# Covariance in models of loop quantum gravity: Gowdy systems

Martin Bojowald and Suddhasattwa Brahma

Phys. Rev. D **92**, 065002 — Published 3 September 2015

DOI: 10.1103/PhysRevD.92.065002

# Covariance in models of loop quantum gravity: Gowdy systems

Martin Bojowald and Suddhasattwa Brahma

*Institute for Gravitation and the Cosmos, The Pennsylvania State University, 104 Davey Lab, University Park, PA 16802*

Recent results in the construction of anomaly-free models of loop quantum gravity have shown obstacles when local physical degrees of freedom are present. Here, a set of no-go properties is derived in polarized Gowdy models, raising the question whether these systems can be covariant beyond a background treatment. As a side product, it is shown that normal deformations in classical polarized Gowdy models can be Abelianized.

PACS numbers: 11.15.-q, 04.60.-m, 04.60.Pp

## I. INTRODUCTION

Covariance is important in cosmological models because it controls the form of partial differential equations for inhomogeneous modes and ensures consistency of the coupled set of equations for a smaller number of free fields. It considerably restricts possible choices of underlying theories, for instance of the dynamics of matter ingredients or higher-derivative corrections to Einstein's equation.

The latter are expected also in effective equations of canonical quantum gravity, but in such approaches covariance is not manifest. In proposed models of loop quantum gravity, as one class of rather widely studied examples, it is then not always clear whether covariance is realized, and to what detriment covariance might unwittingly be broken. The main example of potentially covariance-breaking effects is the replacement of connection components in Hamiltonians by holonomies, a widely studied procedure which captures one of the key ingredients of loop quantizations and gives rise to postulated physical implications such as bounded energy densities. In [1], a systematic analysis of covariance in spherically symmetric or black-hole models with modifications from loop quantum gravity has been started. Partial no-go results have been obtained for covariant holonomy-modified models with local matter degrees of freedom, and to date no such model is known to exist.

Here, we extend the same methods and results to polarized Gowdy models. Also in this context, partial no-go results will be obtained, of a form which resembles those found in spherically symmetric models and can therefore be taken as a sign of genericness. There seem to be obstacles to an implementation of covariant holonomy-modified models with local degrees of freedom, from matter or gravity. In a background treatment, local degrees of freedom can be coupled as inhomogeneous modes to a holonomy-modified homogeneous model. However, irrespective of whether back-reaction on the homogeneous background is included, non-trivial covariance conditions are present but have not been analyzed yet in existing constructions. We will comment on hybrid models [2–4] as one example. Our statements are about holonomy-modified models characteristic of loop quantum cosmology. They do not apply to Wheler–DeWitt type quantizations of Gowdy models as considered for instance in [5–14].

Covariance cannot be seen in homogeneous models, the traditional setting of loop quantum cosmology [15, 16]. At the level of effective equations, there are only ordinary differential equations which are not subject to additional consistency conditions from covariance. And also an equation for a wave function, although it may be a partial differential or difference equation, requires no such restrictions. Dynamical equations of homogeneous cosmological models can therefore be modified at will by any putative quantum effects, but not all versions can be minisuperspace reductions of covariant inhomogeneous models (or of a covariant full theory of modified or quantum gravity).

In this paper we consider polarized Gowdy systems [17] as a class of models with 1-dimensional spatial inhomogeneity and applications to cosmology. As in [1], the canonical definition of covariance we use for modified theories is based on the general form of this condition in classical models: Instead of considering transformations generated by Lie derivatives along space-time vector fields, one has such derivatives only for vector fields $M^a$ tangential to spatial hypersurfaces used for the canonical decomposition of fields. These spatial diffeomorphisms, acting on phase-space variables, are generated by the diffeomorphism constraint $D[N^a]$. For the remaining transformations it is sufficient to have a generator of normal deformations of spatial hypersurfaces, given by the Hamiltonian constraint $H[N]$, the spatial function $N$ determining the extent $Nn^a$ of the deformation along the normal vector field $n^a$. These generators have Poisson brackets

$$\{D[M_1^a], D[M_2^a]\} = D[\mathcal{L}_{M_1} M_2^a] \qquad (1)$$

$$\{H[N], D[M^a]\} = -H[\mathcal{L}_M N] \qquad (2)$$

$$\{H[N_1], H[N_2]\} = D[q^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)] \qquad (3)$$

with structure functions in the last line, given by the inverse spatial metric $q^{ab}$ [18, 19].

A modified or quantized canonical theory must have at least a classical limit in which (1)–(3) are realized. For non-classical solutions, the brackets may be subject to quantum corrections but must still close for an anomaly-free theory: Since the constraints generate gauge transformations, there must be analogs of the classical constraints $D[M^a]$ and $H[N]$ with brackets which are closed under all circumstances (not just in the classical limit). As discussed in [1], there are therefore two conditions for a covariant theory: (i) Anomaly-free gauge generators and (ii) a classical limit in which the hypersurface-deformation brackets (1)–(3) are obtained. As shown in [1], building on results of [20], condition (ii) is not necessarily a consequence of condition (i). (Conditions (i) and (ii) have been shown to be satisfied in suitable holonomy-modified models of cosmological perturbations [21, 22]. Here, we analyze the question of covariance for inhomogeneity which is not perturbative, but restricted by some symmetry and polarization conditions. The situation we find in this context, regarding holonomy modifications, appears to be qualitatively different from the case of perturbative inhomogeneity.)

An important part of the conditions for covariance is that they refer to the off-shell brackets when the constraints are not necessarily zero. This feature is analogous to the usual space-time definition of a covariant theory as one with a Lagrangian covariant under tensor transformations. The conceptual reason for the prominence of off-shell structures is the classical picture of space-time as a background on which different kinds of matter fields can be put. Even if back-reaction is included and one does not restrict equations to those for a field on a fixed background, one thinks of space-time as an independent ingredient which is covariant in its own right, irrespective of the matter fields coupled to it. After all, the Einstein tensor of any space-time, independently of solutions to field equations or the inclusion of back-reaction, obeys the contracted Bianchi identity, which in canonical form is equivalent to a version of (1)–(3) [23]. For a consistent matter coupling, one therefore requires the local conservation law for the matter stress-energy tensor, again independently of solutions to field equations. Also the local conservation law is equivalent to a version of (1)–(3) for matter Hamiltonians [24]. In both cases, the form of off-shell brackets is crucial, which we will analyze for modified Gowdy models in the present paper.

## II. MODIFIED THEORIES WITH LOCAL DEGREES OF FREEDOM?

Since the algebraic structure of modified Gowdy models is closely related to the one of spherically symmetric models discussed in [1], we will begin with a brief review of these existing results.

### A. Spherical symmetry

Using triad variables $E^x$ and $E^\varphi$ with canonically conjugate extrinsic-curvature components $K_x$ and $K_\varphi$, the gravitational contribution to the spherically symmetric Hamiltonian constraint is

$$H[N] = -\frac{1}{2G}\int \mathrm{d}x N(x)\left(|E^x|^{-\frac{1}{2}}E^\varphi K_\varphi^2 + 2|E^x|^{\frac{1}{2}}K_\varphi K_x + |E^x|^{-\frac{1}{2}}(1-\Gamma_\varphi^2)E^\varphi + 2\Gamma_\varphi'|E^x|^{\frac{1}{2}}\right) \tag{4}$$

where $\Gamma_\varphi = -(E^x)'/2E^\varphi$ (see [25, 26]). If one adds to this the matter Hamiltonian, for instance

$$H_\phi[N] = \frac{1}{8G}\int \mathrm{d}x N(x)\frac{1}{\sqrt{|E^x|}E^\varphi}\left(P_\phi^2 + 4(E^x)^2(\phi')^2\right) \tag{5}$$

for a scalar field $\phi$ with momentum $P_\phi$, the hypersurface-deformation brackets are realized in combination with the diffeomorphism constraint

$$D[M] = \frac{1}{G}\int \mathrm{d}x M(x)\left(-\frac{1}{2}(E^x)'K_x + K_\varphi' E^\varphi + GP_\phi\phi'\right)\,. \tag{6}$$

Instead of the full spatial metric $q^{ab}$, the structure functions are given by the radial component $|E^x|/(E^\varphi)^2$ of a spherically symmetric inverse spatial metric.

