Mapping the Chevallier-Polarski-Linder parametrization onto physical dark energy Models

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We examine the Chevallier-Polarski-Linder (CPL) parametrization, in the context of quintessence and barotropic dark energy models, to determine the subset of such models to which it can provide a good fit. The CPL parametrization gives the equation of state parameter w for the dark energy as a linear function of the scale factor a, namely $w = w_0 + w_a(1 - a)$. In the case of quintessence models, we find that over most of the w_0, w_a parameter space the CPL parametrization maps onto a fairly narrow form of behavior for the potential $V(\phi)$, while a one-dimensional subset of parameter space, for which $w_a = \kappa(1 + w_0)$, with κ constant, corresponds to a wide range of functional forms for $V(\phi)$. For barotropic models, we show that the functional dependence of the pressure on the density, up to a multiplicative constant, depends only on $w_i = w_a + w_0$ and not on w_0 and w_a separately. Our results suggest that the CPL parametrization is not optimal for testing either type of model.

I.   INTRODUCTION

Observations of the accelerated expansion of the universe [1–7] indicate that approximately 70% of the energy density in the universe is in the form of a negative-pressure component, called dark energy, with the remaining 30% in the form of nonrelativistic matter (including both baryons and dark matter). The dark energy component can be characterized by its equation of state parameter, w, defined as the ratio of the dark energy pressure to its density:

$$w = p/\rho, \tag{1}$$

where a cosmological constant, Λ, corresponds to $w = -1$ and $\rho = constant$. While a model with a cosmological constant and cold dark matter (ΛCDM) is consistent with current observations, there are many models for dark energy that predict a dynamical equation of state. These include, for example, quintessence models, in which the dark energy arises from a time-dependent scalar field, ϕ [8–12]. (See Ref. [13] for a review), and barotropic models, in which the pressure is simply a prescribed function of the density [14–24].

In practice, an enormous number of dynamical models have been proposed for the dark energy. Hence, it has been considered useful to classify such models in terms of simple parametrizations for w. The simplest possibility is to take w to be a constant in time. However, there are a variety of problems with the assumption of constant w. With the exception of $w = -1$, a constant value for w does not arise naturally in the context of most physically-motivated models. Further, the assumption of constant w can provide wildly inaccurate results in the case that w does, in fact, evolve with time [25].

The next level of complexity is a two-parameter model for w as a function of the scale factor, a, or equivalently, of the redshift, z. By far the most widely-used of such parametrizations is the Chevallier-Linder-Polarski (CPL) parametrization, which takes w to be a linear function of the scale factor, namely [26, 27]:

$$w = w_0 + (1 - a)w_a, \tag{2}$$

where w_0 and w_a are constants. Other two-parameter approximations have been discussed in the literature [29, 30]; for a review of these and related approaches, see Ref. [31]. The CPL parametrization describes fairly gradual evolution from a value of $w = w_0 + w_a$ at early times to a present-day value of $w = w_0$. It has several advantages, not least the fact that it is well-behaved all of the way from $a = 0$ to $a = 1$ and, for a variety of models, it can reproduce the predicted observable quantities (distance or Hubble parameter as a function of redshift) to extraordinarily high accuracy [28]. Despite the variety of proposed parametrizations, the CPL approximation has become the de facto standard two-parameter description for the evolution of the dark energy equation of state parameter (e.g., [4–7]).

Because of its importance, it is worthwhile to examine the applicability of the CPL parametrization in more detail. While it is possible to simply treat the CPL parametrization as a convenient heuristic description of the (unknown) dark energy component, it is also useful to understand the mapping between physical models for dark energy and the CPL parametrization. It has been claimed [27] that Eq. (2) provides a good fit to $w(a)$ for a wide variety of quintessence models. In Ref. [32], it was shown that thawing quintessence models with a nearly flat potential all converge toward the behavior given by Eq. (2), with $w_a \approx -1.5(1 + w_0)$. A similar result using different techniques was derived in Refs. [28, 33], with w_a given by $w_a \approx -1.58(1 + w_0)$. This relation is consistent with results obtained using Monte Carlo sampling of a variety of different quintessence potentials [34]. A linear relation between w_a and

$1 + w_0$ is also evident in the much earlier work of Kallosh et al. [35] for the special case of the linear potential. On the other hand, the evolution of $w(a)$ for evolution near a local maximum of the potential (hilltop quintessence) can deviate strongly from a linear form [36].

Hence, it is interesting to determine precisely what physical models are well-described by the CPL parametrization, and which are not. This is, however, a difficult proposition. There are an infinite number of possible quintessence potentials, and even if one could sample the entire space of possibilities, the question remains of defining a "good" fit to Eq. (2). We therefore take the opposite approach here: we begin with Eq. (2) and determine, for two types of models (quintessence and barotropic fluids) precisely which models this parametrization maps onto. This approach has the advantage that one can, in a systematic way, scan over all possible values of w_0 and w_a, and then look for patterns in the corresponding physical models. This mapping from the CPL parametrization to quintessence models has been previously explored by Padmanabhan and Choudhury [37] and by Guo, Ohta, and Zhang [38], both of which examined a single pair of w_0, w_a values. Barboza et al. [39] investigated a more extensive parameter range, but for a different parametrization. Here we extend this earlier work by scanning over the full range of w_0 and w_a values. We then perform a similar mapping for barotropic models, which have been comparatively less well explored.

