Z-boson pole mass at two-loop order in the pure MS[over ¯] scheme

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Z boson pole mass at two-loop order in the pure $\overline{\mathrm{MS}}$ scheme

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I obtain the complex pole squared mass of the Z boson at full two-loop order in the Standard Model in the pure $\overline{\mathrm{MS}}$ renormalization scheme. The input parameters are the running gauge couplings, the top-quark Yukawa coupling, the Higgs self-coupling, and the vacuum expectation value that minimizes the Landau gauge effective potential. The effects of non-zero Goldstone boson mass are resummed. Within a reasonable range of renormalization scale choices, the scale dependence of the computed pole mass is found to be comparable to the current experimental uncertainty, but the true theoretical error is likely somewhat larger.

I. INTRODUCTION

One of the cornerstone physical observables of the Standard Model is the Z boson mass. The experimental value that is usually quoted is obtained using a Breit-Wigner parametrization with a variable width, and is given in ref. [1] from a fit to LEP data as:

$$M_Z^{\mathrm{exp}} = 91.1876 \pm 0.0021\,\mathrm{GeV}. \tag{1.1}$$

This is related [2–4] to the real part of the complex pole squared mass $s^Z_{\mathrm{pole}} = M_Z^2 - i\Gamma_Z M_Z$ (with Γ_Z a constant width) according to:

$$M_Z = M_Z^{\mathrm{exp}}(1 - \Gamma_Z^2/2M_Z^2 + \ldots) \tag{1.2}$$

$$= 91.1535 \pm 0.0021\,\mathrm{GeV}. \tag{1.3}$$

In general, the complex pole squared mass is a physical observable [5–10], independent of the choice of renormalization scheme and scale and the choice of gauge fixing.

In this paper, I report a calculation, at full 2-loop order, of the complex pole squared mass parameters M_Z and Γ_Z, using the pure $\overline{\mathrm{MS}}$ scheme. The input parameters in this scheme are the running renormalized quantities

$$g, g', g_3, y_t, \lambda, v, \tag{1.4}$$

where the first three are the Standard Model gauge couplings, y_t is the top-quark Yukawa coupling, λ is the Higgs self-coupling, and the vacuum expectation value (VEV) v is defined here to be the minimum of the full radiatively corrected effective potential in the Landau gauge. The normalizations used here for λ and v are fixed by writing the tree-level Higgs potential as

$$V(\Phi, \Phi^\dagger) \;=\; m^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2, \tag{1.5}$$

where the canonically normalized doublet Higgs field has VEV $\langle \Phi \rangle = v/\sqrt{2}$, and m^2 is a negative Higgs squared mass parameter. The minimization condition that relates v to m^2 (allowing the latter to be eliminated) is presently known at full 2-loop order [11] augmented by all 3-loop contributions at leading orders in both g_3 and y_t [12]. Goldstone boson mass effects are resummed in this relation using [13, 14]; this effect is usually numerically small but is conceptually important, and in any case leads to simpler formulas.

Other definitions of the Higgs VEV can be found in the literature. One alternative (for example, see refs. [15–17]) is to instead define the VEV as the minimum of the tree-level potential, so $v_{\rm tree} = \sqrt{-m^2/\lambda}$. This has the disadvantage that one must include tadpole diagrams explicitly. Also, one is then expanding around a point that differs from the true radiatively corrected vacuum, so perturbation theory converges less quickly, at least formally and for generic choices of the renormalization scale. Indeed, in the large y_t limit, the loop expansion parameter is $N_c y_t^4/(16\pi^2\lambda)$ rather than the usual $N_c y_t^2/16\pi^2$. [For more details, see for example refs. [13, 18], and the discussion surrounding eq. (2.34) below.] The reason for the λ in the denominator is that the tadpole diagrams have a Higgs propagator at zero momentum, which is just the reciprocal of the Higgs squared mass.

Another alternative (see for example ref. [19]) is to define the VEV so that the sum of tadpole graphs in Feynman gauge vanishes. However, the Landau gauge effective potential is easier to compute to higher orders, and avoids renormalization of the gauge-fixing parameter, making it arguably a more convenient choice as a standard.

The pure $\overline{\rm MS}$ scheme is an alternative to on-shell and hybrid schemes, which have been used for many precision studies of the Z mass and the electroweak sector. For a selection of some important related results in that approach, see refs. [20–41], and for reviews see refs. [1, 42].