In order to eliminate the structure functions, [20] introduced a linear combination of the constraints so that the normal part of hypersurface deformations is replaced by an Abelian bracket. In this process, $H[N]$ is replaced by a new constraint

$$C[N] \;=\; \frac{1}{G}\int \mathrm{d}x N(x)\Bigg(-\frac{1}{2}\frac{(E^x)'}{\sqrt{|E^x|}}(1+K_\varphi^2) - 2\sqrt{|E^x|}K_\varphi K_\varphi' \tag{7}$$

$$+\frac{(E^x)'}{8\sqrt{|E^x|}(E^\varphi)^2}\left(4E^x(E^x)''+((E^x)')^2\right)-\frac{1}{2}\frac{((E^x)')^2\sqrt{|E^x|}(E^\varphi)'}{(E^\varphi)^3}$$
$$+2\pi G\frac{(E^x)'}{\sqrt{|E^x|}(E^\varphi)^2}\left(P_\phi^2+(E^x)^2(\phi')^2\right)-8\pi G\sqrt{|E^x|}\frac{K_\varphi}{E^\varphi}P_\phi\phi'\Bigg).$$

While the pair $(C[N], D[M])$ does not obey the hypersurface-deformation brackets (1)–(3), it has a reduced phase space equivalent to the one of the original system. Quantizing the partially Abelianized system should be easier, as proposed in [20] in combination with a background treatment.

As part of the loop quantization performed in [20], one modifies the dependence of (7) on $K_\varphi$ by replacing it with some bounded function $f(K_\varphi)$ in order to model the appearance of holonomies in loop quantum gravity. However, as there are three different terms in (7) depending on $K_\varphi$, there could in general be three replacement functions which need not be equal but should be related in some way for a consistent theory in which the brackets still close and implement covariance. In [20], this question has been circumvented by the background treatment in which one first considers only the gravitational part of $C[N]$, which happens to be a total derivative. Upon integrating by parts, there is only one term depending on $K_\varphi$, which can easily be modified by a single function $f(K_\varphi)$ while keeping the constraint bracket Abelian.

However, as shown in [1], the modification is consistent with covariance only if the different $K_\varphi$-dependent terms in the original constraint are modified in strictly related ways, of a form equivalent to what had been found earlier by effective methods [27, 28]: The gravitational part of the modified Hamiltonian constraint then has to be of the form

$$\begin{aligned} H[N] \;=\; &-\frac{1}{2G}\int \mathrm{d}x N(x)\Big(|E^x|^{-\frac{1}{2}}E^\varphi f_1(K_\varphi)+2|E^x|^{\frac{1}{2}}f_2(K_\varphi)K_x \\ &+|E^x|^{-\frac{1}{2}}(1-\Gamma_\varphi^2)E^\varphi+2\Gamma_\varphi'|E^x|^{\frac{1}{2}}\Big) \end{aligned} \tag{8}$$

with

$$2f_2=\frac{\mathrm{d}f_1}{\mathrm{d}K_\varphi}\,. \tag{9}$$

As a consequence, the hypersurface-deformation brackets are modified at large curvature and show signature change [29–31]: The classical structure function is multiplied with

$$\beta=\frac{\mathrm{d}f_2}{\mathrm{d}K_\varphi}=\frac{1}{2}\frac{\mathrm{d}^2f_1}{\mathrm{d}K_\varphi^2} \tag{10}$$

which is negative around a local maximum of the modification function $f_1(K_\varphi)$.

Moreover, while the classical system is still Abelian in the presence of a non-zero matter Hamiltonian, no consistent modification has been found. It is therefore unclear whether modified combined systems of gravity and matter can be covariant. We now turn to Gowdy models in order to test whether the problem rests with the form of matter terms or is implied by the general presence of local degrees of freedom.

### B. Polarized Gowdy models

In contrast to spherically symmetric models, polarized Gowdy models have local physical degrees of freedom even if there is no matter. At the kinematical level, on which off-shell questions about constraints are addressed, the local degree of freedom is included by an additional canonical pair of fields. Nevertheless, the structure of the constraints and their algebraic properties are closely related to those of spherically symmetric models, so that a comparison can easily be done and is quite instructive.

#### 1. *Variables*

In Gowdy models, the inhomogeneous coordinate is traditionally called $\theta$, while $x$, used in spherically symmetric models for the radial coordinate, is part of a pair $(x, y)$ of coordinates along two independent homogeneous directions. In a real connection formulation [32] (see [9] for complex variables), there are three triad variables $(\epsilon, E^x, E^y)$ and canonical momenta $(\mathcal{A}, K_x, K_y)$. They appear in the diffeomorphism constraint in standard form

$$D[N^\theta]=\frac{1}{8\pi G}\int \mathrm{d}\theta N^\theta(\theta)\left(K_x'E^x+K_y'E^y-\varepsilon'\mathcal{A}\right) \tag{11}$$

while the Hamiltonian constraint is

$$
\begin{aligned}
H[N] \quad &= \frac{-1}{8\pi G}\int \mathrm{d}\theta N(\theta)\Bigg[f(K_x,K_y)(E^x)^{1/2}(E^y)^{1/2}\varepsilon^{-1/2}+g_1(K_x,K_y)\mathcal{A}(E^x)^{1/2}(E^y)^{-1/2}\varepsilon^{1/2}\\
&+g_2(K_x,K_y)\mathcal{A}(E^x)^{-1/2}(E^y)^{1/2}\varepsilon^{1/2}-\frac{1}{4}(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{-1/2}(\varepsilon')^2\\
&-\frac{1}{4}(E^x)^{-1/2}(E^y)^{-5/2}\varepsilon^{3/2}(E^{y\prime})^2-\frac{1}{4}(E^x)^{-5/2}(E^y)^{-1/2}\varepsilon^{3/2}(E^{x\prime})^2\\
&+\frac{1}{2}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}(E^{x\prime})(E^{y\prime})+\frac{1}{2}(E^x)^{-3/2}(E^y)^{-1/2}\varepsilon^{1/2}(E^{x\prime})\varepsilon'\\
&+\frac{1}{2}(E^x)^{-1/2}(E^y)^{-3/2}\varepsilon^{1/2}(E^{y\prime})\varepsilon'-(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{1/2}\varepsilon''\Bigg]\,. 
\end{aligned} \tag{12}
$$

Classically, $f(K_x,K_y)=K_xK_y$, $g_1(K_x,K_y)=K_x$ and $g_2(K_x,K_y)=K_y$ but as before, the dependence may be modified based on quantum-geometry effects such as the use of holonomies in loop quantum gravity. The classical structure function in the bracket of two normal deformations is $\epsilon^2/E^xE^y$.

### 2. *Structure of modification functions*

The question of consistent deformations of the classical brackets can be split in two: (a) What are the conditions on modification functions $f$, $g_1$ and $g_2$ for the brackets to be closed? And (b), what are the possible modifications of the classical structure function? In order to address (b), (a) must be solved since meaningful structure functions require a consistent set of brackets. However, at a purely formal level one may analyze (b) without first solving (a), in order to study possible features of interest in deformations of the brackets. The main effect seen in this way is signature change [29–31], given by a change of sign of the structure function, which would always be positive in a classical Lorentzian theory. In the first part of this subsection, we analyze (b) for Gowdy models, postponing detailed derivations of Poisson brackets to the subsequent consideration of (a).

