First law and a variational principle for static asymptotically Randall-Sundrum black holes

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The first law and a variational principle for static asymptotically Randall-Sundrum black holes

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We give a new, intrinsic, mass definition for spacetimes asymptotic to the Randall-Sundrum braneworld models, RS1 and RS2. For this mass, we prove a first law for static black holes, including variations of the bulk cosmological constant, brane tensions, and RS1 interbrane distance. Our first law defines a thermodynamic volume and a gravitational tension that are braneworld analogs of the corresponding quantities in asymptotically AdS black hole spacetimes and asymptotically flat compactifications, respectively. We also prove the following related variational principle for asymptotically RS black holes: instantaneously static initial data that extremizes the mass yields a static black hole, for variations at fixed apparent horizon area, AdS curvature length, cosmological constant, brane tensions, and RS brane warp factors. This variational principle is valid with either two branes (RS1) or one brane (RS2), and is applicable to variational trial solutions.

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I. INTRODUCTION

Static and stationary black holes should obey four classical laws [1]. The first law expresses conservation of energy, and has been proven in spacetimes with four dimensions [1, 2] and higher dimensions [3–5], including a compact extra dimension [6], but not in spacetimes asymptotic to the Randall-Sundrum (RS) braneworld models [7, 8]. In this paper, we close this gap by proving a general first law for static asymptotically RS black holes. The first law relates the variations of mass and other physical quantities. For this purpose, we provide a new, intrinsic, mass definition. Our first law defines a thermodynamic volume and a gravitational tension that are braneworld analogs of thermodynamic volume in asymptotically AdS black hole spacetimes [4, 5] and gravitational tension in asymptotically flat compactifications [6].

The RS models are phenomenologically interesting, and have holographic interpretations [9] in the AdS/CFT correspondence. In the RS models, our observed universe is a brane surrounded by an AdS bulk. The bulk is warped by a negative cosmological constant. The RS1 model [7] has two branes of opposite tension, with our universe on the negative-tension brane. Tuning the interbrane distance appropriately predicts the production of small black holes at TeV-scale collider energies [10], and LHC experiments [11] continue to test this hypothesis. In the RS2 model [8], our universe resides on the positive-tension brane, with the negative-tension brane removed to infinite distance. Perturbations of RS2 reproduce Newtonian gravity at large distance on the brane, while in RS1 this requires a mechanism to stabilize the interbrane distance [12]. In RS2, solutions for static black holes on the brane have been found numerically, for both small black holes [13] and large black holes [14], compared to the AdS curvature length. The only known exact analytic black hole solutions are the static and stationary solutions [15] in a lower-dimensional version of RS2.

In general, the first law for a static or stationary black hole takes the form $\delta M = (\kappa/8\pi G)\delta A + \sum_i p_i\, \delta Q_i$. This relates the variations of mass M, horizon area A, and other physical quantities Q_i. Thus, if a black hole is static or stationary, it extremizes M under variations that hold constant the remaining variables (A, Q_i). The converse of this statement motivates a variational principle: *If a black hole's exterior spatial geometry is initially static (or initially stationary) and extremizes the mass with other physical variables held fixed, then the black hole is static (or stationary).* In this variational principle, the specific variables to hold fixed depend on the form of the first law. The appropriate area to hold fixed is that of the black hole apparent horizon, which is determined by the spatial geometry alone (unlike the event horizon, which is a global spacetime property). The apparent horizon generally lies inside the event horizon, and coincides with it for a static or stationary black hole spacetime.

For asymptotically flat black holes in four spacetime dimensions, a version of the above variational principle was proved by Hawking for stationary black holes [16], and was extended to Einstein-Yang-Mills theory by Sudarsky and Wald [2, 17]. In this paper, we prove a version of the above variational principle for static asymptotically RS black holes. The quantities held fixed in our variational principle are the AdS curvature length, cosmological constant, brane tensions, and RS values (at spatial infinity) of warp factors on each brane. The variations of these quantities appear in the general first law that we prove in this paper.

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This paper is organized as follows. After reviewing the RS spacetimes in section II, we define the mass for an asymptotically RS spacetime in section III, and evaluate the mass for a static asymptotic solution in section IV. We prove the first law for static black holes in section V. We prove the variational principle in section VI, including an explicit application using a trial solution. We conclude in section VII.

Throughout this paper, we use two branes, so our results apply to either RS1 or RS2 in the appropriate limit. We work on the orbifold region (between the branes) and use D spacetime dimensions. A timelike surface has metric γ_{ab}, extrinsic curvature $K_{ab} = \gamma_a{}^c \nabla_c n_b$, and outward unit normal n_b. A spatial hypersurface Σ has unit normal u_a, metric h_{ab}, and covariant derivative D_a. Each boundary B of Σ has metric σ_{ab}, extrinsic curvature $k_{ab} = h_a{}^c D_c n_b$, and outward unit normal n_b. The boundaries B of Σ are illustrated in Fig. 1. These boundaries are: spatial infinity B_∞, the branes (B_1, B_2), and the black hole apparent horizon B_H. If the black hole is static, then B_H coincides with the black hole event horizon in Σ.

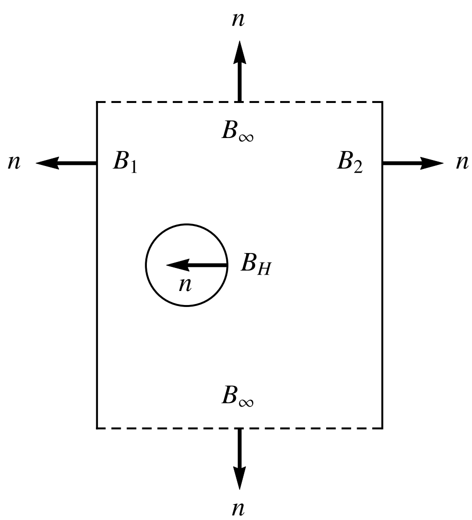

FIG. 1. Illustration of a spatial hypersurface Σ, for a black hole with apparent horizon B_H not intersecting the branes, B_1 and B_2. Spatial infinity B_∞ is a single boundary, transverse to both branes. Each boundary B has outward normal n.

II. THE RANDALL-SUNDRUM SPACETIMES

The RS spacetimes [7, 8] are portions of an anti-de Sitter (AdS) spacetime, with metric

$$ds^2_{\rm RS} = \Omega(Z)^2 \left(-dt^2 + d\rho^2 + \rho^2 d\omega^2_{D-3} + dZ^2\right) \ . \quad (1)$$

Here $d\omega^2_{D-3}$ denotes the unit $(D-3)$-sphere. The warp factor is $\Omega(Z) = \ell/Z$, with values Ω_i on each brane. Here ℓ is the AdS curvature length, related to the bulk cosmological constant $\Lambda < 0$ given below. The RS1 model [7] contains two branes, which are the surfaces $Z = Z_i$ with brane tensions λ_i, where $i = 1, 2$. The brane tensions λ_i and bulk cosmological constant Λ are

$$\lambda_1 = -\lambda_2 = \frac{2(D-2)}{8\pi G_D \ell} \ , \quad \Lambda = -\frac{(D-1)(D-2)}{2\ell^2} \ . \quad (2)$$

The dimension Z is compactified on the orbifold $S^1/\mathbb{Z}_2$ and the branes have orbifold mirror symmetry: in the covering space, symmetric points across a brane are identified. There is a discontinuity in the extrinsic curvature K_{ab} across each brane given by the Israel condition [18]. Using orbifold symmetry, the Israel condition requires the extrinsic curvature at each brane to satisfy

$$2K_{ab} = \left(\frac{8\pi G_D \lambda}{D-2}\right)\gamma_{ab} \ , \quad 2k_{ab} = \left(\frac{8\pi G_D \lambda}{D-2}\right)\sigma_{ab} \ . \quad (3)$$

Using (2), this can also be written as

$$2K_{ab} = \frac{\varepsilon}{\ell}\gamma_{ab} \ , \quad 2k_{ab} = \frac{\varepsilon}{\ell}\sigma_{ab} \ , \quad (4)$$

where $\varepsilon = \pm 1$ is the sign of each brane tension. The RS2 spacetime [8] is obtained from RS1 by removing the negative-tension brane (now a regulator) to infinite distance ($Z_2 \to \infty$) and the orbifold region has $Z \geq Z_1$.

