C_{\mu\nu\rho\sigma}$. For the l.h.s. this is most conveniently seen by virtue of (5.52) and the reduction ansatz (2.1) for $\mathcal{M}_{MN}$, giving rise to

$$\begin{aligned} e\,(\det G)^{-1}\,G_{mk}\,\varepsilon^{kpqrs}\,\widehat{\mathcal{D}}^{\mu}c_{pqrs} \;=\; & 3\sqrt{|\mathrm{g}|}\,\mathcal{K}^{[ab]}{}_{mn}\mathcal{K}_{[cd]}{}^{n}D^{\mu}M_{ab,N}M^{N\,cd} \\ =\; & 6\sqrt{|\mathrm{g}|}\left(\sqrt{2}\,\mathcal{K}^{[cb]}{}_{m}\eta_{ac} - \partial_m(\mathcal{Y}^b\mathcal{Y}_a)\right)D^{\mu}M_{bd,N}M^{N\,da}\,, \end{aligned} \tag{5.53}$$

where we have used (3.55). The derivatives D_μ on the r.h.s. now refer to the $\mathrm{SO}(p,6-p)$ covariant derivatives (4.13). For the terms on the r.h.s. of (5.50), we find with (5.8), (5.12), and (3.48)

$$\begin{aligned} -30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^{\alpha}\partial_n\mathcal{B}_{\rho\sigma]}{}^{\beta} &= 15\sqrt{2}\,\varepsilon_{\alpha\beta}\,B_{[\mu\nu}{}^{a\alpha}B_{\rho\sigma]}{}^{b\beta}\,\mathcal{K}_{[ab]\,n}\,, \\ 6\sqrt{2}\,\mathcal{F}_{[\mu\nu}{}^{k}\mathcal{A}_{\rho}{}^{l}\mathcal{A}_{\sigma]\,lkn} &= -6\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma]}{}^{ef}\,\mathcal{K}_{[ab]}{}^{k}\mathcal{K}_{[cd]}{}^{l}\mathcal{Z}_{[ef]\,nkl}\,, \end{aligned} \tag{5.54}$$

as well as

$$\begin{aligned} 16\,D^{\mathrm{KK}}_{[\mu}C_{\nu\rho\sigma]\,m} &= \frac{1}{2}\,\mathcal{K}^{[ab]}{}_{m}\left(\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,D_\lambda\left(M_{ab,N}F^{\tau\lambda\,N}\right)+\sqrt{2}\,\varepsilon_{abcdef}\,D_{[\mu}\Omega^{cdef}_{\nu\rho\sigma]}\right) \\ &\quad + 4\sqrt{2}\,\mathcal{F}_{[\mu\nu}{}^{k}\mathcal{A}_{\rho}{}^{l}\mathcal{A}_{\sigma]\,mkl}+2\sqrt{2}\,\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\mathcal{F}_{\rho\sigma]\,mkl} \\ &\quad + \sqrt{2}\,\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\left(2\,\mathcal{A}_{\rho}{}^{n}\partial_{|n|}\mathcal{A}_{\sigma]\,klm}+3\,\partial_{[m}\mathcal{A}_{\rho}{}^{n}\mathcal{A}_{\sigma]\,kl]n}-3\,\mathcal{A}_{\rho}{}^{n}\partial_{[m}\mathcal{A}_{\sigma]\,kl]n}\right) \\ &\quad - 2\sqrt{2}\,\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu\,|kmn|}\left(\mathcal{A}_{\rho}{}^{l}\partial_{|l|}\mathcal{A}_{\sigma]}{}^{n}\right)-\sqrt{2}\,\partial_m\left(\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\mathcal{A}_{\rho}{}^{n}\mathcal{A}_{\sigma]\,kln}\right)\,, \end{aligned} \tag{5.55}$$

where we have explicitly evaluated the Kaluza-Klein covariant derivative D_μ on $C_{\mu\nu\rho\,m}$, the latter given by (5.49). Moreover, we have arranged the $\mathcal{A}^4$ terms such that they allow for a convenient evaluation of their reduction formulae. Namely, in the last two lines we have factored out the quadratic polynomials that correspond to the $\mathcal{A}^2$ terms in the non-abelian field strengths (5.20) and thus upon reduction factor in analogy to the field strengths, leaving us with the A^4 terms

$$\begin{aligned} \mathcal{A}\mathcal{A}\mathcal{A}\mathcal{A} \;\longrightarrow\; & -2\,f_{ef,gh}{}^{ij}\,\mathcal{K}^{k}_{[ab]}\left(\mathcal{Z}_{[cd]\,mkl}\mathcal{K}^{l}_{[ij]}+\mathcal{K}^{l}_{[cd]}\mathcal{Z}_{[ij]\,mkl}\right)A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho}{}^{ef}A_{\sigma]}{}^{gh} \\ & -\sqrt{2}\,\partial_m\left(\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\mathcal{A}_{\rho}{}^{n}\mathcal{A}_{\sigma]\,kln}\right) \\ = \; & -\frac{1}{4}\sqrt{2}\,f_{ab,uv}{}^{xy}f_{ef,gh}{}^{ij}\,\varepsilon_{cdijxy}\,\mathcal{K}^{[uv]}_{m}A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho}{}^{ef}A_{\sigma]}{}^{gh} \\ & +\frac{1}{2}\,f_{ef,gh}{}^{ij}\,\varepsilon_{cdijau}\,\partial_m\left(\mathcal{Y}^{u}\mathcal{Y}_{b}\right)A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho}{}^{ef}A_{\sigma]}{}^{gh} \\ & -\sqrt{2}\,\partial_m\left(\mathcal{A}_{[\mu}{}^{k}\mathcal{A}_{\nu}{}^{l}\mathcal{A}_{\rho}{}^{n}\mathcal{A}_{\sigma]\,kln}\right)\,, \end{aligned} \tag{5.56}$$

upon using the identities (3.57), (3.55). While the last two terms are total gradients, the first term cancels against the corresponding contribution from the derivative of the Chern-Simons form $\Omega^{abcd}_{\mu\nu\rho}$ in (5.55)

$$\begin{aligned} D_{[\mu}\Omega^{cdef}_{\nu\rho\sigma]}\,\varepsilon_{abcdef} &= \frac{3}{4}\,F_{[\mu\nu}{}^{cd}F_{\rho\sigma]}{}^{ef}\,\varepsilon_{abcdef}-\frac{1}{2}\sqrt{2}\,A_{[\mu}{}^{cd}A_{\nu}{}^{ef}F_{\rho\sigma]}{}^{gh}\,f_{ab,ef}{}^{uv}\,\varepsilon_{cdghuv} \\ &\quad -\frac{1}{2}\,A_{[\mu}{}^{cd}A_{\nu}{}^{ef}A_{\rho}{}^{gh}A_{\sigma]}{}^{ij}\,f_{cd,ef}{}^{rs}f_{gh,ij}{}^{uv}\varepsilon_{abrsuv}\,. \end{aligned} \tag{5.57}$$

Similarly, the $\mathcal{F}\mathcal{A}\mathcal{A}$ terms in (5.55) combine with those of (5.54) according to

$$\begin{aligned} \mathcal{F}\mathcal{A}\mathcal{A} \;\longrightarrow\; & -2\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma]}{}^{ef}\,\mathcal{K}^{l}_{[cd]}\left(\mathcal{K}^{k}_{[ab]}\mathcal{Z}_{[ef]\,mkl}+\mathcal{K}^{k}_{[ef]}\mathcal{Z}_{[ab]\,mkl}\right) \\ = \; & \frac{1}{2}\,f_{cd,ij}{}^{gh}\,\mathcal{K}^{[ij]}{}_{m}F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma]}{}^{ef}\,\varepsilon_{abefgh}\,-\frac{1}{2}\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma]}{}^{ef}\,\varepsilon_{abefch}\,\partial_m(\mathcal{Y}^{h}\mathcal{Y}_{d}). \end{aligned} \tag{5.58}$$

Again, the first term cancels against the corresponding contribution from the derivative of the Chern-Simons form $\Omega^{abcd}_{\mu\nu\rho}$, given in (5.57).