*a. Deformations and the ubiquity of signature change.* From the relations to be presented soon, it follows that an anomaly-free modification of the Hamiltonian constraint (12) requires the following equation to hold for all values of the canonical fields: We must have

$$
\begin{aligned}
&\left[\frac{1}{2}(E^y)^{-2}\varepsilon E^{y\prime}-\frac{1}{2}(E^y)^{-1}\varepsilon'-\frac{1}{2}(E^x)^{-1}(E^y)^{-1}\varepsilon E^{x\prime}\right](f_{,K_y}-g_1)\\
&+\left[\frac{1}{2}(E^x)^{-2}\varepsilon E^{x\prime}-\frac{1}{2}(E^x)^{-1}\varepsilon'-\frac{1}{2}(E^x)^{-1}(E^y)^{-1}\varepsilon E^{y\prime}\right](f_{,K_x}-g_2)\\
&+\left[\frac{1}{2}\mathcal{A}(E^x)^{-2}(E^y)^{-1}\varepsilon^2E^{x\prime}-\frac{1}{2}\mathcal{A}(E^x)^{-1}(E^y)^{-2}\varepsilon^2E^{y\prime}\right](g_{1,K_x}-g_{2,K_y})\\
&+\frac{1}{2}\mathcal{A}\varepsilon^2\left[\frac{E^{x\prime}}{E^x}-\frac{E^{y\prime}}{E^y}\right]\left[\frac{g_{2,K_x}}{E_x^2}-\frac{g_{1,K_y}}{E_y^2}\right]=0
\end{aligned} \tag{13}
$$

for all terms in the $\{H,H\}$-bracket that cannot contribute to a diffeomorphism constraint to cancel out. (As usual, commas in subscripts indicate partial derivatives by the appended variable(s).) All lines must vanish individually since their coefficients are composed of different functions of the canonical variables and their derivatives. (Otherwise, additional constraints on the phase-space variables would be imposed.) Requiring the first two lines in (13) to be zero gives two conditions,

$$g_1(K_x,K_y)=\frac{\partial f(K_x,K_y)}{\partial K_y}$$

$$g_2(K_x,K_y) = \frac{\partial f(K_x,K_y)}{\partial K_x}\,, \tag{14}$$

for two of the three free modification functions. These conditions automatically make the third line in (13) vanish, owing to the equality of mixed partial derivatives. The last line in (13) is zero if and only if

$$\frac{\partial^2 f}{\partial K_x^2} = \frac{1}{(E^x/E^y)^2}\frac{\partial^2 f}{\partial K_y^2}\,, \tag{15}$$

providing some kind of wave equation for the remaining modification function.

At this stage, we note the first important difference to the spherically symmetric case where it is possible to have modification functions depending only on the curvature variables. In polarized Gowdy models, by contrast, any function $f$, which solves condition (15) and differs from the classical limit by cubic or higher-order terms in curvature variables, must also depend on some of the triad variables. (Spherically symmetric models can be seen as reductions of polarized Gowdy models with $E^x = E^y$, so that the triad dependence disappears from (15).) We must therefore go back and rederive brackets of (12), because in (13) we have assumed that $f$ depends only on curvature variables.

However, we may proceed further without rederiving the more-complete brackets, addressing question (b) without solving problem (a) introduced in the beginning discussion of this subsection. If (13) is assumed to hold, the remaining terms of the $\{H,H\}$-bracket, containing factors that appear in the diffeomorphism constraint, are

$$\begin{aligned}
&-\frac{1}{8\pi G}\int \mathrm{d}\theta (MN'-NM')\frac{\varepsilon^2\mathcal{A}\varepsilon'}{E^xE^y}\left[\frac{\partial^2 f}{\partial K_x\partial K_y}+\frac{1}{2}\frac{\partial^2 f}{\partial K_y^2}\frac{E^x}{E^y}+\frac{1}{2}\frac{\partial^2 f}{\partial K_x^2}\frac{E^y}{E^x}\right] \\
&+\frac{1}{8\pi G}\int \mathrm{d}\theta (MN'-NM')\frac{\varepsilon^2}{E^xE^y}\left[\frac{\partial^2 f}{\partial K_x\partial K_y}(K_x'E^x+K_y'E^y)+\frac{\partial^2 f}{\partial K_y^2}E^xK_y'+\frac{\partial^2 f}{\partial K_x^2}E^yK_x'\right],
\end{aligned} \tag{16}$$

where we have already used condition (14) to simplify the terms. If we insert (15) in (16), we can simplify the structure function in front of terms contributing to the diffeomorphism constraint. The resulting expression is

$$\frac{1}{8\pi G}\int \mathrm{d}\theta (MN'-NM')\frac{\varepsilon^2}{E^xE^y}\left[\frac{\partial^2 f}{\partial K_x\partial K_y}+\frac{\partial^2 f}{\partial K_y^2}\frac{E^x}{E^y}\right]\left[K_x'E^x+K_y'E^y-\mathcal{A}\varepsilon'\right] \tag{17}$$

where, in addition to the classical structure function $\varepsilon^2/(E^xE^y)$, we have a deformation function

$$\beta = \frac{\partial^2 f}{\partial K_x\partial K_y}+\frac{\partial^2 f}{\partial K_y^2}\frac{E^x}{E^y}\,. \tag{18}$$

Although this function is more complicated than its spherically symmetric analog (10), it is still possible to show that for any modification function $f$ with a local maximum, the modified structure function has negative values, $\beta < 0$. In order to do so, we solve (15) by requiring $f$ to have the form $f(K_x,K_y,E^x,E^y) = f_1(E^xK_x+E^yK_y)+f_2(E^xK_x-E^yK_y)$ with two free functions $f_1$ and $f_2$ of one variable. The positions of local maxima of $f$ are determined by properties of the following derivatives:

$$\begin{aligned}
f_{,K_xK_x} &= (E^x)^2\left[\ddot f_1+\ddot f_2\right] \\
f_{,K_yK_y} &= (E^y)^2\left[\ddot f_1+\ddot f_2\right] \\
f_{,K_xK_y} &= E^xE^y\left[\ddot f_1-\ddot f_2\right]\,,
\end{aligned} \tag{19}$$

where a dot over a function denotes a derivative with respect to its argument. At a local maximum, the standard conditions $f_{,K_xK_x}<0$ and $f_{,K_xK_x}f_{,K_yK_y}-(f_{,K_xK_y})^2>0$ imply

$$\ddot f_1+\ddot f_2<0 \quad \text{and} \quad \ddot f_1\ddot f_2>0\,. \tag{20}$$

Therefore, both $\ddot f_1$ and $\ddot f_2$ have to be negative.

The deformation function $\beta$ in (18) is proportional to the first of these expressions,

$$\beta = 2E^xE^y\ddot f_1, \tag{21}$$

so that it turns negative around a local maximum of $f$. The formal aspects of deformation functions, disregarding full anomaly-freedom for now, is therefore in complete agreement with previous investigations in spherically symmetric models [27] and for cosmological perturbations [22]. (See also [33].) Around local maxima of modification functions, the modified structure function in the bracket of normal hypersurface deformations is negative, as it is for Euclidean space. Hyperbolic wave equations are then replaced by elliptic equations which do not allow deterministic propagation through such a region, typically at large curvature. Implications have been studied for cosmological [31] and black-hole models [34].

*b. Closure?* We have seen that we have to generalize the dependence of modification functions on the canonical variables in order to solve part (a) of the question of consistent deformations of the bracket of Hamiltonian constraints. The class of solutions we will find has the classical dependence on curvature variables, so that holonomy modifications are ruled out in modified models as assumed here.

Our more-general ansatz is

$$
\begin{aligned}
H[N] = \frac{-1}{8\pi G}\int \mathrm{d}\theta N(\theta)\Bigg[ & f(K_x,K_y,E^x,E^y,\varepsilon)+g(K_x,K_y,E^x,E^y,\varepsilon)\mathcal{A} \\
& -\frac{1}{4}(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{-1/2}(\varepsilon')^2-\frac{1}{4}(E^x)^{-1/2}(E^y)^{-5/2}\varepsilon^{3/2}(E^{y\prime})^2 \\
& -\frac{1}{4}(E^x)^{-5/2}(E^y)^{-1/2}\varepsilon^{3/2}(E^{x\prime})^2+\frac{1}{2}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}(E^{x\prime})(E^{y\prime}) \\
& +\frac{1}{2}(E^x)^{-3/2}(E^y)^{-1/2}\varepsilon^{1/2}(E^{x\prime})\varepsilon'+\frac{1}{2}(E^x)^{-1/2}(E^y)^{-3/2}\varepsilon^{1/2}(E^{y\prime})\varepsilon' \\
& -(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{1/2}\varepsilon''\Bigg]\,. 
\end{aligned}
\tag{22}
$$

At this stage, we only assume that the modified Hamiltonian constraint is linear in $\mathcal{A}$, motivated by the result that a non-linear dependence on the connection component in the inhomogeneous direction is difficult to achieve in spherically symmetric models even if a derivative expansion is allowed for [28]. We are therefore considering only point-wise holonomy corrections with angular curvature or connection components, setting aside the question of possible non-local modifications that holonomies in the inhomogeneous direction are expected to entail.

