Missing link: A nonlinear post-Friedmann framework for small and large scales

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The missing link: a nonlinear post-Friedmann framework for small and large scales

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We present a nonlinear post-Friedmann framework for structure formation, generalizing to cosmology the weak-field (post-Minkowskian) approximation, unifying the treatment of small and large scales. We consider a universe filled with a pressureless fluid and a cosmological constant Λ, the theory of gravity is Einstein's general relativity and the background is the standard flat ΛCDM cosmological model. We expand the metric and the energy-momentum tensor in powers of $1/c$, keeping the matter density and peculiar velocity as exact fundamental variables. We assume the Poisson gauge, including scalar and tensor modes up to $1/c^4$ order and vector modes up to $1/c^5$ terms. Through a redefinition of the scalar potentials as a resummation of the metric contributions at different orders, we obtain a complete set of nonlinear equations, providing a unified framework to study structure formation from small to superhorizon scales, from the nonlinear Newtonian to the linear relativistic regime. We explicitly show the validity of our scheme in the two limits: at leading order we recover the fully nonlinear equations of Newtonian cosmology; when linearized, our equations become those for scalar and vector modes of first-order relativistic perturbation theory in the Poisson gauge. Tensor modes are non-dynamical at the $1/c^4$ order we consider (gravitational waves only appear at higher order): they are purely nonlinear and describe a distortion of the spatial slices determined at this order by a constraint, quadratic in the scalar and vector variables.

The main results of our analysis are: at leading order a purely Newtonian nonlinear energy current sources a frame-dragging gravitomagnetic vector potential; in the leading-order Newtonian regime and in the linear relativistic regime the two scalar metric potentials are the same, while the nonlinearity of general relativity makes them different.

Possible applications of our formalism include the calculations of the vector potential [1, 2] and the difference between the two scalar potentials from Newtonian N-body simulations, and the extension of Newtonian approximations used in structure formation studies, to include relativistic effects.

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I. INTRODUCTION

The ΛCDM model [3, 4] has emerged in the last few decades as the standard "concordance" model of cosmology [5]. Beyond photons and baryons, the main components of ΛCDM are cold dark matter (CDM), able to cluster and form structures, and the cosmological constant Λ, responsible for the observed acceleration of the universe expansion.

ΛCDM is based on Einstein general relativity (GR), and on the cosmological principle, i.e. a cosmological version of the Copernican principle [6, 7]. This request for the universe to be, *on average*, homogeneous and isotropic translates, in the language of spacetime, in assuming a Robertson-Walker metric. With this Einstein equations reduce to the Friedmann equations; the solutions of these equations are the Friedmann-Lemaitre-Robertson-Walker (FLRW) models. Cosmic microwave background (CMB) anisotropy measurements [8] have established that the universe is, to a great degree, spatially flat, as confirmed by recent Planck [9] and baryon acoustic oscillations (BAO) data [10]. In this

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standard scenario, small primordial inflationary perturbations on top of the FLRW background grow and produce the CMB fluctuations and the large-scale structure that we observe at low redshift.

The theoretical tools that we use to study the growth of the large-scale structure are basically two: *i)* relativistic perturbation theory [11, 12] is used to describe fluctuations in the early universe, in the CMB, and in the matter density field on very large scales; *ii)* Newtonian methods, notably N-body simulations [13], are used to study the growth of structures in the nonlinear regime, at late times and small scales. Lagrangian perturbation theory [14, 15] (LPT, i.e. the Zel'dovich approximation or its second-order extension, 2LPT) is typically used to set up initial conditions for N-body simulations; LPT and other Newtonian approximations are also used to model nonlinear scales, e.g. BAO and CDM halos [16–21].

Observational cosmology has now reached an unprecedented precision, allowing stringent tests on models of the universe. The tightest constraints come from the CMB [22]. A number of probes exist at low redshift, such as supernovae [23], the local measurements of the expansion rate [24] and the 3-dimensional mapping of the large-scale structure [25, 26]. In particular, rich clusters, lensing and redshift space distortion allow the measurement of the growth of clustering [27, 28]. The standard ΛCDM model is very well supported by all these observations. However, a tension emerges in the framework of the base ΛCDM when parameter values measured from low-redshift probes are compared with the values obtained from the CMB, as recently pointed out [29–32]. In particular, this tension shows up in a different growth of clustering in LSS at different epochs [9], as confirmed by the latest Planck release: "as in the 2013 analysis, the amplitude of the fluctuation spectrum is found to be higher than inferred from some analyses of rich cluster counts and weak gravitational lensing" (see [22] and Refs. therein). Measurements of redshift space distortion also suggest less clustering than the CMB [29, 31]. Notably, adding a variable total neutrinos mass as extra parameter to the base ΛCDM model relieves the tension [30, 31].

Given that the presence of a CDM component is widely accepted, while a cosmological constant has its own problems [33], much work is gone into exploring alternatives to Λ. A number of possibilities have been proposed which, in essence, can be divided in three groups. One is to keep GR and modify the energy-momentum content, the dark sector in particular, replacing Λ with some form of dark energy [34]. A second alternative that has been widely explored in the last decade is to replace GR with a modified theory of gravity [35, 36]. A third, more radical, option is that of abandoning the cosmological principle [37], considering inhomogeneous models and the possibility that the observed acceleration of the universe is the result of back-reaction [38, 39], either dynamical or optical [40].

Galaxy surveys are now aiming at the same 1% precision as CMB measurements: for instance, the detection of BAO by the Sloan Digital Sky Survey has recently allowed the first 1% level cosmological constraint by a galaxy survey [41]. In addition, future surveys such as Euclid [42–44] and SKA [45–47] will reach scales of the order of the Hubble horizon. It is therefore crucial that the theory used to make predictions and to interpret observations is developed with a matching accuracy. In particular, while exploring alternatives to the standard ΛCDM scenario is an interesting challenge, it seems timely to refine the theoretical modelling of ΛCDM, bridging the above mentioned gap between the Newtonian treatment of nonlinear small scales and the relativistic description of large scales. Ultimately, going beyond the Newtonian approximation in simulations of large-scale structure should be important in order to take into account causal, retardation and other GR effects that may be non-negligible for simulations that aim at $\sim 1\%$ accuracy, in view of future surveys such as Euclid [42], on scales of the order of the Hubble horizon.

A first step in this direction would be to include GR corrections in the initial conditions for simulations, using a dictionary based on first-order perturbation theory [48], cf. [49]; also, first-order GR effects on horizon scales must be taken into account in interpreting bias and non-Gaussianity [50]. However, while the Poisson equation in Newtonian gravity establishes a linear relation between the gravitational potential and the matter density field, the intrinsic nonlinearity of GR unavoidably generates new effects, even when perturbations are small. For instance, an initially Gaussian curvature inflationary perturbation translates into an effective non-Gaussianity of the density field in the matter era [51–54], a nonlinear effect that should be taken into account in initial conditions for simulations. But the Poisson equation embodies action-at-the-distance, i.e. it is the mathematical representation of the a-causal nature of Newtonian gravity, thus the ultimate step forward would be that of investigating the effects of relativistic nonlinearity in cosmological structure formation, including GR corrections in the evolution of the matter density field in N-body simulations, as well as in approximate treatments of the nonlinear regime [1, 2, 48–60].

Aim of this paper is to present a new nonlinear relativistic post-Friedmann (PF) formalism which, in essence, is a generalisation to cosmology of the post-Minkowski (weak-field) approximation, married with the fundamental assumption of the post-Newtonian (PN) approximation that velocities are small. Our goal is a relativistic framework valid on all scales, and including the full nonlinearity of Newtonian gravity at small scales. In this framework - assuming a flat ΛCDM background and a fluid description of matter - the exact equations of Newtonian cosmology [61, 62] appear as a consistent approximation of the full set of Einstein equations, determining leading order terms (which we call 0PF) in the metric. GR corrections (which we call 1PF) appear next, and are of two types: terms quadratic in the Newtonian variables (we may call these proper PN terms), and proper linear GR terms. Once linearised, the nonlinear approximate equations we obtain for a set of appropriately resummed variables are those for

the scalar and vector sectors of first-order relativistic perturbation theory [63]. Tensor modes are purely nonlinear and non-dynamical, i.e. they don't describe propagating gravitational waves, but rather a distortion of the spatial slices.

Let's clarify how our PF formalism differs from the traditional PN approach [64] and its various application in cosmology. In a contemporary perspective [65], the correct derivation of the post-Newtonian approximation on the flat background space-time follows consistently from the post-Minkowski approximation once the $v/c \ll 1$ assumption is made, which also implies to neglect time derivatives with respect to space derivatives[1]. However, in cosmology the background is a FLRW solution, in our case the flat ΛCDM model, and what we can assume to be small are peculiar velocities, not the change with time of the physical distance between two arbitrary observers. To illustrate the point, adopting a Newtonian perspective, in the absolute space of Newtonian cosmology one uses the background comoving coordinates $\mathbf{x}$ as the Eulerian grid, and the physical position of a fluid element is $\mathbf{r} = a\mathbf{x}$. Then $\dot{\mathbf{r}} = H\mathbf{r} + \mathbf{v}$, where $\mathbf{v} = a\dot{\mathbf{x}}$ is the physical (or proper) peculiar velocity [11], representing the deviation from the Hubble flow; $\dot{\mathbf{x}}$ is the peculiar velocity with respect to the comoving grid [62]. Then, should we assume that $|\dot{\mathbf{r}}| \ll c$, we would end up with an approximation only valid at small scales, well below the Hubble horizon, $|\mathbf{r}| \ll cH^{-1}$. In addition, traditionally the PN formalism has been developed to study GR corrections to the orbits of isolated objects. Thus the focus is on the equations of motion, rather than on a consistent approximation of the full set of Einstein equations. In developing his PN treatment of relativistic hydrodynamics, having in mind applications to relativistic stars, Chandrasekhar [67] also focused on the equations of motion for the fluid, as derived from the conservation equations. In both cases, the metric and other variables are expanded[2] in powers of $1/c$, then the expansion is applied to the equations of motion. As it is well known, a correction to the time-time component of the Minkowski metric is all is needed to obtain the Newtonian equations of motion. All other corrections are then considered post-Newtonian. We will discuss this point in detail in Section VI B where, following [69], we call *passive* this approach focused on the equations of motion. In cosmology however we find desirable to consider an *active* approach where the space-space and time-time components of the metric are equally weighted, leading directly to a consistent treatment, order by order in the expansion parameter $1/c$, of the full set of Einstein equations[3]. In view of applications, a useful byproduct of this active approach is that the equal weighing of the different metric components is what is needed to obtain the correct photon trajectories, i.e. the null geodesics that are at the base of the causal structure of the spacetime.

Various early and more recent works have applied the traditional PN expansion in cosmology: in [70–73] the PN equation of motion for particles in the expanding universe are derived; in [74] the authors clarify the role of the electric and magnetic part of the Weyl tensor in the Newtonian approximation; in [75] and [76] the PN analysis is given in Lagrangian coordinates using respectively the 3+1 and 1+3 framework; in [77] the authors derive a complete set of field and hydrodynamic equations in Newtonian-like forms. Other remarkable works following a PN method are those of Szekeres [78, 79]. Finally, [80] follow a "hybrid approximation scheme", somehow closer in spirit to our PF approach, a mix between standard cosmological perturbation theory and PN approximation, also obtaining equations for the generation of gravitational waves.

