SU(2) gauge field theories, gauge-invariant angular momenta, and a Coulomb theorem: A new viewpoint on the resolution of the nucleon spin crisis

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SU(2) gauge field inner structure, gauge invariant angular momenta and new Coulomb theorem in general field theory: new viewpoint to the final resolution of the nucleon spin crisis

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We investigate the inner structure of a general SU(2) (naturally including SO(3)) symmetry system—the fermion-gauge field interaction system, and achieve naturally a set of gauge invariant spin and orbital angular momentum operators of fermion and gauge fields by Noether theorem in general field theory. Some new relations concerning non-Abelian field strengths are discovered, e.g., the covariant transverse condition, covariant parallel condition (i.e., non-Abelian divergence, non-Abelian curl) and simplified SU(2) Coulomb theorem. And we show that the condition that Chen et al obtained to construct their gauge invariant angular momentum operators is a result of some fundamental equations in the general field theory. The results obtained in this paper present a new perspective to look at the overall structure of the gauge field, and provide a new viewpoint to the final resolution of the nucleon spin crisis in the general field theory. Specially, the achieved theory in this paper can calculate the strong interactions with isospin symmetry and solves the serious problem without gauge invariant angular momenta in strong interaction systems with isospin symmetry, and then the achieved predictions in the calculations can be exactly measured by particle physics experiments due to their gauge invariant properties.

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Introduction: In 1954, C. N. Yang and R. Mills, for the first time, generalized the abelian gauge theory (quantum electrodynamics) to nonabelian SU(2) gauge theory to present an explanation for strong interactions [1]. The idea has been developed as the useful Yang-Mills field theory. Isospin for SU(2) in particle physics is related to the strong interaction, the particles can be treated as different states with isospin values related to the number of charged states.

The nonabelian SU(2) gauge theory has very important uses in different fields, e.g., genuine magnetic monopoles can be created as regular solutions of the field equations with SU(2) or SO(3) gauge symmetry [2]; K. Langfeld and E.M.Ilgenfritz gave the confinement from semiclassical gluon fields in SU(2) gauge theory [3]; P. V. Buividovich, et al presented Magnetic-Field-Induced insulator-conductor transition in SU(2) quenched lattice gauge theory [4].

Ref.[5] studied QCD at finite isospin density, spin-isospin resonances and the neutron skin of nuclei were explored [6], Ref.[7] described isospin dependence in the odd-even staggering of nuclear binding energies, exact solution of the spin-isospin proton-neutron pairing Hamiltonian was presented [8], Ref.[9] investigated spin-isospin resonances: a self-consistent covariant description; isospin splittings in the Light-Baryon octet from lattice QCD and QED are given [10]. But up to now all relevant isospin interaction works, e.g., Refs.[5–10], have not been able to give both their gauge invariant interactions and the corresponding isospin angular momenta algebra relations in the same time, which lead to key difficulties in measuring their relevant observable physics quantities. The situations here are analogous to that of nucleon spin crisis, the method of this paper may solve these key problems and get an important progress in eventually solving nucleon spin crisis due to the exact similarities between the corresponding physics theory structures of nucleon spin crisis and the strong interactions with isospin symmetry.

The renowned nucleon spin crisis concerns the following question: how does the motions of quarks and gluons, as its constituents, contribute to the total spin of the nucleon? Various studies indicate that the spin of the quarks can only be responsible for one third of the nucleon spin [11–13], and hence, the remaining part must come from other internal motions: the orbital movements of the quarks and gluons as well as the spin of the gluons. Theoretically, in order to understand how these motions contribute to the total nucleon spin, the first job is without doubt to define properly the operators that can fully represent these movements. Various good endeavors have been made [14–16].

In this paper, we generalize our discussion in QED [17] to investigate the inner structure of the fermion-gauge field interaction system when the gauge potential is decomposed. This generalization turns out to be nontrivial and six additional conditions are required. However, impressively, all these non-Abelian conditions can be obtained by the proper rewriting of their Abelian counterparts that have been strictly proved in [17]. Two of the conditions are related to the gauge potential and have already been available in literature [18]. The other four depict the properties of the gauge field strength and are discovered here. Based on the six conditions, gauge invariant definitions of four angular momentum operators are naturally and strictly constructed through Noether theorem. In strong interaction systems with isospin symmetry, there still exist the serious problem that gauge invariant angular momenta are still missing [5–10, 19], this paper wants also to solve the serious problem.