III. MASS DEFINITION

For an asymptotically RS spacetime, we will define the mass M using a counterterm method [19]. This is an intrinsic approach, which is well suited to the variations we will perform in section V to prove the first law, and is also useful for spacetimes with nontrivial topology. By comparison, other definitions, such as the Brown-York mass [20], use an auxiliary reference spacetime, which may be awkward when the topology is nontrivial. A reference spacetime approach is also implicit when the asymptotic geometry serves this purpose, as in the Abbott-Deser mass [21] and its counterpart with an asymptotically flat compact dimension, the Deser-Soldate mass [22]. For an asymptotically RS spacetime, it is straightforward to verify that evaluating our mass definition, as in (16) below, reduces to the same result as the Abbott-Deser mass formula [21].

In the counterterm approach [19], for a spacetime with metric g_{ab}, one first evaluates the bare action $\widetilde{S}$. If this diverges, one constructs an action counterterm S_{ct} to render the sum $S = \widetilde{S} + S_{ct}$ finite, as follows. Let the metric g_{ab} asymptote to $g^{(0)}_{ab}$ whose bare action $\widetilde{S}_0$ also diverges. We express $\widetilde{S}_0$ in terms of its intrinsic boundary invariants, and define the action counterterm $S_{ct} = -\widetilde{S}$ where $\widetilde{S}$ is the same functional of its boundary geometry that $\widetilde{S}_0$ is of its boundary geometry. This gives $S_0 = 0$ for $g^{(0)}_{ab}$ and a finite action S for g_{ab}.

To define the mass M, one proceeds from the action to the Hamiltonian (defined on an arbitrary initial value spatial hypersurface Σ), which is given by a bulk term involving initial value constraints, and surface terms. For a solution to the constraints, the bulk term vanishes and the bare mass at spatial infinity is [20]

$$\widetilde{M} = -\frac{1}{8\pi G_D}\int_{B_\infty} d^{D-2}x\, N\sqrt{\sigma}\, k\ . \tag{5}$$

Here k is the extrinsic curvature of the boundary B_∞ and for a static spacetime, the lapse function is $N = \sqrt{-g_{tt}}$. If $\widetilde{M}$ diverges, one constructs a mass counterterm

$$M_{ct} = \int_{B_\infty} d^{D-2}x\, N\sqrt{\sigma}\, u_a u_b \left[\frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{ab}}\right] \tag{6}$$

such that the mass M is finite, defined by

$$M = \widetilde{M} + M_{ct}\ . \tag{7}$$

Following the above procedure, we begin with the RS spacetime. The bare action $\widetilde{S}_{\rm RS}$ consists of a bulk term S_Σ and a Gibbons-Hawking term at each boundary,

$$\widetilde{S}_{\rm RS} = S_\Sigma + S_1 + S_2 + S_\infty\ . \tag{8}$$

Here

$$\begin{aligned} S_\Sigma &= \frac{1}{16\pi G_D}\int dt \int_\Sigma d^{D-1}x\,\sqrt{-g}\,(R-2\Lambda)\ , \\ S_i &= \frac{1}{8\pi G_D}\int dt\int_{B_i} d^{D-2}x\,\sqrt{-\gamma}\left(K - \frac{8\pi G_D\lambda_i}{2}\right)\ , \\ S_\infty &= \frac{1}{8\pi G_D}\int dt\int_{B_\infty} d^{D-2}x\,\sqrt{-\gamma}\,K\ . \end{aligned} \tag{9}$$

For the RS solution (1), the Ricci scalar is $R = 2\Lambda D/(D-2)$ and after integrating S_Σ in Z, we find

$$S_\Sigma + S_1 + S_2 = 0\ . \tag{10}$$

We now specialize to the case $D=5$, for which

$$\widetilde{S}_{\rm RS} = S_\infty = \frac{1}{8\pi G_5}\int d^4x\,\sqrt{-\gamma}\,\frac{2}{\Omega\rho}\ . \tag{11}$$

This diverges as $\rho\to\infty$. The metric γ_{ab} on this boundary is (1) with $\rho =$ constant. Let $\hat{\gamma}_{ab}$ be the submetric with $Z =$ constant, and let $\hat{\sigma}_{ab}$ be the submetric on a 2-sphere (constant ρ, Z, t). Their Ricci scalars are

$$\hat{\mathcal{R}}(\hat{\gamma}) = \hat{\mathcal{R}}(\hat{\sigma}) = \frac{2}{(\Omega\rho)^2}\ . \tag{12}$$

If we express $\widetilde{S}_{\rm RS}$ in terms of $\hat{\mathcal{R}}$, then an asymptotically RS spacetime has action counterterm $S_{ct} = -\widetilde{S}$ where $\widetilde{S}$ is the same functional of its boundary geometry that $\widetilde{S}_{\rm RS}$ is of its geometry. Thus

$$S_{ct} = -\frac{\sqrt{2}}{8\pi G_5}\int d^4x\,\sqrt{-\gamma}\,\sqrt{\hat{\mathcal{R}}}\ . \tag{13}$$

All quantities in (13) refer to the boundary geometry of a general metric g_{ab}, not the RS metric (1). As shown in Appendix A, the mass counterterm (6) then yields our mass definition for an asymptotically RS spacetime,

$$M = \frac{1}{8\pi G_5}\int_{B_\infty} d^3x\, N\sqrt{\sigma}\left(-k + \sqrt{2\hat{\mathcal{R}}}\right)\ . \tag{14}$$

IV. STATIC ASYMPTOTIC SOLUTION

Here we evaluate our mass definition (14) for a static asymptotic solution, which we will use in section V below. We define the branes as the surfaces $Z = Z_1$ and $Z = Z_2$. We consider a static asymptotically RS metric with functions F_ν that fall off at large ρ as follows,

$$\begin{aligned} ds^2 &= \Omega^2\left(-e^{2F_t}dt^2 + e^{2F_\rho}d\rho^2 + e^{2F_\omega}\rho^2 d\omega_2^2 + e^{2F_Z}dZ^2\right) \\ F_\nu &= \frac{a_\nu(Z)}{\rho} + \frac{b_\nu(Z)}{\rho^2} + \frac{c_\nu(Z)}{\rho^3} + O(1/\rho^4)\ . \end{aligned} \tag{15}$$

For these asymptotics, the mass (14) evaluates to

$$M = \frac{1}{G_5}\int_{Z_1}^{Z_2} dZ\,\Omega^3\left(a_\rho + \frac{a_Z}{2}\right)\ . \tag{16}$$

The value of M also appears in the solution to (15). To see this, we find it is necessary to solve the Einstein equations through third order. The first order solutions a_ν are

$$a_t(Z) = a_\rho(Z) + \mu_0\ , \tag{17a}$$
$$a_\omega(Z) = a_\rho(Z) + \mu_1\ , \tag{17b}$$
$$a_Z(Z) = -Z\, a_\rho'\ . \tag{17c}$$

Here μ_0 and μ_1 are integration constants and $' = d/dZ$. The constant μ_1 can be removed by a gauge transformation $\rho\to\rho+\mu_1/2$. Two identities we will need are

$$\int_{Z_1}^{Z_2} dZ\,\Omega^3\,(a_t + a_\rho + a_Z) = 0\ , \tag{18a}$$
$$\int_{Z_1}^{Z_2} dZ\,\Omega^{n+1}\,(n\,a_\rho + a_Z) = -\ell\,(\Omega^n a_\rho)\Big|_{Z_1}^{Z_2}\ . \tag{18b}$$

The identity (18b) also holds with a_ρ replaced by a_t or a_ω. Using (18b), the mass can be written as

$$M = -\frac{\ell}{2G_5}\left(\Omega^2 a_\rho\right)\Big|_{Z_1}^{Z_2}\ . \tag{19}$$

The third order solutions c_ν involve an integration constant q_0. The Israel condition provide one equation at each brane, which can be solved for μ_0 and q_0 as

$$\mu_0\left({\Omega_2}^2 - {\Omega_1}^2\right) = -2\left(\Omega^2 a_\rho\right)\Big|_{Z_1}^{Z_2}\ , \tag{20}$$

$$q_0(\Omega_2^{-2} - \Omega_1^{-2}) = 2a_\rho\Big|_{Z_1}^{Z_2}\ . \tag{21}$$

Here $\Omega_i = \ell/Z_i$ the warp factor at each brane. We see from (18b) and (20) that M is proportional to μ_0,

$$M = \frac{\ell\mu_0}{4G_5}\left({\Omega_2}^2 - {\Omega_1}^2\right)\ . \tag{22}$$