All these works apply the standard iterative approach of the PN expansion. Here instead we focus on defining a set of resummed 1PF variables, and on deriving the set of nonlinear evolution and constraint equations they satisfy. Specifically, assuming the Poisson (or conformal-Newtonian [11, 12, 63, 81]) gauge, we expand the metric in powers of $1/c$ and in the equations we retain all scalar terms up to order $1/c^4$ (and all vector terms up to $1/c^5$), instead of peeling-off the different orders, cf. [78, 79]. This leads to a final set of equations that differs from those of the works mentioned above. Partly following the PN tradition, the 0PF and 1PF orders respectively refer to terms $1/c^2$ and $1/c^4$, relevant for scalar potentials. The 0PF terms are Newtonian, the 1PF terms contain the GR corrections. Finally, we stress that, in order to obtain a system of equations with a well-posed Cauchy problem, we should consider terms of the next 2PF order, i.e. $1/c^6$ [78, 79].

In summary, the main goal of this paper is to present a set of nonlinear resummed equations up to 1PF order which retain in full the nonlinearity of Newtonian theory on small scales and all linear relativistic perturbation theory on large scales. The 1PF scheme is therefore capable of describing, in a unified framework and at the relevant leading orders, the evolution of large-scale structure on all scales of cosmological interest. Our two main results simply follow from the analysis of our nonlinear equations: at leading order a purely Newtonian nonlinear energy current sources a

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[1] The contemporary derivation of the post-Newtonian approximation [65] should be contrasted with the traditional implementation, see e.g. [64], which suffers from various ambiguities and inconsistencies [65, 66]. By and large, cosmological applications of the post-Newtonian approximation follow the traditional approach, see below.

[2] Formally the expansion is with respect to a small dimensionless parameter representing the ratio v/c; by dimensional analysis the ratio of the gravitational field and c^2 is also dimensionless and can be expressed in terms of the same parameter. In this way the parameter, which in practice is the inverse of the speed of light $1/c$, indicates the relativistic weight of each term in the expansion [68].

[3] The approximate conservation equations then consistently follow from the contracted Bianchi identities, as in the exact theory.

frame-dragging gravitomagnetic vector potential; in the leading-order Newtonian regime and in the linear relativistic regime the two scalar metric potentials are the same, while the nonlinear 1PF equations for the resummed scalar potentials imply that the nonlinearity of GR makes them different.

The paper is organized as follows. After defining the various metric terms in Section II, we obtain the stress energy tensor in Section III, the field equations in Section IV and mass and momentum conservation equations in Section V. We then consider our equations in two opposite limits: the Newtonian regime on small scales, neglecting $O(1/c^4)$ terms, see Section VI; the linear regime on large scales through the linearization of the equations, see Section VII. In Section VIII, we first define suitable resummed variables, then we obtain a consistent set of nonlinear equations describing their evolution. In Section IX, we draw our main conclusions. Finally, in Appendix A, we apply the PF expansion to the Riemann and Ricci tensors.

II.   NEWTONIAN AND POST-FRIEDMANN METRIC VARIABLES

We consider a homogeneous and isotropic FLRW flat background where two kinds of perturbation terms are added to the metric, representing two different levels of accuracy. The first will give the Newtonian regime, i.e. an approximate solution of Einstein equations such that the dynamics is described by the exact nonlinear equations of Newtonian cosmology for pressureless matter. The second will give the relativistic corrections, adding terms to the Newtonian equations. In our post-Friedmann framework the goal is to calculate these relativistic terms; in this spirit, we shall refer to the Newtonian approximation as the 0PF (or leading) order, and to the first relativistic corrections as 1PF order.

The expanding parameter for the inhomogeneous perturbations is formally given by $1/c$, so that the components of the metric tensor in the line element

$$-c^2 d\tau^2 = ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

can be written as

$$g_{00} = -\left[1 - \frac{2U_N}{c^2} + \frac{1}{c^4}(2U_N^2 - 4U_P)\right] + O\left(\frac{1}{c^6}\right), \tag{2.2a}$$

$$g_{0i} = -\frac{a}{c^3}B_i^N - \frac{a}{c^5}B_i^P + O\left(\frac{1}{c^7}\right), \tag{2.2b}$$

$$g_{ij} = a^2\left[\left(1 + \frac{2V_N}{c^2} + \frac{1}{c^4}(2V_N^2 + 4V_P)\right)\delta_{ij} + \frac{1}{c^4}h_{ij}\right] + O\left(\frac{1}{c^6}\right), \tag{2.2c}$$

where τ is proper time and $a(t)$ is the scale factor of the FLRW background. Note that we assume that both the scale factor and the metric are dimensionless, while the coordinates have dimension of a length. Greek indices take the values $0, 1, 2, 3$ and refer to spacetime coordinates, Latin indices refer to the spatial coordinates. In particular in our post-Friedmann scheme, for a proper "powers of c" order counting, it is important to note that the time coordinate is $x^0 = ct$. Here the Kronecker δ_{ij} represents the metric on the flat spatial slices of the background; the spatial Cartesian coordinates are understood as an Eulerian system of reference, cf. [82, 83].

Our metric is a generalization to cosmology of Chandrasekhar's metric for post-Newtonian hydrodynamics [67] (cf. [77] for a cosmological application). However, it is important to remark the difference of our post-Friedmann approach from the standard post-Newtonian one [64, 65, 84, 85]. In the latter, the focus is on the equation of motion for matter, hence only the leading order g_{00} metric perturbation is Newtonian, while the g_{ij} is post-Newtonian. In our approach the focus will instead be on the complete set of Einstein equations. We aim at a self-consistent approximate set of equations at each order, similarly to the post-Minkowski (weak field) approximation [65, 86]. Then, consistency of the Einstein equations dictate that the g_{ij} and g_{00} metric perturbations must be of the same order (see section VI for the Newtonian approximation). Anticipating these results, the indices N and P label Newtonian and post-Friedmann quantities. The N quantities are the only relevant one at the 0PF order of approximation of Einstein equations, i.e. in the Newtonian regime. The P quantities appear at the 1PF order.

We assume the Poisson (or conformal Newtonian) gauge [11, 12, 63, 81, 87], so that the three-vectors B_i^N and B_i^P are divergenceless, $B_i^{N,i} = 0$ and $B_i^{P,i} = 0$, and h_{ij} is transverse and tracefree (TT), i.e. it represents pure tensor modes $h^i_{\ i} = h_{ij}{}^{,i} = 0$. Commas in front of indices have the standard meaning of partial derivatives.

Having completely fixed the gauge leaves us with six degrees of freedom at each order: the two scalars U_N and V_N at leading order, and U_P and V_P at 1PF order; the two independent components of B_i^N (B_i^P at 1PF); the two independent components of h_{ij}. However, the latter only appears in the equations at 1PF order. At leading order, Einstein equations impose $U_N = V_N$, i.e. a single scalar gravitational potential in the Newtonian regime, as expected.

At leading order, B_i^N is determined by the (vector part of the) Newtonian energy current, and cannot be set to zero. It is not dynamical, i.e. it doesn't contribute to matter motion, however it appears in the metric and does affect null geodesics and observables. It can therefore be extracted from Newtonian N-body simulations [1, 2] and it contributes to lensing [88].

Note that the 1PF corrections consist of quadratic combinations of Newtonian quantities and intrinsic 1PF variables. We have also included tensorial TT modes but, at 1PF order, they can not be interpreted as gravitational waves; they satisfy a constraint equation rather than an evolution equation, see Section IV C 3, c.f. [83].

III. MATTER VARIABLES

Having defined the metric variables and their weight with respect to the expansion parameter c^{-1}, we now look at the matter quantities. The dimensionless 4-velocity is defined as

$$u^\mu := \frac{dx^\mu}{cd\tau} \tag{3.1}$$

and it satisfies the usual relation

$$g_{\mu\nu}u^\mu u^\nu = -1, \tag{3.2}$$

i.e. is a unitary time-like vector field. In Newtonian cosmology one uses the background comoving coordinates $\mathbf{x}$ as the Eulerian grid, the physical position of a fluid element is $\mathbf{r} = a\mathbf{x}$, and $\mathbf{v} = a\dot{\mathbf{x}}$ and $\dot{\mathbf{x}}$ respectively are is the physical (or proper) peculiar velocity [11] (the deviation from the Hubble flow) and the peculiar velocity with respect to the comoving grid [62]. With this in mind, it is then natural to define the physical peculiar velocity as $v^i := adx^i/dt$. Therefore we also define $v_i := \delta_{ij}v^j$.

Then

$$u^i = \frac{dx^i}{cd\tau} = \frac{dx^i}{cdt}\frac{dt}{d\tau} = \frac{v^i}{ca}u^0, \tag{3.3}$$

and, using (3.2) and keeping terms up to order c^{-4}, the 4-velocity components are

$$u^i = \frac{1}{c}\frac{v^i}{a}u^0, \tag{3.4a}$$

$$u^0 = 1 + \frac{1}{c^2}\left(U_N + \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[\frac{1}{2}U_N^2 + 2U_P + v^2 V_N + \frac{3}{2}v^2 U_N + \frac{3}{8}v^4 - B_i^N v^i\right], \tag{3.4b}$$

$$u_i = \frac{av_i}{c} + \frac{a}{c^3}\left[-B_i^N + v_i U_N + 2v_i V_N + \frac{1}{2}v_i v^2\right], \tag{3.4c}$$

$$u_0 = -1 + \frac{1}{c^2}\left(U_N - \frac{1}{2}v^2\right) + \frac{1}{c^4}\left[2U_P - \frac{1}{2}U_N^2 - \frac{1}{2}v^2 U_N - v^2 V_N - \frac{3}{8}v^4\right]. \tag{3.4d}$$

We consider a universe filled by cold dark matter (CDM), described by a single pressureless (dust) component with energy-momentum tensor

$$T^\mu{}_\nu = c^2\rho u^\mu u_\nu, \tag{3.5}$$

where ρ is the mass density.

In the Poisson gauge the components and trace of $T^\mu{}_\nu$ then are

$$T^0{}_0 = -c^2\rho - \rho v^2 - \frac{1}{c^2}\rho\left[2(U_N + V_N)v^2 - B_i^N v^i + v^4\right], \tag{3.6a}$$

$$T^0{}_i = c\rho av_i + \frac{1}{c}\rho a\left\{v_i[v^2 + 2(U_N + V_N)] - B_i^N\right\}, \tag{3.6b}$$

$$T^i{}_0 = -c\frac{1}{a}\rho v^i - \frac{1}{c}\frac{1}{a}\rho v^2 v^i, \tag{3.6c}$$

$$T^i{}_j = \rho v^i v_j + \frac{1}{c^2}\rho\left\{v^i v_j[v^2 + 2(U_N + V_N)] - v^i B_j^N\right\}, \tag{3.6d}$$

$$T^\mu{}_\mu = T = -\rho c^2. \tag{3.6e}$$

All these quantities are written in order to explicitly show the different contributions in powers of c, where in each expression the first term beyond the background term (if present) represents the leading Newtonian order, the second the first 1PF correction, etc... Note that there is no approximation in the trace: indeed, the mass density ρ plays the role of a fundamental exact quantity that is not expanded into contributions at different orders. In the following it will be useful to use the density contrast δ, defined as usual: $\delta := (\rho - \bar{\rho})/\bar{\rho}$, where $\bar{\rho}$ denotes the background matter density.