In the next section we examine the mapping of the CPL parametrization onto quintessence models and determine the functional form for $V(\phi)$ for this models. In Sec. III, we perform a similar analysis for barotropic models and find the pressure as a function of density for these models. Our conclusions are summarized in Sec. IV.

II.   QUINTESSENCE

In this section, we will consider models in which the dark energy is provided by a minimally-coupled scalar field, ϕ, with equation of motion given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0, \tag{3}$$

where the Hubble parameter H is given by

$$H = \left(\frac{\dot{a}}{a}\right) = \sqrt{\rho_T/3}, \tag{4}$$

and the dot denotes the derivative with respect to time. Here a is the scale factor (taken to be $a = 1$ at the present), ρ_T is the total density, and we work in units for which $8\pi G = 1$. Since we are interested in the evolution of dark energy at relatively late times, we will consider only the contributions of nonrelativistic matter (baryons plus dark matter) to ρ_T, and ignore, e.g., radiation.

The pressure and density of the scalar field are given by

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{5}$$

and

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{6}$$

respectively, and the equation of state parameter, w, is given by equation (1).

Our job is to begin with the CPL parametrization (Eq. 2) and work backwards to derive $V(\phi)$. Such a derivation is provided in Refs. [37–39] (see also the earlier related discussion in Refs. [40, 41]), so we will simply reproduce their results here. First, we note that Eqs. (1), (5), and (6) can be combined to yield

$$V = \frac{1}{2}(1 - w)\rho_\phi. \tag{7}$$

Further, the density of a perfect fluid with equation of state parameter w evolves as

$$\frac{a}{\rho}\frac{d\rho}{da} = -3(1 + w). \tag{8}$$

Taking $w(a)$ to be described by the CPL parametrization (Eq. 2), the density evolves as

$$\rho = \rho_{\phi 0} a^{-3(1+w_0+w_a)} e^{3w_a(a-1)}, \tag{9}$$

where $\rho_{\phi 0}$ is the present-day density. (The 0 subscript will refer to present-day quantities throughout). Note that Eqs. (8) and (9) are not specific to scalar field models; we will use them again when we discuss barotropic fluids in the next section.

Combining Eqs. (7) and (9) gives an expression for the scalar field potential as a function of the scale factor:

$$V(a) = \frac{1}{2}\rho_{\phi 0}(1 - w_0 - w_a + w_a a)a^{-3(1+w_0+w_a)}e^{3w_a(a-1)}. \tag{10}$$

In order to express V as a function of ϕ, we need an expression for $\phi(a)$. From Eqs. (1), (5), and (6), we have

$$\frac{d\phi}{dt} = \sqrt{(1+w)\rho_\phi}, \tag{11}$$

which can be combined with the definition of H (Eq. 4) to give

$$\frac{d\phi}{da} = \frac{\sqrt{(1+w)\rho_\phi}}{aH}. \tag{12}$$

Substituting the CPL expression for w into Eq. (12) and using $H = \sqrt{(\rho_M + \rho_\phi)/3}$, with $\rho_M = \rho_{M0}a^{-3}$, we get

$$\phi = \int \frac{\sqrt{3(1+w_0+w_a-w_a a)}}{\sqrt{1 + (\rho_{M0}/\rho_{\phi 0})a^{3(w_0+w_a)}e^{3w_a(1-a)}}}\frac{da}{a}. \tag{13}$$

Together, Eqs. (10) and (13) give a parametric expression for $V(\phi)$, which is our desired result. Note that Eq. (13) yields an arbitrary additive constant in the expression for $\phi(a)$. However, this simply amounts to a shift in the value of ϕ by an arbitrary constant in the expression for $V(\phi)$, which has no physical significance. We will take $\rho_{M0}/\rho_{\phi 0} \approx 3/7$ in what follows.

Before exploring the functional form for $V(\phi)$ for given values of w_0 and w_a, we first consider the allowed ranges for these two quantities. Limits derived from observational data have tended to converge on a fairly narrow ellipse in the w_0, w_a plane, with negative slope (i.e., smaller w_0 corresponds to larger w_a) [6, 7]. However, the validity of these limits is itself dependent on the assumption that the CPL parametrization is a good fit to the evolution of w. Since we are interested in probing the extent to which the CPL parametrization describes specific dark energy models, we will allow w_0 and w_a to vary outside of these observational limits.

A more fundamental limit comes from Eqs. (1), (5), and (6), which together with the assumption that $\dot{\phi}^2/2$ and $V(\phi)$ are both nonnegative, give the well-known bounds

$$-1 \leq w \leq 1. \tag{14}$$

A less stringent limit comes from the requirement that the evolution correspond to a scalar field rolling downhill in the potential. While upward-rolling fields are physically possible, this behavior is normally a transient phenomenon (unless one is dealing with oscillatory behavior near the minimum of the potential [42]). Hence, a scalar field rolling uphill in a potential today would correspond to yet another coincidence problem to add to the well-known approximate equality of matter and dark energy densities at the present. The requirement that the field be rolling downhall gives the constraint [43]

$$a\frac{dw}{da} > -3(1-w)(1+w). \tag{15}$$

Substituting the CPL parametrization into this equation, and taking $a = 1$ to correspond to the present, we see that this translates into the limit

$$w_a < 3(1 - w_0^2), \tag{16}$$

in order that the field not be rolling uphill today. This limit is not a stringent bound, like Eq. (14), so we will explore the parameter range outside of it, but we note that such models seem somewhat less plausible for the reasons outlined above.