The QED case–a review: For the convenience of the discussion of the non-Abelian system, in this section we will give a brief but comprehensive review of the Abelian gauge potential decomposition [17].

In QED, projection operators are defined as [20]

$$L^j_k = \partial^j \frac{1}{\Delta}\partial_k, T^j_k = \delta^j_k - L^j_k, (\Delta = \partial_k \partial^k) \tag{1}$$

and the Abelian gauge potential A^i is decomposed as

$$A^j_\perp = T^j_k A^k, A^j_\parallel = L^j_k A^k, \tag{2}$$

with its two components satisfying naturally

$$\nabla \cdot \vec{A}_\perp = 0, \tag{3}$$
$$\nabla \times \vec{A}_\parallel = 0. \tag{4}$$

Using Eqs. (3)–(4), the Lagrangian for the electron-photon system can be expanded into a new form [17]

$$\begin{aligned} L^{ep} &= \int d^3x[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}_e \left(i\gamma^\mu D^e_\mu - m\right)\psi_e] \\ &= \int d^3x[\partial_k A_0 \partial^0 A^k_\parallel - \frac{1}{2}\partial_0 A^\perp_k \partial^0 A^k_\perp - \frac{1}{2}\partial_0 A^\parallel_k \partial^0 A^k_\parallel \\ &\quad -\frac{1}{2}\partial_k A_0 \partial^k A^0 - \frac{1}{4}F_{jk}F^{jk} + \overline{\psi}_e \left(i\gamma^\mu D^e_\mu - m\right)\psi_e], \end{aligned} \tag{5}$$

where $D^e_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, the superscripts $'ep'$ stand for the electron and photon, and ψ_e is the electron field. From Eq. (5), the canonical momentum, i.e. the electric field E^i, can be naturally decomposed into two components:

$$E^k_\perp = -\partial^0 A^k_\perp, \tag{6}$$
$$E^k_\parallel = \partial^k A^0 - \partial^0 A^k_\parallel, \tag{7}$$

with $E^i_\perp$ and $E^i_\parallel$, conjugate to $A^k_\perp$ and $A^k_\parallel$, respectively, and satisfying

$$\nabla \cdot \vec{E}_\perp = 0, \tag{8}$$
$$\nabla \times \vec{E}_\parallel = 0, \tag{9}$$
$$\nabla \cdot \vec{E}_\parallel = \rho_e, \tag{10}$$
$$\int d^3x E^k_\perp E^\parallel_k = 0, \tag{11}$$

where ρ_e is the charge density. From the Lagrangian (5) and by Noether theorem, we can get the total angular momentum of the system:

$$\begin{aligned} \vec{J}^{ep}_1 &= \int d^3x \psi^\dagger_e \frac{1}{2} \vec{\Sigma}\, \psi_e + \int d^3x \psi^\dagger_e\, \vec{x} \times \frac{1}{i}\nabla\psi_e \\ &+ \int d^3x\, \vec{E}_\perp \times \vec{A}_\perp + \int d^3x\, \vec{E}_\parallel \times \vec{A}_\parallel \\ &+ \int d^3x E^\perp_k\, \vec{x} \times \nabla A^k_\perp + \int d^3x E^\parallel_k\, \vec{x} \times \nabla A^k_\parallel. \end{aligned} \tag{12}$$

The three terms in the left column are all gauge invariant and to combine the three terms in the right column into a gauge invariant one, we add a surface term $\nabla \cdot [\vec{E}_\parallel\, (\vec{A}_\parallel \times \vec{x})]$ to Eq. (12) and it becomes

$$\begin{aligned} \vec{J}^{ep}_2 &= \int d^3x \psi^\dagger_e \frac{1}{2} \vec{\Sigma}\, \psi_e + \int d^3x \psi^\dagger_e\, \vec{x} \times \frac{1}{i}\, \vec{D}^e_\parallel\, \psi_e \\ &+ \int d^3x\, \vec{E}_\perp \times \vec{A}_\perp + \int d^3x E^\perp_k\, \vec{x} \times \nabla A^k_\perp \\ &= \vec{S}^e + \vec{L}^e_2 + \vec{S}^p_2 + \vec{L}^p_2, \end{aligned} \tag{13}$$