II. COMPLEX POLE MASS OF THE Z BOSON AT 2-LOOP ORDER

To obtain the Z boson pole squared mass, one begins with the symmetric 2×2 matrix of neutral gauge boson transverse self-energy functions, for $V, V' = \gamma, Z$:

$$\Pi_{VV'}(s) \;=\; \frac{1}{16\pi^2}\Pi^{(1)}_{VV'}(s) \,+\, \frac{1}{(16\pi^2)^2}\Pi^{(2)}_{VV'}(s) \,+\, \ldots \tag{2.1}$$

where $s = -p^2$, with p^μ the external momentum, using a metric with Euclidean or $(-,+,+,+)$ signature. These are obtained by calculating, in the theory in $d = 4-2\epsilon$ dimensions with bare parameters, the sum of 1-particle-irreducible 2-point Feynman diagrams for $\Pi^{\mu\nu}_{VV'}$, followed by projecting with $(\eta_{\mu\nu} - p_\mu p_\nu/p^2)/(d-1)$. The pole squared mass is then the solution of

$$s^Z_{\rm pole} \;=\; Z_B + \Pi_{ZZ}(s^Z_{\rm pole}) + [\Pi_{\gamma Z}(s^Z_{\rm pole})]^2 / \left[s^Z_{\rm pole} - \Pi_{\gamma\gamma}(s^Z_{\rm pole})\right]. \tag{2.2}$$

Here,

$$Z_B = (g_B^2 + g_B'^2) v_B^2/4 \tag{2.3}$$

is the bare, tree-level, squared mass of the Z boson. Solving eq. (2.2) iteratively, one obtains to 2-loop order:

$$\begin{aligned} s^Z_{\rm pole} \;=\; & Z_B \,+\, \frac{1}{16\pi^2}\Pi^{(1)}_{ZZ}(Z_B) \,+\, \frac{1}{(16\pi^2)^2}\Big\{\Pi^{(2)}_{ZZ}(Z_B) \,+\, \Pi^{(1)\prime}_{ZZ}(Z_B)\,\Pi^{(1)}_{ZZ}(Z_B) \\ & +[\Pi^{(1)}_{\gamma Z}(Z_B)]^2/Z_B\Big\}. \end{aligned} \tag{2.4}$$

Instead of computing separate counterterm diagrams, the calculation described here was done in terms of only bare quantities g_B, g'_B, g_{3B}, y_{tB}, λ_B, v_B, m^2_B, and then translated to renormalized running $\overline{\rm MS}$ quantities g, g', g_3, y_t, λ, v at the end. Tadpole diagrams need not be calculated, because they automatically sum to zero, due to the defining condition that the VEV is the minimum of the effective potential. Using the minimization condition for the Landau gauge effective potential given in ref. [13], the parameter m^2 (and the Goldstone boson squared mass) are eliminated. These procedures are the same as described in refs. [18, 43], and so most details will not be repeated here. An exception is that the 2-loop translation of the $U(1)_Y$ gauge couplings from bare to renormalized couplings is needed, to go along with eqs. (2.5)-(2.24) of ref. [43] and eqs. (2.3)-(2.10) of ref. [18]:

$$g'_B \;=\; \mu^\epsilon\Big[g' + \frac{1}{16\pi^2}\frac{c^{g'}_{1,1}}{\epsilon} + \frac{1}{(16\pi^2)^2}\Big(\frac{c^{g'}_{2,2}}{\epsilon^2} + \frac{c^{g'}_{2,1}}{\epsilon}\Big) + \ldots\Big] \tag{2.5}$$

where

$$c^{g'}_{1,1} \;=\; \frac{41}{12} g'^3, \tag{2.6}$$

$$c^{g'}_{2,1} \;=\; g'^3\Big(\frac{11}{3}g_3^2 + \frac{9}{8}g^2 + \frac{199}{72}g'^2 - \frac{17}{24}y_t^2\Big), \tag{2.7}$$

$$c^{g'}_{2,2} \;=\; \frac{1681}{96} g'^5, \tag{2.8}$$

and μ is the regularization scale, related to the renormalization scale Q by $\mu^2 = Q^2 e^{\gamma_E}/4\pi$. As in refs. [18, 43], the results are reduced, using the Tarasov recurrence relations [44] to a set of 1-loop basis integrals A, B and 2-loop basis integrals $I, S, T, \overline{T}, U, M$, following the notations and conventions of refs. [45, 46]. The program TSIL [46] can be used to automatically and efficiently evaluate these basis integrals numerically. Where possible, TSIL takes advantage of analytical results in terms of polylogarithms, which were given in refs. [45–53]. In many cases, analytical results for the basis integrals are not available, so TSIL employs Runge-Kutta solution of differential equations in the external momentum invariant [45], similar to that suggested in ref. [54].