Collecting all the remaining terms, equation (5.50) takes the final form

$$
\begin{aligned}
0 \;=\;& \frac{1}{2}\,\mathcal{K}^{[ab]}{}_m\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\left(\frac{1}{2}\,\sqrt{2}\,\eta_{da}\,D^\tau M_{cb,N}M^{N\,cd}+D_\lambda\left(M_{ab,N}F^{\tau\lambda\,N}\right)\right)\\
&+\frac{3}{8}\,\sqrt{2}\,\mathcal{K}^{[ab]}{}_m\left(\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{cd}F_{\rho\sigma]}{}^{ef}+40\,\varepsilon_{\alpha\beta}\,\eta_{ac}\eta_{bd}\,B_{[\mu\nu}{}^{c\alpha}B_{\rho\sigma]}{}^{d\beta}\right)\\
&+\frac{1}{2}\,f_{ef,gh}{}^{ij}\,\varepsilon_{cdijay}\,\partial_m\left(\mathcal{Y}^y\mathcal{Y}_b\right)A_{[\mu}{}^{ab}A_\nu{}^{cd}A_\rho{}^{ef}A_{\sigma]}{}^{gh}\\
&-\frac{1}{4}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,\partial_m(\mathcal{Y}^b\mathcal{Y}_d)\,D^\tau M_{ab,N}M^{N\,ad}-\frac{1}{2}\,\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_\rho{}^{cd}A_{\sigma]}{}^{ef}\,\varepsilon_{abefch}\,\partial_m(\mathcal{Y}^h\mathcal{Y}_d)\\
&-\sqrt{2}\,\partial_m\left(\mathcal{A}_{[\mu}{}^k\mathcal{A}_\nu{}^l\mathcal{A}_\rho{}^n\mathcal{A}_{\sigma]\,kln}\right)+4\,\partial_m C_{\mu\nu\rho\sigma}\;. \qquad (5.59)
\end{aligned}
$$

Now the first two lines of the expression precisely correspond to the vector field equations (4.15) of the $D=5$ theory, which confirms that on-shell this equation reduces to a total gradient in the internal variables. Although guaranteed by the consistency of the generalized Scherk-Schwarz ansatz and the general analysis of [38], it is gratifying that this structure is confirmed by explicit calculation based on the $D=5$ field equations and the non-trivial identities among the Killing vectors. We are thus in position to read off from (5.59) the final expression for the 4-form as

$$
\begin{aligned}
C_{\mu\nu\rho\sigma} \;=\;& -\frac{1}{16}\,\mathcal{Y}_a\mathcal{Y}^b\left(\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D^\tau M_{bc,N}M^{N\,ca}+2\,\sqrt{2}\,\varepsilon_{cdefgb}\,F_{[\mu\nu}{}^{cd}A_\rho{}^{ef}A_{\sigma]}{}^{ga}\right)\\
&+\frac{1}{4}\left(\sqrt{2}\,\mathcal{K}_{[ab]}{}^k\mathcal{K}_{[cd]}{}^l\mathcal{K}_{[ef]}{}^n\mathcal{Z}_{[gh]\,kln}-\mathcal{Y}_h\mathcal{Y}^j\,\varepsilon_{abcegj}\,\eta_{df}\right)A_{[\mu}{}^{ab}A_\nu{}^{cd}A_\rho{}^{ef}A_{\sigma]}{}^{gh}\\
&+\Lambda_{\mu\nu\rho\sigma}(x)\;, \qquad (5.60)
\end{aligned}
$$

in terms of the $D=5$ fields, up to an y-independent term $\Lambda_{\mu\nu\rho\sigma}(x)$, left undetermined by equation (5.50) and fixed by the last component of the IIB self-duality equations (5.3). This equation translates into

$$
\begin{aligned}
4\,D^{\mathrm{KK}}_{[\mu}C_{\nu\rho\sigma\tau]} \;=\;& 30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^\alpha D^{\mathrm{KK}}_\rho\mathcal{B}_{\sigma\tau]}{}^\beta+8\,\mathcal{F}_{[\mu\nu}{}^k C_{\rho\sigma\tau]\,k}\\
&-\frac{1}{120}\,e\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\,(\det G)^{-4/3}X_{klmnp}\;, \qquad (5.61)
\end{aligned}
$$

where X_{klmnp} is a combination of internal derivatives of the scalar fields, c.f. [38], that is most compactly given by

$$
\frac{1}{120}\,\varepsilon^{kpqrs}\,X_{kpqrs} \;=\; -\frac{1}{20}\,\sqrt{2}\,(\det G)\,G^{ml}\,\partial_l\mathcal{M}_{mn,N}\mathcal{M}^{N\,n}\;, \qquad (5.62)
$$

in analogy to (5.52). It can be shown that equation (5.61) can be derived from the external curl of equations (5.50) upon using the EFT field equations and Bianchi identities, up to a y-independent equation that defines the last missing function $\Lambda_{\mu\nu\rho\sigma}$. For the general case this has been worked out in [38]. Alternatively, it can be confirmed by explicit calculation with the Scherk-Schwarz reduction ansatz, that equation (5.61) with the components $C_{\mu\nu\rho\,m}$ and $C_{\mu\nu\rho\sigma}$ from (5.49) and (5.60), respectively, decomposes into a y-dependent part, which vanishes due to the $D=5$ scalar equations of motion, and a y-independent part, that defines the function $\Lambda_{\mu\nu\rho\sigma}$. The calculation is similar (but more lengthy) than the previous steps, requires the same non-trivial identities among Killing vectors derived above, but also some non-trivial algebraic

identities among the components of the scalar $\mathrm{E}_{6(6)}$ matrix M_{MN}. We relegate the rather lengthy details to appendix A and simply report the final result from equation (A.20)

$$
\begin{aligned}
D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} \;=\; & -\frac{1}{480}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_\lambda\left(M^{N\,ac}\,D^\lambda M_{ac,N}\right)\\
& +\frac{1}{240}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,F^{\kappa\lambda\,N}\left(M_{ab,N}F_{\kappa\lambda}{}^{ab}-\frac{1}{2}\,\sqrt{10}\,\varepsilon_{\alpha\beta}\,\eta_{ab}\,M^{a\,\alpha}{}_{N}\,B_{\kappa\lambda}{}^{b\,\beta}\right)\\
& +\frac{1}{600}\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\left(10\,\delta^d_h\delta^a_e+2\,M^{fd,ga}M_{gh,fe}-M_{e\alpha}{}^{ga}M_{gh}{}^{d\alpha}\right)M^{bh,ec}\,\eta_{cd}\eta_{ab}\\
& +\frac{1}{32}\,\sqrt{2}\;\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef}+\frac{1}{16}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh}\\
& +\frac{1}{40}\,\sqrt{2}\;A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho}{}^{ef}A_{\sigma}{}^{gh}A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj}\;. \qquad (5.63)
\end{aligned}
$$

Since there is no non-trivial Bianchi identity for (5.63), this equation can be integrated and yields the last missing term in the four-form potential (5.60). This completes the reduction formulae for the full set of fundamental IIB fields.