IV. EINSTEIN EQUATIONS

A. Expansion of Einstein equations at 1PF order

We now consider Einstein field equations for the metric (2.2), including the cosmological constant Λ:

$$G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}R\delta^{\mu}{}_{\nu} = \frac{8\pi G}{c^4}T^{\mu}{}_{\nu} - \Lambda\delta^{\mu}{}_{\nu} \ . \tag{4.1}$$

Expanding in powers of $1/c$, in all equations we retain the first two terms of the expansion. We then obtain the following equations, where the dot denotes partial differentiation with respect to coordinate time t.

Time-time component

$$G^0{}_0 + \Lambda = \frac{8\pi G}{c^4}T^0{}_0 \ \rightarrow \ \frac{1}{c^2}\left[3\left(\frac{\dot{a}}{a}\right)^2 - 2\frac{\nabla^2 V_N}{a^2}\right] + \frac{1}{c^4}\left[6\frac{\dot{a}}{a}\dot{V}_N + 6\left(\frac{\dot{a}}{a}\right)^2 U_N - \frac{4}{a^2}\nabla^2 V_P + \frac{2}{a^2}\nabla^2 V_N^2 - \frac{5}{a^2}V_N{}^{,i}V_{N,i}\right]$$

$$= \frac{1}{c^2}8\pi G\rho + \frac{1}{c^4}8\pi G\rho v^2 + \Lambda \ . \tag{4.2}$$

Spatial component

$$G^j{}_i + \Lambda\delta^j{}_i = \frac{8\pi G}{c^4}T^i{}_j \ \rightarrow \ \frac{1}{c^2}\left\{\frac{1}{a^2}(V_N - U_N)^{,j}{}_{,i} + \delta^j{}_i\left[\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} - \frac{1}{a^2}\nabla^2(V_N - U_N)\right]\right\} + \frac{1}{c^4}\left\{-\frac{\dot{a}}{a^2}\left(B_i^{N,j} + B^{Nj}{}_{,i}\right)\right.$$

$$- \frac{1}{2a}\left(\dot{B}^{Nj}{}_{,i} + \dot{B}_i^{N,j}\right) - \frac{2}{a^2}U_{P,i}{}^{,j} + \frac{2}{a^2}V_{P,i}{}^{,j} + \frac{1}{a^2}U_{N,i}U_N{}^{,j} - \frac{1}{a^2}V_{N,i}V_N{}^{,j} + \frac{1}{a^2}\left(U_{N,i}V_N{}^{,j}\right.$$

$$\left. + U_N{}^{,j}V_{N,i}\right) - \frac{2}{a^2}V_N(V_N - U_N)^{,j}{}_{,i} + \delta^j{}_i\left[2\frac{\dot{a}}{a}\dot{U}_N + 4\frac{\ddot{a}}{a}U_N + 2\left(\frac{\dot{a}}{a}\right)^2 U_N + 6\frac{\dot{a}}{a}\dot{V}_N + 2\ddot{V}_N\right.$$

$$\left.\left. + \frac{2}{a^2}\nabla^2 U_P - \frac{2}{a^2}\nabla^2 V_P - \frac{1}{a^2}U_{N,k}U_N{}^{,k} + \frac{2}{a^2}V_N\nabla^2(V_N - U_N)\right] + \frac{1}{2a^2}\nabla^2 h^j{}_i\right\}$$

$$= \Lambda\delta^j{}_i - \frac{8\pi G}{c^4}\rho v_i v^j \ . \tag{4.3}$$

Time-space component

$$G^0{}_i = \frac{8\pi G}{c^4}T^0{}_i \ \rightarrow \ \frac{1}{c^3}\left[-\frac{1}{2a}\nabla^2 B_i^N + 2\frac{\dot{a}}{a}U_{N,i} + 2\dot{V}_{N,i}\right] + \frac{1}{c^5}\left[-\frac{1}{2a}\nabla^2 B_i^P + 4\frac{\dot{a}}{a}U_{P,i} + 4\dot{V}_{P,i} + 2\dot{V}_N U_{N,i} + 4\frac{\dot{a}}{a}U_N U_{N,i}\right.$$

$$+ 4\dot{V}_{N,i}V_N + \frac{1}{2a}B_i^N{}_{,k}(V_N - U_N)^{,k} - \frac{1}{2a}B_k^N{}_{,i}(U_N + V_N)^{,k} + \frac{1}{a}\nabla^2 B_i^N(V_N - U_N) + \frac{1}{2a}B_i^N\nabla^2 V_N$$

$$\left. + \frac{1}{a}B^{Nk}V_{N,ki}\right] = \frac{8\pi G}{c^3}\rho a v_i + \frac{8\pi G}{c^5}\rho a\left\{v_i\left[v^2 + 2(U_N + V_N)\right] - B_i^N\right\} \ . \tag{4.4}$$

Finally, neglecting in Eqs. (4.2) and (4.3) all terms representing inhomogeneities, we obtain the background equations:

$$\frac{1}{c^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{c^2}\frac{8\pi G}{3}\bar{\rho} + \frac{\Lambda}{3} \ , \tag{4.5a}$$

$$\frac{1}{c^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a}\right] = \Lambda \ . \tag{4.5b}$$

These are recast into the standard Friedmann and Raychaudhuri equations for the flat FLRW background

$$\frac{1}{c^2}H^2 = \frac{1}{c^2}\frac{8\pi G}{3}\bar{\rho} + \frac{\Lambda}{3} , \tag{4.6a}$$

$$\frac{1}{c^2}\left[\dot{H} + H^2\right] = -\frac{1}{c^2}\frac{4\pi G}{3}\bar{\rho} + \frac{\Lambda}{3} , \tag{4.6b}$$

after substituting the Hubble expansion scalar $H = \dot{a}/a$.

B. 1PF equations for the inhomogeneities

We now subtract the background parts (4.5) from Eqs. (4.2) and (4.3), in order to obtain equations for the inhomogeneous quantities. The time-time component of the field equations then gives a generalized Poisson equation:

$$-\frac{1}{c^2}\frac{1}{3a^2}\nabla^2 V_N + \frac{1}{c^4}\left[\frac{\dot{a}}{a}\dot{V}_N + \left(\frac{\dot{a}}{a}\right)^2 U_N + \frac{1}{3a^2}\nabla^2 V_N^2 - \frac{5}{6a^2}V_{N,i}V_N{}^{,i} - \frac{2}{3a^2}\nabla^2 V_P\right]$$

$$= \frac{1}{c^2}\frac{4\pi G}{3}\bar{\rho}\delta + \frac{1}{c^4}\frac{4\pi G}{3}\bar{\rho}(1+\delta)v^2 . \tag{4.7}$$

Note that the cosmological constant disappears from the equations above; it only directly contributes to the background dynamics, i.e. equations (4.6). Thus perturbations are only affected by Λ through their coupling with the Hubble (background) expansion.

The trace of the space-space component (4.3) gives

$$-\frac{1}{c^2}\left[\frac{2}{a^2}\nabla^2(V_N - U_N)\right] + \frac{1}{c^4}\left\{-\frac{4}{a^2}\nabla^2(V_P - U_P) - \frac{2}{a^2}U_{N,k}U_N{}^{,k} - \frac{1}{a^2}V_{N,k}V_N{}^{,k} + \frac{2}{a^2}U_{N,k}V_N{}^{,k}\right.$$

$$\left. + \frac{4}{a^2}V_N\nabla^2(V_N - U_N) + 6\left[\frac{\dot{a}}{a}(\dot{U}_N + 3\dot{V}_N) + 2\frac{\ddot{a}}{a}U_N + \left(\frac{\dot{a}}{a}\right)^2 U_N + \ddot{V}_N\right]\right\}$$

$$= -\frac{8\pi G}{c^4}\bar{\rho}(1+\delta)v^2 , \tag{4.8}$$

while the trace-free part is

$$\frac{1}{c^2}\left[\frac{1}{a^2}(V_N - U_N)^{,j}{}_{,i} - \frac{1}{3a^2}\nabla^2(V_N - U_N)\delta^j{}_i\right] + \frac{1}{c^4}\left\{-\frac{1}{a}\frac{\dot{a}}{a}\left(B_i^{N,j} + B^{Nj}{}_{,i}\right) - \frac{1}{2a}\left(\dot{B}_i^{N,j} + \dot{B}^{Nj}{}_{,i}\right)\right.$$

$$+ \frac{1}{a^2}\left[2(V_P - U_P)^{,j}{}_{,i} + U_{N,i}U_N{}^{,j} - V_{N,i}V_N{}^{,j} + U_{N,i}V_N{}^{,j} + U_N{}^{,j}V_{N,i} - 2V_N(V_N - U_N)^{,j}{}_{,i} + \frac{1}{2}\nabla^2 h^j{}_i\right]$$

$$+ \frac{1}{a^2}\delta^j{}_i\left[-\frac{2}{3}\nabla^2(V_P - U_P) - \frac{1}{3}(U_{N,k}U_N{}^{,k} - V_{N,k}V_N{}^{,k}) - \frac{2}{3}U_{N,k}V_N{}^{,k} + \frac{2}{3}V_N\nabla^2(V_N - U_N)\right]\right\}$$

$$= -\frac{8\pi G}{c^4}\bar{\rho}(1+\delta)\left(v_i v^j - \frac{1}{3}\delta^j{}_i v^2\right). \tag{4.9}$$

C. Scalar, vector and tensors parts

1. Scalar equations

It is useful to recast the previous equations in order to isolate, in the linear part of the equations, the scalar, vector and tensor contributions. For the scalar sector let us apply the divergence operator on Eq. (4.4):

$$
\frac{1}{c^3}\nabla^2\left(\frac{\dot{a}}{a}U_N + \dot{V}_N\right) + \frac{1}{c^5}\left[2\nabla^2\left(\frac{\dot{a}}{a}U_P + \dot{V}_P\right) + \left(\dot{V}_N U_{,i}\right)^{,i} + 2\frac{\dot{a}}{a}\left(U_N U_{,i}\right)^{,i} + 2\left(\dot{V}_{N,i}V_N\right)^{,i}\right.
$$

$$
\left. + \frac{1}{2a}B^N_{i\ ,k}(V_N - U_N)^{,ki} + \frac{1}{4a}\nabla^2 B^N_i(V_N - 3U_N)^{,i} + \frac{3}{4a}B^N_i\nabla^2 V_N{}^{,i}\right]
$$

$$
= \frac{4\pi G}{c^3}a\bar{\rho}\left[v_i(1+\delta)\right]^{,i} + \frac{4\pi G}{c^5}a\bar{\rho}\left\{\left[(\delta+1)v_i\left(v^2 + 2U_N + 2V_N\right)\right]^{,i} - \delta^{,i}B^N_i\right\}. \tag{4.10}
$$

