Is motion under the conservative self-force in black hole spacetimes an integrable Hamiltonian system?

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Is motion under the conservative self-force in black hole spacetimes an integrable Hamiltonian system?

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A point-like object moving in a background black hole spacetime experiences a gravitational self-force which can be expressed as a local function of the object's instantaneous position and velocity, to linear order in the mass ratio. We consider the worldline dynamics defined by the conservative part of the local self-force, turning off the dissipative part, and we ask: Is that dynamical system a Hamiltonian system, and if so, is it integrable?

In the Schwarzschild spacetime, we show that the system is Hamiltonian and integrable, to linear order in the mass ratio, for generic (but not necessarily all) stable bound orbits. There exist an energy and an angular momentum, being perturbed versions of their counterparts for geodesic motion, which are conserved under the forced motion. We also discuss difficulties associated with establishing analogous results in the Kerr spacetime. This result may be useful for future computational schemes, based on a local Hamiltonian description, for calculating the conservative self-force and its observable effects. It is also relevant to the assumption of the existence of a Hamiltonian for the conservative dynamics for generic orbits in the effective-one-body formalism, to linear order in the mass ratio, but to all orders in the post-Newtonian expansion.

I. INTRODUCTION

Recent years have seen great progress in computing the motion of and gravitational wave signals emitted by point-like objects orbiting massive black holes [1–4], but some key foundational and computational challenges still lie ahead [4]. A sufficiently accurate solution to the gravitational self-force problem will be crucial to the detection and analysis of signals from astrophysical extreme-mass-ratio inspirals by future space-based gravitational wave detectors [5]. Study of the self-force also helps to inform other complementary approaches to the relativistic two-body problem, such as post-Newtonian theory and the effective-one-body formalism [6].

The foundations for computing the first-order self-force acting on a point mass in a curved background spacetime are provided by a result [7, 8] which has been canonized as the "MiSaTaQuWa equation". This is an equation for the self-force experienced by the point mass as a functional of its worldline history, arising from the tail part of its own linearized gravitational field. There are several different formulations of the MiSaTaQuWa result:

1. *A correction to the geodesic deviation equation:* In this formulation, the deviation between the point mass's actual accelerated worldline and a given fiducial background geodesic is characterized by a deviation vector field along the fiducial geodesic. The deviation vector obeys the geodesic deviation equation plus a self-force correction, and the self-force correction is constructed from the self-field sourced by the fiducial geodesic. This formulation is the version that is obtained directly from the most rigorous derivations [9, 10].

2. *A local equation of motion for an accelerated worldline:* In this scheme, the worldline's acceleration is given as a function of its instantaneous position and velocity, by using the geodesic which instantaneously matches the position and velocity of the accelerated worldline (the osculating geodesic) as the source to compute the instantaneous self-force [2].

3. *A non-local equation of motion for an accelerated worldline:* In this family of schemes, one uses the history of the actual accelerated worldline as the source for the self-force[1]. Viewing the self-force as a given functional of the worldline, this corresponds to a non-local equation of motion for the the worldline—the instantaneous acceleration depends on the worldline's entire history.

Schemes 2 and 3 have a clear advantage over scheme 1 in being able to describe inspiraling motion over a radiation reaction timescale. Both 2 and 3 reduce to 1 for small deviations from the fiducial geodesic, over short timescales. Scheme 2 is said to arise from 3 via a reduction of order procedure: because the self-forced worldline and the osculating geodesic agree to zeroth order in the mass ratio, the former can be replaced with the latter in calculating the self-force to first order. The relative accuracies and computational merits of 2 and variants of 3 are subtle issues, entangled with formulating and implementing a consistent second-order analog of the MiSaTaQuWa result [11–14], which will not be addressed here.

This paper addresses a formal question concerning the worldline dynamics defined by scheme 2, using only the conservative part of the self-force, turning off the dissipative/radiative part. We ask: Is the resultant dynamical

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[1] Since the stress energy tensor for an accelerated worldline is not conserved, implementing this sheme requires an ad hoc relaxation of the Lorentz gauge condition.

system, for the worldline degrees of freedom only, an integrable Hamiltonian system, to linear order in the mass ratio?