where $\vec{D}^e_\parallel = \nabla + ie\, \vec{A}_\parallel$ is the new covariant derivative. $\vec{S}^e$, $\vec{L}^e_2$, $\vec{S}^p_2$ and $\vec{L}^p_2$ stand for the spin and orbital angular momenta of the electron and photon, respectively. Because $\vec{A}_\parallel$ has a vanishing curl (Eq. (4)), $\vec{L}^e_2$ satisfies the commutation algebra, $\vec{L}^e_2 \times \vec{L}^e_2 = i\, \vec{L}^e_2$; the gauge independence of $\vec{A}_\perp$ [17] guarantees that $\vec{S}^p_2$ and $\vec{L}^p_2$ are gauge invariant, which is to say that, from the Lagrangian (5) and by Noether theorem, we realize a gauge invariant separation of the photon's total angular momentum. Adding to Eq. (13) another surface term $\nabla \cdot\, [\vec{A}_\perp\, (\vec{E}_\parallel \times \vec{x})]$, we arrive at Chen et al's result [14]

$$\begin{aligned} \vec{J}^{ep}_3 &= \int d^3x \psi^\dagger_e \frac{1}{2} \vec{\Sigma}\, \psi_e + \int d^3x \psi^\dagger_e\, \vec{x} \times \frac{1}{i}\, \vec{D}^e_\parallel\, \psi_e \\ &+ \int d^3x\, \vec{E} \times \vec{A}_\perp + \int d^3x E_k\, \vec{x} \times \nabla A^k_\perp \\ &= \vec{S}^e + \vec{L}^e_3 + \vec{S}^p_3 + \vec{L}^p_3, \end{aligned} \tag{14}$$

the difference of which from Eq. (13) is the substitution of E_i for $E_{\perp i}$. $\vec{S}^p_3$ and $\vec{L}^p_3$ are still gauge invariant.

The SU(2) case: As a generalization of the QED case, in this section, we are going to investigate a general SU(2) (naturally including SO(3) case due to their identical Lie algebra structure) symmetry system by the general method of field theory. The general Lagrangian for the SU(2) fermi-gauge field interaction system is [1, 19]

$$L = \int d^3x[-\frac{1}{4}F^{a\mu\nu}F_{a\mu\nu} + \overline{\psi}\left(i\gamma^\mu \mathbb{D}_\mu - m\right)\psi], \tag{15}$$

where $\mathbb{D}_\mu = \partial_\mu - ig\mathbb{W}_\mu$ is the covariant derivative with $\mathbb{W}_\mu = W^a_\mu T^a$ being the gauge potential and T^a the generator of the $SU(2)$ group, and ψ stands for the fermi field. In SU(2) gauge field theory, the gauge field strength is defined as

$$F^{a\mu\nu} = \partial^\mu W^{a\nu} - \partial^\nu W^{a\mu} + gf^{abc}W^{b\mu}W^{c\nu}. \tag{16}$$

We decompose the spacial part of the gauge potential W^{ak} into two components:

$$W^{ak} = W_\perp^{ak} + W_\parallel^{ak}. \tag{17}$$

Analogous to QED [17], under a $SU(2)$ transformation U, $\mathbb{W}_\perp^k = W_\perp^{ak} T^a$ and $\mathbb{W}_\parallel^k = W_\parallel^{ak} T^a$ transform as [14]

$$\mathbb{W}_\perp^{\prime k} = U \mathbb{W}_\perp^k U^\dagger, \tag{18}$$

$$\mathbb{W}_\parallel^{\prime k} = U \mathbb{W}_\parallel^k U^\dagger - \frac{i}{g} U \partial^k U^\dagger, \tag{19}$$

respectively, which guarantee that

$$\mathbb{W}^{\prime k} = U \mathbb{W}^k U^\dagger - \frac{i}{g} U \partial^k U^\dagger. \tag{20}$$

Due to its nonlinearity, the non-Abelian system is much more complicated than the Abelian one and it is difficult to define a similar pair of projection operators as Eq. (1) and to derive naturally all the conditions that the gauge potentials and its conjugate momenta satisfy as Eqs. (3)–(4) and Eqs. (8)–(10) have shown. However, we just generalize the six Abelian conditions to the non-Abelian case with their forms restricted by the gauge covariance and the theory's consistency. All the generalized conditions are regarded to be valid axiomatically and as our discussion processes, their indispensability will become apparent. The following two are generalizations of Eqs. (3) and (4) and related to the gauge potentials [18]

$$\partial_k W_\perp^{ak} + g f^{abc} W_{\parallel k}^b W_\perp^{ck} = 0. \tag{21}$$

$$\partial^j W_\parallel^{ak} - \partial^k W_\parallel^{aj} + g f^{abc} W_\parallel^{bj} W_\parallel^{ck} = 0, \tag{22}$$

It is easy to prove that Eqs. (21) and (22) are $SU(2)$ gauge covariant.