As before, only the first and second terms give non-zero contributions to the $\{H,H\}$-bracket. Providing more details than before, we list the integrands of all of them, not writing the common factor of smearing functions $(M'N-N'M)$. The first term gives rise to

$$
\begin{aligned}
&\frac{1}{2}f_{,K_y}(E^x)^{-1/2}(E^y)^{-5/2}\varepsilon^{3/2}E^{y\prime}+\frac{1}{2}f_{,K_x}(E^x)^{-5/2}(E^y)^{-1/2}\varepsilon^{3/2}E^{x\prime} \\
&-\frac{1}{2}f_{,K_y}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}E^{x\prime}-\frac{1}{2}f_{,K_x}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}E^{y\prime} \\
&-\frac{1}{2}f_{,K_x}(E^x)^{-3/2}(E^y)^{-1/2}\varepsilon^{1/2}\varepsilon'-\frac{1}{2}f_{,K_y}(E^x)^{-1/2}(E^y)^{-3/2}\varepsilon^{1/2}\varepsilon'\,,
\end{aligned}
\tag{23}
$$

whereas the various commutators with the second term yield

$$
\begin{aligned}
&\frac{1}{2}g(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{-1/2}\varepsilon'+\frac{1}{2}g_{,K_y}\mathcal{A}(E^x)^{-1/2}(E^y)^{-5/2}\varepsilon^{3/2}E^{y\prime} \\
&+\frac{1}{2}g_{,K_x}\mathcal{A}(E^x)^{-5/2}(E^y)^{-1/2}\varepsilon^{3/2}E^{x\prime}-\frac{1}{2}g_{,K_y}\mathcal{A}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}E^{x\prime} \\
&-\frac{1}{2}g_{,K_x}\mathcal{A}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}E^{y\prime}-\frac{1}{2}g_{,K_x}\mathcal{A}(E^x)^{-3/2}(E^y)^{-1/2}\varepsilon^{1/2}\varepsilon' \\
&-\frac{1}{2}g(E^x)^{-3/2}(E^y)^{-1/2}\varepsilon^{1/2}E^{x\prime}-\frac{1}{2}g(E^x)^{-1/2}(E^y)^{-3/2}\varepsilon^{1/2}E^{y\prime} \\
&-\frac{1}{2}g_{,K_y}\mathcal{A}(E^x)^{-1/2}(E^y)^{-3/2}\varepsilon^{1/2}\varepsilon'-\frac{1}{2}g(E^x)^{-3/2}(E^y)^{-1/2}\varepsilon^{1/2}E^{x\prime} \\
&-\frac{1}{2}g(E^x)^{-1/2}(E^y)^{-3/2}\varepsilon^{1/2}E^{y\prime}+\frac{1}{2}g(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{-1/2}\varepsilon' \\
&+g(E^x)^{-3/2}(E^y)^{-1/2}\varepsilon^{1/2}E^{x\prime}+g(E^x)^{-1/2}(E^y)^{-3/2}\varepsilon^{1/2}E^{y\prime}-g(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{-1/2}\varepsilon' \\
&+\left[g_{,K_x}K_x'+g_{,K_y}K_y'+g_{,E^x}E^{x\prime}+g_{,E^y}E^{y\prime}+g_{,\varepsilon}\varepsilon'\right](E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{1/2}\,.
\end{aligned}
\tag{24}
$$

Several of these terms cancel each other so that the combined expression can be simplified. For the bracket to be proportional to the diffeomorphism constraint, terms in (23) and (24) not proportional to $\mathcal{A}\varepsilon'$, $K_x'$ or $K_y'$ must vanish:

$$
E^{y\prime}\left[\frac{1}{2}f_{,K_y}(E^x)^{-1/2}(E^y)^{-5/2}\varepsilon^{3/2}-\frac{1}{2}f_{,K_x}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}+g_{,E^y}(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{1/2}\right]
$$

$$
\begin{aligned}
&+E^{x\prime}\left[\frac{1}{2}f_{,K_x}(E^x)^{-5/2}(E^y)^{-1/2}\varepsilon^{3/2}-\frac{1}{2}f_{,K_y}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}+g_{,E^x}(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{1/2}\right]\\
&-\varepsilon'\left[\frac{1}{2}f_{,K_x}(E^x)^{-3/2}(E^y)^{-1/2}\varepsilon^{1/2}\frac{1}{2}f_{,K_y}(E^x)^{-1/2}(E^y)^{-3/2}\varepsilon^{1/2}+g_{,\varepsilon}(E^x)^{-1/2}(E^y)^{-1/2}\varepsilon^{1/2}\right]\\
&+\mathcal{A}E^{y\prime}\left[g_{,K_y}(E^x)^{-1/2}(E^y)^{-5/2}\varepsilon^{3/2}+g_{,K_x}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}\right]\\
&+\mathcal{A}E^{x\prime}\left[g_{,K_x}(E^x)^{-5/2}(E^y)^{-1/2}\varepsilon^{3/2}+g_{,K_y}(E^x)^{-3/2}(E^y)^{-3/2}\varepsilon^{3/2}\right]=0\,. \qquad (25)
\end{aligned}
$$

As before, all lines in (25) must vanish individually when they have different coefficients. We obtain four independent conditions on the correction functions:

$$
\begin{aligned}
\frac{\partial g}{\partial\varepsilon} &= \frac{1}{E^x}\frac{\partial f}{\partial K_x}+\frac{1}{E^y}\frac{\partial f}{\partial K_y} && (26)\\
\frac{\partial g}{\partial K_x} &= \frac{1}{E^y/E^x}\frac{\partial g}{\partial K_y} && (27)\\
\frac{\partial g}{\partial E^y} &= -\frac{\varepsilon}{(E^y)^2}\frac{\partial f}{\partial K_y}+\frac{\varepsilon}{E^xE^y}\frac{\partial f}{\partial K_x} && (28)\\
\frac{\partial g}{\partial E^x} &= -\frac{\varepsilon}{(E^x)^2}\frac{\partial f}{\partial K_x}+\frac{\varepsilon}{E^xE^y}\frac{\partial f}{\partial K_y}\,. && (29)
\end{aligned}
$$

From (27), $g$ has to be of the form

$$g(K_x,K_y,E^x,E^y,\varepsilon)=g_1(E^xK_x+E^yK_y)\,g_2(E^x,E^y,\varepsilon) \qquad (30)$$

Using this form of the correction function in (28) and (29), respectively, gives

$$
\begin{aligned}
-\frac{1}{E^y}\frac{\partial f}{\partial K_y}+\frac{1}{E^x}\frac{\partial f}{\partial K_x} &= \frac{E^y}{\varepsilon}\left[g_1\frac{\partial g_2}{\partial E^y}+K_yg_2\dot{g}_1\right] && (31)\\
\frac{1}{E^y}\frac{\partial f}{\partial K_y}-\frac{1}{E^x}\frac{\partial f}{\partial K_x} &= \frac{E^x}{\varepsilon}\left[g_1\frac{\partial g_2}{\partial E^x}+K_xg_2\dot{g}_1\right] && (32)
\end{aligned}
$$

Combining (31) and (32),

$$g_1\left[E^y\frac{\partial g_2}{\partial E^y}+E^x\frac{\partial g_2}{\partial E^x}\right]+g_2\dot{g}_1\left[E^xK_x+E^yK_y\right] = 0 \qquad (33)$$

We can try to solve the final differential equation by employing separation of variables. Abbreviating $\Theta:=E^xK_x+E^yK_y$, we have

$$\frac{1}{g_2}\left[E^y\frac{\partial g_2}{\partial E^y}+E^x\frac{\partial g_2}{\partial E^x}\right]=-\frac{\Theta}{g_1}\frac{\mathrm{d}g_1}{\mathrm{d}\Theta}\,. \qquad (34)$$

The left-hand side is a function of the triad components alone whereas the right-hand side depends on a particular combination of triads and connection coefficients. Thus, they must both be equal to some constant, say, $c$. The functions $g_1,g_2$ then satisfy the differential equations

$$
\begin{aligned}
\frac{\mathrm{d}g_1}{g_1} &= c\frac{\mathrm{d}\Theta}{\Theta} && (35)\\
E^y\frac{\partial g_2}{\partial E^y}+E^x\frac{\partial g_2}{\partial E^x} &= -cg_2 && (36)
\end{aligned}
$$

with solutions

$$
\begin{aligned}
g_1\left(E^xK_x+E^yK_y\right) &= c_1\left[E^xK_x+E^yK_y\right]^c\,, && (37)\\
g_2\left(E^x,E^y,\varepsilon\right) &= c_2(\varepsilon,E^x/E^y)\left(E^xE^y\right)^{-c/2} && (38)
\end{aligned}
$$

Here, $c_1$ is an integration constant while $c_2$ can be a function of $\varepsilon$ and the ratio $E^x/E^y$ at most. If $c_2$ is not constant, we have a version of inverse-triad corrections with a restriction on the triad dependence analogous to what has been found in spherically symmetric models [27]. (The two expressions $\epsilon$ and $E^x/E^y$ or functions of them are the only combinations of triad components without density weight.) The curvature dependence is not fully determined yet, but from (37) it could only be of power-law form, already ruling out the usual choice of periodic holonomy-modification functions. We will now show that only the classical case $c=1$ of a linear dependence of $g_1$ on curvature components is allowed.