6 Summary

We have in this paper derived the explicit reduction formulae for the full set of IIB fields in the compactification on the sphere S^5 and the inhomogeneous hyperboloids $H^{p,6-p}$. The fluctuations around the background geometry are described by a $D = 5$ maximal supergravity, with gauge group $\mathrm{SO}(p, 6-p)$. The dependence on the internal variables is explicitly expressed in terms of 1) a set of vectors $\mathcal{K}_{[ab]}{}^m$ which are Killing vectors of a homogeneous metric $\tilde{G}_{mn}$ (3.9), and 2) a four-form $\tilde{C}_{\mu\nu\rho\sigma}$ whose field-strength yields the Lorentzian volume form (3.29). Only for the compact case of S^5, the metric $\tilde{G}_{mn}$ and four-form $\tilde{C}_{klmn}$ coincide with the space-time background geometry. In the non-compact case, they refer to a (virtual) homogeneous Lorentzian geometry which encodes the inhomogeneous space-time background geometry via the formulas provided. This is in accordance with the ansatz proposed and tested for some stationary points of the non-compact $D = 4$ gaugings in [22], see also [20, 21] for earlier work. Only for $p = 6$ and $p = 3$ does the background geometry provide a solution to the IIB field equations. We stress, that also in the remaining cases, the reduction ansatz describes a consistent truncation of the IIB theory to an effectively $D = 5$ supergravity theory, just this theory does not have a simple ground state with all fields vanishing. Still, any stationary point or holographic RG flow of these non-compact gaugings as well as any other solution to their field equations lifts to a IIB solution by virtue of the explicit reduction formulas.

The explicit reduction formulas are derived via the EFT formulation of the IIB theory by evaluating the formulas of the generalized Scherk-Schwarz reduction ansatz for the twist matrices obtained in [19]. The Scherk-Schwarz origin also proves consistency of the truncation in the sense that all solutions of the respective $D = 5$ maximal supergravities lift to solutions of the type IIB fields equations. By virtue of the explicit embedding of the IIB theory into EFT [24, 38] these formulas can be pulled back to read off the reduction formulas for the original type IIB fields. Upon some further computational effort we have also derived the explicit expressions for all the components of the IIB four-form. Along the way, we explicitly

verified the IIB self-duality equations. Although their consistency is guaranteed by the general construction, we have seen that their validation by virtue of non-trivial Killing vector identities still represents a rewarding exercise.

We have in this paper restricted the construction to the bosonic sector of type IIB supergravity. In the EFT framework, consistency of the reduction of the fermionic sector follows along the same lines from the supersymmetric extension of the $\mathrm{E}_{6(6)}$ exceptional field theory [49] which upon generalized Scherk-Schwarz reduction yields the fermionic sector of the $D = 5$ gauged supergravities [19]. In particular, compared to the bosonic reduction ansatz (2.1), the EFT fermions reduce as scalar densities, i.e. their y-dependence is carried by some power of the scale factor, such as $\psi_\mu{}^i(x,y) = \rho^{-\frac{1}{2}}(y)\,\psi_\mu{}^i(x)$, etc.. A derivation of the explicit reduction formulas for the original IIB fermions would require the dictionary of the fermionic sector of EFT into the IIB theory, presumably along the lines of [40]. The very existence of a consistent reduction of the fermionic sector can also be inferred on general grounds [2] combining the bosonic results with the supersymmetry of the IIB theory.

We close by recollecting the full set of IIB reduction formulas derived in this paper. The IIB metric is given by

$$\begin{aligned} ds^2 &= \Delta^{-2/3}(x,y)\,\mathrm{g}_{\mu\nu}(x)\,dx^\mu dx^\nu \\ &\quad + G_{mn}(x,y)\left(dy^m + \mathcal{K}_{[ab]}{}^m(y)A^{ab}_\mu(x)dx^\mu\right)\left(dy^n + \mathcal{K}_{[cd]}{}^n(y)A^{cd}_\nu(x)dx^\nu\right)\,, \end{aligned} \tag{6.1}$$

in standard Kaluza-Klein form, in terms of vectors $\mathcal{K}_{[ab]}{}^m$ from (3.38) that are Killing for the (Lorentzian) metric $\tilde{G}_{mn}$ from (3.9), and the internal block G_{mn} of the metric (6.1) given by the inverse of

$$G^{mn}(x,y) = \Delta^{2/3}(x,y)\,\mathcal{K}_{[ab]}{}^m(y)\mathcal{K}_{[cd]}{}^n(y)\,M^{ab,cd}(x)\,. \tag{6.2}$$

The IIB dilaton and axion combine into the symmetric SL(2) matrix

$$m^{\alpha\beta}(x,y) = \Delta^{4/3}(x,y)\,\mathcal{Y}_a(y)\mathcal{Y}_b(y)\,M^{a\alpha,b\beta}(x)\,, \tag{6.3}$$

in terms of the harmonics $\mathcal{Y}_a$ from (3.45). Since $\det m^{\alpha\beta} = 1$, this equation can also be used as a defining equation for the function $\Delta(x,y)$. The different components of the two-form doublet are given by

$$\begin{aligned} C_{mn}{}^\alpha(x,y) &= -\frac{1}{2}\,\varepsilon^{\alpha\beta}\Delta^{4/3}(x,y)\,m_{\beta\gamma}(x,y)\,\mathcal{Y}_c(y)\,\mathcal{K}^{[ab]}{}_{mn}(y)\,M_{ab}{}^{c\gamma}(x)\,, \\ C_{\mu m}{}^\alpha(x,y) &= 0\,, \\ C_{\mu\nu}{}^\alpha(x,y) &= \sqrt{10}\,\mathcal{Y}_a(y)\,B_{\mu\nu}{}^{a\alpha}(x)\,. \end{aligned} \tag{6.4}$$