In terms of $W_\perp^{ak}$ and $W_\parallel^{ak}$, F^{ak0} can be rewritten as

$$\begin{aligned} F^{ak0} = {} & \partial^k W^{a0} - \partial^0 W_\parallel^{ak} - \partial^0 W_\perp^{ak} \\ & + g f^{abc} W_\parallel^{bk} W^{c0} + g f^{abc} W_\perp^{bk} W^{c0}. \end{aligned} \tag{23}$$

Using Eqs. (21), (23) and $f^{abc} f^{ab'c'} = \delta^{bb'}\delta^{cc'} - \delta^{bc'}\delta^{cb'}$ as well as two other conditions

$$\int d^3x \partial^0 W_\parallel^{ak} (-\partial_0 W_{\perp k}^a + g f^{abc} W_{\perp k}^b W^{c0}) = 0, \tag{24}$$

$$\begin{aligned} & g f^{abc} (\partial_0 W_\parallel^{ak} W_{\perp k}^b W^{c0} - \partial^k W^{a0} W_{\perp k}^b W_0^c) \\ & - g^2 (W_\parallel^{bk} W^{c0} W_{\perp k}^b W_0^c - W_\parallel^{bk} W^{c0} W_{\perp k}^c W_0^b) = 0, \end{aligned} \tag{25}$$

the meanings of which will be discussed later, we can expand the Lagrangian (15) to the following form:

$$\begin{aligned} L = \int d^3x [& -\frac{1}{4} F^{ajk} F_{jk}^a - \frac{1}{2} \partial^k W^{a0} \partial_k W_0^a \\ & - \frac{1}{2} \partial^0 W_\parallel^{ak} \partial_0 W_{\parallel k}^a - \frac{1}{2} \partial^0 W_\perp^{ak} \partial_0 W_{\perp k}^a \\ & + \partial^k W^{a0} \partial_0 W_{\parallel k}^a - g f^{abc} \partial^k W^{a0} W_{\parallel k}^b W_0^c \\ & + g f^{abc} \partial^0 W_\parallel^{ak} W_{\parallel k}^b W_0^c + g f^{abc} \partial^0 W_\perp^{ak} W_{\perp k}^b W_0^c \\ & - \frac{1}{2} g^2 (W_\parallel^{bk} W^{c0} W_{\parallel k}^b W_0^c - W_\parallel^{bk} W^{c0} W_{\parallel k}^c W_0^b) \\ & - \frac{1}{2} g^2 (W_\perp^{bk} W^{c0} W_{\perp k}^b W_0^c - W_\perp^{bk} W^{c0} W_{\perp k}^c W_0^b) \\ & + \overline{\psi} (i\gamma^\mu \mathbb{D}_\mu - m) \psi]. \end{aligned} \tag{26}$$

From the expanded Lagrangian (26), we derive the canonical momenta conjugate to $W_{\parallel k}^a$ and $W_{\perp k}^a$, respectively, as follows

$$\begin{aligned} \pi_\parallel^{ak} & = \frac{\delta L}{\delta(\partial_0 W_{\parallel k}^a)} \\ & = -\partial^0 W_\parallel^{ak} + \partial^k W^{a0} + g f^{abc} W_\parallel^{bk} W^{c0}, \end{aligned} \tag{27}$$

$$\begin{aligned} \pi_\perp^{ak} & = \frac{\delta L}{\delta(\partial_0 W_{\perp k}^a)} \\ & = -\partial^0 W_\perp^{ak} + g f^{abc} W_\perp^{bk} W^{c0}, \end{aligned} \tag{28}$$

which reduce to the QED forms (6) and (7) when the structure constant f^{abc} is zero. What's more, it is apparent that Eqs. (27) and (28) are both gauge covariant and their summation is exactly the conventional π^{ak}. The form of Eq. (28) is the same as the discussion in [21].