We see from (21) that q_0 is proportional to a quantity $\mathcal{Q}$ parametrizing the interbrane distance near $\rho\to\infty$,

$$\mathcal{L}_{\rm branes} = L + \frac{\mathcal{Q}}{\rho} + O(1/\rho^2)\ , \tag{23}$$

where the distance L at infinity and the constant $\mathcal{Q}$ are

$$L = \ell \ln\left(\frac{\Omega_1}{\Omega_2}\right) \quad , \quad \mathcal{Q} = -\ell a_\rho \Big|_{Z_1}^{Z_2} . \tag{24}$$

We will refer to L and $\mathcal{Q}$ in the next section. We will also use the fact that M and $\mathcal{Q}$ appear in the values of a_ρ at each brane, which we find by solving (19) and (24),

$$a_\rho(Z_i) = \frac{2G_5 M - \Omega_j{}^2 \mathcal{Q}}{\ell(\Omega_1{}^2 - \Omega_2{}^2)} \quad , \quad j \neq i . \tag{25}$$

In the RS1 case, M and $\mathcal{Q}$ have lower-dimensional interpretations on each brane. This follows since there is an effective Brans-Dicke gravity on each brane [23], and Brans-Dicke gravity contains two asymptotic quantities, a tensor mass and a scalar mass [24]. On each brane, one can verify that M and $\mathcal{Q}$ are proportional to the effective tensor mass and scalar mass, respectively.

V. FIRST LAW FOR STATIC BLACK HOLES

A. Preliminary form

For a static or stationary black hole, the first law relates the variations of mass, black hole horizon area, and other physical quantities. We will include variations of the bulk cosmological constant Λ and brane tensions λ_i that preserve the RS conditions (2),

$$-\frac{\delta\ell}{\ell} = \frac{\delta\lambda_1}{\lambda_1} = \frac{\delta\lambda_2}{\lambda_2} = \frac{\delta\Lambda}{2\Lambda} . \tag{26}$$

We will also include the variation δL of the interbrane separation. From (24),

$$\delta L = \delta\ell \, \ln\left(\frac{\Omega_1}{\Omega_2}\right) + \ell\,\frac{\delta\Omega_1}{\Omega_1} - \ell\,\frac{\delta\Omega_2}{\Omega_2} . \tag{27}$$

Our setup is general: it applies to a black hole localized on a brane, or isolated in the bulk (away from either brane), and also applies to the asymptotically RS black string [25]. Our method is based on the Hamiltonian approach of [2], with the additional variations (26)–(27). The full Hamiltonian contains a bulk term and boundary terms. The bulk term H_Σ is defined on a spatial hypersurface Σ,

$$H_\Sigma = \int_\Sigma d^{D-1}x \, \left(N\mathcal{C}_0 + N^a \mathcal{C}_a\right) . \tag{28}$$

Here N and N^a are the lapse and shift in the standard decomposition of the spacetime metric. Our focus is the initial data (h_{ab}, p^{ab}) on Σ, where h_{ab} is the spatial metric and p^{ab} is its canonically conjugate momentum,

$$16\pi G_D \, p^{ab} = \sqrt{h}\mathcal{K}^{ab} - \mathcal{K}\, h^{ab} \quad , \quad \mathcal{K}_{ab} = h_a{}^c \nabla_c u_b . \tag{29}$$

Initial data must satisfy constraints, $\mathcal{C}_0 = 0$ and $\mathcal{C}_a = 0$, which we henceforth assume, where

$$\mathcal{C}_0 = \frac{\sqrt{h}}{16\pi G_D}\left(2\Lambda - \mathcal{R}\right) + \frac{16\pi G_D}{\sqrt{h}}\left(p^{ab}p_{ab} - \frac{p^2}{D-2}\right) ,$$
$$\mathcal{C}_a = -2D_b p_a{}^b . \tag{30}$$

Here $\mathcal{R}$ and D_a are the Ricci scalar and covariant derivative associated with h_{ab}. We now consider the change δH_Σ under variations $(\delta h_{ab},\, \delta p^{ab})$. One finds $\delta\mathcal{C}_0$ and $\delta\mathcal{C}_a$ involve derivatives $(D_c\delta h_{ab},\, D_c\delta p^{ab})$. Integrating by parts to remove these derivatives yields surface terms I_B,

$$\delta H_\Sigma = \int_\Sigma d^{D-1}x \left[\mathcal{P}^{ab}\delta h_{ab} + \mathcal{H}_{ab}\,\delta p^{ab}\right] + \frac{\delta\Lambda}{8\pi G_D}\int_\Sigma d^{D-1}x \, N\sqrt{h} \; + \; \sum_B I_B . \tag{31}$$

The sum in (31) is over all of the boundaries B illustrated in Fig. 1. The quantities $\mathcal{P}_{ab}$ and $\mathcal{H}_{ab}$ appear in the time evolution equations,

$$\dot{h}_{ab} = \mathcal{H}_{ab} \ , \quad \dot{p}^{ab} = -\mathcal{P}^{ab} , \tag{32}$$

where the overdot denotes the Lie derivative along the time evolution vector field $t^a = Nu^a + N^a$ with u^a the unit normal to Σ. We will not need the most general forms of $\mathcal{P}_{ab}$, $\mathcal{H}_{ab}$, and I_B. Will give their simplified forms below, after implementing some of our key assumptions.

We now assume our variations take one solution of the constraints to another solution of the constraints, so we take $\delta\mathcal{C}_0 = 0$ and $\delta\mathcal{C}_a = 0$. Then the variation of (28) immediately gives

$$\delta H_\Sigma = 0 . \tag{33}$$

We henceforth assume the initial data is instantaneously static, for which $p^{ab} = \delta p^{ab} = 0$ and we take $N^a = 0$. One then explicitly finds $\mathcal{H}_{ab} = 0$, so (31) and (33) give

$$0 = \int_\Sigma d^{D-1}x \left[\mathcal{P}^{ab}\delta h_{ab} + \frac{\delta\Lambda}{8\pi G_D} N\sqrt{h}\right] + \sum_B I_B . \tag{34}$$

This result will be the primary equation for proving our variational principle in section VI. Here $\mathcal{P}^{ab}$ is given by

$$\mathcal{P}^{ab} = \frac{\sqrt{h}}{16\pi G_D}\left(\mathcal{R}^{ab} + h^{ab}D_c D^c - D^a D^b\right) N . \tag{35}$$

We now assume a static black hole with timelike Killing field ξ^a and choose $t^a = \xi^a$. For a static solution, $\mathcal{P}^{ab} = 0$ by (32). Then (34) gives

$$0 = \frac{\delta\Lambda}{8\pi G_D}\int_\Sigma d^{D-1}x\, N\sqrt{h} \; + \; \sum_B I_B . \tag{36}$$

This equation is our preliminary form of the first law. It simply remains to express (36) in terms of physical quantities. Each surface term I_B can be written [26]

$$I_B = \int_B d^{D-2}x \, N\left[\frac{\delta\left(\sqrt{\sigma}k\right)}{8\pi G_D} - \frac{\sqrt{\sigma}}{2}s^{ab}\delta\sigma_{ab}\right] - J_B , \tag{37}$$

where

$$8\pi G_D\, s^{ab} = -k^{ab} + [k + n^c(D_c N)/N]\,\sigma^{ab} \ , \tag{38}$$

$$16\pi G_D J_B = \sum_{B'\neq B} \int_{B\cap B'} d^{D-3}x\, N\sqrt{\hat\sigma}\, n'_a\, \sigma^a{}_b\, \delta n^b \ . \tag{39}$$

Here $\hat\sigma$ denotes the metric on $B\cap B'$. In what follows, we will have $J_B = 0$. This is due to $N = 0$ on B_H, and due to orthogonality ($n'_a n^a = 0$) at the other boundaries B.