We insert our solution for the correction function $g$ in (26) and obtain

$$\frac{1}{E^x}\frac{\partial f}{\partial K_x}+\frac{1}{E^y}\frac{\partial f}{\partial K_y}=c_1\frac{\partial c_2}{\partial\varepsilon}\left[\sqrt{\frac{E^x}{E^y}}K_x+\sqrt{\frac{E^y}{E^x}}K_y\right]^c \qquad (39)$$

Doing the same in (31) yields

$$
\begin{aligned}
\frac{1}{E^x}\frac{\partial f}{\partial K_x}-\frac{1}{E^y}\frac{\partial f}{\partial K_y} &= \frac{cc_1c_2}{2\varepsilon}\left[\sqrt{\frac{E^x}{E^y}}K_x+\sqrt{\frac{E^y}{E^x}}K_y\right]^c && (40)\\
&\quad\times\left[\frac{E^yK_y-E^xK_x}{E^xK_x+E^yK_y}-\frac{2}{c}\frac{E^x}{(E^y)^2}\frac{1}{c_2}\frac{\partial c_2}{\partial(E^x/E^y)}\right]\,.
\end{aligned}
$$

From these two relations, we identify the partial derivatives

$$\begin{aligned}\frac{\partial f}{\partial K_x} &= \frac{c_1 E^x}{2}\left[\sqrt{\frac{E^x}{E^y}}K_x+\sqrt{\frac{E^y}{E^x}}K_y\right]^c \\ &\qquad\times\left[\frac{\partial c_2}{\partial\varepsilon}+\frac{cc_2}{2\varepsilon}\frac{E^yK_y-E^xK_x}{E^xK_x+E^yK_y}-\frac{c_2}{\varepsilon}\frac{E^x}{(E^y)^2}\frac{1}{c_2}\frac{\partial c_2}{\partial(E^x/E^y)}\right], \quad (41)\\ \frac{\partial f}{\partial K_y} &= \frac{c_1 E^y}{2}\left[\sqrt{\frac{E^x}{E^y}}K_x+\sqrt{\frac{E^y}{E^x}}K_y\right]^c \\ &\qquad\times\left[\frac{\partial c_2}{\partial\varepsilon}+\frac{cc_2}{2\varepsilon}\frac{E^xK_x-E^yK_y}{E^xK_x+E^yK_y}+\frac{c_2}{\varepsilon}\frac{E^x}{(E^y)^2}\frac{1}{c_2}\frac{\partial c_2}{\partial(E^x/E^y)}\right]. \quad (42)\end{aligned}$$

At this point, we still have a consistent system of equations. We can calculate the left-hand side of (32) using the expressions above in (41) and (42) and verify that it gives the same result as the right-hand side of (32).

We now calculate the second-order mixed partial derivative by operating on (41) with $\partial/\partial K_y$:

$$\begin{aligned}\frac{\partial^2 f}{\partial K_y\partial K_x} = \frac{cc_1}{2}\Bigg[&\left\{\sqrt{\frac{E^x}{E^y}}K_x+\sqrt{\frac{E^y}{E^x}}K_y\right\}^{c-1}\sqrt{E^xE^y}\left\{\frac{\partial c_2}{\partial\varepsilon}-\frac{1}{\epsilon}\frac{E^x}{(E^y)^2}\frac{\partial c_2}{\partial(E^x/E^y)}\right.\\ &\left.+\frac{cc_2}{2\varepsilon}\left[\frac{-E^xK_x+E^yK_y}{E^xK_x+E^yK_y}\right]\right\}\\ &+\frac{c_2E^x}{\varepsilon}\left\{\sqrt{\frac{E^x}{E^y}}K_x+\sqrt{\frac{E^y}{E^x}}K_y\right\}^c\left\{\frac{E^yE^xK_x}{(E^xK_x+E^yK_y)^2}\right\}\Bigg]. \quad (43)\end{aligned}$$

We operate on (42) with $\partial/\partial K_x$ to obtain

$$\begin{aligned}\frac{\partial^2 f}{\partial K_x\partial K_y} = \frac{cc_1}{2}\Bigg[&\left\{\sqrt{\frac{E^x}{E^y}}K_x+\sqrt{\frac{E^y}{E^x}}K_y\right\}^{c-1}\sqrt{E^xE^y}\left\{\frac{\partial c_2}{\partial\varepsilon}+\frac{1}{\epsilon}\frac{E^x}{(E^y)^2}\frac{\partial c_2}{\partial(E^x/E^y)}\right.\\ &\left.+\frac{cc_2}{2\varepsilon}\left[\frac{E^xK_x-E^yK_y}{E^xK_x+E^yK_y}\right]\right\}\\ &+\frac{c_2E^y}{\varepsilon}\left\{\sqrt{\frac{E^x}{E^y}}K_x+\sqrt{\frac{E^y}{E^x}}K_y\right\}^c\left\{\frac{E^xE^yK_y}{(E^xK_x+E^yK_y)^2}\right\}\Bigg]. \quad (44)\end{aligned}$$

Requiring that these two quantities must be equal to each other results in one fixed value of the constant, $c = 1$. (Also, $\partial c_2/\partial(E^x/E^y) = 0$, so that $c_2$ depends only on $\epsilon$.)

Therefore, all modification functions that are consistent with anomaly freedom have the classical dependence on curvature variables. It is impossible to include holonomy modifications for these models with the parameterization used. The only possibility left is to include holonomy-correction functions modifying the dependence on all three variables, $K_x, K_y$ and $\mathcal{A}$. It is not possible to factorize the holonomy function to give separate point-wise correction functions and non-local ones. Moreover, obstructions to this last possibility have been found in the related expressions of spherically symmetric models [28].

It is instructive to look back at the spherically symmetric models and ask how it is possible to introduce point-wise holonomy modifications in that case. The answer lies in additional symmetries that ensure $E^x = E^y$. The obstructions noted here can then be by-passed, but, as it appears, only as an artifact of the more-symmetric nature of this model. Quantizing a symmetry-reduced model is different from symmetry-reducing a more general quantum system, and accordingly we find additional obstructions to covariance in our less-symmetric holonomy-modified models.

### 3. *Abelianization of normal deformations*

In the vacuum spherically symmetric model, an Abelianization of normal hypersurface deformations has been found, making it easier to see consistent modifications of the constraint [35]: One can use the construction to eliminate most derivatives in the constraint, so that no non-zero Poisson brackets remain with or without modified dependence on the angular curvature component. If there is scalar matter, it is no longer possible to eliminate as many spatial derivatives, and finding consistent modifications is more complicated; in fact, so far only obstructions to consistent modification have been seen [1]. We now demonstrate the analogous features for polarized Gowdy models: Classical Abelianization of normal deformations is possible, but no consistent holonomy modification seems to exist.

We write the constraints as

$$H[N] = -\frac{1}{8\pi G}\int \mathrm{d}\theta\; N\left[K_xK_y\varepsilon^{-1/2}\sqrt{E^xE^y} + \varepsilon^{1/2}\left(\sqrt{\frac{E^x}{E^y}}K_x + \sqrt{\frac{E^y}{E^x}}K_y\right)\mathcal{A} + \frac{1}{4\sqrt{\varepsilon E^xE^y}}\left([\varepsilon']^2 - \left[\varepsilon\left(\ln\frac{E^y}{E^x}\right)'\right]^2\right) - \left(\frac{\varepsilon^{1/2}\varepsilon'}{\sqrt{E^xE^y}}\right)'\right] \tag{45}$$

$$D[N^\theta] = \frac{1}{8\pi G}\int \mathrm{d}\theta\; N^\theta\left[K_x'E^x + K_y'E^y - \varepsilon'\mathcal{A}\right]\,. \tag{46}$$

They can be combined to the total constraint

$$\mathrm{H_T}[N,N^\theta] = \frac{1}{\kappa}\int \mathrm{d}\theta\;\left[-N(\theta)\mathcal{H}(\theta) + N^\theta(\theta)\mathcal{D}(\theta)\right]\,, \tag{47}$$

where $\mathcal{H}$ and $\mathcal{D}$ are the unsmeared local versions of the gravitational constraints (45).