The final result for the 2-loop Z boson complex pole mass can be written as:

$$s^Z_{\rm pole} \;=\; M_Z^2 - i\Gamma_Z M_Z \;=\; Z + \frac{1}{16\pi^2}\Delta_Z^{(1)} + \frac{1}{(16\pi^2)^2}\left[\Delta_Z^{(2),{\rm QCD}} + \Delta_Z^{(2),{\rm non-QCD}}\right]. \tag{2.9}$$

In the following,

$$Z \;=\; (g^2+g'^2)v^2/4, \tag{2.10}$$
$$W \;=\; g^2v^2/4, \tag{2.11}$$
$$t \;=\; y_t^2 v^2/2, \tag{2.12}$$
$$h \;=\; 2\lambda v^2 \tag{2.13}$$

are the tree-level $\overline{\rm MS}$ squared masses of the Z boson, W boson, top quark, and Higgs boson, respectively, and the couplings of the quarks and leptons to the Z boson are:

$$a_f \;=\; \sqrt{g^2+g'^2}\left[T_3^f - Q_f\, g'^2/(g^2+g'^2)\right], \tag{2.14}$$

for $f = u_L, u_R, d_L, d_R, e_L, e_R, \nu_L$, where

$$T_3^{u_L} = T_3^{\nu_L} = -T_3^{d_L} = -T_3^{e_L} = 1/2, \tag{2.15}$$
$$T_3^{u_R} = T_3^{d_R} = T_3^{e_R} = 0, \tag{2.16}$$
$$Q_{u_L} = Q_{u_R} = 2/3, \tag{2.17}$$
$$Q_{d_L} = Q_{d_R} = -1/3, \tag{2.18}$$
$$Q_{e_L} = Q_{e_R} = -1, \tag{2.19}$$
$$Q_{\nu_L} = 0. \tag{2.20}$$

Also, $N_c = 3$, and

$$n_Q = n_u = n_d = n_L = n_e = 3 \tag{2.21}$$

are the the numbers of flavors of two-component quarks and leptons of each gauge transformation type, (u_L, d_L) and u_R and d_R and (ν_L, e_L) and e_R, respectively. The quantities N_c, n_Q, n_u, n_d, n_L and n_e are kept general in the following as a way of tagging different fermion contributions, although they are all equal to 3 in the Standard Model.

The 1-loop contribution is then:

$$\begin{aligned}
\Delta_Z^{(1)} &= N_c(a_{u_L}^2 + a_{u_R}^2)f_1(t) + N_c 2a_{u_L}a_{u_R}f_2(t) + \Big[N_c(n_Q-1)a_{u_L}^2 \\
&\quad + N_c(n_u-1)a_{u_R}^2 + N_c n_Q a_{d_L}^2 + N_c n_d a_{d_R}^2 + n_L(a_{e_L}^2 + a_{\nu_L}^2) + n_e a_{e_R}^2\Big] f_1(0) \\
&\quad + g^2\Big\{(4W-Z)\Big(\frac{W}{Z} + \frac{5}{3} + \frac{Z}{12W}\Big)B(W,W) + \Big(\frac{4W}{Z} - \frac{4}{3} - \frac{Z}{6W}\Big)A(W) \\
&\quad + \Big(\frac{4hZ - 12Z^2 - h^2}{12W}\Big)B(h,Z) + \Big(\frac{h-2Z}{12W}\Big)A(Z) + \Big(\frac{3Z-h}{12W}\Big)A(h) \\
&\quad + \frac{4W^2}{Z} - \frac{4W}{3} + \frac{5Z}{9} + \frac{hZ}{6W} + \frac{Z^2}{18W}\Big\}, \qquad (2.22)
\end{aligned}$$

where the fermion 1-loop integral functions are:

$$f_1(t) = \frac{2}{3}(t-Z)B(t,t) - \frac{4}{3}A(t) + \frac{2}{9}Z - \frac{4}{3}t, \qquad (2.23)$$

$$f_1(0) = -\frac{2}{3}ZB(0,0) + \frac{2}{9}Z, \qquad (2.24)$$

$$f_2(t) = -2tB(t,t). \qquad (2.25)$$

The basis integrals $B(0,0)$, $B(t,t)$, $B(h,Z)$, and $B(W,W)$, and other integral functions below, are always evaluated at the external momentum invariant $s = Z$ and renormalization scale Q. The bottom-quark, tau-lepton, and other fermion masses have been neglected for simplicity, because even at 1-loop order they make a difference of less than 1 MeV in the real Z pole mass. However, they can easily be restored in the 1-loop part by following the example of the top-quark terms in the obvious way.