We now evaluate the boundary terms (37) for $D = 5$. The results at the horizon B_H and the branes B_i are

$$I_{B_H} = \frac{\kappa}{8\pi G_5}\,\delta A \quad , \quad I_{B_i} = \frac{\delta\lambda_i}{2}\int_{B_i} d^3x\, N\sqrt{\sigma} \ , \tag{40}$$

with A the black hole horizon area. These results are straightforward to derive. At the black hole horizon B_H, we have $N = 0$ and $D_a N = -\kappa\, n_a$ where κ is the constant surface gravity [2]. This gives $8\pi G_5 N s^{ab} = -\kappa\,\sigma^{ab}$ and the result in (40) follows. At each brane B_i, we use $n^c(D_c N)/N = K - k$, which is a general result [20] valid when $u^a n_a = 0$. Using (3) gives $s^{ab} = (\lambda_i/2)\sigma^{ab}$ which yields the result in (40). In Appendix B, we show the boundary term I_{B_∞} is

$$I_{B_\infty} = -\delta M + \mathcal{F}_\infty\,\delta\ell + \mathcal{U}_1\frac{\delta\Omega_1}{\Omega_1} - \mathcal{U}_2\frac{\delta\Omega_2}{\Omega_2} \ , \tag{41}$$

where the boundary quantity $\mathcal{F}_\infty$ at infinity is

$$\mathcal{F}_\infty = -\frac{1}{2G_5\ell}\int_{Z_1}^{Z_2} dZ\,\Omega^3\,(a_t + 2a_Z) \ , \tag{42}$$

and the coefficients $\mathcal{U}_i$ are

$$\mathcal{U}_i = M\left(\frac{{\Omega_i}^2}{{\Omega_1}^2 - {\Omega_2}^2}\right) - \frac{3\mathcal{Q}}{2G_5}\left(\frac{{\Omega_1}^2{\Omega_2}^2}{{\Omega_1}^2 - {\Omega_2}^2}\right) \ . \tag{43}$$

Here $\mathcal{Q}$ parametrizes the asymptotic interbrane separation (24). We also define $\mathcal{F}$ by the following sum,

$$\mathcal{F}\,\delta\ell = \frac{\delta\Lambda}{8\pi G_5}\int_\Sigma d^4x\, N\sqrt{h} + I_{B_1} + I_{B_2} + \mathcal{F}_\infty\delta\ell \ . \tag{44}$$

Here the two brane terms I_{B_i} render the volume integral finite, as one can verify. The term $\mathcal{F}_\infty$ renders $\mathcal{F}$ gauge invariant, as shown in Appendix C. We also define $\mathcal{V}$ by

$$\mathcal{V} = \left(\frac{\ell}{2P}\right)\mathcal{F} \quad , \quad -\mathcal{V}\,\delta P = \mathcal{F}\,\delta\ell \ , \tag{45}$$

where $P = -\Lambda/(8\pi G_5)$ is the pressure due to the cosmological constant. We have now evaluated all the terms needed to rewrite the preliminary first law (36).

B. The first law

We will give four versions of the first law, corresponding to different choices of variations. Substituting (40), (41), and (45) into (36) gives the first law in the form

$$\delta M = \frac{\kappa\,\delta A}{8\pi G_5} - \mathcal{V}\,\delta P + \mathcal{U}_1\frac{\delta\Omega_1}{\Omega_1} - \mathcal{U}_2\frac{\delta\Omega_2}{\Omega_2} \ . \tag{46}$$

The area term is standard. The last two terms are changes in mass due to changes in the branes' gravitational field, since $\delta\Omega_i$ are variations of gravitational redshift factors on each brane. The last term is absent in RS2, which removes the negative-tension brane to $Z_2 \to \infty$, for which $\Omega_2 \to 0$ and $\mathcal{U}_2/\Omega_2 \to 0$ by (43).

For discussion purposes, we will take $\mathcal{V} > 0$. This is easily verified for the static asymptotically RS black string [25], which is the only known exact solution for an asymptotically RS black object in 5-dimensional spacetime. We will also see in (47) below that $\mathcal{V} > 0$ if and only if a gravitational tension $\mathcal{T}_0$ is positive.

The coefficient of δP defines a thermodynamic volume $V_{\rm eff}$ in a black hole first law [4, 5, 27]. For a static asymptotically AdS black hole, it was found in [4] that $V_{\rm eff} > 0$ is the volume *removed* by the black hole (the volume of pure AdS space minus the volume outside the black hole). In our first law, $V_{\rm eff} = -\mathcal{V} < 0$ suggests that net volume is *added* outside the black hole (compared to the case with no black hole). Added volume makes sense physically: in RS2 the black hole repels the positive-tension brane, and in RS1 we would expect a version of the black hole Archimedes effect [28, 29], where the black hole increases the size of the compact dimension (here the interbrane distance). We also note that $V_{\rm eff} < 0$ occurs, with a natural interpretation as an added volume, in AdS-Taub-NUT-AdS spacetime [30].

In RS1, there are three ways the variation δL of the interbrane distance can be introduced into the first law, using (27). In each case, the coefficient of δL defines a gravitational tension $\mathcal{T}$ that depends on which quantities are held fixed. The three gravitational tensions we refer to below are

$$\mathcal{T}_0 = \frac{2P\,\mathcal{V}}{L} \quad , \quad \mathcal{T}_1 = \frac{\mathcal{U}_1}{\ell} \quad , \quad \mathcal{T}_2 = \frac{\mathcal{U}_2}{\ell} \ . \tag{47}$$

Using (27) in (46) to change variables from $\delta\ell$ to δL gives

$$\delta M = \frac{\kappa\,\delta A}{8\pi G_5} + \mathcal{T}_0\,\delta L + \sum_{i=1,2} \pm\Big(\mathcal{U}_i - \mathcal{T}_0\,\ell\Big)\frac{\delta\Omega_i}{\Omega_i} \ . \tag{48}$$

Here $\pm$ is the sign of each brane tension λ_i and $\mathcal{T}_0$ is a gravitational tension at fixed values of (A, Ω_1, Ω_2). Using (27) in (46) to eliminate $\delta\Omega_1$ or $\delta\Omega_2$ gives the first law as

$$\delta M = \frac{\kappa\,\delta A}{8\pi G_5} + \mathcal{T}_1\,\delta L + \left(-\mathcal{V} + \frac{\mathcal{T}_1\,L}{2P}\right)\delta P + M\frac{\delta\Omega_2}{\Omega_2} \tag{49}$$

and

$$\delta M = \frac{\kappa\,\delta A}{8\pi G_5} + \mathcal{T}_2\,\delta L + \left(-\mathcal{V} + \frac{\mathcal{T}_2\,L}{2P}\right)\delta P + M\frac{\delta\Omega_1}{\Omega_1} \ . \tag{50}$$

Each term $\mathcal{T}\delta L$ in (48)$-$(50) is the work needed to vary the RS1 interbrane distance (with different quantities held fixed), analogous to the work terms in the first law in the case of a compact dimension without branes [6]. Our gravitational tensions (47) are easily verified to be positive for the asymptotically RS1 black string [25]. We would also expect our gravitational tensions to be positive due the black hole's attraction to its images in the covering space, which has been shown [29, 31, 32] in the case of a compact dimension without branes.

Since each version of the first law reparametrizes the geometry, in (49) and (50) each thermodynamic volume $V_{\rm eff} = -\mathcal{V} + \mathcal{T}_i L/P$ differs from $-\mathcal{V}$. For $-\mathcal{V} < 0$, this indicates that positive gravitational tension $\mathcal{T}_i$ opposes the black hole Archimedes effect, and the sign of $V_{\rm eff}$ depends on their relative strengths.

Reparametrizing the geometry also transforms the brane terms in the first law, but with the interesting property that the coefficients of $\delta\Omega_i/\Omega_i$ always add to M, in each version of the first law. A brane term in (46) or (48) shifts into both $V_{\rm eff}\,\delta P$ and $\mathcal{T}_i\,\delta L$ in (49) and (50), and this shift incorporates the brane's orbifold symmetry into the gravitational tension, since $\mathcal{T}_i$ is due to the black hole's attraction to its orbifold mirror images.

VI. VARIATIONAL PRINCIPLE

The first law we proved in section V includes the variations of the AdS curvature length, cosmological constant, brane tensions, and RS brane warp factors. This motivates the following variational principle, which we prove in this section.