Multiplying Eq. (4.10) by H/c, and applying ∇^2 on Eq. (4.7), we obtain the following constraint equation:

$$
\frac{1}{c^2}\nabla^2\nabla^2 V_N - \frac{1}{c^4}\left[\nabla^2\nabla^2(V_N^2) - \frac{5}{2}\nabla^2(V_{N,i}V_N{}^{,i}) - 2\nabla^2\nabla^2 V_P\right] = -4\pi G(a^3\bar{\rho})\left\{\frac{1}{c^2}\frac{1}{a}\nabla^2\delta\right.
$$

$$
\left. + \frac{1}{c^4}\left[\frac{1}{a}\nabla^2((1+\delta)v^2) - 3\frac{\dot{a}}{a}(v_i(1+\delta))^{,i}\right]\right\}. \tag{4.11}
$$

Now, we can obtain a second scalar constraint by applying the operator $\partial_j\partial^i$ on both sides of Eq. (4.9). This then gives:

$$
\frac{1}{c^2}\frac{2}{3}\nabla^2\nabla^2(V_N - U_N) + \frac{1}{c^4}\left\{\frac{4}{3}\nabla^2\nabla^2(V_P - U_P) + \left(U_{N,i}U_N{}^{,j}\right)_{,j}^{,i} - \left(V_{N,i}V_N{}^{,j}\right)_{,j}^{,i} + 2\left(U_{N,i}V_N{}^{,j}\right)_{,j}^{,i}\right.
$$

$$
\left. - \frac{1}{3}\nabla^2\left(U_{N,k}U_N{}^{,k}\right) + \frac{1}{3}\nabla^2\left(V_{N,k}V_N{}^{,k}\right) - \frac{2}{3}\nabla^2\left(U_{N,k}V_N{}^{,k}\right) - 2\left[V_N(V_N - U_N)_{,i}{}^{,j}\right]_{,j}^{,i} + \frac{2}{3}\nabla^2\left[V_N\nabla^2(V_N - U_N)\right]\right\}
$$

$$
= -\frac{8\pi G}{c^4}a^2\bar{\rho}\left[(1+\delta)\left(v_iv^j - \frac{1}{3}\delta^j_iv^2\right)\right]_{,j}^{,i}. \tag{4.12}
$$

These equations will be useful in Section VIII.

2. Vector equations

The vectorial part of equation (4.4) can be found using the curl operator:

$$
-\frac{1}{c^3}\nabla\times\nabla^2 B^N_i + \frac{1}{c^5}\nabla\times\left[-\nabla^2 B^P_i + B^N_{i\ ,k}(V_N - U_N)^{,k} - B^N_{k\ ,i}(U_N + V_N)^{,k} + 2\nabla^2 B^N_i(V_N - U_N) + B^N_i\nabla^2 V_N\right.
$$

$$
\left. + 2B^N_k V_{N,i}{}^{,k}\right] = \frac{16\pi Ga^2}{c^3}\nabla\times(\rho v_i) + \frac{16\pi Ga^2}{c^5}\nabla\times\left(\rho\left\{v_i\left[v^2 + 2(U_N + V_N)\right] - B^N_i\right\}\right). \tag{4.13}
$$

Alternatively, we can also obtain a constraint equation for the vector part by applying the operators ∇^2 to Eq. (4.4) and ∂_j to Eq. (4.10), finding:

$$
\frac{1}{c^3}\nabla^2\nabla^2 B^N_i + \frac{1}{c^5}\left\{\nabla^2\nabla^2 B^P_i - \nabla^2\left[B^N_{i\ ,k}(V_N - U_N)^{,k}\right] + \nabla^2\left[B^N_{k\ ,i}(V_N + U_N)^{,k}\right] - 2\nabla^2\left[\nabla^2 B^N_i(V_N - U_N)\right]\right.
$$

$$
- \nabla^2\left(B^N_i\nabla^2 V_N\right) - 2\nabla^2\left(B^N_k V_{N,i}{}^{,k}\right) + 2\left[B^N_{j\ ,k}(V_N - U_N)^{,jk}\right]_{,i} + \left[\nabla^2 B^N_{j\ ,k}(V_N - 3U_N)^{,j}\right]_{,i}
$$

$$
+ 3\left(B^N_j\nabla^2 V_N{}^{,j}\right)_{,i} + 16\pi Ga^2\bar{\rho}\left(\delta^{,j}B^N_j\right)_{,i} - 16\pi Ga^2\bar{\rho}\,\nabla^2\left[(1+\delta)B^N_j\right]\right\} = \frac{1}{c^3}16\pi Ga^2\bar{\rho}\left\{[v_j(1+\delta)]_{,i}^{,j} - \nabla^2\left[v_i(1+\delta)\right]\right\}
$$

$$
+ \frac{1}{c^5}\left\{4a\left[\nabla^2\left(\dot{V}_N U_{N,i}\right) - \left(\dot{V}_N U_{N,k}\right)_{,i}^{,k} + 2\frac{\dot{a}}{a}\nabla^2\left(U_N U_{N,i}\right) - 2\frac{\dot{a}}{a}(U_N U_{N,k})_{,i}^{,k} + 2\nabla^2\left(\dot{V}_{N,i}V_N\right) - 2\left(\dot{V}_{N,k}V_N\right)_{,i}^{,k}\right]\right.
$$

$$
\left. + 16\pi Ga^2\bar{\rho}\left[v_j(1+\delta)\left(v^2 + 2U_N + 2V_N\right)^{,j}\right]_{,i} - 16\pi Ga^2\bar{\rho}\,\nabla^2\left[v_i(1+\delta)\left(v^2 + 2U_N + 2V_N\right)\right]\right\}. \tag{4.14}
$$

These vectorial equations also depend on scalar quantities, because of nonlinearity; however, the divergence of these equations would give zero, as it should.

An evolution equation for B_i can be obtained by applying the operator $\partial_j \nabla^2$ on Eq. (4.9) and subtracting the equation obtained applying ∂_i on Eq. (4.12). This procedure leads to

$$
\frac{1}{c^4}\nabla^2\nabla^2\left(\frac{1}{2}\dot{B}_i^N + \frac{\dot{a}}{a}B_i^N\right) = \frac{1}{c^4}\frac{1}{a}\left\{\nabla^2\left(U_{N,i}U_N{}^{,j}\right)_{,j} - \left(U_{N,k}U_N{}^{,j}\right)_{,ji}^{,k} - \nabla^2\left(V_{N,i}V_N{}^{,j}\right)_{,j} + \left(V_{N,k}V_N{}^{,j}\right)_{,ji}^{,k}\right.
$$
$$
\left. +\nabla^2\left(U_{N,i}V_N{}^{,j}\right)_{,j} + \nabla^2\left(U_N{}^{,j}V_{N,i}\right)_{,j} - 2\left(U_{N,k}V_N{}^{,j}\right)_{,ji}^{,k} - 2\nabla^2\left[(V_N - U_N)_{,i}^{,j}V_N\right]_{,j} + 2\left[(V_N - U_N)_{,k}^{,j}V_N\right]_{,ji}^{,k}\right\}
$$
$$
+\frac{8\pi G}{c^4}a\bar{\rho}\left\{\nabla^2\left[(1+\delta)(v_i v^k - \frac{1}{3}\delta_i^k v^2)\right]_{,k} - \left[(1+\delta)(v_k v^j - \frac{1}{3}\delta_k^j v^2)\right]_{,ji}^{,k}\right\}. \tag{4.15}
$$

This shows that an evolution term for B_i^N only appears at order $1/c^4$.

3. Tensor equations

In order to isolate the TT part of the metric h_{ij} we define the following nonlinear quantities:

$$
\mathcal{A}_i{}^j = U_{N,i}U_N{}^{,j} - V_{N,i}V_N{}^{,j} + U_{N,i}V_N{}^{,j} + U_N{}^{,j}V_{N,i} - 2V_N(V_N - U_N)_{,i}{}^{,j}
$$
$$
+ \delta_i^j\left[-\frac{1}{3}U_{N,k}U_N{}^{,k} + \frac{1}{3}V_{N,k}V_N{}^{,k} - \frac{2}{3}U_{N,k}V_N{}^{,k} + \frac{2}{3}V_N\nabla^2(V_N - U_N)\right], \tag{4.16}
$$

$$
\mathcal{S}_i{}^j = (1+\delta)\left(v_i v^j - \frac{1}{3}\delta_i^j v^2\right). \tag{4.17}
$$

With these definitions, Eq.(4.15) becomes

$$
\frac{1}{c^4}\nabla^2\nabla^2\left(\frac{1}{2a}\dot{B}_i^N + \frac{1}{a}\frac{\dot{a}}{a}B_i^N\right) = \frac{1}{c^4}\left[\frac{1}{a^2}\left(\nabla^2\mathcal{A}_{i,j}^j - \mathcal{A}_{k,ji}^{i,k}\right) + 8\pi G\bar{\rho}\left(\nabla^2\mathcal{S}_{i,j}^j - \mathcal{S}_{k,ji}^{i,k}\right)\right]. \tag{4.18}
$$

Finally, using Eqs. (4.9), (4.12) and (4.18), we obtain the following constraint equation for h_{ij}:

$$
\frac{1}{c^4}\nabla^2\nabla^2\nabla^2 h_i^j = \frac{1}{c^4}\left[-\mathcal{A}_{k,li}^{l,kj} - \nabla^2\mathcal{A}_{k,l}^{l,k}\delta_i^j + 2\nabla^2\mathcal{A}_{i,k}^{k,j} + 2\nabla^2\mathcal{A}_{l,ki}^k\delta^{lj} - 2\nabla^2\nabla^2\mathcal{A}_i^j\right.
$$
$$
\left. +8\pi Ga^2\bar{\rho}\left(-\mathcal{S}_{k,li}^{l,kj} - \nabla^2\mathcal{S}_{k,l}^{l,k}\delta_i^j + 2\nabla^2\mathcal{S}_{l,ki}^k\delta^{lj} + 2\nabla^2\mathcal{S}_{i,k}^{k,j} - 2\nabla^2\nabla^2\mathcal{S}_i^j\right)\right]. \tag{4.19}
$$

V. CONSERVATION EQUATIONS

The field equations in GR are constructed in order to imply the conservation equations through the contracted Bianchi identities [89]:

$$
T^\nu{}_{\mu;\nu} = 0. \tag{5.1}
$$

From this, considering the time component and keeping all terms up to $1/c^2$ order, we obtain the energy conservation equation

$$
\frac{(a^3\rho)^{\cdot}}{a^3} + \frac{(v^i\rho)_{,i}}{a} + \frac{1}{c^2}\left[\frac{(a^4\rho v^2)^{\cdot}}{a^4} + 3\dot{V}_N\rho + \frac{(\rho v^2 v^i)_{,i}}{a} + \rho v^i\frac{(3V_N - U_N)_{,i}}{a}\right] = 0. \tag{5.2}
$$