In the case where the background spacetime is Schwarzschild, we show that this system is indeed a Hamiltonian dynamical system, in the classic sense [15]. We also show that it is integrable; in other words, there exist three functions (the rest mass, an energy, and an angular momentum) on the effectively six-dimensional phase space whose values are instantaneously conserved along the system's trajectories. These results hold to linear order in the mass ratio, for generic stable bound orbits (with some caveats for orbits in the zoom-whirl regime).

We also carry out a partial analysis of the case of the Kerr spacetime, following the same line of reasoning which leads to a successful construction in Schwarzschild. In Kerr, we encounter complications which are associated with orbital resonances [16], when the ratio of the frequencies of the radial and polar motions is a rational number. These complications prevent us from drawing definite conclusions about the Kerr case.

There is, however, a heuristic argument which suggests that the motion in Kerr should be integrable and Hamiltonian [17, 18]. Namely, the ambiguities in the definition of angular momentum related to the BMS group are associated with the dissipative sector of the linear perturbations, so the conservative sector should admit three independent conserved components of angular momentum [19], a sufficient number to make the system integrable. This general picture is consistent with what has been found in the post-Newtonian approximation, where the system is integrable at successive orders [18].

The conservative self-force dynamics in Kerr has recently been cast in a Hamiltonian-like formulation in Ref. [20]. However, that formulation is not Hamiltonian in the classic sense used here. Specifically, in that formalism, the equations of motion are obtained by differentiating the Hamiltonian with respect to one set of variables while holding a second set of variables fixed, and then taking the two sets of variables to coincide.

The existence of a Hamiltonian system for the conservative dynamics is a foundational assumption in the effective-one-body formalism [21], as well as in other fruitful treatments of the relativistic two-body problem, e.g. [22–25]. While the results of Refs. [23, 25, 26] make it clear that such a Hamiltonian exists for circular orbits in the extreme-mass-ratio limit (to linear order in the mass ratio, but to all orders in the post-Newtonian expansion), our result confirms the validity of this assumption for generic (non-circular) orbits, in the non-spinning case.

This paper is organized as follows. Our construction relies heavily on the fact that geodesic motion in the Schwarzschild (and Kerr) spacetime is completely integrable, and thus admits a representation in terms of generalized action-angle variables [17, 27]. We review relevant properties of these variables in Sec. II, and we review relevant properties of the local conservative self-force in Sec. III. We develop sufficient conditions for the forced system to be Hamiltonian and integrable in Sec. IV, and we conclude in Sec. V.

II. GEODESIC MOTION IN KERR AND ACTION-ANGLE VARIABLES

Geodesic motion in a spacetime with metric $g_{\mu\nu}$ is generated by the Hamiltonian

$$H_0(z,p) = \frac{1}{2} g^{\mu\nu}(z) p_\mu p_\nu, \tag{2.1}$$

where $z^\mu(\lambda)$ are the coordinates of a worldline, and $p_\mu(\lambda)$ are the components of its momentum, together with the canonical symplectic form $\Omega_0 = \mathrm{d}z^\mu \wedge \mathrm{d}p_\mu$ on the worldline phase space (z^μ, p_μ) [15, 17]. Hamilton's equations read

$$\frac{dz^\mu}{d\lambda} = \frac{\partial H_0}{\partial p_\mu}, \qquad \frac{dp_\mu}{d\lambda} = -\frac{\partial H_0}{\partial z^\mu}, \tag{2.2}$$

and imply the geodesic equation in affine parameterization, $p^\nu \nabla_\nu p_\mu = 0$. Identifying the affine parameter as $\lambda = \tau/m$, where τ is the proper time along the worldline, gives the usual expression $p_\mu = m g_{\mu\nu} dz^\nu/d\tau$ for the momentum of a particle of mass m, and also yields $H_0 = -m^2/2$ on shell.