We now determine $V(\phi)$ for a range of values for w_0 and w_a. We normalize $V(\phi)$ to its present day value, V_0, and we use the freedom to shift ϕ by an arbitrary constant to take $\phi = 0$ to correspond to the present. Because the CPL parametrization represents a heuristic fit to the data, we will not require it to hold at arbitrarily early epochs, but only between some initial scale factor a_i and the present. The choice for a_i is somewhat arbitrary, but we will take
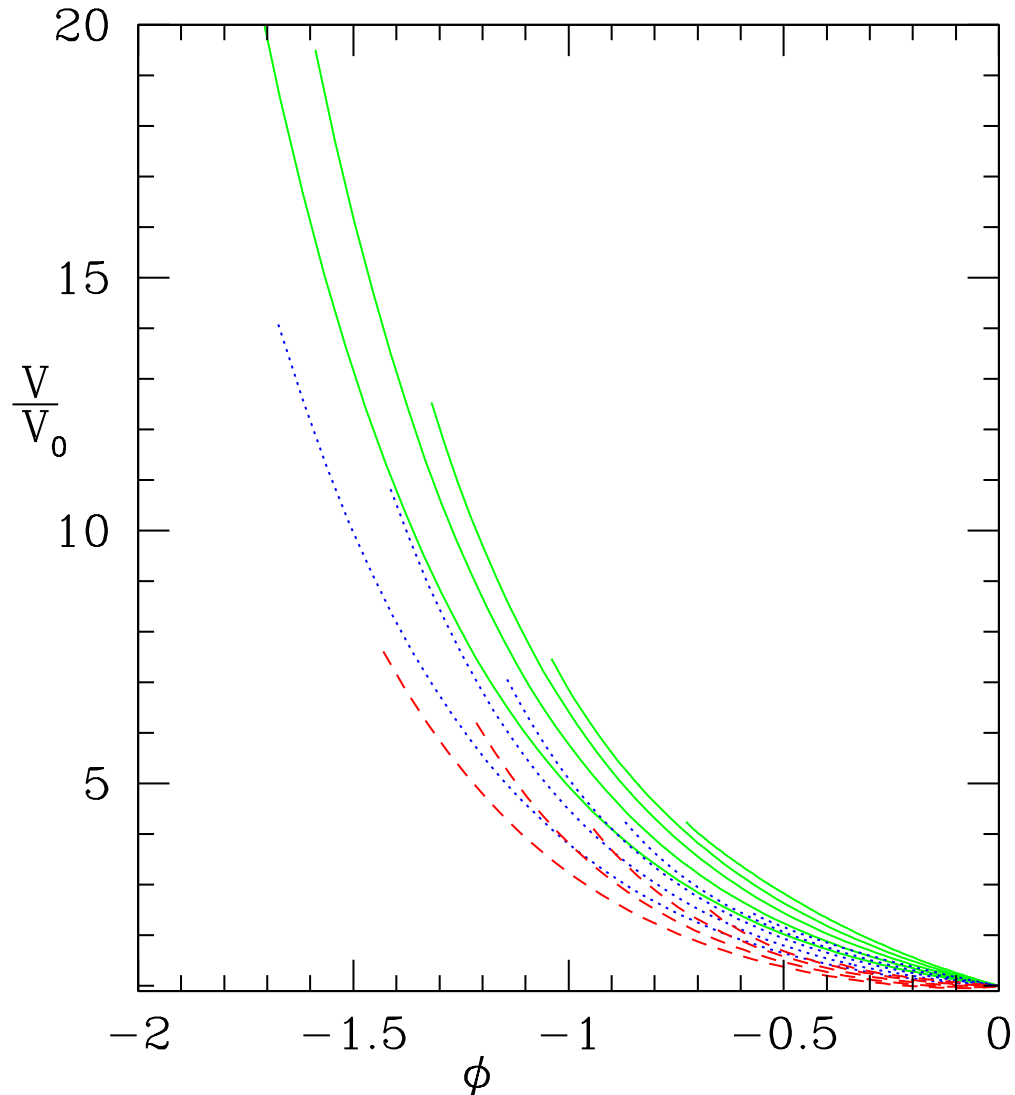
FIG. 1: For quintessence models described by the CPL parametrization, the scalar field potential $V(\phi)$ normalized to its present day value, V_0, is plotted as a function of ϕ, for $w_0 = -0.9$ (dashed, red), and $w_a =$ (top to bottom) 0, 0.5, 1, 1.5, 1.9; $w_0 = -0.6$ (dotted, blue) and $w_a =$ (top to bottom) -0.5, 0, 0.5, 1.0, 1.5; $w_0 = -0.3$ (solid, green) and $w_a =$ (top to bottom) -1.0, -0.5, 0, 0.5, and 1.0, where we restrict the scale factor a to lie in the range $0.3 \le a \le 1$, and ϕ is translated such that its present-day value is $\phi_0 = 0$.

$a_i = 0.3$ in what follows, so that all of the current supernova data lie in the interval $a_i \le a \le 1$. The results for $V(\phi)$ are shown in Figs. 1-2. Note that for the particular case of $w_a = 0$ (so that $w = w_0$), there is an exact solution to Eqs. (10) and (13), namely $V(\phi)$ is a power of a hyperbolic sine [44, 45].