Setting $v^i = U_N = V_N = 0$, and $\rho = \bar{\rho}$, this equation reduces to the background continuity equation for cold dark matter, i.e. $\dot{\bar{\rho}} = -3H\bar{\rho}$. Subtracting this equation from (5.2) and defining the total, or convective, derivative for any quantity Q as

$$
\frac{dQ}{dt} = \dot{Q} + \frac{v^i Q_{,i}}{a}, \tag{5.3}
$$

Eq. (5.2) becomes

$$\frac{d\delta}{dt} + \frac{v^i{}_{,i}}{a}(\delta+1) + \frac{1}{c^2}\left\{(\delta+1)\left[\frac{\dot{a}}{a}v^2 + \frac{d(v^2)}{dt} + 3\frac{dV_N}{dt} - \frac{v^i}{a}U_{N,i}\right]\right\} = 0 \ . \tag{5.4}$$

Note that we have used the Newtonian part of this equation to simplify the 1PF order part. The space part of (5.1) gives the momentum conservation equation:

$$\frac{(a^4\rho v_i)^{\cdot}}{a^4} - \frac{\rho U_{N,i}}{a} + \frac{(v^j\rho v_i)_{,j}}{a} + \frac{1}{c^2}\left\{\rho v_i v^j\frac{(3V_N - U_N)_{,j}}{a} - 2\frac{\rho U_{P,i}}{a} + \rho v_i(3V_N - U_N)^{\cdot} + \frac{(\rho v_i v^j v^2)_{,j}}{a} + \frac{(a^4\rho v^2 v_i)^{\cdot}}{a^4} + \right.$$

$$\left. + \frac{[2a^4\rho v_i(V_N + U_N)]^{\cdot}}{a^4} + 2\frac{[\rho v^j v_i(V_N + U_N)]_{,j}}{a} - \rho v^2\frac{(V_N + U_N)_{,i}}{a} - \frac{(a^4 B_i^N \rho)^{\cdot}}{a^4} - \frac{(B_i^N \rho v^j)_{,j}}{a} + \frac{B_{j,i}^N}{a}\rho v^j\right\} = 0 \ . \tag{5.5}$$

Note that Eqs. (5.2) and (5.5) are equivalent to Eqs. (57) and (58) in [77]: as far as the conservation equations are concerned, there is no difference between the standard post-Newtonian approach used in [77] and our post-Friedmann approach. The difference becomes relevant for the consistency of the full set of Einstein equations; we discuss this point in Section VI B. Simplifying this equation by using the Newtonian part of Eq. (5.4) and the background continuity equation for $\bar{\rho}$, we derive the 1PF Euler equation:

$$\frac{dv_i}{dt} + \frac{\dot{a}}{a}v_i - \frac{U_{N,i}}{a} + \frac{1}{c^2}\left[v_i\frac{d}{dt}(U_N + 2V_N) + \frac{2}{a}U_{N,i}(U_N + V_N) - \frac{1}{a}v^2 V_{N,i} - \frac{2}{a}U_{P,i} + \frac{1}{a}v_i v^j U_{N,j} - \frac{\dot{a}}{a}v^2 v_i \right.$$

$$\left. - \frac{1}{a}\frac{d}{dt}(aB_i^N) + \frac{B_{j,i}^N v^j}{a}\right] = 0 \ . \tag{5.6}$$

Finally, using the Newtonian part of Eq. (5.6), the continuity equation can be recast in the following way:

$$\frac{d\delta}{dt} + \frac{v^i{}_{,i}}{a}(\delta+1) + \frac{1}{c^2}\left[(\delta+1)\left(\frac{1}{a}v^j U_{N,j} - \frac{\dot{a}}{a}v^2 + 3\frac{dV_N}{dt}\right)\right] = 0 \ . \tag{5.7}$$

The last two equations are the hydrodynamic equations of motion of the dust component in the post-Friedmann approximation, at 1PF order. Let us stress that, in Eqs. (5.7) and (5.6), the post-Friedmann corrections contain nonlinear coupling terms between metric variables and matter variables, ρ and v_i. This is the reason why the vector potential B_i can not be decoupled from the scalar modes in the equations of motion.

In the next two section we will check the consistency of our approach with our goal of obtaining a set of equation valid at all scales. To this end we will first consider the leading order, the Newtonian regime, which is a good approximation on scales much smaller than the Hubble radius, then we will linearise our equation to check consistency with linear relativistic perturbation theory in the Poisson gauge [11, 63], which is a good approximation on large scales.

VI.   LEADING ORDER: THE NEWTONIAN REGIME

A.   Newtonian dynamics from consistency of Einstein equations, with a bonus

Retaining the leading order terms in the c^{-1} expansion, i.e. the 0PF order, we recover the equations of Newtonian cosmology and we obtain the corresponding spacetime metric. We call this the Newtonian regime: as we are going to see, the dynamics is purely Newtonian, yet we have a spacetime metric that is a well defined approximate solution of Einstein equations.

We obtain the Newtonian continuity and Euler equation from the hydrodynamic equations (5.4) and (5.6):

$$\dot{\delta} + \frac{v^i\delta_{,i}}{a} + \frac{v^i{}_{,i}}{a}(\delta+1) = 0 \ , \tag{6.1}$$

$$\dot{v}_i + \frac{v^j v_{i,j}}{a} + \frac{\dot{a}}{a}v_i = \frac{1}{a}U_{N,i} \ . \tag{6.2}$$

At leading order, from Einstein equations (4.7), (4.4), (4.8), (4.9), we obtain

$$G^0{}_0 + \Lambda = \frac{8\pi G}{c^4}T^0{}_0 \;\; \rightarrow \;\; \frac{1}{c^2}\frac{1}{a^2}\nabla^2 V_N = -\frac{4\pi G}{c^2}\bar{\rho}\delta \;, \tag{6.3a}$$

$$G^0{}_i = \frac{8\pi G}{c^4}T^0{}_i \;\; \rightarrow \;\; \frac{1}{c^3}\left[-\frac{1}{2a^2}\nabla^2 B_i^N + 2\frac{\dot{a}}{a^2}U_{N,i} + \frac{2}{a}\dot{V}_{N,i}\right] = \frac{8\pi G}{c^3}\bar{\rho}(1+\delta)v_i \;, \tag{6.3b}$$

$$\text{trace of} \;\; G^i{}_j + \Lambda\delta^i{}_j = \frac{8\pi G}{c^4}T^i{}_j \;\; \rightarrow \;\; \frac{1}{c^2}\frac{2}{a^2}\nabla^2(V_N - U_N) = 0 \;, \tag{6.3c}$$

$$\text{traceless part of} \;\; G^i{}_j + \Lambda\delta^i{}_j = \frac{8\pi G}{c^4}T^i{}_j \;\; \rightarrow \;\; \frac{1}{c^2}\frac{1}{a^2}\left[(V_N - U_N)_{,i}{}^{,j} - \frac{1}{3}\nabla^2(V_N - U_N)\delta_i^j\right] = 0 \;. \tag{6.3d}$$

Equation (6.3a) is the Poisson equation, from which the Newtonian character of the spatial metric potential V_N is apparent as long as we identify it with the Newtonian gravitational potential generated by the matter field, $V_N = -\phi_N$. As noted by Bertschinger [86], it is an interesting fact that at leading order only g_{ij} contributes to $G^0{}_0$. On the other hand, as it is well known, it is the time-time metric potential U_N that tells matter how to move in the Euler equation (6.2) (or to particles in the geodesic equation, see e.g. [69]). Equations (6.3c) and (6.3d) show that Eqs. (4.8) (4.9) reduce, at leading order, to constraint equations: they give the consistency relations between the scalar metric potentials U_N and V_N in a GR context at this order. Modulo residual gauge modes [86] that leave (6.3c) and (6.3d) invariant, we must have $U_N = V_N = -\phi_N$: in summary, Einstein equations reduce, at leading order, to the standard equations of Newtonian cosmology. The metric tensor generated from a self-consistent expansion of the full set of Einstein equations at leading order is the cosmological version of the weak field metric, with the FLRW metric replacing Minkowski as background.

Our metric (2.2), however, also contains the frame dragging vector potential B_i and the TT part h_{ij}. This TT part only appears at 1PF order in $G^i{}_j$ and so it is irrelevant at leading order, consistently with the fact that, in this Newtonian regime, we only retain the V_N term in g_{ij}. On the other hand, the leading order of the $G^0{}_i$ equation is Eq. (6.3b), thus it can not be neglected. This is a new equation that arises in the relativistic context and determines the leading-order frame dragging potential B_i^N from the purely Newtonian term on the right hand side, the energy current that is determined by the other equations. Therefore, even in the Newtonian regime, the frame dragging term B_i^N can not be set to zero. Some authors consider this term of higher order, however Eq. (6.3b) tell us that this is inconsistent with the other Newtonian equations. Indeed, taking the divergence of (6.3b) (i.e. its scalar part) and using the Poisson equation implies the continuity equation, while taking the curl (i.e. the vector part) shows that the curl of B_i^N, a gravito-magnetic field [86], is sourced by the curl of the energy current ρv_i, and there is no reason why the transverse part of this current should vanish. Alternatively [86], one can split vectors into longitudinal and transverse parts. Then one can see from (6.3b) that a purely Newtonian transverse energy current sources B_i^N. Thus, setting $B_i^N = 0$ does imply, even in the Newtonian regime of dynamics, a purely longitudinal energy current, i.e. imposing the condition $B_i^N = 0$ does imply an artificial constraint on the general Newtonian dynamics. While in the linear regime vectors modes can be separated from scalar modes, in the nonlinear regime of Newtonian dynamics there is no reason why the energy current should be longitudinal, and indeed it is well known that transverse vector modes are generated by nonlinearity, and in particular vorticity is generated after shell crossing [90]. The nonlinearly generated gravito-magnetic potential B_i^N can actually be extracted from standard Newtonian N-body simulations [1, 2], and its contribution to lensing has been computed in [88]. The bottom line is that in the Newtonian regime of GR the gravitational field is not purely scalar, it does contain a gravito-magnetic vector part [74, 91].

As we are going to show in the next Section, the above considerations on the frame dragging B_i are entirely consistent with linear perturbation theory. In the perturbative regime one naturally splits vectors into longitudinal and transverse parts and finds that B_i decays [78–80, 92, 93]. Nonetheless, in the nonlinear Newtonian regime the transverse part of the energy current sources B_i^N [1, 2]. At this order, however, B_i^N is a non-dynamical variable, rather it is determined non-locally through the a-causal equation (6.3b). Furthermore, a calculation of the magnetic Weyl tensor shows that, at leading order in powers of c^{-1},

$$H_{ij} = \frac{1}{2c^3}\left[B^N_{\mu,\nu(i}\varepsilon_{j)}{}^{\mu\nu} + 2v_\mu(U_N + V_N)_{,\nu(i}\varepsilon_{j)}{}^{\mu\nu}\right], \tag{6.4}$$

so that its linear part, i.e. the first term in the square bracket, is precisely the curl of B_i^N. As discussed in detail by Kofman and Pogosyan [74], although the magnetic Weyl tensor is zero in Newtonian theory (i.e. starting from a scalar theory of gravity), it can not be neglected in deriving the "covariant equations" [6, 94] in the Newtonian regime from relativistic theory. Nonetheless H_{ij} is not an independent dynamical variable in the Newtonian regime, rather Eq. (6.4) shows that it is locally determined by other variables.