In the Kerr spacetime, with Boyer-Lindquist coordinates $z^\mu = (t, r, \theta, \phi)$, geodesic motion is completely integrable thanks to the existence of four first integrals of motion. These are the Hamiltonian H_0, the energy $E = -(\partial_t)^\mu p_\mu$, the axial angular momentum $L_z = (\partial_\phi)^\mu p_\mu$, and the Carter constant $Q = K^{\mu\nu} p_\mu p_\nu$, where $(\partial_t)^\mu$ and $(\partial_\phi)^\mu$ are the timelike and axial Killing vectors, and $K^{\mu\nu}$ is the non-trivial Killing tensor.

These four first integrals are independent and in involution, which implies the existence of generalized action-angle variables[2] for bound geodesics in the Kerr geometry [17, 27]. These are canonical coordinates $\big(q^\alpha, J_\alpha\big) = (q^t, q^r, q^\theta, q^\phi, J_t, J_r, J_\theta, J_\phi)$ on the worldline phase space, for which the geodesic Hamiltonian H_0 depends only on the action variables J_α,

$$H_0(z,p) = H_0(J), \tag{2.3}$$

and not on the angle variables q^α. They are obtained from the (z^μ, p_μ) coordinates via a canonical transformation

$$q^\alpha = q^\alpha(z^\mu, p_\mu), \qquad J_\alpha = J_\alpha(z^\mu, p_\mu), \tag{2.4}$$

[2] The action-angle variables discussed here are associated with the affine parameter $\lambda = \tau/m$ along the worldline; there are also other action-angle coordinates on the phase space associated with Mino time [28] and Boyer-Lindquist coordinate time.

so that $\Omega_0 = \mathrm{d}z^\mu \wedge \mathrm{d}p_\mu = \mathrm{d}q^\alpha \wedge \mathrm{d}J_\alpha$. Hamilton's equations then take the particularly simple form

$$\frac{dq^\alpha}{d\lambda} = \frac{\partial H_0}{\partial J_\alpha} \equiv \omega^\alpha(J), \tag{2.5a}$$

$$\frac{dJ_\alpha}{d\lambda} = -\frac{\partial H_0}{\partial q^\alpha} = 0. \tag{2.5b}$$

The action variables J_α are all independent constants of motion, and the angle variables q^α all increase linearly, at the constant rates ω^α known as the fundamental frequencies. The angle variables q^r, q^θ, and q^ϕ are each periodic with period 2π, while q^t has an infinite range. The action variables J_α are functions of the geodesic first integrals $P_\alpha \equiv (H_0, E, L_z, Q)$; in particular, $J_t = -E$ and $J_\phi = L_z$.

In the Schwarzschild limit of the Kerr geometry, both geodesic motion and self-forced motion are confined to a plane, which can be taken without loss of generality to be the equatorial plane $\theta = \pi/2$. We can then ignore the θ-motion, working in the reduced phase space with coordinates $(z^\mu, p_\mu) = (t, r, \phi, p_t, p_r, p_\phi)$. We have the three first integrals $P_\alpha = (H_0, E, L_z)$, and action-angle variables $\big(q^\alpha, J_\alpha\big) = (q^t, q^r, q^\phi, J_t, J_r, J_\phi)$, defined just as in the general Kerr case.

III. CONSERVATIVE-SELF-FORCE PERTURBATION TO GEODESIC MOTION

Instead of geodesic motion, we now consider a point mass m with worldline $z^\mu(\lambda)$ experiencing a local linear perturbing force,

$$\frac{dz^\mu}{d\lambda} = p^\mu, \qquad p^\nu \nabla_\nu p_\mu = \epsilon \mathcal{F}_\mu(z,p), \tag{3.1}$$

where ϵ is a small parameter. We are interested in the case where the forcing function $\mathcal{F}_\mu(z,p)$ is given by the conservative part of the osculating-geodesic-sourced first-order gravitational self-force in the Kerr spacetime, with the parameter ϵ being the small mass ratio m/M. (The form of the self-force is given more explicitly in Appendix A 1.)