Variational principle for asymptotically RS black holes: *Instantaneously static initial data that extremizes the mass M is initial data for a static black hole, for variations that leave fixed the apparent horizon area A, the AdS curvature length ℓ, cosmological constant Λ, brane tensions λ_i, and RS values (at spatial infinity) of the warp factors Ω_i on each brane.*

A. Main proof

Our proof of the variational principle proceeds in two steps. Here we perform the main step, which reduces the proof to two auxiliary boundary value problems. These boundary value problems are the topics of section VI B. Our setup is general: it applies to a black hole localized on a brane, or isolated in the bulk (away from either brane), and also applies to the asymptotically RS black string [25].

Our key assumptions will be the following. We assume our initial data h_{ab} is instantaneously static. We also assume the variations δh_{ab} extremize the mass ($\delta M = 0$), while holding fixed the apparent horizon area A and the remaining quantities (ℓ, Λ, λ_i, Ω_i).

Our proof closely follows the proof of the first law in section V. The initial steps are the same, as we indicated after the result (34). Thus, for an instantaneously static geometry h_{ab}, we proceed exactly as in section V through (34). Now taking $\delta\Lambda = 0$ in (34) gives

$$\int_\Sigma d^{D-1}x\,\mathcal{P}^{ab}\delta h_{ab} + \sum_B I_B = 0\ . \tag{51}$$

In what follows, (51) will be our primary equation, where

$$\mathcal{P}^{ab} = \frac{\sqrt{h}}{16\pi G_D}\left(\mathcal{R}^{ab} + h^{ab}D_cD^c - D^aD^b\right)N\ . \tag{52}$$

For instantaneously static initial data, the constraint $\mathcal{C}_0 = 0$ in (30) simplifies to $\mathcal{R} = 2\Lambda$ and $\delta\mathcal{C}_a$ vanishes identically. The remaining linearized constraint ($\delta\mathcal{C}_0 = 0$) simplifies to

$$\left(\mathcal{R}^{ab} + h^{ab}D^cD_c - D^aD^b\right)\delta h_{ab} = 0\ . \tag{53}$$

We now evaluate the boundary terms I_B in (51). The boundaries B are illustrated in Fig. 1. In section V, we evaluated I_{B_i} (at each brane) and I_{B_∞} (at spatial infinity), including the variations of quantities (ℓ, Λ, λ_i, Ω_i) held constant here by assumption. In this case, (40) and (41) reduce to

$$I_{B_i} = 0\ ,\quad I_{B_\infty} = -\delta M\ . \tag{54}$$

Additionally, we have $\delta M = 0$, by our assumption of a mass extremum, so $I_{B_\infty} = 0$. At the apparent horizon B_H, we use an alternate form to that given in (37),

$$I_{B_H} = \int d^{D-2}x\sqrt{\sigma}\mathcal{A}^{bcd}\left[(D_bN)\delta h_{cd} - ND_b\delta h_{cd}\right]\ , \tag{55}$$

where

$$16\pi G_D\,\mathcal{A}^{bcd} = n_a(h^{ac}h^{bd} - h^{ab}h^{cd})\ . \tag{56}$$

The boundary condition on the lapse is $N = 0$, whence $D_aN = -fn_a$, where $f^2 = (D^bN)(D_bN)$. Then (55) is

$$I_{B_H} = \frac{1}{8\pi G_D}\int d^{D-2}x\,f\,\delta\sqrt{\sigma}\ , \tag{57}$$

using $\sigma_{ab} = h_{ab} - n_an_b$ and $\delta\sqrt{\sigma} = \sqrt{\sigma}\sigma^{ab}\delta\sigma_{ab}/2$. For convenience, we now choose to set $I_{B_H} = 0$ using the following gauge transformation,

$$\delta\sigma_{ab} \to \delta\sigma_{ab} + 2\mathcal{D}_{(a}\xi_{b)}\ ,\quad \sigma^{ab}\delta\sigma_{ab} \to 0\ , \tag{58}$$

where $\mathcal{D}_a$ is the covariant derivative associated with σ_{ab}. If we let $\xi_a = \mathcal{D}_aF$, then $\sigma^{ab}\delta\sigma_{ab} \to 0$ requires

$$-\sqrt{\sigma}\,\mathcal{D}^a\mathcal{D}_aF = \delta\sqrt{\sigma}\ . \tag{59}$$

Note the apparent horizon is a closed surface (this is most clearly seen in the covering space, if the apparent horizon intersects a brane). A solution F to (59) on a closed surface is well known to exist if and only if the surface integral of the right-hand side of (59) vanishes. This integral is simply δA, which indeed vanishes since we hold

A constant. Thus a solution ξ_a exists to achieve (58), and we henceforth set $I_{B_H} = 0$. Since $I_{B_i} = I_{B_\infty} = 0$ and we set $I_{B_H} = 0$, our primary equation (51) simplifies to

$$\int_\Sigma d^{D-1}x \, \mathcal{P}^{ab}\delta h_{ab} = 0 \ . \tag{60}$$

Our goal is to conclude that the initial geometry h_{ab} evolves to a static spacetime. The well known condition for this is that $\mathcal{P}^{ab} = 0$ on Σ. We cannot, however, immediately conclude that $\mathcal{P}^{ab} = 0$ from (60), because not all of the variations δh_{ab} are arbitrary: the linearized constraint (53) removes one degree of freedom, which can be taken as $h^{ab}\delta h_{ab}$ or as the variation δh of the determinant h. These are related by $h^{ab}\delta h_{ab} = \delta h/h$. As an identity, we may decompose δh_{ab} into a trace-free (TF) part and a part proportional to δh:

$$\delta h_{ab} = (\delta h_{ab})^{TF} + \frac{1}{D-1}\left(\frac{\delta h}{h}\right) h_{ab} \ . \tag{61}$$

Using (61), our primary equation (60) then becomes

$$\int_\Sigma d^{D-1}x \left[(\mathcal{P}^{ab})^{TF}(\delta h_{ab})^{TF} + \frac{\mathcal{P}\,\delta h}{(D-1)h} \right] = 0 \ , \tag{62}$$

where

$$\mathcal{P}_{ab} = (\mathcal{P}_{ab})^{TF} + \frac{\mathcal{P}}{D-1} h_{ab} \ , \tag{63}$$

$$\mathcal{P} = h_{ab}\mathcal{P}^{ab} \ . \tag{64}$$

The arbitrary variations are $(\delta h_{ab})^{TF}$, subject to smoothness at the apparent horizon, $(\delta h_{ab})^{TF} \to 0$ at B_∞, and boundary conditions at the branes that we will specify in the next section. As a completeness check, the arbitrary variations $(\delta h_{ab})^{TF}$ alone should determine the dependent quantity δh, which we verify below by showing the linearized constraint (53) is a well posed boundary value problem for δh.

Our proof then reduces to showing $\mathcal{P} = 0$, which allows us to conclude from (62) that $(\mathcal{P}^{ab})^{TF} = 0$, since $(\delta h_{ab})^{TF}$ are arbitrary variations. It then follows from (63) that $\mathcal{P}^{ab} = 0$, which is the desired result. The statement $\mathcal{P} = 0$ is a boundary value problem for N that we demonstrate is solvable in the following section, which completes our proof of the variational principle.

B. Auxiliary boundary value problems

The boundary value problem for N is

$$D^a D_a N - \frac{(D-1)}{\ell^2} N = 0 \ , \tag{65a}$$

$$\left(n^a D_a N - \frac{\varepsilon}{\ell} N \right)\Big|_{B_i} = 0 \ , \tag{65b}$$

$$N\Big|_{B_H} = 0 \ , \tag{65c}$$

$$N\Big|_{B_\infty} \to \Omega \ . \tag{65d}$$

Here, Ω is the warp factor of the asymptotic RS solution (1) and $\varepsilon = \pm 1$ is the sign of each brane tension. The result (65a) follows from setting $\mathcal{P} = h_{ab}\mathcal{P}^{ab} = 0$, using (52) and $\mathcal{R} = 2\Lambda$. The boundary conditions (65c) and (65d) are straightforward. Our main concern is the brane boundary condition (65b), which results from using

$$2K_{ab} = n^c\partial_c\gamma_{ab} + \gamma_{ac}\partial_b n^c + \gamma_{bc}\partial_a n^c \ . \tag{66}$$

Now $\gamma_{tt} = -N^2$ and $\gamma_{ta} = 0$ gives $2K_{tt} = n^c\partial_c(-N^2)$, and the Israel condition $K_{tt} = (\varepsilon/\ell)\gamma_{tt}$ then gives (65b).