It is clear from Figs. 1-2 that the potentials corresponding to the CPL parametrization are all quite similar to each other. In particular, fixing w_0 and varying w_a produces a family of functions that are nearly indistinguishable, and the freezing ($w_a > 0$) and thawing ($w_a < 0$) models correspond to potentials which appear very similar. (Note that some of the curves with $w_0 = -0.9$ violate the bound in Eq. (16) and correspond to scalar fields rolling uphill at the present; this can be seen in Fig. 2.)

All of the potentials in Figs. 1-2 appear to be convex functions ($d^2V/d\phi^2 > 0$). We can test this hypothesis using the direct expression for $d^2V/d\phi^2$ from Ref. [46] (see also Ref. [47]), namely

$$\frac{1}{H^2}\frac{d^2V}{d\phi^2} = (2 + 3w + 2q)\frac{w'}{1+w} + \frac{1}{4}\left(\frac{w'}{1+w}\right)^2 - \frac{w''}{2(1+w)} + \frac{3}{4}(1-w)(5+3w+2q). \tag{17}$$
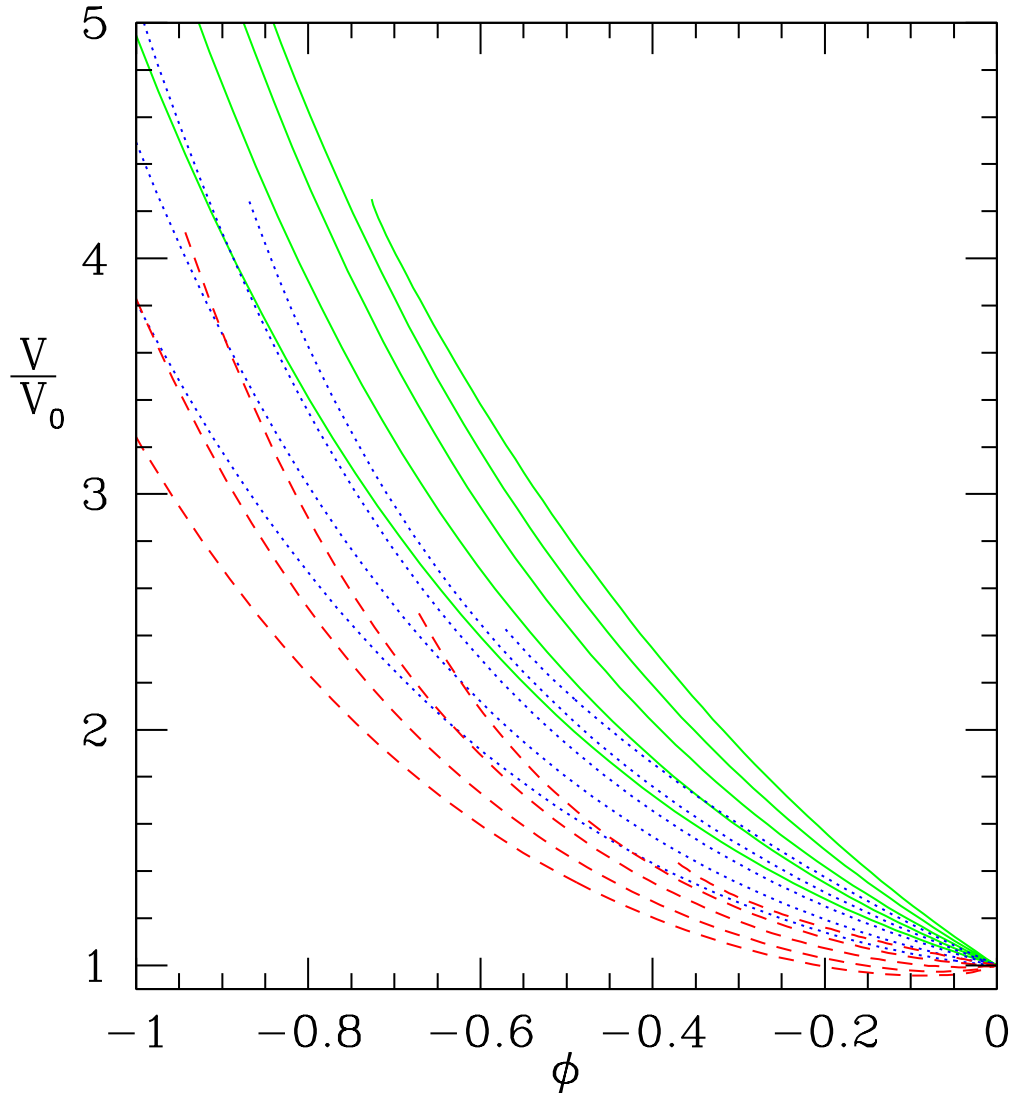
FIG. 2: Expanded view of the region of Fig. 1 near $\phi = 0$.

Here the prime denotes the derivative with respect to $\ln a$, and q is the deceleration parameter, which, for a universe containing matter and a quintessence component with equation of state parameter given by Eq. (2), is

$$q = \frac{1}{2} + \frac{3}{2} \left(\frac{w_0 + (1-a)w_a}{1 + (\rho_{M0}/\rho_{\phi 0})a^{3(w_0+w_a)}e^{3w_a(1-a)}} \right). \tag{18}$$

Taking $0.3 \leq a \leq 1$, we scan over w_0 and w_a and use Eqs. (17)-(18) to determine the sign of $d^2V/d\phi^2$. We find that $d^2V/d\phi^2 > 0$ except for two regimes. For small values of a, $d^2V/d\phi^2 < 0$ for $w_a > 1.5$, which is outside of the parameter range explored in this paper. The other case for which $d^2V/d\phi^2 < 0$ is for $w_0 \leq -0.8$, w_a small and negative, and a near 1. We will examine this regime in more detail shortly.