The worldline dynamics (3.1) can be re-expressed in terms of the geodesic action-angle variables (q^α, J_α) as

$$\frac{dq^\alpha}{d\lambda} = \omega^\alpha(J) + \epsilon f^\alpha(q,J), \tag{3.2a}$$

$$\frac{dJ_\alpha}{d\lambda} = \epsilon F_\alpha(q,J), \tag{3.2b}$$

following Ref. [17]. Here we use the same phase space coordinate transformation (2.4) as for geodesic motion to obtain Eqs. (3.2) from Eqs. (3.1). The forcing functions $f^\alpha(q,J)$ and $F_\alpha(q,J)$ are determined from the self-force components $\mathcal{F}_\mu(z,p)$ via $f^\alpha = (\partial q^\alpha/\partial p_\mu)_z\, \mathcal{F}_\mu$ and $F_\alpha = (\partial J_\alpha/\partial p_\mu)_z\, \mathcal{F}_\mu$. These functions have the important property that they are independent of the angle variables q^t and q^ϕ, because of the symmetries of the Kerr spacetime. Thus, they can be written as functions of q^r, q^θ and the four variables J_α [17]:

$$f^\alpha(q,J) = f^\alpha\big(q^r, q^\theta, J\big), \tag{3.3a}$$

$$F_\alpha(q,J) = F_\alpha\big(q^r, q^\theta, J\big). \tag{3.3b}$$

In the Schwarzschild case, these functions depend only on q^r and the three J_α.

The forcing functions f^α and F_α are periodic functions of q^r and of q^θ, each with period 2π. They can thus be expanded as Fourier series in q^r and q^θ, according to

$$F_\alpha\big(q^r, q^\theta, J\big) = \sum_{k_r,k_\theta=-\infty}^{\infty} \hat{F}_\alpha(k_r, k_\theta, J)\, e^{ik_r q^r + ik_\theta q^\theta}, \tag{3.4}$$

and similarly for f^α. As shown in Ref. [28], the $(0,0)$ Fourier mode of each F_α vanishes,

$$\hat{F}_\alpha(0,0,J) = \frac{1}{(2\pi)^2}\int_0^{2\pi} dq^r \int_0^{2\pi} dq^\theta\, F_\alpha(q^r, q^\theta, J) = 0, \tag{3.5}$$

due to reflection properties of Kerr geodesics and to the time-reversal symmetry of the conservative self-force [17, 28].

In the Schwarzschild case, we lose the dependence on the θ-motion. Equations (3.3) and (3.4) are replaced by

$$F_\alpha\big(q^r, J\big) = \sum_{k_r=-\infty}^{\infty} \hat{F}_\alpha(k_r, J)\, e^{ik_r q^r}, \tag{3.6}$$

and similarly with F_α replaced by f^α. Equation (3.5) becomes

$$\hat{F}_\alpha(0,J) = \frac{1}{2\pi}\int_0^{2\pi} dq^r\, F_\alpha(q^r, J) = 0. \tag{3.7}$$

Recalling that the forcing functions F_α give the rates of change of the action variables J_α, Eqs. (3.5) and (3.7) express the fact that the conservative first-order self-force causes no net change in the geodesic first integrals, when the force is evaluated along a geodesic and suitably averaged [28]. In the Schwarzschild case (3.7), this is an orbital average, or time average, over one period of radial motion. In the Kerr case (3.5), the average is over the (q^r, q^θ) torus in phase space, which is equivalent to a time average only over an infinite time and only for non-resonant orbits [16, 17, 29].

IV. IS THE PERTURBED SYSTEM HAMILTONIAN AND INTEGRABLE?

The perturbed system (3.2) will be Hamiltonian and integrable, to linear order in ϵ, if there exist new phase space coordinates $(\bar{q}^\alpha, \bar{J}_\alpha)$ and a new Hamiltonian function $\bar{H}(\bar{J})$ for which Eqs. (3.2) are equivalent to

$$\frac{d\bar{q}^\alpha}{d\lambda} = \frac{\partial \bar{H}(\bar{J})}{\partial \bar{J}_\alpha} + O(\epsilon^2), \qquad \frac{d\bar{J}_\alpha}{d\lambda} = O(\epsilon^2). \tag{4.1}$$