The 2-loop QCD contribution can also be written in terms of the basis integral functions in a few lines:

$$\begin{aligned}
\Delta_Z^{(2),\mathrm{QCD}} &= g_3^2\Big(\frac{N_c^2-1}{4}\Big)\Big[(a_{u_L}^2 + a_{u_R}^2)F_1(t) + 2a_{u_L}a_{u_R}F_2(t) \\
&\quad + [(n_Q-1)a_{u_L}^2 + (n_u-1)a_{u_R}^2 + n_Q a_{d_L}^2 + n_d a_{d_R}^2]F_1(0)\Big], \qquad (2.26)
\end{aligned}$$

where:

$$\begin{aligned}
F_1(t) &= \frac{8}{3}(Z-t)(2t-Z)M(t,t,t,t,0) + \frac{16}{3}(Z-t)\overline{T}(0,t,t) \\
&\quad + \frac{1}{3Z(4t-Z)}\Big[(24t^2Z - 24t^3 + 20tZ^2 - 8Z^3)B(t,t)^2 \\
&\quad + (32Z^2 - 32tZ - 48t^2)A(t)B(t,t) + (56Z - 24t + 16Z^2/t)A(t)^2 \\
&\quad - 4(t-Z)(12t^2 - 30tZ + 7Z^2)B(t,t) + (296tZ - 48t^2 - 104Z^2)A(t)
\end{aligned}$$

$$
\begin{aligned}
& -24t^3 + 220t^2 Z - 141 t Z^2 + 23 Z^3 \Big], && (2.27) \\
F_1(0) &= -\frac{8}{3} Z^2 M(0,0,0,0,0) - 4 Z B(0,0) - \frac{31}{3} Z, && (2.28) \\
F_2(t) &= 8t(2t - Z) M(t,t,t,t,0) + 16 t \overline{T}(0,t,t) \\
& + \frac{1}{Z(4t-Z)} \Big[(8t^3 + 4tZ^2) B(t,t)^2 + (16t^2 + 80tZ) A(t) B(t,t) \\
& + (8t + 64Z) A(t)^2 + (16t^3 - 200t^2 Z + 36 t Z^2) B(t,t) \\
& + (16t^2 - 104 t Z) A(t) + 8t^3 - 140 t^2 Z + 43 t Z^2 \Big]. && (2.29)
\end{aligned}
$$

The 2-loop non-QCD contribution to the Z boson pole squared mass has the form:

$$
\Delta_Z^{(2),\text{non-QCD}} = \sum_i c_i^{(2)} I_i^{(2)} + \sum_{j \leq k} c_{j,k}^{(1,1)} I_j^{(1)} I_k^{(1)} + \sum_j c_j^{(1)} I_j^{(1)} + c^{(0)}. \qquad (2.30)
$$

where the list of 1-loop basis integrals is

$$
I^{(1)} = \big\{ A(h),\ A(t),\ A(W),\ A(Z),\ B(0,0),\ B(t,t),\ B(h,Z),\ B(W,W) \big\}, \qquad (2.31)
$$

and the list of necessary 2-loop basis integrals is:

$$
\begin{aligned}
I^{(2)} = \big\{ & I(0,0,h),\ I(0,0,t),\ I(0,0,W),\ I(0,0,Z),\ I(0,h,W),\ I(0,h,Z), \\
& I(0,t,W),\ I(0,W,Z),\ I(h,h,h),\ I(h,t,t),\ I(h,W,W),\ I(h,Z,Z), \\
& I(t,t,Z),\ I(W,W,Z),\ M(0,0,0,0,0),\ M(0,0,0,0,W),\ M(0,0,0,0,Z), \\
& M(0,t,0,t,W),\ M(0,W,0,W,0),\ M(0,W,0,W,t),\ M(h,h,Z,Z,h), \\
& M(h,t,Z,t,t),\ M(h,W,Z,W,W),\ M(h,Z,Z,h,Z),\ M(t,t,t,t,0), \\
& M(t,t,t,t,h),\ M(t,t,t,t,Z),\ M(t,W,t,W,0),\ M(W,W,W,W,0), \\
& M(W,W,W,W,h),\ M(W,W,W,W,Z),\ S(0,0,h),\ S(0,0,W),\ S(0,t,W), \\
& S(h,h,Z),\ S(h,t,t),\ S(h,W,W),\ S(t,t,Z),\ S(W,W,Z),\ S(Z,Z,Z), \\
& T(h,0,0),\ T(h,h,Z),\ T(h,t,t),\ T(h,W,W),\ T(t,0,W),\ T(t,h,t), \\
& T(t,t,Z),\ T(W,0,0),\ T(W,0,t),\ T(W,h,W),\ T(W,W,Z),\ T(Z,0,0), \\
& \overline{T}(0,t,t),\ \overline{T}(0,W,W),\ U(h,Z,0,0),\ U(h,Z,h,Z),\ U(h,Z,t,t), \\
& U(h,Z,W,W),\ U(t,t,0,W),\ U(t,t,h,t),\ U(t,t,t,Z),\ U(W,W,0,0), \\
& U(W,W,0,t),\ U(W,W,h,W),\ U(W,W,W,Z),\ U(Z,h,h,h), \\
& U(Z,h,t,t),\ U(Z,h,W,W),\ U(Z,h,Z,Z) \big\}. \qquad (2.32)
\end{aligned}
$$

The coefficients $c_i^{(2)}$ and $c_{j,k}^{(1,1)}$ and $c_j^{(1)}$ and $c^{(0)}$ are quite lengthy, so they will not be listed in print here. Instead, they are listed in electronic form in an ancillary file provided with the arXiv source for this article, called `coefficients.txt`. They are ratios of polynomials of Z, W, t, h, and v. As usual, these coefficients are not unique, because of special identities that relate different basis integrals in cases where the masses are not generic.