We keep $\mathcal{D}$ as a constraint but replace $\mathcal{H}$ by the linear combination

$$\mathcal{C} = \frac{\epsilon'}{\sqrt{E^xE^y}}\mathcal{H} + \sqrt{\epsilon}\left(\frac{K_x}{E^y} + \frac{K_y}{E^x}\right)\mathcal{D}\,, \tag{48}$$

smeared to a new constraint

$$C[L] = -\frac{1}{8\pi G}\int \mathrm{d}\theta\; L\left[K_xK_y\varepsilon^{-1/2}\varepsilon' + \varepsilon^{1/2}\left(K_xK_y' + K_yK_x' + \left[\frac{E^x}{E^y}\right]K_xK_x' + \left[\frac{E^y}{E^x}\right]K_yK_y'\right) + \frac{\varepsilon'}{4\sqrt{\varepsilon}E^xE^y}\left([\varepsilon']^2 - \left[\varepsilon\left(\ln\frac{E^y}{E^x}\right)'\right]^2\right) - \left(\frac{\varepsilon^{1/2}\varepsilon'}{\sqrt{E^xE^y}}\right)'\right]. \tag{49}$$

As in Abelianizations of normal deformations in spherically symmetric models [20, 35], an important feature of the new constraint is that the inhomogeneous curvature component, here $\mathcal{A}$, has been eliminated.

Computing the brackets of constraints $(C[L], D[N^\theta])$, it is clear that the $\{D,D\}$-bracket has the original form. Also the $\{C,D\}$-bracket has the same form as the original $\{H,D\}$-bracket because $\mathcal{C}$ has the same spatial density weight as $\mathcal{H}$. The $\{C,C\}$-bracket must be computed explicitly, and turns out to be zero as shown in App. A. See also [36] for a related result. The set of brackets of the constraints takes the form

$$\begin{aligned}\{D[N^\theta], D[M^\theta]\} &= D[\mathcal{L}_{N^\theta}M^\theta]\\ \{C[L], D[M^\theta]\} &= -C[\mathcal{L}_{M^\theta}L]\\ \{C[L_1], C[L_2]\} &= 0\,.\end{aligned} \tag{50}$$

As in spherically symmetric models, one cannot consistently modify the curvature dependence of the constraints without destroying properties relevant for closure of the brackets.

### C. Relation to hybrid models

A Gowdy system has been proposed and analyzed in the context of loop quantum gravity in a hybrid version [2–4]: There is a homogeneous background with modifications suggested by loop quantum cosmology, coupled to inhomogeneous Gowdy modes quantized in the standard way on a Fock space. Concrete realizations make use of gauge fixings of space-time transformations, but nevertheless the framework should be expected to be covariant: It is an example of a covariant quantum-field theory (the Fock-represented Gowdy modes) on a Riemannian background (the loop-modified homogeneous model). Since quantum-field theory has an established covariant formulation on any curved background, not just on those satisfying Einstein's equation, there is no reason why covariance should be broken in hybrid models, interpreted as systems of quantum fields on a background. Indeed, different choices of gauge fixings have been shown to lead to compatible results [37].

However, going beyond the background setting is more difficult. (See [1] for a detailed discussion of the difference between background treatments and background-independent models in the context of modified or quantized canonical theories.) To do so, one would have to show that the modified background can be part of a covariant inhomogeneous model of Gowdy type. Our no-go results show that this condition is difficult to achieve. It therefore seems unlikely that hybrid models can be reductions of a covariant background-independent system with the same symmetries (leaving aside the much harder question of a reduction from a covariant full theory). Such an extension would be important not just on conceptual grounds, but also for a uniform treatment of modifications: In hybrid models, the background dynamics is modified by loop effects (holonomies), but inhomogeneous mode equations have no such modifications (except indirect ones via background variables in their coefficients). When holonomy effects are significant for the background dynamics (near a "bounce" at large curvature), they should be expected to contribute to the dynamics of inhomogeneities as well. (Interestingly, numerical investigations in hybrid models have revealed instabilities [38] reminiscent of some effects related to signature change [29–31], an apparently generic consequence of consistent holonomy modifications of inhomogeneous gravitational equations [22, 27, 33].) Consistently including these terms in inhomogeneous equations requires a covariant Gowdy model with holonomy modifications, which has failed to materialize in our attempts shown here. Using our partial no-go results, several non-trivial modifications would be required to ensure covariance, which go well beyond those included in our already rather general functions $f$, $g_1$ and $g_2$.

## III. CONCLUSIONS

In this paper, we have continued the discussion of covariance in holonomy-modified models with local degrees of freedom, started in [1] for spherically symmetric models with matter. Also here, partial no-go results but no consistent covariant versions have been found. One cannot draw final conclusions from partial no-go results, but they do show that holonomy modifications in inhomogeneous models cannot be as simple as they had been anticipated in homogeneous models. In the models studied here and in [1], covariance is therefore shown to be a restrictive criterion, capable of limiting the quantization choices that exist without the condition (as emphasized for instance in [39]). However, at present it is not clear whether holonomy modifications in models with local degrees of freedom can lead to covariant theories at all. Further study into this question and the related problem of anomalies in canonical quantum gravity is needed before the effects proposed in homogeneous models can be considered generic. As in [1], it is encouraging that the analysis of Poisson brackets of modified constraints leads to the same result as attempts to Abelianize the generators of normal hypersurface deformations, which has been shown in Sec. II B 3 to be possible for classical polarized Gowdy models, but not for the proposed modified ones.

At present, no consistent holonomy-modified model of non-perturbative inhomogeneity is known, while perturbative inhomogeneity has led to consistent modified versions [21, 22] which are being analyzed for their possible phenomenology [40]. A consistent fundamental theory should produce covariant models with all kinds of ingredients, including local degrees of freedom with non-perturbative inhomogeneity, and one may wonder whether the obstructions found by us could mean that perturbative cosmological models, along with their phenomenology, cannot be embedded within a consistent more-general theory. However, we do not think that investigations of the anomaly problem in loop quantum gravity, by effective or operator methods, are advanced enough to make such a statement at the present stage.

Even though the modifications used here did not lead to fully covariant models, we were able to confirm certain structural properties of constraint brackets in the extension to Gowdy systems. If conditions for anomaly freedom are only partially solved so as to allow for non-trivial modifications, as analyzed in the first part of Sec. II B 2, the multiplier of the diffeomorphism constraint in the bracket of two modified Hamiltonian constraints receives a factor (18) which is negative around a local maximum of the holonomy-modification function. The presence of anomalies means that this statement cannot be a physical one as long as no consistent set of modified constraints has been found. Nevertheless, the dependence of modification functions on two independent variables makes the behavior of local maxima less trivial than in the case of spherically symmetric models. The fact that the same formal behavior is found is an indication that the sign of the multiplier around local maxima may be generic, as would be the consequence of signature change.

### Acknowledgments

We are grateful to Guillermo Mena Marugán for discussions and comments on a draft of this paper. This work was supported in part by NSF grant PHY-1307408.

### Appendix A: Abelian constraints

In order to confirm the Abelian nature of (49), we list all non-trivial terms in the $\{C, C\}$-bracket, split in different types according to the "kinetic" terms in $C$

(that is, those containing extrinsic curvature components). The non-zero Poisson brackets from terms of the form $\{K_i, E^i\}$ cancel out by antisymmetry. The only remaining non-zero terms come from the $\{K_i, E^{i\prime}\}$, $\{K_i', E^i\}$ and $\{K_i', E^{i\prime}\}$-types of brackets, where $i$ can be either $x$ or $y$.