Without loss of generality, we can express the new coordinates as linear perturbations of the geodesic action-angle coordinates (q^α, J_α):

$$\bar{q}^\alpha(q,J) = q^\alpha + \epsilon\chi^\alpha(q,J), \quad (4.2a)$$
$$\bar{J}_\alpha(q,J) = J_\alpha + \epsilon\zeta_\alpha(q,J), \quad (4.2b)$$

for some functions χ^α and ζ_α to be determined. Note that (4.2) is not assumed to be a canonical transformation. Similarly we can express the new Hamiltonian as

$$\bar{H}(\bar{J}) = H_0(\bar{J}) + \epsilon H_1(\bar{J}), \quad (4.3)$$

where H_0 is the geodesic Hamiltonian function, for some function $H_1(\bar{J})$ to be determined.

Combining Eqs. (4.1), (4.2) and (4.3) now yields that Eqs. (4.1) will be equivalent to Eqs. (3.2) to linear order in ϵ if

$$f^\alpha(q,J) = -\omega^\beta\frac{\partial\chi^\alpha}{\partial q^\beta} + \frac{\partial\omega^\alpha}{\partial J_\beta}\zeta_\beta + \frac{\partial H_1(J)}{\partial J_\alpha}, \quad (4.4a)$$
$$F_\alpha(q,J) = -\omega^\beta\frac{\partial\zeta_\alpha}{\partial q^\beta}, \quad (4.4b)$$

where $\omega^\alpha = \omega^\alpha(J)$ are the geodesic fundamental frequencies (2.5). Thus, the conservative-self-force dynamics (3.2) will be Hamiltonian and integrable if there exist functions $\chi^\alpha(q,J)$, $\zeta_\alpha(q,J)$, and $H_1(J)$ satisfying Eqs. (4.4a) and (4.4b).

In light of Eqs. (3.3), it is natural to consider solutions for χ^α and ζ_α which, like f^α and F_α, are independent of q^t and q^ϕ, and which are periodic functions of q^r and q^θ [or of just q^r in Schwarzschild]. We can then decompose all of these functions into discrete Fourier series for the q^r and q^θ dependence, just as for F_α in Eq. (3.4) [or Eq. (3.6)]. This defines Fourier mode amplitudes $\hat{f}^\alpha$, $\hat{F}_\alpha$, $\hat{\chi}^\alpha$, and $\hat{\zeta}_\alpha$ which are functions of the two integers k_r and k_θ [or just k_r] and all of the action variables J_α.

We then have the following Fourier transforms of Eqs. (4.4a) and (4.4b):

$$\hat{f}^\alpha = -i(\omega\cdot k)\hat{\chi}^\alpha + \frac{\partial\omega^\alpha}{\partial J_\beta}\hat{\zeta}_\beta + \delta_{k_r,0}\delta_{k_\theta,0}\frac{\partial H_1}{\partial J_\alpha}, \quad (4.5a)$$
$$\hat{F}_\alpha = -i(\omega\cdot k)\hat{\zeta}_\alpha, \quad (4.5b)$$

where

$$(\omega\cdot k) = \begin{cases} \omega^r k_r + \omega^\theta k_\theta & \text{Kerr} \\ \omega^r k_r & \text{Schwarzschild} \end{cases}, \quad (4.6)$$

and with $\delta_{k_\theta,0} \to 1$ in Schwarzschild. If we restrict attention to Fourier modes for which $(\omega\cdot k) \neq 0$, then Eqs. (4.5a) and (4.5b) admit the simple solutions

$$\hat{\zeta}_\alpha = \frac{i\hat{F}_\alpha}{(\omega\cdot k)}, \qquad \hat{\chi}^\alpha = \frac{i}{(\omega\cdot k)}\left(\hat{f}^\alpha - \frac{\partial\omega^\alpha}{\partial J_\beta}\hat{\zeta}_\beta\right). \quad (4.7)$$

In the general Kerr case, the quantity $\omega\cdot k = \omega^r k_r + \omega^\theta k_\theta$ can vanish at locations in phase space where ω^r/ω^θ is a rational number, corresponding to an orbital resonance in the r and θ motions [16]. The solutions (4.7) are clearly not valid in such cases, and so our analysis does not allow us to draw any definite conclusions about the Kerr case.