For each of the five-propagator M integrals for which analytical results are not available, the main TSIL Runge-Kutta evaluation function `TSIL_Evaluate` simultaneously computes all of the subordinate integrals S, T, U obtained by removing one or more propagator lines. Therefore, only 11 calls of `TSIL_Evaluate` are required (in addition to the relatively fast evaluation of the integrals that are known in terms of polylogarithms), and in total the numerical computation takes well under 1 second on modern computer hardware.

I performed a number of stringent analytical checks on the calculation, similar to those described for the calculations of the Higgs and W boson pole masses in [18, 43]. First, $s^Z_{\rm pole}$ is free of poles in ϵ. The cancellation of these poles relies on agreement between the divergent parts of the loop integrals performed here and the counterterm coefficients which can be obtained from the 2-loop scalar anomalous dimension and β functions from refs. [55–58]. Second, poles and logs of the Goldstone boson squared mass $G = m^2 + \lambda^2 v^2$ were checked to cancel after the resummation described in [13, 14]. Third, I checked the cancellations between contributions from unphysical vector propagator components with poles at 0 squared mass and the corresponding Landau gauge Goldstone boson propagators. This ensures the absence of unphysical imaginary parts of the complex pole squared mass. Note that $\Gamma_Z = 0$ in the case $N_c = n_L = n_e = 0$. Next, I checked the absence of singularities in various formal limits (none of which are close to being realized in the actual parameters of the Standard Model), in which one or more of the following quantities vanish: Z, W, t, h, $t - W$, $4t - Z$, $4W - h$, and $4Z - h$. This is despite the fact that many of the individual 2-loop coefficients do have singularities in one or more of those cases; non-trivial relations between basis integrals are responsible for the smooth limits of the total. Finally, I checked analytically that the complex pole squared mass is renormalization group scale-invariant up to and including all terms of 2-loop order, using

$$Q\frac{d}{dQ}s^Z_{\rm pole} \;=\; \left[Q\frac{\partial}{\partial Q} - \gamma v\frac{\partial}{\partial v} + \sum_X \beta_X \frac{\partial}{\partial X}\right] s^Z_{\rm pole} \;=\; 0, \tag{2.33}$$

where γ is the Higgs anomalous dimension, and $X = \{g, g', g_3, y_t, \lambda\}$. In the conventions used here, the derivatives of the 1-loop basis integrals with respect to squared masses are listed in eqs. (A.5) and (A.6) of ref. [43], while the derivatives of the 1-loop and 2-loop basis integrals with respect to the renormalization scale Q can be found in eqs. (4.7)-(4.13) of ref. [45]. The beta functions and scalar anomalous dimension are listed in refs. [11, 55–58]. In the next section, a numerical check of the Q invariance will be shown.

In refs. [15, 16], a calculation of the Z boson pole mass in the pure $\overline{\rm MS}$ scheme has already been given. However, unlike the present paper, they expanded around the tree-level definition of the VEV, as discussed in the Introduction above. This means that even at 1-loop order, the results take different forms. The expression for the 1-loop pole squared mass contribution $\Delta_Z^{(1)}/16\pi^2$ given in eq. (2.22) above appears to differ from the result of eq. (B.4) of ref. [15] and eq. (B.3) of ref. [16] by an amount

$$\frac{Z}{16\pi^2 v^2 h}\left[-8N_c t A(t) + 3hA(h) + 12WA(W) + 8W^2 + 6ZA(Z) + 4Z^2\right], \tag{2.34}$$

in the notation of the present paper. There is of course no contradiction; this merely reflects the difference between the tree-level contributions, which are $(g^2+g'^2)v^2/4$ in this paper and $(g^2+g'^2)v_{\rm tree}^2/4$ in refs. [15, 16]. Note in particular the presence of $1/h \propto 1/\lambda$ in eq. (2.34); at loop order ℓ, the use of the tree-level VEV results in terms proportional to $1/\lambda^\ell$. In contrast, there are no $\lambda \to 0$ singularities in the present paper. A detailed comparison would be much more difficult at 2-loop order, as refs. [15, 16] also relied on doing high-order expansions in Z/h and Z/t and $1/4 - \sin^2\theta_W$.