Terms of the first kind are:

$$\begin{aligned}
&\frac{\varepsilon(\varepsilon')^2 K_x E^{y\prime}}{2E^x(E^y)^3} + \frac{\varepsilon(\varepsilon')^2 K_y E^{x\prime}}{2(E^x)^3 E^y} - \frac{\varepsilon(\varepsilon')^2 K_y E^{y\prime}}{2(E^x)^2(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 K_x E^{x\prime}}{2(E^x)^2(E^y)^2} - \frac{(\varepsilon')^3 K_y}{2(E^x)^2 E^y} \\
&- \frac{(\varepsilon')^3 K_x}{2E^x(E^y)^2} + \frac{(\varepsilon)^2\varepsilon' K_y' E^{x\prime}}{2(E^x)^3 E^y} - \frac{(\varepsilon)^2\varepsilon' K_y' E^{y\prime}}{2(E^x)^2(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 K_y'}{2(E^x)^2 E^y} + \frac{(\varepsilon)^2\varepsilon' K_x' E^{y\prime}}{2E^x(E^y)^3} \\
&- \frac{(\varepsilon)^2\varepsilon' K_x' E^{x\prime}}{2(E^x)^2(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 K_x'}{2E^x(E^y)^2} + \frac{(\varepsilon)^2\varepsilon' K_x' E^{x\prime}}{2(E^x)^2(E^y)^2} - \frac{(\varepsilon)^2\varepsilon' K_x' E^{y\prime}}{2E^x(E^y)^3} - \frac{\varepsilon(\varepsilon')^2 K_x'}{2E^x(E^y)^2} \\
&+ \frac{(\varepsilon)^2\varepsilon' K_y' E^{y\prime}}{2(E^x)^2(E^y)^2} - \frac{(\varepsilon)^2\varepsilon' K_y' E^{x\prime}}{2(E^x)^3 E^y} - \frac{\varepsilon(\varepsilon')^2 K_y'}{2(E^x)^2 E^y}\,.
\end{aligned} \tag{A1}$$

Terms of the second kind are:

$$\begin{aligned}
&- \frac{\varepsilon K_x^2 K_x' E^x}{(E^y)^2} + \frac{\varepsilon K_x K_y K_y'}{E^x} + \frac{(\varepsilon')^3 K_x}{4E^x(E^y)^2} + \frac{3(\varepsilon)^2\varepsilon'(E^{y\prime})^2 K_x}{4E^x(E^y)^4} + \frac{(\varepsilon)^2\varepsilon'(E^{x\prime})^2 K_x}{4(E^x)^3(E^y)^2} \\
&- \frac{(\varepsilon)^2\varepsilon' E^{x\prime}E^{y\prime}K_x}{(E^x)^2(E^y)^3} - \frac{\varepsilon(\varepsilon')^2 E^{x\prime}K_x}{2(E^x)^2(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 E^{y\prime}K_x}{E^x(E^y)^3} + \frac{\varepsilon\varepsilon'\varepsilon'' K_x}{E^x(E^y)^2} \\
&+ \frac{\varepsilon K_y K_x K_x'}{E^y} - \frac{\varepsilon E^y K_y^2 K_y'}{(E^x)^2} + \frac{(\varepsilon')^3 K_y}{4(E^x)^2 E^y} + \frac{(\varepsilon)^2\varepsilon'(E^{y\prime})^2 K_y}{4(E^x)^2(E^y)^3} + \frac{3(\varepsilon)^2\varepsilon'(E^{x\prime})^2 K_y}{4(E^x)^4 E^y} \\
&- \frac{(\varepsilon)^2\varepsilon' E^{x\prime}E^{y\prime}K_y}{(E^x)^3(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 E^{y\prime}K_y}{2(E^x)^2(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 E^{x\prime}K_y}{(E^x)^3 E^y} + \frac{\varepsilon\varepsilon'\varepsilon'' K_y}{(E^x)^2 E^y} \\
&+ \frac{\varepsilon K_x^2 K_x' E^x}{(E^y)^2} - \frac{\varepsilon K_x K_y K_y'}{E^x} + \frac{(\varepsilon')^3 K_x}{4E^x(E^y)^2} + \frac{3(\varepsilon)^2\varepsilon'(E^{x\prime})^2 K_x}{4(E^x)^3(E^y)^2} + \frac{(\varepsilon)^2\varepsilon'(E^{y\prime})^2 K_x}{4E^x(E^y)^4} \\
&- \frac{(\varepsilon)^2\varepsilon' E^{x\prime}E^{y\prime}K_x}{(E^x)^2(E^y)^3} - \frac{\varepsilon(\varepsilon')^2 E^{x\prime}K_x}{(E^x)^2(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 E^{y\prime}K_x}{2E^x(E^y)^3} + \frac{\varepsilon\varepsilon'\varepsilon'' K_x}{E^x(E^y)^2} \\
&- \frac{\varepsilon K_y K_x K_x'}{E^y} + \frac{\varepsilon E^y K_y^2 K_y'}{(E^x)^2} + \frac{(\varepsilon')^3 K_y}{4(E^x)^2 E^y} + \frac{3(\varepsilon)^2\varepsilon'(E^{y\prime})^2 K_y}{4(E^x)^2(E^y)^3} + \frac{(\varepsilon)^2\varepsilon'(E^{x\prime})^2 K_y}{4(E^x)^4 E^y} \\
&- \frac{(\varepsilon)^2\varepsilon' E^{x\prime}E^{y\prime}K_y}{(E^x)^3(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 E^{y\prime}K_y}{(E^x)^2(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 E^{x\prime}K_y}{2(E^x)^3 E^y} + \frac{\varepsilon\varepsilon'\varepsilon'' K_y}{(E^x)^2 E^y}\,.
\end{aligned} \tag{A2}$$

And finally, the most complicated terms come from brackets of the form $\{K_i', E^{i\prime}\}$. Since there are many terms of this form, we first list those from the contributions proportional to $K_x K_y'$ and $K_y K_x'$:

$$\begin{aligned}
&- \frac{\varepsilon(\varepsilon')^2 E^{y\prime}K_x}{2E^x(E^y)^3} - \frac{(\varepsilon)^2\varepsilon' E^{y\prime}K_x'}{2E^x(E^y)^3} + \frac{(\varepsilon)^2\varepsilon'' E^{y\prime}K_x}{2E^x(E^y)^3} + \frac{(\varepsilon)^2\varepsilon' E^{y\prime\prime}K_x}{2E^x(E^y)^3} + \frac{(\varepsilon)^2\varepsilon' E^{x\prime}E^{y\prime}K_x}{2(E^x)^2(E^y)^3} \\
&- \frac{3(\varepsilon)^2\varepsilon'(E^{y\prime})^2 K_x}{2E^x(E^y)^4} + \frac{(\varepsilon)^2\varepsilon' E^{x\prime}K_x'}{2(E^x)^2(E^y)^2} - \frac{(\varepsilon)^2\varepsilon'' E^{x\prime}K_x}{2(E^x)^2(E^y)^2} - \frac{(\varepsilon)^2\varepsilon' E^{x\prime\prime}K_x}{2(E^x)^2(E^y)^2} + \frac{(\varepsilon)^2\varepsilon'(E^{x\prime})^2 K_x}{(E^x)^3(E^y)^2} \\
&+ \frac{\varepsilon(\varepsilon')^2 K_x'}{2E^x(E^y)^2} - \frac{\varepsilon\varepsilon'\varepsilon'' K_x}{E^x(E^y)^2} \\
&- \frac{\varepsilon(\varepsilon')^2 E^{x\prime}K_y}{2E^y(E^x)^3} - \frac{(\varepsilon)^2\varepsilon' E^{x\prime}K_y'}{2E^y(E^x)^3} + \frac{(\varepsilon)^2\varepsilon'' E^{x\prime}K_y}{2E^y(E^x)^3} + \frac{(\varepsilon)^2\varepsilon' E^{x\prime\prime}K_y}{2E^y(E^x)^3} + \frac{(\varepsilon)^2\varepsilon' E^{x\prime}E^{y\prime}K_y}{2(E^y)^2(E^x)^3} \\
&- \frac{3(\varepsilon)^2\varepsilon'(E^{x\prime})^2 K_y}{2E^y(E^x)^4} + \frac{(\varepsilon)^2\varepsilon' E^{y\prime}K_y'}{2(E^y)^2(E^x)^2} - \frac{(\varepsilon)^2\varepsilon'' E^{y\prime}K_y}{2(E^x)^2(E^y)^2} - \frac{(\varepsilon)^2\varepsilon' E^{y\prime\prime}K_y}{2(E^x)^2(E^y)^2} + \frac{(\varepsilon)^2\varepsilon'(E^{y\prime})^2 K_y}{(E^y)^3(E^x)^2} \\
&+ \frac{\varepsilon(\varepsilon')^2 K_y'}{2E^y(E^x)^2} - \frac{\varepsilon\varepsilon'\varepsilon'' K_y}{E^y(E^x)^2}\,.
\end{aligned} \tag{A3}$$