For stable bound orbits in Schwarzschild, the quantity $\omega\cdot k = \omega^r k_r$ vanishes only when $k_r = 0$, since $\omega^r = 0$ occurs only in the limit of unbound or unstable orbits. Equations (4.7) thus provide valid solutions for all Fourier modes of χ^α and ζ_α, except for the $k_r = 0$ modes. Given the fact (3.7) that $\hat{F}_\alpha(0,J) = 0$, we see that a separate solution to Eqs. (4.5a) and (4.5b) for the case $k_r = 0$ is given by

$$\hat{\zeta}_\alpha(0,J) = \left(\frac{\partial\omega^\beta}{\partial J_\alpha}\right)^{-1}\hat{f}^\beta(0,J), \qquad H_1 = 0, \quad (4.8)$$

[with $\hat{\chi}^\alpha(0,J)$ unconstrained], provided that the matrix $\partial\omega^\beta/\partial J_\alpha$ is invertible. It follows from the results of Ref. [30], which discovered "isofrequency pairing" in Schwarzschild, that $\partial\omega^\beta/\partial J_\alpha$ is invertible for all stable bound geodesics, except for those along the singular curve associated with isofrequency pairing of Schwarzschild geodesics in the zoom-whirl regime (and those along the separatrix defining the boundary of stable orbits); see Figure 1 of Ref. [30].[3] Thus, we have constructed a solution of Eqs. (4.4a) and (4.4b), and so the perturbed motion is Hamiltonian and integrable (off the singular curve).

We should note that the construction presented here is an example of what is referred to as a "near-identity averaging transformation" in the context of dynamical systems [31]. The application of such transformations to self-forced motion in Kerr has been previously discussed in Ref. [32], which investigated the conditions for sustained orbital resonances in extreme-mass-ratio inspirals.[4]

[3] More accurately, the results of Ref. [30] show that the Jacobian $\partial(\Omega_r,\Omega_\phi)/\partial(E,L_z)$ is regular for all stable bound orbits except those along the "singular curve", where (Ω_r,Ω_ϕ) are the frequencies with respect to coordinate time t. Our statements here involve instead the Jacobian $\partial\omega^\beta/\partial J_\alpha = \partial(\omega^t,\omega^r,\omega^\phi)/\partial(-E,J_r,L_z)$, with the frequencies with respect to proper time (divided by mass). However, one can show (e.g., via manipulations of the formulae in Appendix A of Ref. [17] and in Ref. [30]) that the determinants of these two Jacobians are proportional, with a proportionality factor which is regular for all stable bound orbits.

[4] It was also pointed out in Ref. [32] that one can avoid the isofrequency pairing found in Ref. [30] by using frequencies $(\Upsilon_r,\Upsilon_\phi)$ with respect to Mino time (rather than coordinate time or proper time), which uniquely label all stable bound geodesics in Schwarzschild. It is thus conceivable that one could reformulate our construction here in terms of Mino time and thereby extend it to all stable bound geodesics. This would require a Hamiltonian description of Schwarzschild geodesics with Mino time as the parameter.

V. CONCLUSION

We have shown that the local first-order conservative self-force dynamics in the Schwarzschild spacetime is an integrable Hamiltonian system, to linear order in the mass ratio, for generic stable bound orbits outside the zoom-whirl regime (more specifically, for all orbits to the right of the "singular curve" in Figure 1 of Ref. [30]). The Hamiltonian system is defined by Eqs. (4.1), with the coordinates $(\bar{q}^\alpha, \bar{J}_\alpha)$ defined in terms of the geodesic action-angle coordinates by Eqs. (4.2), and with the Fourier modes of the functions χ^α and ζ_α given by Eqs. (4.7) and (4.8). The quantities $-\bar{J}_t$ and $\bar{J}_\phi$ are well-defined functions on the worldline phase space, which are perturbed versions of the geodesic energy E and angular momentum L_z, which are instantaneously conserved to linear order under the conservative-self-forced motion.