By Noether theorem, the Lagrangian (26) leads to the angular momentum of the SU(2) fermi-gauge field interaction system:

$$\begin{aligned} \vec{J_1} = & \int d^3x \psi^\dagger \frac{1}{2} \vec{\Sigma} \psi + \int d^3x \psi^\dagger \vec{x} \times \frac{1}{i} \nabla \psi \\ & + \int d^3x \, \vec{\pi_\perp^a} \times \vec{W_\perp^a} + \int d^3x \, \vec{\pi_\parallel^a} \times \vec{W_\parallel^a} \\ & + \int d^3x \pi_{\perp k}^a \vec{x} \times \nabla W_\perp^{ak} + \int d^3x \pi_{\parallel k}^a \vec{x} \times \nabla W_\parallel^{ak}. \end{aligned} \tag{29}$$

Actually, Eq. (29) is a straightforward generalization of Eq. (12). Let us consider the surface term

$$\begin{aligned} & \nabla \cdot [\vec{\pi_\parallel^a} (\vec{W_\parallel^a} \times \vec{x})] \\ & = - \vec{\pi_\parallel^a} \times \vec{W_\parallel^a} - \pi_{\parallel i}^a \vec{x} \times \nabla W_\parallel^{ai} \\ & \quad - (\partial_k \pi_\parallel^{ak} + g f^{abc} W_{\parallel k}^b \pi_\parallel^{ck}) (\vec{x} \times \vec{W_\parallel^a}), \end{aligned} \tag{30}$$

where Eq. (22) is used. As a gauge covariant generalization of Eq. (10), we deduce the simplified SU(2) Coulomb theorem:

$$\partial_k \pi_\parallel^{ak} + g f^{abc} W_{\parallel k}^b \pi_\parallel^{ck} = \rho^a = g \psi^\dagger T^a \psi, \tag{31}$$

where ρ^a is the non-Abelian charge density. Adding Eqs. (30) and (31) to $\vec{J_1}$, we have

$$\begin{aligned} \vec{J_2} = & \int d^3x \psi^\dagger \frac{1}{2} \vec{\Sigma} \psi + \int d^3x \psi^\dagger \vec{x} \times \frac{1}{i} \vec{\mathbb{D}}_\parallel \psi \\ & + \int d^3x \, \vec{\pi_\perp^a} \times \vec{W_\perp^a} + \int d^3x \pi_{\perp k}^a \vec{x} \times \nabla W_\perp^{ak} \\ = & \vec{S^q} + \vec{L_2^q} + \vec{S_2^g} + \vec{L_2^g}, \end{aligned} \tag{32}$$

where, due to Eq. (19), $\vec{\mathbb{D}}_{\parallel} = \nabla - ig \vec{\mathbb{W}}_{\parallel}$ is the new covariant derivative. $\vec{S^q}$, $\vec{L_2^q}$, $\vec{S_2^g}$ and $\vec{L_2^g}$ stand for the spin and orbital angular momenta of the fermi field and gauge field respectively. Due to the properties of $W_{\parallel}^{ak}$ shown in Eqs. (19) and (22), $\vec{L_2^g}$ satisfies the commutation law $\vec{L_2^q} \times \vec{L_2^q} = i \vec{L_2^q}$ and is gauge invariant [14]. The gauge invariance of $\vec{S^q}$ and $\vec{S_2^g}$ is obvious; however, unlike in QED, rather than invariant, $W_{\perp}^{ak}$ is gauge covariant, which means that $\vec{L_2^g}$ is not trivially gauge invariant, because there will be additional terms containing the spatial derivatives of $U(x)$.

In general, Noether's theorem requires the invariances of the Lagrangian density and Hamiltonian density of the system under the Lorentz transformation of the fundamental fields [20, 22], e.g., the system is invariant not under $W_{\perp}^{\mu}$'s Lorentz transformation but under the Lorentz transformations of W^{μ}'s all components, even though $W_{\perp}^{\mu}$'s Lorentz transformation is complicated, because the relevant complicated terms of the transformations of their all components can be canceled each other in the whole system according to the symmetric invariance property of this system, which is a general rule, see Ref.[20, 22]. Besides, the frame-dependence issue has important physics meaning, Refs.[23–25] have given the very good investigations and descriptions relevant to the issue and the other problems.