III. NUMERICAL RESULTS

Consider a benchmark set of Standard Model $\overline{\rm MS}$ parameters defined at the input renormalization scale $Q = M_t = 173.34$ GeV:

$$g(M_t) = 0.647550, \tag{3.1}$$
$$g'(M_t) = 0.358521, \tag{3.2}$$
$$y_t(M_t) = 0.93690, \tag{3.3}$$
$$g_3(M_t) = 1.1666, \tag{3.4}$$
$$v(M_t) = 246.647 \text{ GeV}, \tag{3.5}$$
$$\lambda(M_t) = 0.12597, \tag{3.6}$$

The gauge couplings g and g' are taken to agree with ref. [19], while y_t and g_3 are from eqs. (57) and (60) of version 4 of ref. [59]. The VEV $v(M_t)$ and the Higgs self-coupling were then chosen so that M_Z agrees with the central value of eq. (1.3), when computed at $Q = M_Z$, and M_h agrees with the current experimental central value [60] of $M_h = 125.09$ GeV, when computed at $Q = 160$ GeV using the program SMH [61] as described in ref. [43]. With this set of input parameters, one also obtains $m^2(M_t) = -(92.890 \text{ GeV})^2$ from minimization of the Higgs potential using SMH at $Q = M_t$. In this way, the experimental measurements of M_Z and M_h can be used to obtain the Higgs potential parameters. The choice of $Q = 160$ GeV for computing M_h was explained in ref. [43]; at this scale the effects of top-quark loops in the neglected electroweak 3-loop parts should be not too large. The lower choice of $Q = M_Z$ for computing M_Z is somewhat arbitrary. One also obtains a W boson pole mass of $M_W = 80.329$ GeV, when computed at $Q = M_W$, using the calculation described in [18]. This translates into a Breit-Wigner mass of $M_W^{\rm exp} = 80.356$ GeV, using the analog of eq. (1.2) above. (Somewhat coincidentally, this agrees with the value found in ref. [19] to within 1 MeV, although that calculation uses a different scheme.)

The dependences of the computed pole mass parameters M_Z and Γ_Z on the choice of Q are shown in figures 3.1 and 3.2, in various approximations. These graphs are made by running the input parameters g, g', y_t, g_3, λ, and v, using their 3-loop beta functions [62, 63], from the input scale M_t to the scale Q on the horizontal axis, where $s^Z_{\rm pole}$ is computed. In an idealized case that $s^Z_{\rm pole}$ is computed to sufficiently high order in perturbation theory, M_Z and Γ_Z would be independent of Q. Therefore the Q-independence is a check on the calculation. I find that the calculated 2-loop value of M_Z varies by only about ± 2 MeV from its median value, over the range 70 GeV $< Q <$ 200 GeV. Below Q = 70 GeV, the scale dependence is much stronger. The scale dependence is smallest for Q near 100 GeV, where the computed M_Z has its minimum, but this does not necessarily mean that this is the best renormalization scale; only a higher-order calculation can reduce the theoretical uncertainty.

With regard to the width Γ_Z, the scale dependence of the full 2-loop result is again about ± 2 MeV from the median value over the same range 70 GeV $< Q <$ 200 GeV. Note that here, including only the QCD part of the 2-loop contribution does not actually reduce the scale dependence much compared to the 1-loop result. This is because most of the Q dependence in the width arises from the runnings of the VEV and the electroweak couplings of the Z boson to the fermions into which it decays, and these are independent of QCD at the leading (1-loop) order. The result for Γ_Z is consistent with, and slightly lower than the central value of, the experimental range [1] $\Gamma_Z = 2.4952 \pm 0.0023$ GeV. Of course, there are much better ways to calculate Γ_Z, because the imaginary part of the 2-loop complex pole mass really corresponds to only a 1-loop calculation of the Z width. (Moreover, the inclusion of bottom-quark mass effects, neglected above for simplicity, has a larger effect on Γ_Z than on M_Z, and will decrease the former by an amount of order 2 MeV due to kinematics. There is a significant uncertainty in estimating this reduction in the imaginary part of the Z complex pole mass, because of the large difference between the pole and running bottom quark masses.)