B. Passive and active approach

Starting with Einstein [89], the usual way to consider the Newtonian limit in GR consists in demanding that the time-like geodesic equation agrees with the Newtonian equation of motion for a particle [64, 69]. This naturally sets $g_{00} = -(1 - 2U_N/c^2)$, identifying $-U_N$ with the gravitational potential ϕ_N. For a fluid, as we can see from (6.1) and (6.2), the continuity and the Euler equations do not contain V_N, so that the latter could be interpreted as a post-Newtonian variable. This way of thinking considers the "passive" aspect of gravitation, namely the response of matter to gravity [69]: one wants to determine the equation of motion of a particle in a *given* gravitational potential.

In addition, the Newtonian limit is again considered using the field equations with matter, in order to determine the gravitational coupling constant by recovering the Poisson equation [89]. Starting from the following form of Einstein equations

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}T\delta_{\mu\nu}) + \Lambda\delta_{\mu\nu} \tag{6.5}$$

one finds that the time-time component only contains g_{00}, so that, at $O(c^{-2})$, we obtain the Poisson equation for U_N

$$\frac{1}{c^2}\frac{1}{3a^2}\nabla^2 U_N = -\frac{4\pi G}{3c^2}\bar{\rho}\delta \tag{6.6}$$

by identifying $\kappa = 8\pi G/c^4$. Again, one would be tempted to conclude that only U_N is needed at leading Newtonian order.

However, in our approach we are interested in the "active" aspect of gravitation, i.e. the generation of gravity by the matter distribution. To this end, we need to determine the metric using the field equation in a fully consistent manner, even in the Newtonian regime. In particular, in cosmology we deal with a self-gravitating fluid, so that the complete set of field equations is as important as the equations of motion.

In the passive Newtonian approach of the standard post-Newtonian formalism the geodetic equation determines the order of the metric variables, U_N as Newtonian and V_N as the post-Newtonian correction. In our post-Friedmann scheme it is the leading order of the field equations, derived from the complete set of Einstein equation, that establishes the order of the metric perturbations, giving that both U_N and V_N have a Newtonian character. This emerges from (6.3a), (6.3c) and (6.3d) and naturally also from the spatial components of (6.5), whose trace is

$$\frac{1}{c^2}\frac{1}{a^2}\nabla^2(U_N - 4V_N) = \frac{12\pi G}{c^2}\bar{\rho}\delta \ . \tag{6.7}$$

This clearly shows that neglecting V_N with respect to U_N would not yield the Poisson equation.

In summary, if we want to obtain the equations of Newtonian cosmology from GR, in the active approach and consistently considering all components of Einstein equations, we do need both scalar potentials U_N and V_N as 0PF terms in the metric, with the result

$$U_N = V_N = -\phi_N \ . \tag{6.8}$$

VII. LINEARIZATION: RECOVERING FIRST-ORDER PERTURBATION THEORY

Standard relativistic perturbation theory [11, 12, 87, 92, 93, 95], where density, velocity and metric variables are all assumed to be small, is a fundamental tool in theoretical cosmology. It provides the framework to develop predictions from inflation for the matter and metric fluctuations in the early universe and to work out their imprint on the CMB and the matter fluctuations at the beginning of the matter dominated era [53].

Our post-Friedmann scheme has been developed to generalise the equations of nonlinear Newtonian cosmology to include relativistic corrections. It would however be a great bonus if, in linearising our equations, we could recover standard first-order relativistic perturbation theory. Given that the observed universe is remarkably well described by perturbation theory at large scales, up to the Hubble horizon and larger, recovering at least the first order would imply that our scheme is not only valid at small nonlinear Newtonian scales, as shown in the previous sections, but also at the largest scales of interests in cosmology. Note that it is not a-priory obvious that this recovery is possible; aim of this section is to explicitly show that, by linearising our equations and by defining resummed metric variables that contain a Newtonian part and a relativistic 1PF correction, we obtain first-order perturbation theory. This resummation will also be at the base of the following sections.

In first-order perturbation theory it is standard to decouple the scalar, vector and tensor modes [11, 12, 63, 92, 93, 95]. It is therefore convenient to separately look at the scalar and vector type equations of Sec. IV C (as already noted, at 1PF order the tensorial modes are not dynamical, i.e. we can't recover first-order gravitational waves). In particular, the first-order velocity perturbation v^i can also be split into a scalar and vector (solenoidal) part, $v^i = v^{\parallel\,,i} + v^i_\perp$, where $v^i_{\perp\,,i} = 0$.

Starting from the scalar sector and considering only the linear terms, we obtain from equations (4.7), (4.8), (4.10), and (4.12):

$$\frac{1}{c^2}\frac{1}{3}\nabla^2 V_N - \frac{1}{c^4}a^2\left[\frac{\dot{a}}{a}\dot{V}_N + \left(\frac{\dot{a}}{a}\right)^2 U_N - 2\frac{\nabla^2 V_P}{3a^2}\right] = -\frac{1}{c^2}\frac{4\pi G}{3}a^2\bar{\rho}\delta \ , \tag{7.1a}$$

$$\frac{1}{c^2}\left[\nabla^2(U_N - V_N)\right] + \frac{1}{c^4}\left\{2\nabla^2(U_P - V_P) + 3a^2\left[\frac{\dot{a}}{a}\left(\dot{U}_N + 3\dot{V}_N\right) + 2\frac{\ddot{a}}{a}U_N + \left(\frac{\dot{a}}{a}\right)^2 U_N + \ddot{V}_N\right]\right\} = 0 \ , \tag{7.1b}$$

$$\frac{1}{c^3}\nabla^2\left(\frac{\dot{a}}{a}U_N + \dot{V}_N\right) + \frac{1}{c^5}\left(2\frac{\dot{a}}{a}\nabla^2 U_P + 2\nabla^2\dot{V}_P\right) = \frac{1}{c^3}4\pi Ga\bar{\rho}\theta \ , \tag{7.1c}$$

$$\frac{1}{c^2}\nabla^2\nabla^2(V_N - U_N) + \frac{1}{c^4}2\nabla^2\nabla^2(V_P - U_P) = 0 \ , \tag{7.1d}$$

where, in Eq. (7.1c), we have defined the spatial velocity divergence as $\theta = v^i_{\ ,i} = \nabla^2 v^\parallel$. The continuity equation and the divergence of the Euler equation from (5.7) and (5.6), after linearization, are

$$\dot{\delta} + \frac{\theta}{a} + \frac{3}{c^2}\dot{V}_N = 0 \ , \tag{7.2}$$

$$\dot{\theta} + \frac{\dot{a}}{a}\theta - \frac{\nabla^2 U_N}{a} - \frac{2}{c^2}\frac{\nabla^2 U_P}{a} = 0 \ , \tag{7.3}$$

where the convective term in the total time derivatives is neglected and partial and total derivatives coincide. Finally, defining the following resummed scalar metric variables

$$\phi_P := -(U_N + \frac{2}{c^2}U_P), \tag{7.4}$$

$$\psi_P := -(V_N + \frac{2}{c^2}V_P), \tag{7.5}$$

the previous equations become

$$\nabla^2\psi_P - \frac{3}{c^2}a^2\left[\frac{\dot{a}}{a}\dot{\psi}_P + \left(\frac{\dot{a}}{a}\right)^2\phi_P\right] = 4\pi G\bar{\rho}a^2\delta \ , \tag{7.6a}$$

$$-\nabla^2(\psi_P - \phi_P) + \frac{3}{c^2}a^2\left[\frac{\dot{a}}{a}(\dot{\phi}_P + 3\dot{\psi}_P) + 2\frac{\ddot{a}}{a}\phi_P + \left(\frac{\dot{a}}{a}\right)^2\phi_P + \ddot{\psi}_P\right] = 0 \ , \tag{7.6b}$$

$$\nabla^2\left(\frac{\dot{a}}{a}\phi_P + \dot{\psi}_P\right) = -4\pi Ga\bar{\rho}\theta \ , \tag{7.6c}$$

$$\frac{1}{c^2}\nabla^2\nabla^2(\phi_P - \psi_P) = 0 \ , \tag{7.6d}$$

$$\dot{\delta} + \frac{\theta}{a} - \frac{3}{c^2}\dot{\psi}_P = 0 \ , \tag{7.6e}$$

$$\dot{\theta} + \frac{\dot{a}}{a}\theta + \frac{1}{a}\nabla^2\phi_P = 0 \ . \tag{7.6f}$$

Note that, from Eq. (7.6d), $\psi_P = \phi_P$; this follows from the perfect fluid matter model we are considering (i.e. null anisotropic stress; see e.g. [92, 93]). The equations above coincide with those for scalar fluctuations of standard perturbation theory in the Newtonian gauge, see e.g. [11, 63, 81]. The conclusion of this analysis is important: the 1PF approximation, in the linear regime, completely includes first-order scalar relativistic perturbation theory, so that all the linear GR terms are just 1PF. On the other hand, the 1PF order does not involve just linear terms, indeed there are contributions of second and higher order in the standard perturbative expansion.

This result is due to our active approach, where U_N and V_N are both Newtonian variables, to the derivation of 1PF equations that retain the first two orders in the $1/c$ expansion (rather than proceeding iteratively, order by order), and to the definition of the resummed variables (7.4) and (7.5) that turn out to coincide with the standard first-order potentials in Poisson gauge.

We now briefly consider the linear vector sector. Again defining a resummed variable, let us consider the vectorial potential ω_i:

$$\omega_i = B_i^N + \frac{1}{c^2}B_i^P. \tag{7.7}$$

Linearising Eqs. (4.13) and (4.9), we immediately obtain

$$\frac{1}{c^3}\nabla^2\omega^i = -\frac{1}{c^3}16\pi Ga^2\bar{\rho}\left(v_\perp^i - \frac{1}{c^2}\omega^i\right), \tag{7.8}$$

$$2\frac{\dot{a}}{a}\left(\omega_i^{,j} + \omega_{,i}^j\right) + \left(\dot{\omega}_i^{,j} + \dot{\omega}_{,i}^j\right) = 0. \tag{7.9}$$

Therefore, the linearisation of the 1PF approximation also reproduces the equations of linear relativistic perturbations theory in the vector sector. Note that Eq. (7.8) is the first-order analogue of the nonlinear Newtonian Eq. (6.3b). Eq. (7.9) instead gives the evolution of the first-order metric vector potential, which decay. In the nonlinear case, it is indeed shell-crossing that generates vorticity [90]. In general, nonlinearity generates the vector modes in the energy current that sources the gravitomagnetic frame-dragging potential [1, 2].

Finally, a remark on the tensor sector. Had we extended our post-Friedmann scheme to the 2PF order, in the linear regime we would have recovered the wave equation for free gravitational waves. In the nonlinear regime gravitational radiation would be generated by nonlinear sources terms, of higher order in scalar and vector modes, much in the same way that is generated at second order in standard perturbation theory [12, 81, 96, 97].