To make $\pi_{\perp k}^{a} \vec{x} \times \nabla W_{\perp}^{ak}$ in the last term of Eq.(32) gauge invariant, one should impose a new condition [14]:

$$g f^{abc} W_{\perp k}^{b} \pi_{\perp}^{ck} = 0, \tag{33}$$

which, as well as Eqs. (24) and (25), can be naturally derived as follows.

First, the Coulomb law in QCD is [22]

$$\partial_k \pi^{ak} + g f^{abc} W_k^b \pi^{ck} = \rho^a, \tag{34}$$

and referring to the structure of Eq. (21), we generalize Eq. (8) to

$$\partial_k \pi_{\perp}^{ak} + g f^{abc} W_{\parallel k}^{b} \pi_{\perp}^{ck} = 0. \tag{35}$$

As a consequence of Eqs. (31), (34) and (35), we have

$$g f^{abc} W_{\perp k}^{b} \pi^{ck} = 0. \tag{36}$$

Then, substituting Eq. (28) into Eq. (35) and using Eq. (21), we can prove that

$$\begin{aligned} &\partial_k \pi_{\perp}^{ak} + g f^{abc} W_{\parallel k}^{b} \pi_{\perp}^{ck} \\ &= g^2 W_{\parallel k}^{a} W_{\perp}^{bk} W^{b0} - g^2 W_{\parallel k}^{b} W_{\perp}^{bk} W^{a0} \\ &\quad + g f^{abc} W_{\perp}^{bk} \partial_k W^{c0} - g f^{abc} \partial^0 W_{\parallel k}^{c} W_{\perp}^{bk}. \end{aligned} \tag{37}$$

Multiplying its right side by W_0^a and rearranging the SU(2) group indices, Eq. (37) results in Eq. (25).

Moreover, we rewrite the right side of Eq. (37) as

$$\begin{aligned} &g f^{abc} W_{\perp}^{bk} (g f^{ca'b'} W_{\parallel k}^{a'} W^{b'0} + \partial_k W^{c0} - \partial^0 W_{\parallel k}^{c}) \\ &= g f^{abc} W_{\perp}^{bk} \pi_{\parallel k}^{c} = 0. \end{aligned} \tag{38}$$

It is easy to see that Eq. (33) can be obtained by detracting Eq. (38) from Eq. (36), i.e. we show that Eq. (33) is a natural result of Eqs. (31), (34) and (35).

Besides, using Eqs. (27), (28) and (35) as well as the partial integral, we prove that Eq. (24) is just the orthogonal relation (in the sense of the whole space) of the two components of the canonical momentum, i.e.

$$\int d^3x \pi_{\parallel}^{ak} \pi_{\perp k}^{a} = 0. \tag{39}$$

which is a non-Abelian generalization of Eq. (11).

As a result, all the assumptions we have made so far to keep our discussion consistent, i.e. Eqs. (24), (25) and (33), can be deduced from Eqs. (21), (31), (35) and (39), which are directly generalized from the Abelian conditions in a gauge covariant way.

Now, let's consider the following surface term,

$$\begin{aligned} &\nabla \cdot [\vec{W_{\perp}^{a}} (\vec{\pi_{\parallel}^{a}} \times \vec{x})] \\ &= \vec{\pi_{\parallel}^{a}} \times \vec{W_{\perp}^{a}} + \pi_{\parallel}^{ak} \vec{x} \times \nabla W_{\perp k}^{a} \\ &\quad + W_{\perp k}^{a} \varepsilon^{lmn} x_n \vec{e_l} (\mathbb{D}_{\parallel}^{k} \pi_{\parallel m}^{a} - \mathbb{D}_{\parallel m} \pi_{\parallel}^{ak}), \end{aligned} \tag{40}$$

where Eqs. (21), (22) and (38) are used ($\vec{e_l}$ is the spatial unit vector), and

$$\mathbb{D}_{\parallel}^{k} \pi_{\parallel m}^{a} = \partial^k \pi_{\parallel m}^{a} + g f^{abc} W_{\parallel}^{bk} \pi_{\parallel m}^{c}. \tag{41}$$