It is important to keep in mind that the renormalization scale dependence only provides a lower bound on the theoretical error. Another way of investigating the robustness of the calculation is to take the running top-quark squared mass t in the 1-loop part eq. (2.22) and perform an expansion around an arbitrary value T that can be considered to differ from t by an amount that is parametrically of 1-loop order. An obvious choice is to take T to be the (real part of the) top-quark pole squared mass. It makes sense to do this in particular for the 1-loop contribution, because the top quark mass appears only in propagators at this order, not as a vertex Yukawa coupling. Expanding, one finds:

$$f_1(t) \;=\; f_1(T) + (t-T)\left[(4T-2Z)B(T,T) - 4A(T) - 12T + 4Z\right]/(4T-Z) + \ldots, \tag{3.7}$$

$$f_2(t) \;=\; f_2(T) + (t-T)\left[(2Z-12T)B(T,T) - 4A(T) + 4T\right]/(4T-Z) + \ldots. \tag{3.8}$$

I have checked that if these expansions were continued to include order $(t-T)^3$, then the results for the Z pole squared mass would be nearly indistinguishable from the original

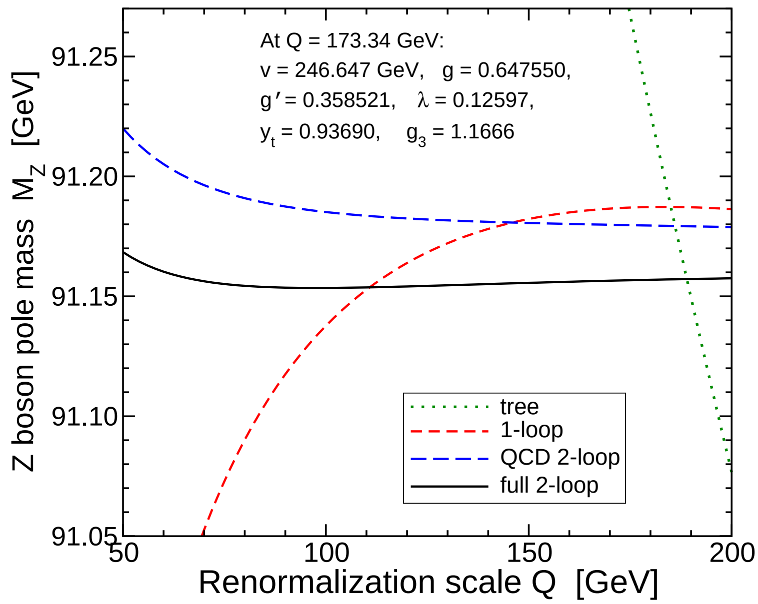

FIG. 3.1: The computed pole mass M_Z of the Z boson, defined by $s^Z_{\rm pole} = M_Z^2 - i\Gamma_Z M_Z$, as a function of the renormalization scale Q at which it is computed, in various approximations. The dotted (green) line is the tree-level result Z, the short-dashed (red) line is the 1-loop result, the long-dashed (blue) line is the result from the 1-loop and 2-loop QCD contribution, while the solid (black) line is the full 2-loop order result. The input parameters g, g', y_t, g_3, λ, and v at the renormalization scale Q are obtained by running 3-loop renormalization group running, starting from eqs. (3.1)-(3.6). Note that the usual Breit-Wigner mass $M_Z^{\rm exp}$ is 0.0341 GeV larger than the M_Z shown here.

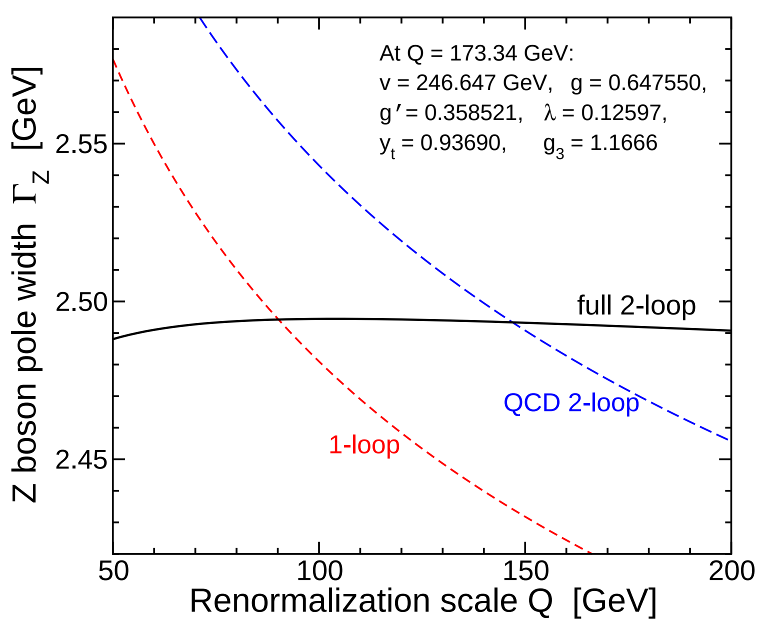

FIG. 3.2: The computed width Γ_Z of the Z boson, defined in terms of the complex pole squared mass $s^Z_{\rm pole} = M_Z^2 - i\Gamma_Z M_Z$, as in Figure 3.1. The short-dashed (red) line is the 1-loop result, the long-dashed (blue) line is the result from the 1-loop and 2-loop QCD contribution, and the solid (black) line is the full 2-loop order result.