Figs. 1-2 show that the CPL parametrization maps onto a very narrow range of functions $V(\phi)$. However, this result seems paradoxical, since previous work [28, 32, 33] suggests that a wide range of functional forms for $V(\phi)$ are well-approximated by the CPL parametrization. The key to resolving this apparent contradiction is to note that these previous studies did not indicate that quintessence models map to the full w_0, w_a plane. Instead, Refs. [28, 32, 33] show that thawing quintessence maps to a one-dimensional subspace of this plane, namely, the line defined by the relation

$$w_a = \kappa(1 + w_0), \tag{19}$$

with κ variously estimated to be in the range from -1.6 to -1.5.

To understand this result, we assume the validity of Eq. (19) and derive the corresponding forms for $V(\phi)$ that satisfy it. These forms for $V(\phi)$ are shown in Fig. 3 for κ ranging from -0.5 to -3. Note that in this case, we choose a_i to be the value of a for which $w(a) = -1$; this is the earliest allowed initial value for a for which this parametrization can be valid. This yields $a_i = 1 + 1/\kappa$, independent of w_0 and w_a.
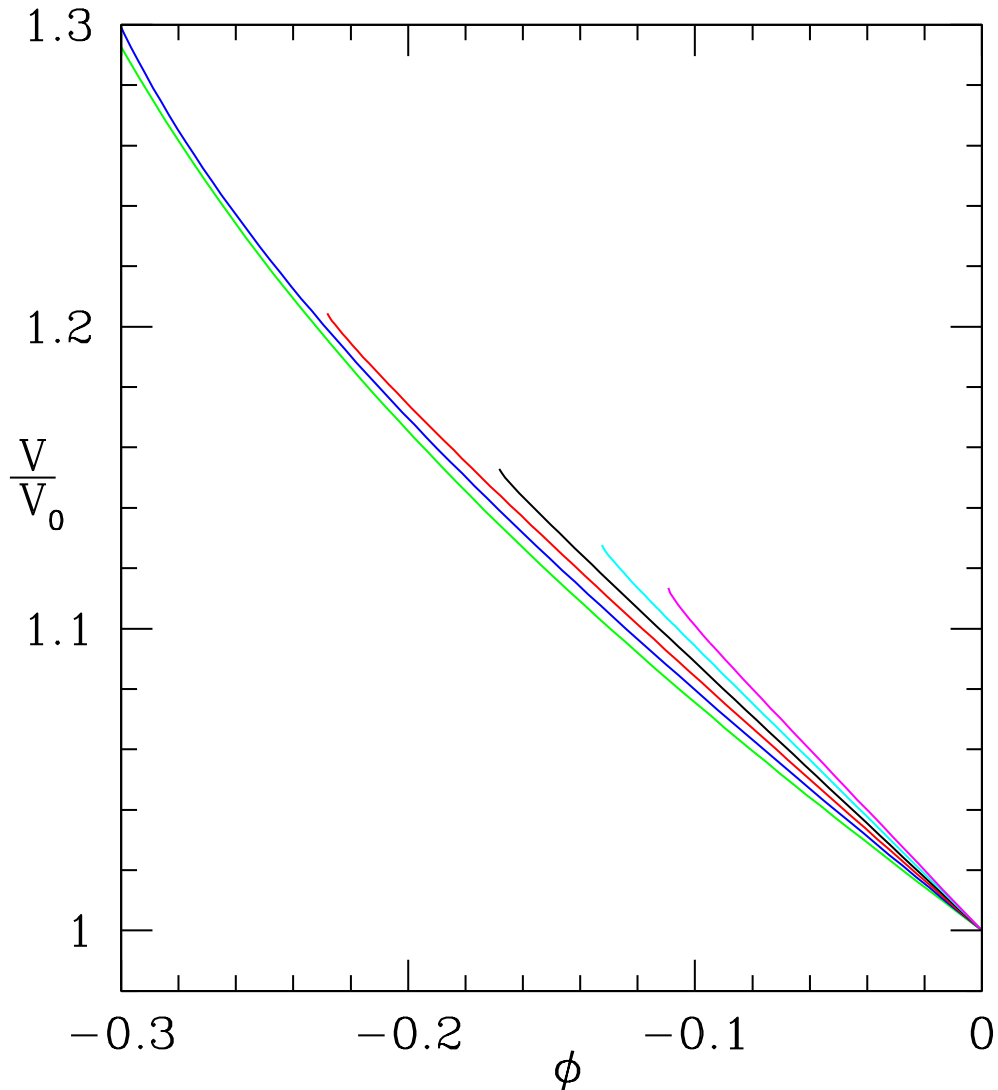

FIG. 3: As Fig. 1, for $w_a = \kappa(1 + w_0)$, with $w_0 = -0.9$ and $\kappa = -0.5$ (green), -1 (blue), -1.5 (red), -2.0 (black), -2.5 (cyan), -3.0 (magenta), where ϕ is translated such that its present-day value is $\phi_0 = 0$, and a_i is the smallest value of a for which $w(a) \geq -1$.

These forms for $V(\phi)$ become nearly linear for $\kappa = -1.5$, while diverging from linear behavior on either side of this value. This suggests that the particular choice of Eq. (19) corresponds to a nearly linear potential. Further, when w is close to -1, (which is always the case when $w_0 \approx -1$ and $w_a < 0$), ϕ rolls only a short distance down the potential, and any sufficiently smooth potential will appear to be linear on small enough scales. This explains why thawing potentials with w near -1 are described so well by Eq. (19).