Finally, we remark that the obstacles encountered by our construction in the general Kerr case do not show that the system is not Hamiltonian or integrable in Kerr. As mentioned in the introduction, there is a heuristic argument indicating that the Kerr dynamics should be integrable and Hamiltonian.

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Appendix A: Comments

1. Detweiler-Whiting formulation of the self-force; non-local and local versions

As shown by Detweiler and Whiting [33], self-forced motion in a background spacetime with metric $g_{\mu\nu}$ is equivalent to geodesic motion in a perturbed spacetime with metric $g_{\mu\nu} + h^{\rm R}_{\mu\nu}$. The self-forced worldline equation of motion (3.1) can be written as

$$u^\nu \nabla_\nu u^\mu = -\left(g^{\mu\nu} + u^\mu u^\nu\right) u^\rho u^\sigma \left(\nabla_\rho h^{\rm R}_{\sigma\nu} - \tfrac{1}{2}\nabla_\nu h^{\rm R}_{\rho\sigma}\right), \tag{A1}$$

which is the geodesic equation for the perturbed metric $g_{\mu\nu} + h^{\rm R}_{\mu\nu}$. The appropriate "Regular" metric perturbation $h^{\rm R}_{\mu\nu}$ is given as both a function a field point x and a functional of a source worldline $z(\tau')$ with velocity $u^{\mu'}$, by $h^{\rm R}_{\mu\nu} = \bar{h}^{\rm R}_{\mu\nu} - \frac{1}{2}\bar{h}^{\rm R}_{\rho\sigma} g^{\rho\sigma} g_{\mu\nu}$ and

$$\bar{h}^{\rm R}_{\mu\nu}(x)[z(\tau')] = 4m \int d\tau'\, G^{\rm R}_{\mu\nu\mu'\nu'}\big(x, z(\tau')\big)\, u^{\mu'} u^{\nu'}, \tag{A2}$$

where $G^{\rm R}_{\mu\nu\mu'\nu'}(x, x')$ is the Detweiler-Whiting Regular Green's function. Importantly, in the geodesic equation (A1), the covariant derivatives (associated to the background metric) are understood to differentiate with respect to the field point x, while holding the source worldline $z(\tau')$ fixed, and then to be evaluated at a point $z(\tau)$ on the source worldline:

$$\text{“}\nabla_\mu h^{\rm R}_{\nu\rho}\text{”} = \Big(\nabla^{(x)}_\mu h^{\rm R}_{\nu\rho}(x)[z(\tau')]\Big)_{x=z(\tau)}. \tag{A3}$$

Motion under the conservative part of the self-force is obtained by replacing the Regular Green's function with its symmetric part, $G^{\rm R,\,cons.}_{\mu\nu\mu'\nu'}(x, x') = \frac{1}{2}\big(G^{\rm R}_{\mu\nu\mu'\nu'}(x, x') + G^{\rm R}_{\mu'\nu'\mu\nu}(x', x)\big)$, and is equivalent to geodesic motion in the corresponding perturbed metric $g_{\mu\nu} + h^{\rm R,\,cons.}_{\mu\nu}$. Similarly, one obtains the dissipative dynamics from the antisymmetric part of the Green's function.

In the "local" formulation of the self-force (the dynamics considered in Secs. III-V, referred to as Scheme 2 in Sec. I), one replaces the source worldline $z(\tau')$ in Eq. (A2) with the osculating geodesic (o.g.), i.e. the geodesic of the background spacetime determined by the instantaneous position $z(\tau)$ and velocity $u^\mu(\tau)$ of the self-forced worldline, given by

$$z(\tau') \to z_{\rm o.g.}(\tau', z(\tau), u(\tau)) = \exp\left(z(\tau), \tau' u(\tau)\right), \tag{A4}$$

where $\exp(x, v)$ is the (background) exponential map of a vector v^μ at a point x. Via Eq. (A2), this defines a metric perturbation as a function of a field point x and, separately, of the position $z(\tau)$ and velocity $u^\mu(\tau)$,

$$h^{\rm R,\,o.g.}_{\mu\nu}(x, z(\tau), u(\tau)) = h^{\rm R}_{\mu\nu}(x)[z_{\rm o.g.}]. \tag{A5}$$

The local self-force equation of motion is then given again by Eq. (A1) with $h^{\rm R}_{\mu\nu} \to h^{\rm R,\,o.g.}_{\mu\nu}$, with the understanding that ∇_μ differentiates only with respect to x, holding $z(\tau)$ and $u^\mu(\tau)$ fixed, evaluating after differentiation at $x = z(\tau)$. Conservative and dissipative parts are defined just as above.