Next, we give the uplift formulas for the four-form components in terms of the Killing vectors $\mathcal{K}_{[ab]}{}^m(y)$, Killing tensors $\mathcal{K}_{[ab]\,mn}(y)$, the sphere harmonics $\mathcal{Y}_a(y)$ given in (3.45), the function $\mathcal{Z}_{[ab]\,kmn}(y)$ given by (3.21), and the four-form $\tilde{C}_{klmn}(y)$ from (3.49). In order not to clutter the formulas, in the following we do not display the dependence on the arguments x and y as it is always clear from the definition of the various objects whether they depend on the external

or internal coordinates or both. The final result reads

$$
\begin{aligned}
C_{klmn} &= \tilde{C}_{klmn} + \tilde{\omega}_{klmnp}\,\Delta^{4/3}\,m_{\alpha\beta}\,\tilde{G}^{pq}\,\partial_q\left(\Delta^{-4/3}\,m^{\alpha\beta}\right)\,,\\
C_{\mu kmn} &= \frac{\sqrt{2}}{4}\,\mathcal{Z}_{[ab]\,kmn}\,A_\mu{}^{ab}\,,\\
C_{\mu\nu\,mn} &= \frac{\sqrt{2}}{4}\,\mathcal{K}_{[ab]}{}^{k}\,\mathcal{Z}_{[cd]\,kmn}\,A_{[\mu}{}^{ab}A_{\nu]}{}^{cd}\,,\\
C_{m\,\mu\nu\rho} &= -\frac{1}{32}\,\mathcal{K}_{[ab]\,m}\left(2\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,M_{ab,N}F^{\sigma\tau\,N}+\sqrt{2}\,\varepsilon_{abcdef}\,\Omega^{cdef}_{\mu\nu\rho}\right)\\
&\quad -\frac{1}{4}\sqrt{2}\,\mathcal{K}_{[ab]}{}^{k}\mathcal{K}_{[cd]}{}^{l}\,\mathcal{Z}_{[ef]\,mkl}\left(A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho]}{}^{ef}\right)\,,\\
C_{\mu\nu\rho\sigma} &= -\frac{1}{16}\,\mathcal{Y}_a\mathcal{Y}^b\left(\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D^\tau M_{bc,N}M^{N\,ca}\,+2\sqrt{2}\,\varepsilon_{cdefgb}\,F_{[\mu\nu}{}^{cd}A_\rho{}^{ef}A_{\sigma]}{}^{ga}\right)\\
&\quad +\frac{1}{4}\left(\sqrt{2}\,\mathcal{K}_{[ab]}{}^{k}\mathcal{K}_{[cd]}{}^{l}\mathcal{K}_{[ef]}{}^{n}\,\mathcal{Z}_{[gh]\,kln}-\mathcal{Y}_h\mathcal{Y}^j\,\varepsilon_{abcegj}\,\eta_{df}\right)A_{[\mu}{}^{ab}A_\nu{}^{cd}A_\rho{}^{ef}A_{\sigma]}{}^{gh}\\
&\quad +\Lambda_{\mu\nu\rho\sigma}(x)\,.
\end{aligned}
\tag{6.5}
$$

We recall, that the curved indices on these objects are raised and lowered with the x-independent metric $\tilde{G}_{mn}(y)$ from (3.9) and not with the background metric G_{mn}. The function $\Lambda_{\mu\nu\rho\sigma}(x)$ is defined by equation (5.63). All p-form components are given in the basis after standard Kaluza-Klein decomposition, explicitly related to the original IIB fields by (5.6).

With the reduction ansatz (6.1)–(6.5), the type IIB field equations reduce to the $D = 5$ field equations derived from the Lagrangian (4.11). As a consequence, these formulas lift every solution of $D = 5$, SO(p, q) gauged supergravity to a solution of IIB supergravity.

Acknowledgments

We would like to thank Bernard de Wit, Edvard Musaev, and Mario Trigiante for helpful discussions. The work of O.H. is supported by the U.S. Department of Energy (DoE) under the cooperative research agreement DE-FG02-05ER41360 and a DFG Heisenberg fellowship.

Appendix

A Finding $\Lambda_{\mu\nu\rho\sigma}$

In order to find the last missing contribution $\Lambda_{\mu\nu\rho\sigma}$ in the expression (5.60) for the four-form component $C_{\mu\nu\rho\sigma}$ let us study the reduction of the different terms of equation (5.61)

$$
\begin{aligned}
\frac{1}{120}\,e\varepsilon_{\mu\nu\rho\sigma\tau}\varepsilon^{klmnp}\,(\det G)^{-4/3}X_{klmnp} &= 30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^{\alpha}D^{\mathrm{KK}}_{\rho}\mathcal{B}_{\sigma\tau]}{}^{\beta}+8\,\mathcal{F}_{[\mu\nu}{}^{k}C_{\rho\sigma\tau]\,k}\\
&\quad -4\,D^{\mathrm{KK}}_{[\mu}C_{\nu\rho\sigma\tau]}\,.
\end{aligned}
\tag{A.1}
$$

By construction, after imposing the generalized Scherk-Schwarz ansatz this equation should split into a y-dependent part proportional to the $D = 5$ scalar field equations (4.16), and a y-independent part which determines the function $\Lambda_{\mu\nu\rho\sigma}$.

The first term on the r.h.s. simply reduces according to the reduction ansatz (5.12)

$$30\,\varepsilon_{\alpha\beta}\,\mathcal{B}_{[\mu\nu}{}^{\alpha}D_{\rho}^{\mathrm{KK}}\mathcal{B}_{\sigma\tau]}{}^{\beta} \;=\; 30\,\varepsilon_{\alpha\beta}\,\mathcal{Y}_a\mathcal{Y}_b\,B_{[\mu\nu}{}^{a\alpha}D_{\rho}B_{\sigma\tau]}{}^{b\beta}\,. \tag{A.2}$$

Note that the Kaluza-Klein covariant derivative turns into the $\mathrm{SO}(p,6-p)$ covariant derivative by virtue of (3.46). With (5.49) and the identity (3.56), we find for the second term on the r.h.s. of (A.1)

$$\begin{aligned} 8\,\mathcal{F}_{[\mu\nu}{}^{k}C_{\rho\sigma\tau]\,k} \;&=\; -\frac{1}{2}\,\mathcal{Y}_b\mathcal{Y}^a\,F_{[\mu\nu}{}^{cb}\left(2\sqrt{|\mathrm{g}|}\,\varepsilon_{\rho\sigma\tau]\kappa\lambda}\,M_{ac,N}F^{\kappa\lambda\,N}+\sqrt{2}\,\Omega^{efgh}_{\rho\sigma\tau]}\,\varepsilon_{acefgh}\right)\\ &\quad+2\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}\,\mathcal{K}_{[ab]}{}^{m}\mathcal{K}_{[cd]}{}^{k}\mathcal{K}_{[ef]}{}^{l}\mathcal{Z}_{[gh]\,mkl}\,. \end{aligned} \tag{A.3}$$