As a generalization of the Abelian Eq. (9), we set

$$\mathbb{D}_{\parallel}^{k} \pi_{\parallel m}^{a} - \mathbb{D}_{\parallel m} \pi_{\parallel}^{ak} = 0. \tag{42}$$

Though slightly different from Eq. (22), the form of Eq. (42) guarantees its gauge covariance. Adding Eqs. (40) and (42) to Eq. (32), we get Chen et al's separation of the angular momentum [14]:

$$\begin{aligned} \vec{J_3} &= \int d^3x \psi^{\dagger} \frac{1}{2} \vec{\Sigma} \psi + \int d^3x \psi^{\dagger} \vec{x} \times \frac{1}{i} \vec{\mathbb{D}}_{\parallel} \psi \\ &\quad + \int d^3x \, \vec{\pi^a} \times \vec{W_{\perp}^{a}} + \int d^3x \pi_k^a \vec{x} \times \nabla W_{\perp}^{ak} \\ &= \vec{S^q} + \vec{L_3^q} + \vec{S_3^g} + \vec{L_3^g}, \end{aligned} \tag{43}$$

However, we here show that Eq. (43) can be deduced strictly and naturally from the Lagrangian (15) by Noether theorem in general field theory, i.e. Eq. (43) is spontaneously involved in our theory. Similar to Eq. (33), the gauge invariance of $\vec{L_3^g}$ requires the condition (36) (or more precisely, Eqs. (31) and (35)), which also plays a key role in transforming Eq. (43) to the conventional form [14]

$$\begin{aligned}\vec{J}_4 &= \int d^3x\psi^\dagger \frac{1}{2}\vec{\Sigma}\,\psi + \int d^3x\psi^\dagger\,\vec{x}\times\frac{1}{i}\vec{\nabla}\,\psi \\ &+ \int d^3x\,\vec{\pi}^a\times\vec{W}^a + \int d^3x\pi_k^a\,\vec{x}\times\nabla W^{ak} \\ &= \vec{S}^q + \vec{L}_4^q + \vec{S}_4^g + \vec{L}_4^g, \end{aligned} \tag{44}$$

which means that, under certain conditions, $\vec{J}_2$, $\vec{J}_3$ and $\vec{J}_4$ will give the same value of the total angular momentum. Apparently, however, the last three terms of $\vec{J}_4$ are all gauge dependent, so $\vec{J}_4$ is not properly defined. By contrast, both $\vec{J}_2$ and $\vec{J}_3$ are physically sound. Nevertheless, $\vec{J}_2$ is the simplest and the most rational one for the sake of simplicity, since it eliminates the terms that fail to contribute to the total angular momentum, as Eq. (40) shows.

Discussion and conclusion: The nonlinearity of the non-Abelian system causes considerable complexity and the theory's consistency spontaneously demands the establishments of six equations: (21), (22), (31), (35), (39) and (42), the rationality of which lies in the fact that they can be straightforwardly and uniformly generalized from their Abelian counterparts with the consideration of gauge covariance, and will reduce to the Abelian forms when f^{abc} is zero. To illustrate this more explicitly, referring to Eq. (41), we rewrite the six conditions in a compact form:

$$\partial^j W_\parallel^{ak} - \partial^k W_\parallel^{aj} + g f^{abc} W_\parallel^{bj} W_\parallel^{ck} = 0, \tag{45}$$

$$\mathbb{D}_\parallel^j \pi_\parallel^{ak} - \mathbb{D}_\parallel^k \pi_\parallel^{aj} = 0, \tag{46}$$

$$\mathbb{D}_{\parallel k} W_\perp^{ak} = 0, \tag{47}$$

$$\mathbb{D}_{\parallel k} \pi_\perp^{ak} = 0, \tag{48}$$

$$\mathbb{D}_{\parallel k} \pi_\parallel^{ak} = \rho^a, \tag{49}$$

$$\int d^3x \pi_\parallel^{ak} \pi_{\perp k}^a = 0, \tag{50}$$

Unlike Eqs. (4) and (9) in QED, Eq. (45) does not have the same formation as Eq. (46), and because of the gauge dependence of $W_\parallel^{ak}$, Eq. (45) must possess the same structure as the definition of the gauge field strength (16). Actually, it is just a coincidence that Eq. (9) has the same structure as Eq. (4), since the form of the field strength in QED is a curl.