2. Actions and Hamiltonians

Given a *fixed* metric perturbation $h_{\mu\nu}(x)$, geodesic motion of a worldline $z(\tau)$ in the metric $g_{\mu\nu} + h_{\mu\nu}$ follows from the action functional

$$S[z(\tau)] = \frac{1}{2}\int d\tau \Big(g_{\mu\nu}(z) + h_{\mu\nu}(z)\Big) u^\mu u^\nu, \tag{A6}$$

with a Hamiltonian formulation following from a straightforward Legendre transformation. This remains true for the case of the self-force, as described above, only if one holds fixed the source-worldline dependence of the metric perturbation, varying only with respect to the field-point dependence. (Note that such a scheme can accommodate both the conservative and dissipative parts of the self-force.) However, this scheme does not provide a "true" action or Hamiltonian, in the usual sense, for the self-force, due to the need to hold fixed the source worldline, identifying it with the dynamical worldline only after variation.

It is natural to ask what happens when one replaces $h_{\mu\nu}(z)$ in the action (A6) with the non-local or local metric perturbation evaluated at $x = z(\tau)$, i.e. $h^{\mathrm{R}}_{\mu\nu}(z(\tau))[z(\tau')]$ or $h^{\mathrm{R,\,o.g.}}_{\mu\nu}(z(\tau), z(\tau), u(\tau))$, and then *fully* varies with respect to the worldline $z(\tau)$. In both cases, one does not recover the correct equations of motion.

In the non-local case, one can find a non-local action functional which yields the correct (conservative) dynamics by simply inserting a factor of 1/2 in front of the metric perturbation,

$$S_{\text{non-local}}[z(\tau)] = \frac{1}{2}\int d\tau \left(g_{\mu\nu}(z) + \frac{1}{2}h^{\mathrm{R}}_{\mu\nu}(z)[z]\right) u^\mu u^\nu, \tag{A7}$$

with the caveat that, even if one uses here the full metric perturbation or Green's function, its symmetric part is singled out by the structure of the double integral [from substituting Eq. (A2)], and one thus obtains only the conservative part of the dynamics. This 1/2 prescription works because of the just-mentioned symmetry; the variations with respect to the field- and source-point dependences give equal contributions, adding to twice the desired result. While such a non-local action principle is formally satisfying, it is not clear how it might be useful in self-force calculations, or if there exist associated versions of Noether's theorem or Hamiltonian formulations, due to the non-locality.

To obtain a "true" *local* action principle, one is then tempted to apply to the non-local action (A7) the same "reduction of order" procedure which yields the local equation of motion from the non-local one, namely, replacing the source worldline with the osculating geodesic, as in Eq. (A4),

$$S_{\text{local}}[z(\tau)] \neq \frac{1}{2}\int d\tau \left(g_{\mu\nu}(z) + \frac{1}{2}h^{\mathrm{R,\,o.g.}}_{\mu\nu}(z,z,u)\right) u^\mu u^\nu. \tag{A8}$$

The equations of motion resulting from this action principle have been discussed in Ref. [34], where it was shown via numerical calculations that they do not reproduce the correct conservative self-force, by $O(1)$ errors. The order-reduction procedure and the variation of the action do not commute.

While it is conceivable that further investigations along these general lines of thought (which do not depend on the specific background spacetime) could lead to a true local action principle (and thus a Hamiltonian) for the conservative self-force, it might also be the case that such a construction exists only in specific spacetimes. We have contented ourselves in the body of this paper to show that a true local Hamiltonian exists in the Schwarzschild spacetime.

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