Next, we have to work out the covariant curl of $C_{\mu\nu\rho\sigma}$ with the explicit expression (5.60). To this end, we first note that for all terms with y-dependence proportional to $\mathcal{Y}^a\mathcal{Y}^b$, the Kaluza-Klein covariant derivative reduces to

$$D_{\mu}^{\mathrm{KK}}\left(\mathcal{Y}^a\mathcal{Y}^b\,X_{ab}\right) \;=\; \mathcal{Y}^a\mathcal{Y}^b\,D_{\mu}X_{ab}\,, \tag{A.4}$$

in view of the property (3.46) of the harmonics $\mathcal{Y}^a$. We thus find

$$\begin{aligned} -4\,D_{[\mu}^{\mathrm{KK}}C_{\nu\rho\sigma\tau]} \;&=\; \frac{1}{20}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_{\lambda}\left(M^{N\,ca}\,D^{\lambda}M_{bc,N}\right)-4\,D_{[\mu}\Lambda_{\nu\rho\sigma\tau]}\\ &\quad+\frac{1}{2}\,\sqrt{2}\,\mathcal{Y}_b\mathcal{Y}^a\,\varepsilon_{acdefg}\,D_{[\mu}\left(F_{\nu\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{bg}+\sqrt{2}\,A_{\nu}{}^{cd}A_{\rho}{}^{eh}A_{\sigma}{}^{fj}A_{\tau]}{}^{bg}\,\eta_{hj}\right)\\ &\quad-\sqrt{2}\,D_{[\mu}^{\mathrm{KK}}\left(\mathcal{K}_{[ab]}{}^{k}\mathcal{K}_{[cd]}{}^{l}\mathcal{K}_{[ef]}{}^{n}\mathcal{Z}_{[gh]\,kln}A_{\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}\right)\,. \end{aligned} \tag{A.5}$$

In order to evaluate the last term it is important to note that unlike in (A.4), the Kaluza-Klein covariant derivative here cannot just be pulled through the (non-covariant) y-dependent functions but has to be evaluated explicitly leading to

$$\begin{aligned} -\sqrt{2}\,D_{[\mu}^{\mathrm{KK}}\left(\mathcal{A}_{\nu}{}^{k}\mathcal{A}_{\rho}{}^{l}\mathcal{A}_{\sigma}^{n}\mathcal{A}_{\tau]\,kln}\right) \;&=\; -\frac{3}{2}\,\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}\,\mathcal{K}_{[ab]}{}^{k}\mathcal{K}_{[cd]}{}^{l}\mathcal{K}_{[ef]}{}^{n}\,\mathcal{Z}_{[gh]\,kln}\\ &\quad+\frac{1}{2}\,\sqrt{2}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}\,\mathcal{K}_{[cd]}{}^{k}\mathcal{K}_{[ef]}{}^{l}\mathcal{K}_{[gh]}{}^{n}\,\mathcal{Z}_{[ab]\,kln}\\ &\quad+\frac{3}{10}\,\sqrt{2}\,A_{[\mu}{}^{rs}A_{\nu}{}^{uv}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}f_{cd,rs}{}^{ab}\varepsilon_{abuvge}\mathcal{Y}_f\mathcal{Y}_h\,, \end{aligned}$$

after some manipulation of the functions $\mathcal{K}_{[ab]}$, $\mathcal{Z}_{[ab]}$. Putting everything together and again using once more the identity (3.57), the full r.h.s. of equation (A.1) is given by

$$\begin{aligned} (\mathrm{A.1})_{\mathrm{r.h.s}} \;&=\; \frac{1}{20}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\mathrm{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_{\lambda}\left(M^{N\,ca}\,D^{\lambda}M_{bc,N}\right)-4\,D_{[\mu}\Lambda_{\nu\rho\sigma\tau]}\\ &\quad+\frac{1}{2}\,\sqrt{2}\,\mathcal{Y}_a\mathcal{Y}^b\,\varepsilon_{bcdefg}\,D_{[\mu}\left(F_{\nu\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{ag}+\sqrt{2}\,A_{\nu}{}^{cd}A_{\rho}{}^{eh}A_{\sigma}{}^{fj}A_{\tau]}{}^{ag}\,\eta_{hj}\right)\\ &\quad+\frac{1}{2}\,\varepsilon_{dfghce}\mathcal{Y}_a\mathcal{Y}_b\,F_{[\mu\nu}{}^{df}A_{\rho}{}^{ac}A_{\sigma}{}^{be}A_{\tau]}{}^{gh}+30\,\varepsilon_{\alpha\beta}\,\mathcal{Y}_a\mathcal{Y}_b\,B_{[\mu\nu}{}^{a\,\alpha}D_{\rho}B_{\sigma\tau]}{}^{b\,\beta}\\ &\quad+\frac{3}{5}\,\sqrt{2}\,\varepsilon_{csuvge}\,\mathcal{Y}_a\mathcal{Y}_b\,\eta_{dr}\,A_{[\mu}{}^{rs}A_{\nu}{}^{uv}A_{\rho}{}^{cd}A_{\sigma}{}^{ae}A_{\tau]}{}^{bg}\\ &\quad-\frac{1}{2}\,\mathcal{Y}_b\mathcal{Y}^a\,F_{[\mu\nu}{}^{cb}\left(2\sqrt{|\mathrm{g}|}\,\varepsilon_{\rho\sigma\tau]\kappa\lambda}\,M_{ac,N}F^{\kappa\lambda\,N}+\sqrt{2}\,\Omega^{efgh}_{\rho\sigma\tau]}\,\varepsilon_{acefgh}\right)\,. \end{aligned} \tag{A.6}$$

Some calculation and use of the Schouten identity shows that all terms carrying explicit gauge fields add up precisely such that their y-dependence drops out due to $\mathcal{Y}_a\mathcal{Y}^a$. Specifically, we find

$$\begin{aligned}
(\text{A.1})_{\text{r.h.s}}\Big|_{FFA} &= \frac{1}{8}\sqrt{2}\,\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef}\,,\\
(\text{A.1})_{\text{r.h.s}}\Big|_{FAAA} &= \frac{1}{4}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh}\,,\\
(\text{A.1})_{\text{r.h.s}}\Big|_{AAAAA} &= \frac{1}{10}\sqrt{2}\,A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho}{}^{ef}A_{\sigma}{}^{gh}A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj}\,.
\end{aligned}\tag{A.7}$$

In addition, we use the $D=5$ duality equation (4.10) in order to rewrite the BDB term of (A.1) and arrive at

$$\begin{aligned}
(\text{A.1})_{\text{r.h.s}} &= -\frac{1}{20}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\text{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}D_\lambda\left(M^{N\,ac}\,D^\lambda M_{bc,N}\right)-4\,D_{[\mu}\Lambda_{\nu\rho\sigma\tau]}\\
&\quad+\frac{1}{10}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|\text{g}|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,F^{\kappa\lambda\,N}\left(M_{bc,N}F_{\kappa\lambda}{}^{ac}-\frac{1}{2}\,\sqrt{10}\,\varepsilon_{\alpha\beta}\,\eta_{db}\,M^{d\,\alpha}{}_N\,B_{\kappa\lambda}{}^{a\,\beta}\right)\\
&\quad+\frac{1}{8}\,\sqrt{2}\,\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef}+\frac{1}{4}\,F_{[\mu\nu}{}^{ab}A_{\rho}{}^{cd}A_{\sigma}{}^{ef}A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh}\\
&\quad+\frac{1}{10}\,\sqrt{2}\,A_{[\mu}{}^{ab}A_{\nu}{}^{cd}A_{\rho}{}^{ef}A_{\sigma}{}^{gh}A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj}\,.
\end{aligned}\tag{A.8}$$

Structurewise, the r.h.s. of equation (A.1) is thus of the form

$$(\text{A.1})_{\text{r.h.s}} = \left(\mathcal{Y}_a(y)\mathcal{Y}_b(y)-\frac{1}{6}\,\eta_{ab}\right)\mathcal{E}_{1\,ab}(x)+\mathcal{E}_2(x)\,.\tag{A.9}$$

Consistency of the reduction ansatz then implies that also the l.h.s. of (A.1) organizes into the same structure. The coefficients multiplying the y-dependent factor $\left(\mathcal{Y}_a(y)\mathcal{Y}_b(y)-\frac{1}{6}\,\eta_{ab}\right)$ must combine into a $D=5$ field equation in order to reduce (A.1) to an y-independent equation which then provides the defining equation for $\Lambda_{\mu\nu\rho\sigma}$.