As shown in [33], the following approach can put a Robin boundary condition (65b) in a standard form while keeping its associated elliptic equation (65a) in a divergence form. Let w_a be any vector field and define $\mathcal{W}_a N = (D_a - w_a)N$. Then (65a) and (65b) become

$$D^a\mathcal{W}_a N + w^a D_a N + \left[D_a w^a - \frac{(D-1)}{\ell^2} \right] N = 0 \tag{67}$$

and

$$\left[n^a\mathcal{W}_a N + \left(n^a w_a - \frac{\varepsilon}{\ell} \right) N \right]\Big|_{B_i} = 0 \ . \tag{68}$$

As in [33], we now choose w_a so $(n^a w_a - \varepsilon/\ell) \geq 0$ at B_i, which is the usual prerequisite for applying an existence theorem to a boundary value problem of the form (67)–(68). For example, we choose $w_a = -\tilde{n}_a/\ell$, where $\tilde{n}_a$ is any vector field, pointing from B_1 to B_2, that interpolates from the inward unit normal $(-n_a)$ of B_1 to the outward unit normal n_a of B_2. Then $n^a\tilde{n}_a = -\varepsilon$ at each brane B_i, and $w_a = -\tilde{n}_a/\ell$ gives $(n^a w_a - \varepsilon/\ell) = 0$ in (68). With the brane boundary conditions in standard form, and the remaining standard (Dirichlet) boundary conditions, (65c) and (65d), we then readily infer that the boundary value problem (65) for N is solvable.

We now turn to the boundary value problem for δh, which we will state in terms of a scalar quantity $(\delta h/h)$,

$$D^a D_a(\delta h/h) - \frac{(D-1)}{\ell^2}(\delta h/h) = f_\Sigma \ , \tag{69a}$$

$$\left[n^a D_a(\delta h/h) - \frac{\varepsilon}{\ell}(\delta h/h) \right]\Big|_{B_i} = f_i \ , \tag{69b}$$

$$n^a D_a(\delta h/h)\Big|_{B_H} = f_H \ , \tag{69c}$$

$$(\delta h/h)\Big|_{B_\infty} \to 0 \ . \tag{69d}$$

As above, $\varepsilon = \pm 1$ is the sign of each brane tension. The result (69a) follows from substituting (61) into the linearized constraint (53) with $\mathcal{R} = 2\Lambda$. We will give the source terms and derive the boundary conditions below.

The key point is that (69) is a well posed boundary value problem. The elliptic equation (69a) and the boundary conditions (69b) are similar in form to (65a) and (65b) in the previous boundary value problem (65). The remaining boundary conditions, (69c) and (69d), are well known types: Neumann and Dirichlet, respectively.

In the remainder of this section, we provide the details of the source terms and the boundary conditions in (69). The source terms in (69) are

$$
\begin{aligned}
f_\Sigma &= \frac{D-1}{D-2}\left[D^a D^b (\delta h_{ab})^{TF} - \mathcal{R}^{ab}(\delta h_{ab})^{TF}\right] \ , \\
-f_i &= \frac{D-1}{D-2}\left[\sigma^{ab} n^c D_c (\delta h_{ab})^{TF} + \frac{\varepsilon D}{\ell} n^a n^b (\delta h_{ab})^{TF}\right] \ , \\
f_H &= \frac{1}{D-2}\left[2k^{ab}(\delta h_{ab})^{TF} - \sigma^{ab} n^c D_c (\delta h_{ab})^{TF}\right] \ .
\end{aligned}
$$

The boundary conditions on δh are given by varying those on h_{ab}, which at the apparent horizon and the branes involve the extrinsic curvature $k_{ab} = \sigma_a{}^c D_c n_b$,

$$
k\Big|_{B_H} = 0 \ , \quad k_{ab}\Big|_{B_i} = \frac{\varepsilon}{\ell}\,\sigma_{ab} \ . \tag{70}
$$

By varying these, we obtain

$$
\delta k\Big|_{B_H} = 0 \ , \quad \delta k\Big|_{B_i} = 0 \ , \quad \delta k_{ab}\Big|_{B_i} = \frac{\varepsilon}{\ell}\,\delta\sigma_{ab} \ . \tag{71}
$$

To evaluate these, we use the general results

$$
\begin{aligned}
2\delta k_{ab} &= \left(n^c n^d \delta h_{cd}\right) k_{ab} - \sigma_a{}^c \sigma_b{}^d n^f J_{cdf} \ , && \text{(72a)} \\
-2\delta k &= 2k^{ab}\delta\sigma_{ab} - k\, n^a n^b \delta h_{ab} + \sigma^{ab} n^c J_{abc} \ , && \text{(72b)} \\
J_{abc} &= D_a \delta h_{bc} + D_b \delta h_{ac} - D_c \delta h_{ab} \ . && \text{(72c)}
\end{aligned}
$$

The boundary conditions at the branes (69b) and the apparent horizon (69c) result from evaluating $\delta k = 0$ using (61), (70), and (72b). The last relation in (71) expresses brane boundary conditions for $(\delta h_{ab})^{TF}$, since it reduces to a form independent of δh after substituting (3), (61), (69b), and (72a).

C. An application of the variational principle

Here we demonstrate the utility of the variational principle, by applying it to a trial solution and reproducing the static asymptotically RS black string [25], which is the only known exact solution for an asymptotically RS black object in 5-dimensional spacetime. We first specify a trial geometry for an initially static a black string. After evaluating the apparent horizon area A and mass M, we then apply the variational principle.

A black string is a set of lower dimensional black holes stacked in an extra dimension Z, which is how we will construct the trial geometry. We take

$$
ds^2 = \Omega(Z)^2\left[\Psi^4(\mathbf{x}, Z)\, d\mathbf{x}^2 + dZ^2\right] \ , \tag{73}
$$

where $\Omega = \ell/Z$ and the branes are the surfaces $Z = Z_1$ and $Z = Z_2$. We will take

$$
\Psi = 1 + \frac{\rho_0}{2}\left(\frac{1}{|\mathbf{x}+\mathbf{x}_0|} + \frac{1}{|\mathbf{x}-\mathbf{x}_0|}\right) \ . \tag{74}
$$

Here $\rho_0 > 0$ is a constant and $\mathbf{x}_0 = (0, 0, d)$, using Cartesian coordinates $\mathbf{x} = (x, y, z)$ with origin at $\mathbf{x} = 0$. We choose a function $d(Z)$ as follows. The constraint, $\mathcal{R} = -12/\ell^2$, after linearizing in d and its derivatives, has the solution $d(Z) = d_0 + c_0 Z^4$, where c_0 and d_0 are constants. For the case $c_0 = 0$, (74) is an exact solution and the RS1 limit ($\Omega_2 \to 0$) is easily taken. We will work in RS2 and take c_0 as a small nonzero parameter.

On each slice Z=constant, we now transform to spherical coordinates centered at $\mathbf{x} = 0$, with $z = \rho\cos\theta$, and expand Ψ in Legendre polynomials $P_k(\cos\theta)$ for $\rho > d$,

$$
\Psi = 1 + \frac{\rho_0}{\rho} + \frac{\rho_0}{\rho}\sum_{j=1}^{\infty}\left(\frac{d}{\rho}\right)^{2j} P_{2j}(\cos\theta) \ . \tag{75}
$$

On each slice Z=constant, we will take $\rho_0 \gg d$, which describes a 3-dimensional black hole [34], and d parametrizes the 2-dimensional apparent horizon's distortion from the sphere $\rho = \rho_0$. The full 3-dimensional apparent horizon (the union of the 2-dimensional apparent horizons) therefore describes a distorted black string.