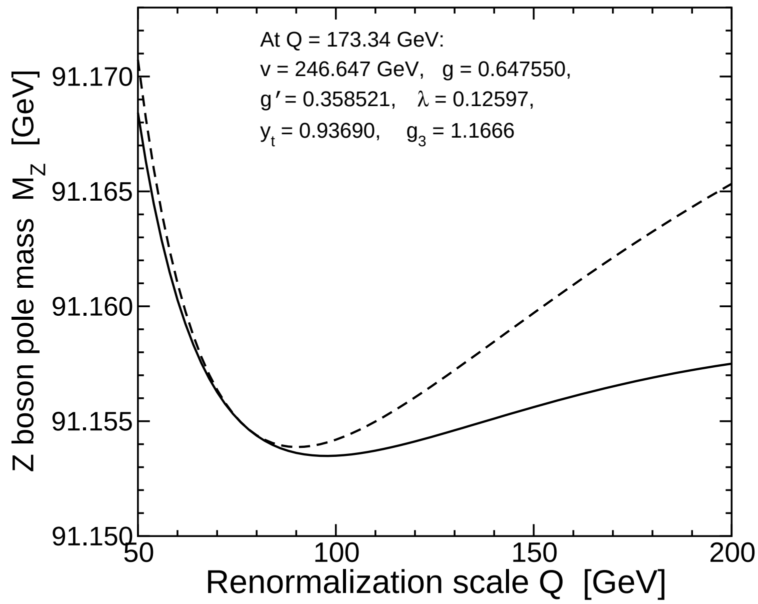

FIG. 3.3: A close-up of the renormalization scale dependence of the computed pole mass M_Z. The solid line is the 2-loop result, just as in Figure 3.1. The input parameters were chosen so that at $Q = M_Z$ the computed pole mass agrees with the experimental central value $M_Z = 91.1535$ GeV from eq. (1.3). The dashed line is the same, but after expanding the 1-loop part in the $\overline{\text{MS}}$ squared mass t to linear order about the value $T = (173.34 \text{ GeV})^2$, using eqs. (3.7) and (3.8). This provides an alternate consistent 2-loop order result. The two approximations agree near the scale $Q = 77$ GeV where $t = T$ (the top-quark running and pole masses coincide). Note that the Breit-Wigner mass M_Z^{exp} is 0.0341 GeV larger than the pole mass M_Z shown here.

result obtained directly from the un-expanded $f_1(t)$ and $f_2(t)$. However, by instead keeping the expansion only at first order in $(t - T)$ as shown, one obtains an alternative consistent 2-loop order result for the Z pole squared mass, since $t - T$ is to be treated as formally of 1-loop order. This alternative consistent 2-loop order result is numerically different, with the difference giving an indication of the magnitude of the error made in terminating perturbation theory at 2-loop order. The result of using eqs. (3.7) and (3.8) compared to the original un-expanded $f_1(t)$ and $f_2(t)$ is shown in Figure 3.3. We see that the alternate consistent 2-loop result, shown as the dashed line, has a significantly worse scale dependence, especially at larger Q. This suggests that the scale dependence of M_Z found in the original calculation (the solid line) is actually accidentally small, and probably underestimates the theoretical error. A very similar behavior was found for the W boson mass in ref. [18].

IV. OUTLOOK

In this paper I have provided a full 2-loop calculation of the Z boson complex pole square mass in the pure $\overline{\text{MS}}$ scheme, to go along with similar results for the W boson [18] and the Higgs boson [43] using the same renormalization scheme and the same definition of the VEV.

These calculations are an alternative to the on-shell scheme results that have been widely used for precision studies in the Standard Model, in which M_Z instead plays the role of an input parameter.

The ultimate goal should be to obtain results in which the theoretical error is very small compared to present and projected experimental errors. The previous section shows that this is certainly not obtained using just the full 2-loop calculation, as the scale dependence is comparable to the experimental errors, and the theoretical error is probably somewhat larger. There is no compelling evidence or argument that the subset of 3-loop contributions that are QCD and top-Yukawa enhanced will be enough to ensure the dominance of experimental errors over theoretical errors. At 2-loop order, one can see from the benchmark example of Figure 3.1 that the QCD contribution has a much larger scale dependence, but not a much larger magnitude, than the non-QCD contributions, except for smaller choices of the renormalization scale Q where the top-enhanced QCD corrections are big. The same thing was noted in the comparable results for the W boson in [18]. It is therefore reasonable to conclude that complete 3-loop calculations will be necessary, providing a worthy challenge for future work.

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