In order to see this explicitly, we recall, that the l.h.s. of (A.1) is defined by (5.62), which together with the reduction ansatz (2.1) for $\mathcal{M}_{MN}$ may be used to read off the form of this term after reduction. After some manipulation of the Killing vectors and tensors and use of the identities collected in section 3.3, we obtain

$$\begin{aligned}
\frac{1}{120}\,e\,\varepsilon^{klmnp}\,(\det G)^{-4/3}X_{klmnp} &= -\frac{1}{10}\,\sqrt{2}\,\sqrt{|\text{g}|}\,\mathcal{Y}_a\mathcal{Y}_b\,\mathcal{X}^{(ab)cd,e}{}_f\,(U^{-1})_e{}^q\mathcal{K}_{[cd]}{}^m\partial_m U_q{}^f\\
&\quad-\frac{2}{5}\,\sqrt{|\text{g}|}\,\mathcal{Y}_a\mathcal{Y}_b\,\eta_{cd}\,M^{ac,bd}\,.
\end{aligned}\tag{A.10}$$

in terms of the SL(6) twist matrix (2.8), and the combination

$$\mathcal{X}^{(ab)cd,e}{}_f = \mathcal{X}^{(ab)[cd],e}{}_f \equiv 2\,M^{je,g(a}M^{b)h,cd}M_{gh,jf}-M_{f\alpha}{}^{g(a}M^{b)h,cd}M_{gh}{}^{e\alpha}\,,\tag{A.11}$$

of matrix components of (4.12). At first view, the structure of this expression in no way ressembles the form of (A.9), with a far more complicated y-dependence in its first term. This seemingly jeopardizes the consistency of the reduction of equation (A.1), which after all should be guaranteed by consistency of the ansatz. What comes to the rescue is some additional

properties of the twist matrix together with some highly non-trivial non-linear identities among the components of an $\mathrm{E}_{6(6)}$ matrix. Namely the last factor in the first term of (A.10) drastically reduces upon certain index projections

$$\begin{aligned}(U^{-1})_a{}^q \mathcal{K}_{[bc]}{}^m \partial_m U_q{}^c + (U^{-1})_b{}^q \mathcal{K}_{[ac]}{}^m \partial_m U_q{}^c &= -\sqrt{2}\,\eta_{ab}\,,\\ (U^{-1})_a{}^q \mathcal{K}_{[bc]}{}^m \partial_m U_q{}^d + (U^{-1})_b{}^q \mathcal{K}_{[ca]}{}^m \partial_m U_q{}^d + (U^{-1})_c{}^q \mathcal{K}_{[ab]}{}^m \partial_m U_q{}^d &= 0\,,\end{aligned} \tag{A.12}$$

as may be verified by explicit computation. Moreover, the tensor $\mathcal{X}^{(ab)cd,e}{}_f$ defined in (A.11) is of quite restricted nature and satisfies

$$\mathcal{X}^{(ab)cd,e}{}_f = \mathcal{X}^{(ab)[cd,e]}{}_f - \frac{2}{5}\,\delta_f{}^{[c}\,\mathcal{X}^{(ab)d]g,e}{}_g - \frac{2}{45}\,\delta_f{}^{[c}\,\mathcal{X}^{(ab)d]e,g}{}_g + \frac{1}{9}\,\delta_f{}^e\,\mathcal{X}^{(ab)cd,g}{}_g\,, \tag{A.13}$$

implying in particular that

$$\mathcal{X}^{(ab)e[c,d]}{}_e = -\frac{1}{6}\,\mathcal{X}^{(ab)cd,e}{}_e\,. \tag{A.14}$$

The identity (A.13) is far from obvious and hinges on the group properties of the matrix (4.12). It can be verified by choosing an explicit parametrization of this matrix (e.g. as given in [38]), at least with the help of some computer algebra [50–52]. Combining this identity with the properties (A.12) of the twist matrix, we conclude that the first term on the r.h.s. of (A.10) simplifies according to

$$\begin{aligned}\mathcal{X}^{(ab)cd,e}{}_f\,(U^{-1})_e{}^q \mathcal{K}_{[cd]}{}^m \partial_m U_q{}^f &= \frac{2}{5}\,\mathcal{X}^{(ab)g(d,e)}{}_g\,(U^{-1})_e{}^q \mathcal{K}_{[fd]}{}^m \partial_m U_q{}^f\\ &= \frac{1}{5}\,\sqrt{2}\,\mathcal{X}^{(ab)gd,e}{}_g\,\eta_{de}\,,\end{aligned} \tag{A.15}$$

such that its y-dependence reduces to the harmonics $\mathcal{Y}_a\mathcal{Y}_b$.

As a consequence, together with (A.12), we conclude that the penultimate term in (A.10) reduces to

$$-\frac{1}{10}\,\sqrt{2}\,\sqrt{|g|}\,\mathcal{Y}_a\mathcal{Y}_b\,\mathcal{X}^{(ab)cd,e}{}_f\,(U^{-1})_e{}^q \mathcal{K}_{[cd]}{}^l \partial_l U_q{}^f = -\frac{1}{25}\,\sqrt{|g|}\,\mathcal{Y}_a\mathcal{Y}_b\,\mathcal{X}^{(ab)gc,d}{}_g\,\eta_{cd}\,. \tag{A.16}$$