On each slice Z=constant, as in [34], the surface $\rho(\theta)$ of the 2-dimensional apparent horizon can be found as the sum of Legendre polynomials that minimizes the area A_2,

$$
A_2 = 2\pi\int_0^\pi d\theta\, \Psi^4 \rho \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2} \ . \tag{76}
$$

To lowest order in $(d/\rho_0) \ll 1$, we find the apparent horizon on each slice Z=constant is

$$
\rho(\theta) = \rho_0\left[1 + \frac{5}{7}\left(\frac{d}{\rho_0}\right)^2 P_2(\cos\theta)\right] \ . \tag{77}
$$

This agrees with the numerical results of [34], and gives, to lowest order in (d/ρ_0),

$$
A_2(Z) = 64\pi\rho_0^2\left[1 - \frac{5}{7}\left(\frac{d}{\rho_0}\right)^4\right] \ . \tag{78}
$$

Integrating in Z gives the 3-dimensional area of the full apparent horizon in the black string geometry,

$$
A = \int_{Z_1}^{Z_2} dZ\, \Omega^3 A_2 = 32\pi w\left(\rho_0^2 - \frac{5Q}{7\rho_0^2}\right) \ , \tag{79}
$$

where, with Ω_i the warp factor at each brane,

$$
w = \ell\left(\Omega_1^2 - \Omega_2^2\right) \ , \quad Q = d_0^4 + \frac{8\, c_0\, d_0^3\, \ell^4}{\Omega_1\Omega_2(\Omega_1 + \Omega_2)} \ . \tag{80}
$$

We defined the mass M in (14), which for (75) gives

$$
G_5 M = w\rho_0 \ . \tag{81}
$$

To lowest order in Q, combining (79)–(81) gives

$$
\left(\frac{G_5 M}{w}\right)^2 = \left(\frac{A}{32\pi w}\right) + \frac{5}{7}\left(\frac{32\pi w}{A}\right) Q \ . \tag{82}
$$

We now apply our variational principle: we extremize M in (82) at fixed A, ℓ, and Ω_i. This yields the conditions $c_0 = 0$ and $d_0 = 0$, which we conclude describes a static black string. We can verify this directly, since $d = 0$ in (75) gives $\Psi = 1 + \rho_0/\rho$ and the apparent horizon (77) is located at $\rho = \rho_0$. This is indeed the initial geometry of the static black string [25] in isotropic coordinates.

We can also deduce that the evolution of the distorted black string, with $d \neq 0$, will not be static, since each slice Z=constant is the initial geometry for an attracting two-body problem [34], and the 2-dimensional apparent horizon considered above describes a black hole formed by, and surrounding, two closely separated smaller black holes (with a small minimal surface surrounding each point $\mathbf{x} = \pm\mathbf{x}_0$). From the perspective in each slice Z=constant, as in [34], the two small interior black holes will coalesce as the initial data evolves, due to mutual gravitational attraction. This results in a time-dependent geometry on each slice Z=constant, and results in a time-dependent black string geometry in the bulk perspective.

VII. CONCLUSION

We have derived the first law for a static asymptotically RS black hole, whose mass M is defined in (14) and (16). Four versions of this law are given in (46)–(50) for different choices of variations. In both RS1 and RS2, the general first law contains brane terms and a thermodynamic volume. In RS1, we can define both a thermodynamic volume and a gravitational tension, due to the presence of both a cosmological constant and a compact interbrane distance. This differs from the first law in previously studied spacetimes (with either a cosmological constant or a compact dimension), where the analogs of our thermodynamic volume and gravitational tension are isolated from each other, appearing in the separate first laws of separate spacetimes.

The variational principle we developed in this paper states that for an asymptotically RS black hole initially at rest, initial data that extremizes the mass yields a static black hole, for variations at fixed values of the apparent horizon area and the remaining physical variables in the first law (L, Ω_i, ℓ, Λ, λ_i). It would be interesting to investigate the consequences of holding fewer variables fixed. An example of this in four-dimensional spacetime is Hawking's proof [16] that the static (Schwarzschild) black hole is an extremum of mass at fixed apparent horizon area but arbitrary angular momentum.

Our example application of the variational principle to a trial solution serves as a prelude to the approach we will take in a sequel paper [35]. In [35], we will conclude that solutions exist for small static black holes in RS2, both on and off the brane, as special members of a general family of initially static black holes. This family of black hole initial data will also indicate that a small black hole on an orbifold-symmetric brane in RS2 is stable against leaving the brane, which generalizes to other models with an orbifold-symmetric brane. If we inhabit such a brane, then small black holes, if produced in high energy collider experiments on the brane, could be studied directly (instead of leaving behind a signature of missing energy), which is an important result for future experiments at the LHC.

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Appendix A: Mass counterterm M_{ct}

In this appendix, we derive the mass counterterm M_{ct} from (6) and (13), and thereby prove the mass formula (14). We begin with the variation of (13),

$$\delta S_{ct} = -\int d^4x \frac{\sqrt{-\gamma}}{8\sqrt{2}\pi G_5}\left(\sqrt{\hat{\mathcal{R}}}\gamma^{ab}\delta\gamma_{ab} + \frac{\delta\hat{\mathcal{R}}}{\sqrt{\hat{\mathcal{R}}}}\right) . \quad \text{(A1)}$$

The standard variation of the Ricci scalar is

$$\delta\hat{\mathcal{R}} = -\hat{\mathcal{R}}^{ab}\delta\hat{\sigma}_{ab} + \hat{d}_a v^a \quad \text{(A2)}$$

where $v^a = 2\hat{\sigma}^{b[a}\hat{d}^{c]}\delta\hat{\sigma}_{bc}$ and $\hat{d}_a$ is the covariant derivative associated with $\hat{\sigma}_{ab}$. This gives

$$\delta S_{ct} = \int d^4x\, \mathcal{S}^{ab}\delta\gamma_{ab} - (8\sqrt{2}\pi G_5)I , \quad \text{(A3)}$$

where $\mathcal{S}^{ab}$ is given below and, with $J = \sqrt{-\gamma_{tt}\gamma_{ZZ}/\hat{\mathcal{R}}}$,

$$I = \int dt\, dZ \int d^2x \sqrt{\hat{\sigma}} \left[\hat{d}_a(Jv^a) - v^a\hat{d}_a J\right] . \quad \text{(A4)}$$

We conclude that $I = 0$, as follows. The first term is a total divergence, but the 2-sphere has no boundary. Also, $\hat{d}_a J = 0$ since we take J independent of the angular coordinates. Then (A3) gives

$$\mathcal{S}^{ab} = \frac{\delta S_{ct}}{\delta\gamma_{ab}} . \quad \text{(A5)}$$

In (A1) we now use

$$\gamma^{ab}\delta\gamma_{ab} = \gamma^{tt}\delta\gamma_{tt} + \hat{\sigma}^{ab}\delta\hat{\sigma}_{ab} + \sigma^{ZZ}\delta\sigma_{ZZ} \quad \text{(A6)}$$

which gives

$$\frac{\delta S_{ct}}{\delta\gamma_{tt}} = -\frac{1}{8\sqrt{2}\pi G_5}\sqrt{-\gamma\hat{\mathcal{R}}}\,\gamma^{tt} , \quad \text{(A7a)}$$

$$\frac{\delta S_{ct}}{\delta\hat{\sigma}_{ab}} = -\frac{1}{8\sqrt{2}\pi G_5}\sqrt{-\gamma\hat{\mathcal{R}}}\left(\hat{\sigma}^{ab} - \frac{\hat{\mathcal{R}}^{ab}}{\hat{\mathcal{R}}}\right) , \quad \text{(A7b)}$$

$$\frac{\delta S_{ct}}{\delta\sigma_{ZZ}} = -\frac{1}{8\sqrt{2}\pi G_5}\sqrt{-\gamma\hat{\mathcal{R}}}\,\sigma^{ZZ} . \quad \text{(A7c)}$$

The mass counterterm from (6) and (A7a) is then

$$M_{ct} = \frac{\sqrt{2}}{8\pi G_5} \int d^3x \sqrt{-\gamma \hat{\mathcal{R}}} \ . \tag{A8}$$

Combining this with (5) now gives the mass formula (14).