VIII.   NONLINEAR SET OF 1PF EQUATIONS

A.   Resummed metric potentials and new conservation equations

We now recast the nonlinear equations obtained in the previous sections, using a suitable change of variables. In the previous section we have introduced the potentials ϕ_P and ψ_P: we have shown that they coincide in both the linear and the Newtonian regime. It is then convenient to consider the following combinations:

$$\phi_G := \frac{1}{2}(\phi_P + \psi_P), \tag{8.1}$$

$$\frac{1}{c^2}D_P := \frac{1}{2}(\phi_P - \psi_P); \tag{8.2}$$

the first generalizes the definition of the gravitational potential given in Sec. VI, equation (6.8), the second defines a new nonlinear post-Friedmannian quantity, D_P, negligible in the Newtonian and in the linear regimes[4]. With these new variables, up to $O(1/c^4)$, the metric can be written as

$$ds^2 = -c^2\left[1 + 2\left(\frac{\phi_G}{c^2} + \frac{\phi_G^2}{c^4} + \frac{D_P}{c^4}\right)\right]dt^2 - 2a\frac{\omega_i}{c^2}dtdx^i + a^2\left\{\left[1 + 2\left(-\frac{\phi_G}{c^2} + \frac{\phi_G^2}{c^4} + \frac{D_P}{c^4}\right)\right]\delta_{ij} + \frac{1}{c^4}h_{ij}\right\}dx^idx^j. \tag{8.3}$$

Moreover, it is convenient to define a new velocity variable [67],

$$v_i^* = v_i - \frac{1}{c^2}\omega_i, \tag{8.4}$$

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[4] A difference in the two scalar potentials is expected from nonlinearity in GR, consistently with second-order perturbation theory results [12, 81, 87].

representing the velocity of matter with respect to observers moving along the normal to the slicing. With these new variables, Eqs. (4.7), (4.8), (4.9) and (4.4) become:

$$G^0{}_0 + \Lambda = \frac{8\pi G}{c^4} T^0{}_0 \quad \rightarrow \quad \frac{1}{c^2}\nabla^2\phi_G - \frac{1}{c^4}\left[\nabla^2 D_P + 3a^2\left(\frac{\dot{a}}{a}\dot{\phi}_G + \left(\frac{\dot{a}}{a}\right)^2\phi_G\right) - \nabla^2\phi_G^2 + \frac{5}{2}\phi_{G,i}\phi_G{}^{,i}\right]$$

$$= \frac{1}{c^2}4\pi G a^2\bar{\rho}\delta + \frac{1}{c^4}4\pi G a^2\bar{\rho}(1+\delta)v^{*2}, \tag{8.5a}$$

$$\left[G^j{}_i + \Lambda\delta^j_i = \frac{8\pi G}{c^4}T^j{}_i\right]_{\text{trace}} \quad \rightarrow \quad \frac{1}{c^4}\left[4\nabla^2 D_P + \phi_{G,k}\phi_G{}^{,k} + 6a^2\left(4\frac{\dot{a}}{a}\dot{\phi}_G + 2\frac{\ddot{a}}{a}\phi_G + \left(\frac{\dot{a}}{a}\right)^2\phi_G + \ddot{\phi}_G\right)\right]$$

$$= \frac{1}{c^4}8\pi G a^2\bar{\rho}(1+\delta)v^{*2}, \tag{8.5b}$$

$$\left[G^j{}_i + \Lambda\delta^j_i = \frac{8\pi G}{c^4}T^j{}_i\right]_{\text{tracefree}} \quad \rightarrow \quad \frac{1}{c^4}\left\{2\left(D_{P,i}{}^{,j} - \frac{1}{3}\nabla^2 D_P\,\delta^j_i\right) + 2\phi_{G,i}\phi_G{}^{,j} - \frac{2}{3}\phi_{G,k}\phi_G{}^{,k}\,\delta^j_i\right.$$

$$\left. -a\left[\frac{\dot{a}}{a}\left(\omega_i{}^{,j} + \omega^j{}_{,i}\right) + \frac{1}{2}\left(\dot{\omega}_i{}^{,j} + \dot{\omega}^j{}_{,i}\right)\right] + \frac{1}{2}\nabla^2 h^j_i\right\}$$

$$= -\frac{1}{c^4}8\pi G a^2\bar{\rho}(1+\delta)\left(v_i^* v^{*j} - \frac{1}{3}v^{*2}\,\delta^j_i\right), \tag{8.5c}$$

$$G^0{}_i = \frac{8\pi G}{c^4}T^0{}_i \quad \rightarrow \quad \frac{1}{c^3}\left[\frac{1}{2a}\nabla^2\omega_i + 2\frac{\dot{a}}{a}\phi_{G,i} + 2\dot{\phi}_{G,i}\right] - \frac{1}{c^5}\left\{-2\frac{\dot{a}}{a}D_{P,i} + 2\dot{D}_{P,i}\right.$$

$$\left. +\frac{1}{2a}\left[4a\left(\dot{\phi}_G\phi_{G,i} + 2\phi_G\dot{\phi}_{G,i}\right) + 8\dot{a}\phi_G\phi_{G,i} + 2\omega_{k,i}\phi_{G,k} - \omega_i\nabla^2\phi_G - 2\omega_k\phi_{G,ki}\right]\right\}$$

$$= -\frac{1}{c^3}8\pi G a\bar{\rho}(1+\delta)v_i^* - \frac{1}{c^5}8\pi G a\bar{\rho}(1+\delta)\left[v_i^*\left(v^{*2} - 4\phi_G\right)\right]. \tag{8.5d}$$

The continuity equation (5.7) and the Euler equation (5.6) become

$$\frac{d\delta}{dt} + \frac{v^{*i}{}_{,i}}{a}(\delta+1) - \frac{1}{c^2}\left[(\delta+1)\left(3\frac{d\phi_G}{dt} + \frac{v_k^*\phi_{G,k}}{a} + \frac{\dot{a}}{a}v^{*2}\right) - \frac{1}{a}\omega^j\delta_{,j}\right] = 0. \tag{8.6}$$

$$\frac{dv_i^*}{dt} + \frac{\dot{a}}{a}v_i^* + \frac{1}{a}\phi_{G,i} + \frac{1}{c^2}\left[\frac{1}{a}\phi_{G,i}(4\phi_G + v^{*2}) - 3v_i^*\frac{d\phi_G}{dt} + \frac{1}{a}D_{P,i} - \frac{1}{a}v_i^*v_j^*\phi_G{}^{,j} - \frac{\dot{a}}{a}v^{*2}v_i^* + \frac{1}{a}\omega_{j,i}v^{*j} + \frac{1}{a}\omega^j v^{*i}{}_{,j}\right] = 0, \tag{8.7}$$

where here and in the following the convective derivative of a quantity Q is defined in terms of v^*:

$$\frac{dQ}{dt} = \dot{Q} + \frac{v_*^i Q_{,i}}{a}. \tag{8.8}$$

B. Constraint type and evolution type equations

Starting from the new definitions given above, we want to obtain a final set of equations, providing a unified description of gravitational instability in both the linear and the nonlinear regimes, valid for fluctuations on all scales.

From Eqs.(8.5a) and (8.5b) we obtain an equation involving only the potential ϕ_G, sourced by the matter and velocity perturbations:

$$\frac{1}{c^2}\frac{2}{3}\nabla^2\phi_G + \frac{1}{c^4}\left[a^2\left(\ddot{\phi}_G + 2\frac{\dot{a}}{a}\dot{\phi}_G + 2\frac{\ddot{a}}{a}\phi_G - \left(\frac{\dot{a}}{a}\right)^2\phi_G\right) + \frac{2}{3}\nabla^2\phi_G^2 - \frac{3}{2}\phi_{G,i}\phi_G{}^{,i}\right] = \frac{1}{c^2}\frac{8\pi G}{3}a^2\bar{\rho}\delta$$

$$+\frac{1}{c^4}4\pi G a^2\bar{\rho}(1+\delta)v^{*2}. \tag{8.9}$$

Note that the linearized version of this equation is a combination of Eqs. (7.6a) and (7.6b).

A constraint equation for D_P is obtained from Eq. (4.12) [or, equivalently, by applying the operator $\partial_j \partial^i$ on both sides of Eq. (8.5c)]:

$$\frac{1}{c^4}\frac{2}{3}\nabla^2\nabla^2 D_P = -\frac{1}{c^4}\left[\left(\phi_{G,i}\phi_G{}^{,j}\right)_{,j}^{,i} - \frac{1}{3}\nabla^2\left(\phi_{G,i}\phi_G{}^{,i}\right)\right] - \frac{1}{c^4}4\pi Ga^2\bar{\rho}\left[(1+\delta)\left(v_i^* v^{*j} - \frac{1}{3}v^{*2}\,\delta_i^j\right)\right]_{,j}^{,i} . \quad (8.10)$$

Rewriting the definitions of $\mathcal{A}_i^j$ and $\mathcal{S}_i^j$, Eqs. (4.16) and (4.17), in terms of ϕ_G and v^{*i}, also neglecting a higher order contribution for D_P, we obtain

$$\mathcal{A}_i^j = 2\phi_{G,i}\phi_G{}^{,j} - \frac{2}{3}\delta_i^j\phi_{G,k}\phi_G{}^{,k} , \quad (8.11)$$

$$\mathcal{S}_i^j = (1+\delta)(v_i^* v^{*j} - \frac{1}{3}v^{*2}\,\delta_i^j) . \quad (8.12)$$

With these definitions Eq. (8.10) becomes

$$\frac{1}{c^4}\frac{4}{3}\nabla^2\nabla^2 D_P = -\frac{1}{c^4}\mathcal{A}_{k,j}^{j,k} - \frac{1}{c^4}8\pi Ga^2\bar{\rho}\mathcal{S}_{k,j}^{j,k} . \quad (8.13)$$

Finally, let us rewrite Eq. (4.18) as an evolution equation for the frame dragging potential ω_i

$$\frac{1}{c^4}\nabla^2\nabla^2\left(\frac{1}{2a}\dot{\omega}_i + \frac{1}{a}\frac{\dot{a}}{a}\omega_i\right) = \frac{1}{c^4}\left[\frac{1}{a^2}\left(\nabla^2\mathcal{A}_{i,j}^j - \mathcal{A}_{k,ji}^{i,k}\right) + 8\pi G\bar{\rho}\left(\nabla^2\mathcal{S}_{i,j}^j - \mathcal{S}_{k,ji}^{i,k}\right)\right] , \quad (8.14)$$

while the constraint Eq. (4.19) for h_{ij} remains formally unchanged,

$$\frac{1}{c^4}\nabla^2\nabla^2\nabla^2 h_i^j = \frac{1}{c^4}\left[-\mathcal{A}_{k,li}^{l,kj} - \nabla^2\mathcal{A}_{k,l}^{l,k}\delta_i^j + 2\nabla^2\mathcal{A}_{i,k}^{k,j} + 2\nabla^2\mathcal{A}_{l,ki}^{k}\delta^{lj} - 2\nabla^2\nabla^2\mathcal{A}_i^j\right.$$
$$\left. + 8\pi Ga^2\bar{\rho}\left(-\mathcal{S}_{k,li}^{l,kj} - \nabla^2\mathcal{S}_{k,l}^{l,k}\delta_i^j + 2\nabla^2\mathcal{S}_{l,ki}^{k}\delta^{lj} + 2\nabla^2\mathcal{S}_{i,k}^{k,j} - 2\nabla^2\nabla^2\mathcal{S}_i^j\right)\right] . \quad (8.15)$$

At 1PF order the variables h_{ij} and D_P are not dynamical, i.e. they do not satisfy an evolution equation, rather they are given in terms of other variables by constraint equations. We can therefore conclude that the 1PF correction to Newtonian gravity introduces three non-dynamical geometrical degrees of freedom (two d.o.f. in h_{ij} and 1 d.o.f. in D_P) and provides the evolution equation for the gravitational potential ϕ_G and the frame dragging vector potential ω_i. For comparison, at 0PF Newtonian order we only have three non-dynamical d.o.f.: one in the scalar gravitational potential and two in the gravitomagnetic vector potential. In full relativistic theory we would have six dynamical d.o.f., as it is already apparent in first-order perturbation theory. Therefore, in order to recover a fully dynamical theory we should extend our post-Friedmann scheme to the 2PF order, cf. [78]. However the equations above, given that they extend the fully nonlinear 0PF Newtonian equations and (as shown in Sec VII) include the scalar and vector first-order perturbation equations, are sufficient to study structure formation at all scales.