Together with (A.8), equation (A.1) then eventually reduces to

$$\begin{aligned}D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} &= -\frac{1}{80}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau} D_\lambda\left(M^{N\,ac}\,D^\lambda M_{bc,N}\right)\\ &\quad + \frac{1}{40}\,\mathcal{Y}_a\mathcal{Y}^b\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,F^{\kappa\lambda\,N}\left(M_{bc,N}F_{\kappa\lambda}{}^{ac} - \frac{1}{2}\,\sqrt{10}\,\varepsilon_{\alpha\beta}\,\eta_{db}\,M^{d\,\alpha}{}_N\,B_{\kappa\lambda}{}^{a\,\beta}\right)\\ &\quad + \frac{1}{100}\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\,\mathcal{Y}_a\mathcal{Y}^b\left(10\,M^{ac,fd} + \mathcal{X}^{(af)ec,d}{}_e\right)\eta_{cd}\eta_{bf}\\ &\quad + \frac{1}{32}\,\sqrt{2}\,\varepsilon_{abcdef}\,F_{[\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}A_{\tau]}{}^{ef} + \frac{1}{16}\,F_{[\mu\nu}{}^{ab}A_\rho{}^{cd}A_\sigma{}^{ef}A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh}\\ &\quad + \frac{1}{40}\,\sqrt{2}\,A_{[\mu}{}^{ab}A_\nu{}^{cd}A_\rho{}^{ef}A_\sigma{}^{gh}A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj}\,,\end{aligned} \tag{A.17}$$

such that the y-dependence of the entire equation organizes into the form (A.9). Now the x-dependent coefficient of the traceless combination $\left(\mathcal{Y}_a\mathcal{Y}_b - \frac{1}{6}\,\eta_{ab}\right)$ precisely reproduces the $D = 5$ scalar equations of motion (4.16). In particular, the third line of (A.17) coincides with

the SL(6) variation of the scalar potential (4.14). This match requires an additional non-trivial relations among the components of an $E_{6(6)}$ matrix (4.12)

$$\begin{aligned}
\eta_{ef} M_{d\alpha}{}^{h(a} M^{b)c,de} M^{f\alpha}{}_{ch} &= \eta_{ef} M_{g\alpha}{}^{de} M^{fc,g(a} M^{b)\alpha}{}_{cd} \,, \qquad \text{(A.18)}\\
\eta_{ef} M^{de,c(a} M^{b)\gamma,f\alpha} M_{d\alpha,c\gamma} &= 2\,\eta_{ef}\, M^{de,c(a} M^{b)h,fg} M_{dg,ch} + \eta_{ef} M_{d\alpha}{}^{h(a} M^{b)c,de} M^{f\alpha}{}_{ch} \,,
\end{aligned}$$

which can be proven similar to (A.13). From these it is straightforward to deduce that

$$\begin{aligned}
\mathcal{X}^{(af)ec,d}{}_{e} &= -\frac{4}{3} M^{de,c(a} M^{b)h,fg} M_{dg,ch} \eta_{ef} - \frac{1}{3}\,\eta_{ef} M^{de,c(a} M^{b)\gamma,f\alpha} M_{d\alpha,c\gamma}\\
&\quad + \frac{2}{3}\,\eta_{de} M^{cd,g(a} M^{b\alpha}{}_{cf} M_{g\alpha}{}^{ef} + \frac{2}{3}\,\eta_{ef} M^{de,c(a} M^{b)h}{}_{d\alpha} M^{f\alpha}{}_{ch} \,, \qquad \text{(A.19)}
\end{aligned}$$

thus matching the expression obtained from variation of the scalar potential in (4.16). As a consequence, the y-dependent part of equation (A.17) vanishes on-shell, such that the equation reduces to

$$\begin{aligned}
D_{[\mu}\Lambda_{\nu\rho\sigma\tau]} &= -\frac{1}{480}\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau} D_\lambda \left(M^{N\,ac}\, D^\lambda M_{ac,N} \right)\\
&\quad + \frac{1}{240}\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau}\, F^{\kappa\lambda\,N} \left(M_{ab,N} F_{\kappa\lambda}{}^{ab} - \frac{1}{2}\,\sqrt{10}\,\varepsilon_{\alpha\beta}\,\eta_{ab}\, M^{a\,\alpha}{}_N\, B_{\kappa\lambda}{}^{b\,\beta} \right)\\
&\quad + \frac{1}{600}\,\sqrt{|g|}\,\varepsilon_{\mu\nu\rho\sigma\tau} \left(10\, M^{ac,fd} + \mathcal{X}^{(af)ec,d}{}_{e} \right) \eta_{cd}\eta_{af}\\
&\quad + \frac{1}{32}\,\sqrt{2}\;\varepsilon_{abcdef}\, F_{[\mu\nu}{}^{ab} F_{\rho\sigma}{}^{cd} A_{\tau]}{}^{ef} + \frac{1}{16}\, F_{[\mu\nu}{}^{ab} A_\rho{}^{cd} A_\sigma{}^{ef} A_{\tau]}{}^{gh}\,\varepsilon_{abcdeh}\eta_{fh}\\
&\quad + \frac{1}{40}\,\sqrt{2}\;A_{[\mu}{}^{ab} A_\nu{}^{cd} A_\rho{}^{ef} A_\sigma{}^{gh} A_{\tau]}{}^{ij}\,\varepsilon_{abcegi}\,\eta_{df}\eta_{hj} \,. \qquad \text{(A.20)}
\end{aligned}$$

This equation can be integrated to yield the function $\Lambda_{\mu\nu\rho\rho}$. This yields the last missing part in the reduction ansatz of the IIB four form (5.60) and establishes the full type IIB self-duality equation.

References

[1] M. Duff, B. Nilsson, C. Pope, and N. Warner, On the consistency of the Kaluza-Klein ansatz, Phys.Lett. B149 (1984) 90.

[2] M. Cvetic, H. Lu, and C. N. Pope, Consistent Kaluza-Klein sphere reductions, Phys. Rev. D62 (2000) 064028, [hep-th/0003286].

[3] M. Gunaydin, L. J. Romans and N. P. Warner, Gauged $N = 8$ supergravity in five-dimensions, Phys. Lett. B 154 (1985) 268.

[4] M. Pernici, K. Pilch and P. van Nieuwenhuizen, Gauged $N = 8$, $D = 5$ supergravity, Nucl. Phys. B 259 (1985) 460.

[5] M. Günaydin, L. J. Romans, and N. P. Warner, Compact and noncompact gauged supergravity theories in five-dimensions, Nucl. Phys. B272 (1986) 598.

[6] B. de Wit and H. Nicolai, The consistency of the S^7 truncation in $D = 11$ supergravity, Nucl.Phys. B281 (1987) 211.

[7] H. Nastase, D. Vaman, and P. van Nieuwenhuizen, Consistency of the $\mathrm{AdS}_7 \times S^4$ reduction and the origin of self-duality in odd dimensions, Nucl. Phys. B581 (2000) 179–239, [hep-th/9911238].

[8] M. Cvetic, M. Duff, P. Hoxha, J. T. Liu, H. Lu, J. Lu, R. Martinez-Acosta, C. Pope, H. Sati, and T. A. Tran, Embedding AdS black holes in ten-dimensions and eleven-dimensions, Nucl.Phys. B558 (1999) 96–126, [hep-th/9903214].

[9] H. Lu, C. Pope, and T. A. Tran, Five-dimensional $N = 4$, $SU(2) \times U(1)$ gauged supergravity from type IIB, Phys.Lett. B475 (2000) 261–268, [hep-th/9909203].

[10] M. Cvetic, H. Lu, C. Pope, A. Sadrzadeh, and T. A. Tran, Consistent $SO(6)$ reduction of type IIB supergravity on S^5, Nucl.Phys. B586 (2000) 275–286, [hep-th/0003103].

[11] A. Khavaev, K. Pilch and N. P. Warner, New vacua of gauged N=8 supergravity in five-dimensions, Phys. Lett. B 487 (2000) 14 [hep-th/9812035].

[12] H. Nastase and D. Vaman, On the nonlinear KK reductions on spheres of supergravity theories, Nucl. Phys. B 583 (2000) 211 [hep-th/0002028].