Appendix B: Boundary term I_B at infinity

Here we evaluate the term I_{B_∞} given by (37) at the boundary $\rho \to \infty$. Throughout this appendix, $\simeq$ denotes evaluating at leading order and neglecting terms of higher order in $1/\rho$. We will relate I_{B_∞} to the mass variation δM and additional terms. The mass is a sum of two terms, $M = \widetilde{M} + M_{ct}$, whose individual variations are

$$\delta\widetilde{M} = -\frac{1}{8\pi G_5} \int_{B_\infty} d^3x \left[N\delta\left(\sqrt{\sigma}\, k\right) + \sqrt{\sigma}\, k\, \delta N\right] \tag{B1}$$

and

$$\delta M_{ct} = \int_{B_\infty} d^3x \left[\frac{\delta M_{ct}}{\delta\sigma_{ab}}\,\delta\sigma_{ab} + \frac{\sqrt{2\sigma\hat{\mathcal{R}}}}{8\pi G_5}\,\delta N\right] \ . \tag{B2}$$

Note $\delta M_{ct}/\delta\sigma_{ab} = -\delta S_{ct}/\delta\sigma_{ab}$ since $\int dt\, M_{ct} = -S_{ct}$ by (A8) and (13). From (A7b) and (A7c), we find

$$\frac{\delta M_{ct}}{\delta\sigma_{ab}} - \frac{N}{2}\sqrt{\sigma} s^{ab} = X^{ab} \tag{B3}$$

at large ρ, where the quantities X^{ab} are given below. Using (B1)–(B3), we rewrite the boundary term (37) as

$$I_{B_\infty} = -\delta M + \int_{B_\infty} d^3x\, X^{ab}\delta\sigma_{ab} - \frac{1}{8\pi G_5}\int_{B_\infty} d^3x\,\sqrt{\sigma}\left(k - \sqrt{2\hat{\mathcal{R}}}\right)\delta N \ . \tag{B4}$$

For the metric asymptotics (15), the quantities X^{ab} are

$$16\pi G_5\, X^{\chi\chi} \simeq \frac{\Omega\sin\chi}{\rho^2}\left(a_t + a_\rho + a_Z\right) \ , \tag{B5a}$$

$$16\pi G_5\, X^{\phi\phi} \simeq \frac{\Omega}{\rho^2\sin\chi}\left(a_t + a_\rho + a_Z\right) \ , \tag{B5b}$$

$$16\pi G_5\, X^{ZZ} \simeq \Omega\sin\chi\left(a_t + 2a_\rho\right) \ . \tag{B5c}$$

Here χ is the polar angle on the 2-sphere with radius ρ. We now proceed to evaluate (B4). We begin with three convenient variables (ℓ, Z_1, Z_2) and then express results in terms of three physical variables (ℓ, Ω_1, Ω_2). We first consider the variation $\delta\ell$ at fixed (Z_1, Z_2). At large ρ, we have $\delta g_{ab} \simeq 2(\delta\ell/\ell)g_{ab}$. Then

$$\int_{B_\infty} d^3x\, X^{ab}\delta\sigma_{ab} = \mathcal{F}_\infty\delta\ell \tag{B6}$$

where

$$\mathcal{F}_\infty = \frac{1}{2G_5\ell}\int_{Z_1}^{Z_2} dZ\,\Omega^3\left(a_t + 2a_\rho\right) \ . \tag{B7}$$

This is entirely due to $\delta\sigma_{ZZ}$ since the integral contributions from $\delta\sigma_{\phi\phi}$ and $\delta\sigma_{\chi\chi}$ vanish by the identity (18a). This identity can also be used to rewrite $\mathcal{F}_\infty$ in the form given in (42). The last line in (B4) yields $M\delta\ell/\ell$. Hence (B4) yields, at fixed (Z_1, Z_2),

$$\left(I_{B_\infty}\right)_{Z_1 Z_2} = -\delta M + \left(\mathcal{F}_\infty + \frac{M}{\ell}\right)\delta\ell \ . \tag{B8}$$

We now consider variations (δZ_1, δZ_2) at fixed ℓ. We then perform a coordinate transformation

$$Z \to \widetilde{Z} = (1-\epsilon)Z - \zeta \tag{B9}$$

such that the branes again reside at $Z = Z_i$. The required transformation is

$$\epsilon = \frac{\delta Z_2 - \delta Z_1}{Z_2 - Z_1} \quad , \quad \zeta = \frac{Z_2\,\delta Z_1 - Z_1\,\delta Z_2}{Z_2 - Z_1} \ . \tag{B10}$$

At large ρ, the resulting metric perturbation is

$$\delta g_{ab} \simeq 2\Omega^2\left[\epsilon\left(\delta_a^Z\delta_b^Z - \eta_{ab}\right) - \frac{\zeta}{\ell}\,\Omega\eta_{ab}\right] \ , \tag{B11}$$

where η_{ab} is the 5-dimensional Minkowski metric. Then (B4) becomes, at fixed ℓ,

$$\left(I_{B_\infty}\right)_\ell = -\delta M - \epsilon M - \zeta\,\mathcal{I} \ , \tag{B12}$$

where the integral $\mathcal{I}$ is

$$\mathcal{I} = \frac{3}{2G_5\ell}\int_{Z_1}^{Z_2} dZ\,\Omega^4\left(a_t + 2a_\rho + a_Z\right) \ . \tag{B13}$$

For the case when all three quantities (ℓ, Z_1, Z_2) are varied, we combine (B8) and (B12) to obtain

$$I_{B_\infty} = -\delta M + \left(\mathcal{F}_\infty + \frac{M}{\ell}\right)\delta\ell - \epsilon M - \zeta\,\mathcal{I} \ . \tag{B14}$$

We can evaluate the integral $\mathcal{I}$ using (17a) and the identity (18b). We then express the result in terms of M and $\mathcal{Q}$ using (22) and (25). This gives

$$\mathcal{I}\ell = M\left(\frac{{\Omega_1}^3 - {\Omega_1}^3}{{\Omega_1}^2 - {\Omega_2}^2}\right) - \frac{3\mathcal{Q}}{2G_5}\left(\frac{{\Omega_1}^2{\Omega_2}^2}{{\Omega_1}^2 - {\Omega_2}^2}\right) \ . \tag{B15}$$

We also express ϵ and ζ in terms of three physical variables (ℓ, Ω_1, Ω_2) using

$$\delta Z_i = \frac{1}{\Omega_i}\left(\delta\ell - \frac{\ell}{\Omega_i}\,\delta\Omega_i\right) \ . \tag{B16}$$

Using (B15) and (B16) in (B14) then yields the result for I_{B_∞} given in (41).

Appendix C: Gauge invariance

It is important to confirm that our quantities (M, $\mathcal{Q}$, $\mathcal{V}$, L, $\mathcal{T}_0$, $\mathcal{T}_i$, $\mathcal{U}_i$) are gauge invariant at infinity. As one can verify, these quantities are invariant under the following metric transformation that leaves the branes fixed,

$$a_\nu \to a_\nu - \frac{\Omega'}{\Omega} w - \delta^Z_\nu w' \quad , \quad w(Z_1) = w(Z_2) = 0 \ , \quad \text{(C1)}$$

with $' = d/dZ$. This is generated by the coordinate transformation $x^a \to x^a + \varepsilon^a$, where to leading order in $1/\rho$,

$$\varepsilon^Z = \frac{w}{\rho} \quad , \quad \varepsilon^\rho = \frac{W}{\rho^2} \quad , \quad w = W' \ . \quad \text{(C2)}$$

In particular, we consider the quantity $\mathcal{F}$, and write

$$\mathcal{F} = \mathcal{F}_\Sigma + \mathcal{F}_\infty \quad \text{(C3)}$$

where $\mathcal{F}_\Sigma$ is the sum of bulk and brane terms in (44),

$$\mathcal{F}_\Sigma \, \delta\ell \ \equiv \ \frac{\delta\Lambda}{8\pi G_5} \int_\Sigma d^4x \, N\sqrt{h} \ + \ I_{B_1} \ + \ I_{B_2} \ . \quad \text{(C4)}$$

We note that $\mathcal{F}$ is gauge invariant, but neither $\mathcal{F}_\Sigma$ nor $\mathcal{F}_\infty$ is separately invariant, since they transform as

$$\mathcal{F}_\Sigma \to \mathcal{F}_\Sigma - \varphi \quad , \quad \mathcal{F}_\infty \to \mathcal{F}_\infty + \varphi \ , \quad \text{(C5)}$$

where

$$\varphi = \frac{3}{2G_5\ell^2} \int_{Z_1}^{Z_2} dZ \, \Omega^4 w \ . \quad \text{(C6)}$$

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