Further, Fig. 3 corresponds precisely to the potentials for which $V(\phi)$ is concave ($d^2V/d\phi^2 < 0$) for a near 1. This corresponds to ϕ near 0 in Fig. 3. It is clear from this figure that $d^2V/d\phi^2$ is negligibly small in these cases. Thus, we can conclude that for the parameter ranges examined in this paper, $d^2V/d\phi^2$ is never large and negative. Thus, it makes sense that $w(a)$ for the hilltop models examined in Ref. [36] (for which $d^2V/d\phi^2$ is large and negative) strongly diverges from linear evolution in a. This does *not* mean that the CPL parametrization is a poor fit for all models with

$d^2V/d\phi^2 < 0$. If $d^2V/d\phi^2$ is sufficiently small, then our argument from the previous paragraph can apply, yielding evolution well-described by the CPL parametrization and Eq. (19) [32, 33, 36].

Thus, we conclude that over most of the range in w_0 and w_a, the CPL parametrization maps onto a very narrow form of behavior for $V(\phi)$, while a narrow range in the w_0, w_a parameter space (given by Eq. 19) maps to a wide range of quintessence models, namely thawing quintessence with w near -1. However, it is precisely these models that appear to be allowed by current observations, as emphasized by Ref. [33].

III.   BAROTROPIC MODELS

In this section, we examine barotropic models, for which the pressure is a fixed function of the density:

$$p = f(\rho). \tag{20}$$

Particular models of this form include the Chaplygin gas [14, 15] and the generalized Chaplygin gas [16], the linear equation of state [17, 18] and the affine equation of state [19, 20] (note these are actually the same model), the quadratic equation of state [19], and the Van der Waals equation of state [21, 22]. A general study of the properties of barotropic models for dark energy was undertaken in Ref. [23] and further extended in Ref. [24]. Note that there is a simple mapping between the barotropic models discussed here and kinetic k-essence models [23], so the results presented here can be extended in a straightforward way to the latter set of models.

Ref. [23] provided two reasonable constraints on the behavior of $w(a)$ in barotropic models. To prevent instabilities, the sound speed $c_s^2 = dp/d\rho$ must obey $c_s^2 \geq 0$, while causality requires $c_s^2 \leq 1$. While each of these limits translates into a constraint on $w(a)$, it is easier in this case to derive the form for p as a function of ρ and then apply the direct limits on $dp/d\rho$:

$$0 \leq \frac{dp}{d\rho} \leq 1. \tag{21}$$

If the equation of state for the barotropic fluid is given by Eq. (2), then the expression for the density ρ given by Eq. (9) is valid as well, i.e.,

$$\rho = \rho_0 a^{-3(1+w_0+w_a)} e^{3w_a(a-1)}, \tag{22}$$

Further, the pressure will be given by $w\rho$, so

$$p = \rho_0[w_0 + (1-a)w_a]a^{-3(1+w_0+w_a)} e^{3w_a(a-1)} \tag{23}$$

Eqs. (22) and (23) together provide an implicit expression for $f(\rho)$ in Eq. (20). As in the previous section, we will not require the CPL parametrization to hold all of the back to $a = 0$, but only for $a_i \leq a \leq 1$, where we take $a_i = 0.3$. In Figs. 4-5, we plot p/ρ_0 as a function of ρ/ρ_0 for the scale factor lying in this interval.

The behavior of p as a function of ρ for these barotropic models obeying the CPL parametrization displays a much wider set of functional behaviors than is the case for $V(\phi)$ in the quintessence models. Note, however, that all of the curves converge on the same value of p/ρ_0 ($= w_0$) at $\rho/\rho_0 = 1$, corresponding to $a = 1$.

We can apply the limits from Ref. [23] given by Eq. (21) directly to the models displayed in Fig. 4. It is clear that causality ($dp/d\rho \leq 1$) is satisfied for all of the models under consideration. However, the requirement that $dp/d\rho \geq 0$ is violated by some of these models. From Figs. $4-5$, it is clear that causality requires relatively large positive values of w_a (i.e., freezing models rather than thawing models).

We can make a stronger statement about the allowed forms for $f(\rho)$ and their dependence on w_0 and w_a. Eq. (2) can be used to express the scale factor as a function of w, w_0, and w_a:

$$a = \frac{w_a + w_0 - w}{w_a}, \tag{24}$$

and this expression can be substituted into equation (22) to give

$$\rho = \rho_0|w_a|^{3(1+w_i)} e^{-3w_a} |w_i - w|^{-3(1+w_i)} e^{3(w_i-w)}, \tag{25}$$

where we define w_i to be the value of w at $a = 0$ in the CPL approximation, namely, $w_i = w_0 + w_a$. Note that Eq. (25) is an implicit equation for $p(\rho)$, since $w = p/\rho$. Now define