The other two kinetic terms proportional to $K_x K_x'$ and $K_y K_y'$ also give contributions via the $\{K_i', E^{i\prime}\}$-bracket:

$$
\begin{aligned}
&\frac{2\varepsilon(\varepsilon')^2 E^{x\prime} K_x}{(E^x)^2(E^y)^2} - \frac{2(\varepsilon)^2\varepsilon'(E^{x\prime})^2 K_x}{(E^x)^3(E^y)^2} + \frac{3(\varepsilon)^2\varepsilon' E^{x\prime} E^{y\prime} K_x}{2(E^x)^2(E^y)^3} - \frac{(\varepsilon)^2\varepsilon' E^{x\prime} K_x'}{2(E^x)^2(E^y)^2} + \frac{(\varepsilon)^2\varepsilon'' E^{x\prime} K_x}{2(E^x)^2(E^y)^2}\\
&+\frac{(\varepsilon)^2\varepsilon' E^{x\prime\prime} K_x}{2(E^x)^2(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 E^{y\prime} K_x}{2E^x(E^y)^3} + \frac{(\varepsilon)^2\varepsilon'(E^{y\prime})^2 K_x}{2E^x(E^y)^4} + \frac{(\varepsilon)^2\varepsilon' E^{y\prime} K_x'}{2E^x(E^y)^3} - \frac{(\varepsilon)^2\varepsilon'' E^{y\prime} K_x}{2E^x(E^y)^3}\\
&-\frac{(\varepsilon)^2\varepsilon' E^{y\prime\prime} K_x}{2E^x(E^y)^3} + \frac{\varepsilon(\varepsilon')^2 K_x'}{2E^x(E^y)^2} - \frac{\varepsilon\varepsilon'\varepsilon'' K_x}{E^x(E^y)^2}\\
&\frac{2\varepsilon(\varepsilon')^2 E^{y\prime} K_y}{(E^x)^2(E^y)^2} - \frac{2(\varepsilon)^2\varepsilon'(E^{y\prime})^2 K_y}{(E^y)^3(E^x)^2} + \frac{3(\varepsilon)^2\varepsilon' E^{x\prime} E^{y\prime} K_y}{2(E^y)^2(E^x)^3} - \frac{(\varepsilon)^2\varepsilon' E^{y\prime} K_y'}{2(E^x)^2(E^y)^2} + \frac{(\varepsilon)^2\varepsilon'' E^{y\prime} K_y}{2(E^x)^2(E^y)^2}\\
&+\frac{(\varepsilon)^2\varepsilon' E^{y\prime\prime} K_y}{2(E^x)^2(E^y)^2} - \frac{\varepsilon(\varepsilon')^2 E^{x\prime} K_y}{2E^y(E^x)^3} + \frac{(\varepsilon)^2\varepsilon'(E^{x\prime})^2 K_y}{2E^y(E^x)^4} + \frac{(\varepsilon)^2\varepsilon' E^{x\prime} K_y'}{2E^y(E^x)^3} - \frac{(\varepsilon)^2\varepsilon'' E^{x\prime} K_y}{2E^y(E^x)^3}\\
&-\frac{(\varepsilon)^2\varepsilon' E^{x\prime\prime} K_y}{2E^y(E^x)^3} + \frac{\varepsilon(\varepsilon')^2 K_y'}{2E^y(E^x)^2} - \frac{\varepsilon\varepsilon'\varepsilon'' K_y}{E^y(E^x)^2}\,.
\end{aligned}
\tag{A4}
$$

These are all non-zero terms, and in spite of their large number it is straightforward to observe that they all cancel one another when combined.


[1] M. Bojowald, S. Brahma, and J. D. Reyes, arXiv:1507.00329.
[2] M. Martín-Benito, L. J. Garay, and G. A. Mena Marugán, Phys. Rev. D **78**, 083516 (2008), arXiv:0804.1098.
[3] L. J. Garay, M. Martín-Benito, and G. A. Mena Marugán, Phys. Rev. D **82**, 044048 (2010), arXiv:1005.5654.
[4] M. Martín-Benito, D. Martín-de Blas, and G. A. Mena Marugán, Phys. Rev. D **83**, 084050 (2011), arXiv:1012.2324.
[5] C. W. Misner, Phys. Rev. D **8**, 3271 (1973).
[6] B. K. Berger, Phys. Rev. D **11**, 2770 (1975).
[7] B. K. Berger, Ann. Phys. **156**, 155 (1984).
[8] V. Husain, Class. Quantum Grav. **4**, 1587 (1987).
[9] V. Husain and L. Smolin, Nucl. Phys. B **327**, 205 (1989).
[10] G. Mena Marugán, Phys. Rev. D **56**, 908 (1997), gr-qc/9704041.
[11] M. Pierri, Int. J. Mod. Phys. D **11**, 135 (2002), gr-qc/0101013.
[12] A. Corichi, J. Cortez, and H. Quevedo, Phys. Rev. D **67**, 087502 (2003).
[13] C. G. Torre, Phys. Rev. D **66**, 084017 (2002), gr-qc/0206083.
[14] C. G. Torre, gr-qc/0607084.
[15] M. Bojowald, Living Rev. Relativity **11**, 4 (2008), gr-qc/0601085.`http://www.livingreviews.org/lrr-2008-4`.
[16] M. Bojowald, Rep. Prog. Phys. **78**, 023901 (2015), arXiv:1501.04899.
[17] R. H. Gowdy, Ann. Phys. **83**, 203 (1974).
[18] P. A. M. Dirac, Proc. Roy. Soc. A **246**, 333 (1958).
[19] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), reprinted in [41].
[20] R. Gambini and J. Pullin, arXiv:1312.3595.
[21] T. Cailleteau, J. Mielczarek, A. Barrau, and J. Grain, Class. Quant. Grav. **29**, 095010 (2012), arXiv:1111.3535.
[22] T. Cailleteau, L. Linsefors, and A. Barrau, Class. Quantum Grav. **31**, 125011 (2014), arXiv:1307.5238.
[23] K. V. Kuchař, J. Math. Phys. **17**, 777 (1976).
[24] M. Bojowald, G. Hossain, M. Kagan, and C. Tomlin, Quantum Matter **2**, 436 (2013), arXiv:1302.5695.
[25] M. Bojowald, Class. Quantum Grav. **21**, 3733 (2004), gr-qc/0407017.
[26] M. Bojowald and R. Swiderski, Class. Quantum Grav. **23**, 2129 (2006), gr-qc/0511108.
[27] J. D. Reyes, Ph.D. thesis, The Pennsylvania State University, 2009.
[28] M. Bojowald, G. M. Paily, and J. D. Reyes, Phys. Rev. D **90**, 025025 (2014), arXiv:1402.5130.
[29] M. Bojowald and G. M. Paily, Phys. Rev. D **86**, 104018 (2012), arXiv:1112.1899.
[30] J. Mielczarek, Springer Proc. Phys. **157**, 555 (2014), arXiv:1207.4657.
[31] M. Bojowald and J. Mielczarek, arXiv:1503.09154.
[32] K. Banerjee and G. Date, Class. Quantum Grav. **25**, 105014 (2008), arXiv:0712.0683.
[33] A. Barrau *et al.*, arXiv:1404.1018.
[34] M. Bojowald, Front. Phys. **3**, 33 (2015), arXiv:1409.3157.
[35] R. Gambini and J. Pullin, Phys. Rev. Lett. **110**, 211301 (2013), arXiv:1302.5265.
[36] D. Martín-de Blas, J. Olmedo, and T. Pawlowski, in preparation.
[37] L. Castelló Gomar, M. Fernández-Méndez, G. A. Mena Marugán, and J. Olmedo, Phys. Rev. D **90**, 064015 (2014), arXiv:1407.0998.
[38] D. Brizuela, G. A. Mena Marugán, and T. Pawlowski, Class. Quantum Grav. **27**, 052001 (2010), arXiv:0902.0697.
[39] A. Perez, Phys. Rev. D **73**, 044007 (2006), gr-qc/0509118.
[40] A. Barrau, T. Cailleteau, J. Grain, and J. Mielczarek,

Class. Quant. Grav. **31**, 053001 (2014), arXiv:1309.6896.
[41] R. Arnowitt, S. Deser, and C. W. Misner, Gen. Rel. Grav. **40**, 1997 (2008).