As generalizations of Eqs. (3) and (4), Eqs. (45) and (47) are obtained by Wang et al [18]. Except Eqs. (45) and (47), which are related to the gauge potentials, all the other four conditions that concern the gauge field strengths are discovered here. Corresponding to Eqs. (9) and (8), Eqs. (46) and (48) are the non-Abelian curl and divergence conditions (i.e., the non-Abelian parallel and transverse conditions), respectively; Eq. (49) is generalized from Eq. (10) and is the new SU(2) Coulomb law; the orthogonal relation (50) has the similar form as the Abelian one (11). In addition, we also show that the condition (36) imposed by Chen et al [14], which is essential to keep $\vec{L}_3^g$ gauge invariant, is just a direct consequence of Eqs. (48) and (49). Under the six conditions, we obtain naturally the simplest gauge invariant separation of the total angular momentum (32) of a general SU(2) system—the fermion-gauge field interaction system by Noether theorem in general field theory.

The discovery of the inner-structure similarity of the Abelian and non-Abelian systems depicts vividly the consistency of the gauge field theory and will deepen our understanding of some fundamental problems available in physics. Because of the universality of Eqs. (45)-(50), they can be applied to treat the inner structure and other aspects of any gauge system.

As in general quantum field theory, because the achieved Eq.(32) is just a general expression of angular momentum of a Lorentz vector and is deduced by a general method in a general field theory. Furthermore, the general expression satisfies not only gauge invariant property but also angular momentum commutation relation, which satisfy the demand of the consistence of a general physical system in a general field theory, make physical measurement no dependent on gauge transformation and insure the consistence of angular momentum commutation relation. These are not satisfied simultaneously in a general field theory in the past, however, this paper, for the first time, makes these satisfied in the same time in a general field theory, without artificial choice.

In Eq. (32), $\vec{\pi}_\perp$ is used to construct the gluon spin and orbital angular momentum operators, which is the equivalent to using $\vec{\pi}$ to do the same thing, while their differences are adding the two terms of equating zero (surface term Eq.(40) that is equal to zero in its integration expression and the natural and exact non-curl condition (42) of any longitudinal component vector field) to Eq.(32) in deducing Eq.(43), so they have practical physics meanings according to general field theory, e.g., as Eq.(43) does. Namely, we not only give the simplest expression (32) but also achieve the current expression (43), and uncover the relation between Eq.(32) and Eq.(43).

Because SU(2) gauge group has the exact contraction relation of structure constants: $f^{abc}f^{ab'c'} = \delta^{bb'}\delta^{cc'} - \delta^{bc'}\delta^{cb'}$, which has been used in deducing the theory of this paper, but for SU(3) gauge theory there is no such simple or exact relation, therefore, generalizing the SU(2) gauge theory to SU(3) gauge theory etc will have to be written in our following works. And one can see that after the general generalization, the non-Abelian SU(2) gauge theory is very different from the original Abelian QED theory, thus this paper opens a door of investigating lots of strong interactions with isospin quantum numbers.

In this paper, especially, the usefulness of the achieved expressions and their very important physical implications are bringing hope to the resolution of the nucleon spin crisis. There are two main reasons. First it is because the general generalization of the SU(2) theory to SU(3) QCD theory is very direct and their mathematical expressions are very similar, while the second reason is that the theory of longitudinal and transverse fields of Abelian gauge fields exactly reflects the real physics in QED, so does the theory of longitudinal and transverse fields of non-Abelian gauge fields in a general SU(2) symmetry system, i.e., the presented theory of the fermion-gauge field interaction system in this paper does very similarly. Many research works, including the different gauge theories of the fundamental interactions of the universe, need to be renewed by using the new general theory of decomposed non-Abelian gauge fields. And lots of applications of this papers theory are being written in following papers.

This paper solves the serious problem that there is no gauge invariant angular momenta in strong interaction systems with isospin symmetry, which is the very key problem that is very similar to the nucleon spin crisis. Therefore, this paper can unifyingly give solutions to the both of the two very critical problems. Specially, we want to stress that the achieved theory in this paper can be utilized to calculate the strong interactions with SU(2) isospin symmetry theory and give the precise predictions, and further the achieved predictions in the calculations can be exactly measured by current particle physics experiments due to their gauge invariant properties, because any physical quantity (relative to gauge transformations) without gauge invariant property cannot be exactly measured [14].

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