$$\widetilde{\rho} = \rho/\rho_C, \tag{26}$$
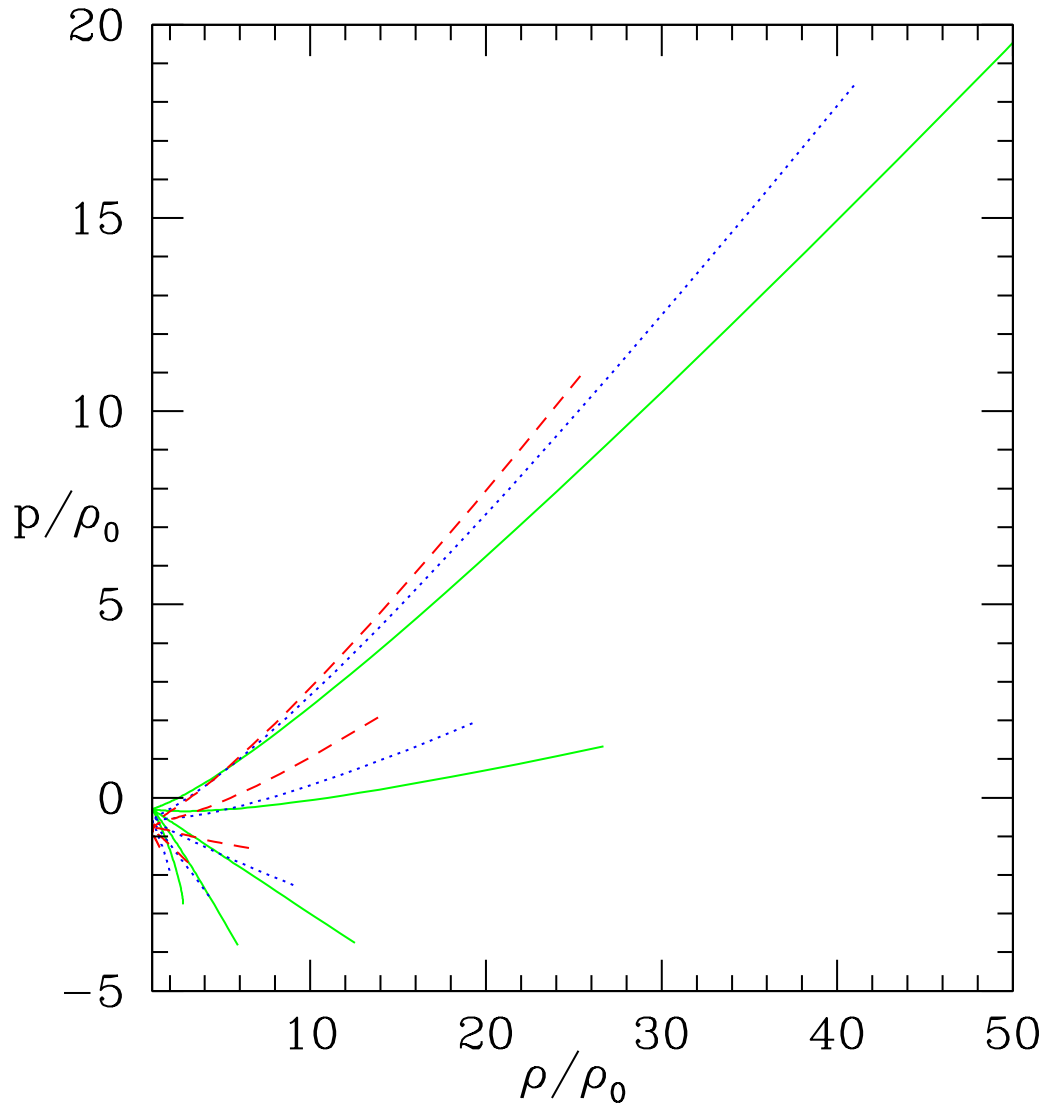$$\widetilde{p} = p/\rho_C, \tag{27}$$
$$\tag{28}$$

FIG. 4: For barotropic models described by the CPL parametrization, the pressure p, given as p/ρ_0, is plotted as a function of the density ρ, given as ρ/ρ_0, where ρ_0 is the present-day dark energy density, for $w_0 = -0.9$ (dashed, red), and $w_a =$ (bottom to top) 0, 0.5, 1, 1.5, 1.9; $w_0 = -0.6$ (dotted, blue) and $w_a =$ (bottom to top) -0.5, 0, 0.5, 1.0, 1.5; $w_0 = -0.3$ (solid, green) and $w_a =$ (bottom to top) -1.0, -0.5, 0, 0.5, and 1.0, where we restrict the scale factor a to lie in the range $0.3 \leq a \leq 1$.

with the constant density ρ_C given by

$$\rho_C = \rho_0 |w_a|^{3(1+w_i)} e^{-3w_a} \tag{29}$$

Then Eq. (25) becomes

$$\widetilde{\rho} = |w_i - \widetilde{p}/\widetilde{\rho}|^{-3(1+w_i)} e^{3(w_i - \widetilde{p}/\widetilde{\rho})}. \tag{30}$$