C.   New matter variables and simplified conservation equations

There exists a further change of variable in the matter density, such that the 1PF continuity equation formally takes the usual Newtonian form [67].

Defining

$$\hat{\rho} := a^{-3}\rho(-g)^{1/2}u^0 = \rho\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^{*2} - 3\phi_G\right)\right] , \quad (8.16)$$

and $\hat{\delta} := (\hat{\rho} - \bar{\rho})/\bar{\rho}$, equation (8.6) becomes

$$\frac{d\hat{\delta}}{dt} + \frac{v^{*i}_{\ ,i}}{a}(\hat{\delta} + 1) = 0 ; \quad (8.17)$$

therefore "the mass defined in terms of the density $\hat{\rho}$ is conserved" [67] in the usual Newtonian sense[5]. Similarly, we can introduce a new velocity field $\hat{v}_i$, such that the 1PF Euler equation simplifies. Defining

$$\hat{v}_i = v_i^*\left[1 + \frac{1}{c^2}\left(\frac{1}{2}v^{*2} - 3\phi_G\right)\right] , \quad (8.18)$$

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[5] Of course, mass is covariantly conserved in GR [6, 94]. For a discussion in second-order perturbation theory see [98].

equation (8.7) takes the form

$$\frac{d\hat{v}_i}{dt} + \frac{\dot{a}}{a}\hat{v}_i + \frac{1}{a}\phi_{G,i} + \frac{1}{c^2}\left[\frac{\phi_{G,i}}{a}\left(\phi_G + \frac{3}{2}\hat{v}^2\right) + \frac{D_{P,i}}{a} + \frac{\omega_{j,i}\hat{v}^j}{a}\right] = 0 \; . \tag{8.19}$$

Let us note, however, that using this definition of velocity in the continuity equation (8.17) we would get some extra terms.

IX.   CONCLUSIONS

Nonlinear structure formation in cosmology is traditionally studied with Newtonian N-body simulations and various approximation methods, e.g. Lagrangian perturbation theory. Relativistic perturbation theory and other approximations such as a gradient expansion are used to study small fluctuations in the early universe, in the CMB and on very large scales in the matter era. Thus there is a gap between methods used to study large and small scales. Various authors [1, 2, 48–60] have recently pointed out that, given the high precision of current and future galaxy surveys, it is timely to investigate possible GR effects on structure formation at all scales.

Assuming a standard flat ΛCDM cosmology, and a fluid approximation, in this Paper (Section VIII) we have developed a unified resummed nonlinear post-Friedmann formalism to study structure formation in the universe. This relativistic scheme reduces to fully nonlinear Newtonian cosmology at leading order (Section VI) and, if linearised (Section VII), to standard first-order relativistic perturbation theory. Thus our post-Friedmann formalism is valid on all scales, bridging the existing gap in current studies of nonlinear structure formation.

We have focused on obtaining a set of approximate nonlinear equation, consistently using the complete set of Einstein equations and the conservation equations. Rather than using the standard iterative procedure of the perturbative analysis where the equations are derived and solved order by order, we have constructed a resummed approximation scheme, which includes the first relativistic corrections to the equations of Newtonian cosmology, where a set of appropriately defined resummed variables satisfies a set of nonlinear equations.

Our equations could be used to implement GR corrections in N-body simulations. This would be important in order to take into account causal, retardation and other GR effects that may be non-negligible for simulations aiming at a 1% accuracy [42] on scales of the order of the Hubble horizon. Indeed, we should consider that the relevant fluctuations for the formation of large-scale structure have not always been much smaller than the Hubble scale in the past. For instance, the present physical length of the horizon scale at decoupling is of the order of $ct_{dec}(1 + z_{dec}) \approx 80h^{-1}$Mpc. Since galaxy surveys are going to cover a large proportion of the Hubble volume and aim at high precision measurements [42], we should consider GR corrections for the large and intermediate scales where the Newtonian approximation is not good enough. A first step in this direction has been done in [1, 2], computing the frame-dragging gravitomagnetic vector potentials from N-body Newtonian simulations, and in [88], where weak-lensing lensing has been considered in the post-Friedmann framework. Our equations could also be used to include GR corrections in approximate Newtonian studies of various nonlinear effects such as BAO and CDM halos [16–21]. A numerical implementation of our resummed equations is now a timely as well as doable goal: codes going beyond the quasi-static approximation have now been successfully developed for modified gravity [99], and codes specifically including GR corrections are in the making [56]. In this paper we have provided the theoretical framework aimed at this goal.

Appendix A

In this Appendix, for completeness we collect some useful expression for the energy-momentum tensor and some geometrical quantities.

$$T_{00} = \rho c^2 + \rho(v^2 - 2U_N) + \frac{\rho}{c^2}\left[v^4 - 4U_P + 2v^2 V_N + 2U_N{}^2\right] \tag{A1a}$$

$$T_{0i} = -\rho a c v_i + \frac{\rho a}{c}[B^N{}_i - v_i(v^2 + 2V_N)] \tag{A1b}$$

$$T_{ij} = \rho a^2 v_i v_j + \frac{a^2\rho}{c^2}\left[(4V_N + 2U_N + v^2)v_i v_j - 2B^N{}_{(i}v_{j)}\right] \tag{A1c}$$

Riemann Tensor:

$$
\begin{aligned}
R_{i00j} =& \frac{1}{c^2}[a\ddot{a}\delta_{ij} + U_{N,ij}] + \frac{1}{c^4}\left[a^2\delta_{ij}\left(\frac{\dot{a}}{a}\dot{U}_N + \ddot{V}_N + 2\frac{\dot{a}}{a}\dot{V}_N + 2\frac{\ddot{a}}{a}V_N + U_{N,k}V_N{}^{,k}\right)\right.\\
&\left. + a\dot{B}^N{}_{(i,j)} + \dot{a}B^N{}_{(i,j)} + 2U_{P,ij} - U_{N,i}U_{N,j} - 2U_{N,(i}V_{N,j)} - 2U_N U_{N,ij}\right]
\end{aligned}
\tag{A2a}
$$

$$R_{0ijk} = \frac{1}{c^3}\left[2a^2\left(\dot{V}_{N,[k} + \frac{\dot{a}}{a}U_{N,[k}\right)\delta_{j]i} + aB^N{}_{[j,k]i}\right] \tag{A2b}$$

$$
\begin{aligned}
R_{ijkl} =& \frac{1}{c^2}\left[2a^2\dot{a}^2\delta_{i[k}\delta_{l]j} + 2a^2(V_{N,j[k}\delta_{l]i} - V_{N,i[k}\delta_{l]j})\right] + \frac{1}{c^4}\left[\delta_{i[l}\delta_{k]j}a^2 2\left(-4\dot{a}^2 V_N - 2a\dot{a}\dot{V}_N - 2\dot{a}^2 U_N + V_{N,n}V_{N,n}\right)\right.\\
&+ 2a^2\left(\frac{1}{2}\dot{a}B^N{}_{i,[k} + \frac{1}{2}\dot{a}B^N{}_{[k,|i|} + V_{N,i}V_{N,[k} - 2V_{N,i[k}V_N - 2V_{P,i[k}\right)\delta_{l]j} - 2a^2\left(\frac{1}{2}\dot{a}B^N{}_{j,[k} + \frac{1}{2}\dot{a}B^N{}_{[k,|j|} + V_{N,j}V_{N,[k}\right.\\
&\left.\left. - 2V_{N,j[k}V - 2V_{B,j[k}\right)\delta_{l]i}\right]
\end{aligned}
\tag{A2c}
$$

Ricci Tensor:

$$R_{00} = -\frac{1}{c^2 a^2}(\nabla^2 U_N + 3a\ddot{a}) - \frac{1}{c^4 a^2}\left[V_{N,k}U_{N,k} - 2\nabla^2 U_N(U_N + V_N) - U_{N,k}U_N{}^{,k} + 2\nabla^2 U_P + 3a^2\ddot{V}_N + 6a\dot{a}\dot{V}_N + 3a\dot{a}\dot{U}_N\right] \tag{A3a}$$

$$R_{0i} = -\frac{1}{2c^3 a}(4\dot{a}U_{N,i} + 4a\dot{V}_{N,i} - \nabla^2 B^N{}_i) \tag{A3b}$$

$$
\begin{aligned}
R_{ij} =& \frac{1}{c^2}\left[\left(a\ddot{a} + 2\dot{a}^2 - \nabla^2 V_N\right)\delta_{ij} + (U_N - V_N)_{,ij}\right] + \frac{1}{c^4}\left[2\dot{a}B^N{}_{(i,j)} + a\dot{B}^N{}_{(i,j)} + V_{N,i}V_{N,j} - U_{N,i}U_{N,j} - 2V_{N,(i}U_{N,j)}\right.\\
&+ 2(U_P - V_P)_{,ij} + \delta_{ij}\left(V_{N,k}U_N{}^{,k} - V_{N,k}V_N{}^{,k} + 4\dot{a}^2(V_N + U_N) - 2\nabla^2 V_P + 6a\dot{a}\dot{V}_N\right.\\
&\left.\left. + 2a\ddot{a}(U_N + V_N) + a^2\ddot{V}_N + a\dot{a}\dot{U}_N - \frac{1}{2}\nabla^2 h_{ij}\right)\right].
\end{aligned}
\tag{A3c}
$$

Ricci scalar:

$$
\begin{aligned}
R =& \frac{1}{c^2}\left[6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) + 2\frac{\nabla^2}{a^2}(U_N - 2V_N)\right] + \frac{1}{c^4}\left\{6\ddot{V}_N + 6\frac{\dot{a}}{a}\left(\dot{U}_N + 4\dot{V}_N\right)\right.\\
&\left. - \frac{2}{a^2}(V_{N,i} - U_{N,i})^2 - 2V\frac{\nabla^2}{a^2}(U_N - 2V_N) + \frac{2}{a^2}V_{N,i}U_{N,i} + 4\frac{\nabla^2}{a^2}(U_P - 2V_P)\right\}
\end{aligned}
\tag{A4}
$$

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