[13] K. Pilch and N. P. Warner, $N = 2$ supersymmetric RG flows and the IIB dilaton, Nucl.Phys. B594 (2001) 209–228, [hep-th/0004063].

[14] D. Cassani, G. Dall'Agata and A. F. Faedo, Type IIB supergravity on squashed Sasaki-Einstein manifolds, JHEP 1005 (2010) 094, [1003.4283].

[15] J. T. Liu, P. Szepietowski and Z. Zhao, Consistent massive truncations of IIB supergravity on Sasaki-Einstein manifolds, Phys. Rev. D 81 (2010) 124028, [1003.5374].

[16] J. P. Gauntlett and O. Varela, Universal Kaluza-Klein reductions of type IIB to N=4 supergravity in five dimensions, JHEP 1006 (2010) 081, [1003.5642].

[17] K. Skenderis, M. Taylor, and D. Tsimpis, A consistent truncation of IIB supergravity on manifolds admitting a Sasaki-Einstein structure, JHEP 1006 (2010) 025, [1003.5657].

[18] A. Guarino, D. L. Jafferis and O. Varela, The String Origin of Dyonic $\mathcal{N} = 8$ Supergravity and its Simple Chern-Simons Duals, [1504.08009].

[19] O. Hohm and H. Samtleben, Consistent Kaluza-Klein truncations via exceptional field theory, JHEP 1501 (2015) 131, [1410.8145].

[20] C. M. Hull and N. P. Warner, Noncompact gaugings from higher dimensions, Class. Quant. Grav. 5 (1988) 1517.

[21] M. Cvetic, G. Gibbons, and C. Pope, Ghost free de Sitter supergravities as consistent reductions of string and M theory, Nucl.Phys. B708 (2005) 381–410, [hep-th/0401151].

[22] W. H. Baron and G. Dall'Agata, Uplifting non-compact gauged supergravities, JHEP 1502 (2015) 003, [1410.8823].

[23] O. Hohm and H. Samtleben, Exceptional form of $D = 11$ supergravity, Phys.Rev.Lett. 111 (2013) 231601, [1308.1673].

[24] O. Hohm and H. Samtleben, Exceptional field theory I: $E_{6(6)}$ covariant form of M-theory and type IIB, Phys.Rev. D89 (2014) 066016, [1312.0614].

[25] O. Hohm and H. Samtleben, Exceptional field theory II: $\mathrm{E}_{7(7)}$, Phys.Rev. D89 (2014) 066017, [1312.4542].

[26] O. Hohm and H. Samtleben, Exceptional field theory III: $E_{8(8)}$, Phys.Rev. D90 (2014) 066002, [1406.3348].

[27] D. S. Berman and M. J. Perry, Generalized geometry and M theory, JHEP 1106 (2011) 074, [1008.1763].

[28] A. Coimbra, C. Strickland-Constable, and D. Waldram, $E_{d(d)} \times \mathbb{R}^+$ generalised geometry, connections and M theory, JHEP 1402 (2014) 054, [1112.3989].

[29] G. Aldazabal, M. Graña, D. Marqués, and J. Rosabal, Extended geometry and gauged maximal supergravity, JHEP 1306 (2013) 046, [1302.5419].

[30] O. Hohm and Y.-N. Wang, Tensor hierarchy and generalized Cartan calculus in SL(3) $\times$ SL(2) exceptional field theory, JHEP 1504 (2015) 050, [1501.01600].

[31] W. Siegel, Superspace duality in low-energy superstrings, Phys.Rev. D48 (1993) 2826–2837, [hep-th/9305073].

[32] C. Hull and B. Zwiebach, Double field theory, JHEP 0909 (2009) 099, [0904.4664].

[33] O. Hohm, C. Hull, and B. Zwiebach, Background independent action for double field theory, JHEP 1007 (2010) 016, [1003.5027].

[34] O. Hohm, C. Hull, and B. Zwiebach, Generalized metric formulation of double field theory, JHEP 1008 (2010) 008, [1006.4823].

[35] M. Cvetic, G. Gibbons, H. Lu, and C. Pope, Consistent group and coset reductions of the bosonic string, Class.Quant.Grav. 20 (2003) 5161–5194, [hep-th/0306043].

[36] J. Scherk and J. H. Schwarz, How to get masses from extra dimensions, Nucl. Phys. B153 (1979) 61–88.

[37] D. S. Berman, M. Cederwall, A. Kleinschmidt, and D. C. Thompson, The gauge structure of generalised diffeomorphisms, JHEP 1301 (2013) 064, [1208.5884].

[38] A. Baguet, O. Hohm, and H. Samtleben, $\mathrm{E}_{6(6)}$ exceptional field theory: Review and embedding of type IIB, Contribution to the Proceedings of the Workshop on Quantum Fields and Strings, Corfu 2014, [1506.01065].

[39] K. Lee, C. Strickland-Constable, and D. Waldram, Spheres, generalised parallelisability and consistent truncations, 1401.3360.

[40] F. Ciceri, B. de Wit, and O. Varela, IIB supergravity and the $E_{6(6)}$ covariant vector-tensor hierarchy, JHEP 1504 (2015) 094, [1412.8297].

[41] B. de Wit, H. Samtleben, and M. Trigiante, The maximal $D = 5$ supergravities, Nucl. Phys. B716 (2005) 215–247, [hep-th/0412173].

[42] P. K. Townsend, K. Pilch, and P. van Nieuwenhuizen, Selfduality in odd dimensions, Phys. Lett. 136B (1984) 38.

[43] J. H. Schwarz and P. C. West, Symmetries and transformations of chiral $N = 2$ $D = 10$ supergravity, Phys. Lett. B126 (1983) 301.

[44] J. H. Schwarz, Covariant field equations of chiral $N = 2$, $D = 10$ supergravity, Nucl.Phys. B226 (1983) 269.

[45] P. S. Howe and P. C. West, The complete $N = 2$, $D = 10$ supergravity, Nucl. Phys. B238 (1984) 181.

[46] H. Godazgar, M. Godazgar, and H. Nicolai, Nonlinear Kaluza-Klein theory for dual fields, Phys.Rev. D88 (2013) 125002, [1309.0266].

[47] B. de Wit, H. Nicolai, and H. Samtleben, Gauged supergravities, tensor hierarchies, and M-theory, JHEP 0802 (2008) 044, [arXiv:0801.1294].

[48] M. Duff, B. Nilsson, and C. Pope, Kaluza-Klein supergravity, Phys.Rept. 130 (1986) 1–142.

[49] E. Musaev and H. Samtleben, Fermions and supersymmetry in $E_{6(6)}$ exceptional field theory, JHEP 1503 (2015) 027, [1412.7286].

[50] K. Peeters, A field-theory motivated approach to symbolic computer algebra, Comput. Phys. Commun. 176 (2007) 550–558, [cs/0608005].

[51] K. Peeters, Introducing Cadabra: A symbolic computer algebra system for field theory problems, hep-th/0701238.

[52] Wolfram Research, Inc., Mathematica. Champaign, Illinois, version 8.0 ed., 2010.