From this equation, it is clear that the expression for $\widetilde{p}$ as a function of $\widetilde{\rho}$ depends only on w_i and not on w_a and w_0 independently. Writing this function as $\widetilde{p} = g(\widetilde{\rho})$, we then have

$$p/p_C = g(\rho/\rho_C) \tag{31}$$

where the function g depends only on w_i. Thus, aside from an overall multiplicative constant in both ρ and p (which is the same for ρ and p, so that w is unaltered), $f(\rho)$ in Eq. (20) depends only on w_i, and not on w_a and w_0 separately.
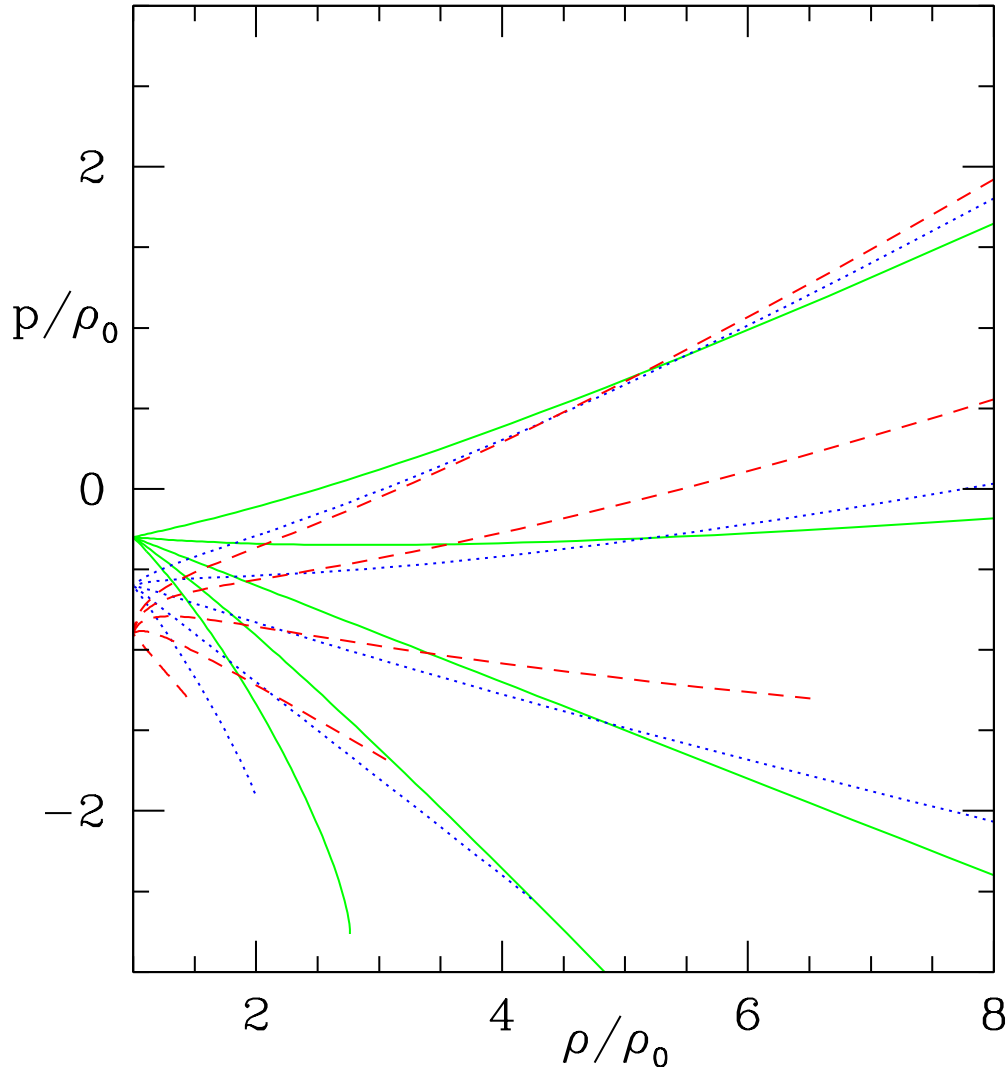
FIG. 5: Expanded view of the region of Fig. 3 near $\rho/\rho_0 = 1$.

IV. CONCLUSIONS

Lacking a priori knowledge of the actual physical model underlying the accelerated expansion of the universe, it is impossible to make blanket statements about the utility of of the CPL parametrization for approximating such models. However, we can make statements about individual dark energy models; the implicit question is the extent to which a linear parametrization of w provides useful information about the parameters of a given physical model.

For the case of quintessence, we have seen that the full parameter space of the CPL parametrization maps to only a fairly narrow form of functional behavior for $V(\phi)$. However, a one-dimensional subset of CPL parameter space, namely $w_a = \kappa(1+w_0)$, with $\kappa \approx -1.6$ - -1.5 corresponds to a wide range of functional forms for $V(\phi)$. This issue has been noted by Linder [33], who pointed out that such models are better fit by the one-parameter model of Eq. (19) than by the full CPL parametrization. Further, Ref. [33] introduced a more accurate two-parameter approximation, using Eq. (19) as a starting point, that better fits the behavior of thawing quintessence models. Our results provide further evidence in favor of this approach.

We note a similar situation for barotropic models. For these models, the CPL parametrization does a much better job of capturing the full range of possible functional forms for $p = f(\rho)$ (the pressure as a function of density). However, we have shown explicitly that the barotropic models that correspond exactly to the CPL parametrization

belong to a one-dimensional subset of the full CPL parameter space; namely, they are fully determined (up to a multiplicative constant) by the value of $w_i = w_0 + w_a$. Thus, the full CPL approximation may not be the most useful way to characterize such models.

Considered as a straightforward linear approximation for the equation of state parameter as a function of the expansion factor, the CPL parametrization is undeniably useful, particularly when attempting to detect deviations from the ΛCDM model. However, as a probe of physical models which might correspond to non-ΛCDM behavior (or at least the types of physical models considered here), the CPL parametrization is not optimal, and, at least for the case of quintessence, it seems more appropriate to move toward the kind of parametrization discussed in Ref. [33]. It would, of course, be useful to extend this study to some of the other classes of models for dark energy that have been proposed in the literature.

V. ACKNOWLEDGMENTS

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