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Linear Newtonian perturbation theory from the Schrödinger-Poisson equations

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Abstract

We obtain solutions to the coupled Schrödinger-Poisson equations. The solutions describe the evolution of cold dark matter density perturbations in an otherwise homogeneous expanding Friedmann universe. We discuss the relationships between descriptions of cold dark matter in terms of a pressureless fluid, in terms of a wavefunction, of a classical scalar field, and a quantum scalar field. We identify the regimes where the various descriptions coincide and where they differ.

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I. INTRODUCTION

The identity of dark matter is among the most tantalizing questions in science today [1]. Fortunately we live in an era where a large number of observations directly or indirectly bear upon this question. Foremost among these are measurements of galactic rotation curves, observations of gravitational lensing by dark matter clumps on various scales, the cosmic microwave background anisotropy observations, and broad surveys of the visible matter distribution such as the Sloan Digital Sky Survey.

The observations require the dark matter to be, in first approximation, *cold* and *collisionless*. Collisionless means that the main force acting on dark matter and therefore the main force by which dark matter manifests its presence is gravity. The dark matter may have in addition weak non-gravitational interactions but the observations are consistent with the absence of non-gravitational interactions. That the dark matter is cold means that its primordial velocity dispersion is small. By primordial velocity dispersion we mean the velocity dispersion that the dark matter particles have even in the absence of density perturbations. An upper limit of order $10^{-8}c$ on the primordial velocity dispersion follows from the requirement that free streaming of the dark matter particles does not erase density perturbations on the smallest scales on which they are observed.

Particle candidates for the dark matter are also meaningfully constrained by the requirement that they must fit comfortably with the particles that are already known to exist, i.e. those described by the Standard Model. There are three broad categories of dark matter candidates that are thought to fit well into the existing scheme of particle physics: weakly interacting massive particles (WIMPs), axions or axion-like particles, and sterile neutrinos. WIMPs are motivated by supersymmetric extensions of the Standard Model. Their mass is typically 100 GeV, and their primordial velocity dispersion of order $10^{-12}c$. Axions are motivated by the Peccei-Quinn solution of the strong CP problem, the puzzle within the Standard Model why the strong interactions are P and CP invariant. The axion mass is thought to be of order 10^{-5} eV/c^2 and the axion primordial velocity dispersion of order $10^{-17}c$. Sterile neutrinos have mass of order a few keV/c^2 and primordial velocity dispersion of order $10^{-8}c$, at the limit of what is allowed. For this reason, sterile neutrinos are sometimes called "warm dark matter". Axions and WIMPs are definitely cold dark matter.

Cold collisionless dark matter may be described in the linear regime of the evolution of density perturbations as a pressureless fluid. This is the description of cold dark matter in calculations of the cosmic microwave background anisotropies [2]. Since cold dark matter plays an important role in this context and the calculations agree very well with the observations, the pressureless fluid description has high credibility. To obtain the cosmic microwave background anisotropies a full general relativistic treatment [3] is necessary because the relevant evolution occurs in part on length scales of order the horizon. However, on length scales much less than the horizon (i.e. for wavevectors much larger than the Hubble rate) dark matter density perturbations are correctly described by Newtonian gravity. Linear Newtonian perturbation theory [4] is simple, well understood and agrees with the general relativistic description on length scales much less than the horizon, where many of the interesting phenomena in large scale structure formation occur. It is therefore a very useful tool.

L. Widrow and N. Kaiser [5] pointed out that, on scales much less than the horizon, cold collisionless dark matter can be described by a wavefunction satisfying the Schrödinger-Poisson equations. As is discussed below, in Section III, the wavefunction description is in many ways more powerful than the pressureless fluid description. It allows the introduction of velocity dispersion whereas the fluid description allows none. It can be used to describe multi-streaming and caustics in the non-linear regime, whereas the fluid description breaks down in that regime. Indeed, Widrow and Kaiser carried out numerical simulations of structure formation using a wavefunction satisfying the Schrödinger equation, in lieu of N bodies satisfying Newton's force law equation. Several such simulations have since been carried out [6].

In the present paper, we reproduce the results of Newtonian linear perturbation theory using a wavefunction solving the Schrödinger-Poisson (sometimes called Schrödinger-Newton) equations. As far as we are aware, this had not been done before although related work, also using the Schrödinger equation to analyze the growth of density perturbations in the early universe, can be found in Refs. [7–9]. One of our motivations is to show that the formalism does indeed work as expected. However, our main motivation is to prepare the ground for an in-depth study of the dynamical evolution of axion dark matter.

The proposal that the dark matter may be axions originates with the papers of Ref.[10] which showed that axions are copiously produced during the QCD phase transition, at a temperature of order 1 GeV. The estimate of the cosmological energy density of the axions thus produced was obtained by a simple classical treatment of the axion field. The axions are extremely weakly interacting and therefore collisionless. They are non-relativistic shortly after being produced and subsequently red-shifted by the expansion of the universe. Thus they are very cold, as was already mentioned. It was emphasized in Ref. [11] that the axions produced during the QCD phase transition behave as cold dark matter on all scales much longer than the wavelength of the axion field, hereafter called the de Broglie wavelength. So, although axions are much lighter than WIMPs, they behave in many circumstances in the same way as WIMPs. However, there are differences.

Obviously, axions behave differently from WIMPs on the length scale of their de Broglie wavelength. One manifestation of this is the existence of a Jeans' length for axion dark matter. This was originally pointed out by the authors of Refs.[12, 13]. The formula for the Jeans' length is given in Eq. (2.31) below. For QCD axions, those that solve the strong CP problem and have masses conservatively in the range 10^{-2} to 10^{-12} eV, the Jeans length is far too short to affect structure formation in an observable way. However, we may also have axion-like particles (ALPs) with much smaller masses. Indeed, string theory predicts the existence of numerous axion and axion-like fields [14, 15]. If the ALP mass is in the 10^{-21} to 10^{-24} eV mass range and below, the Jeans' length is large enough (kpc and larger) to have observable effects. Structure formation on length scales less than the Jeans' length is suppressed. There is a long standing discrepancy by which observations show less structure on small scales than is predicted by N-body simulations. Many authors have proposed to resolve this by hypothesizing that the dark matter is an extremely light scalar field, with mass of order 10^{-21} eV or less [16].

Less obviously, axions differ from WIMPs because they thermalize and form a Bose-Einstein condensate (BEC) [17, 18]. Axions thermalize as a result of their gravitational self-interactions when the photon temperature is of order 500 eV. Their thermalization time becomes shorter than the age of the universe then. When they thermalize, almost all axions go to the lowest energy state available to them. In this they differ from the other dark matter candidates. It was shown in Ref. [17] that, on all scales of observational interest, density perturbations in axion BEC behave in exactly the same way as those in ordinary cold dark matter provided the density perturbations are within the horizon and in the linear regime. On the other hand, when density perturbations enter the horizon, and in second order of perturbation theory, axions generally behave differently from ordinary cold dark matter because the axions rethermalize so that the state most axions are in tracks the lowest energy available state. Axion BEC explains the occurrence of caustic rings of dark matter in galactic halos and their observed radii [19]. It also solves the galactic angular momentum problem [20]. The observations require that at least 35% of the dark matter is axions [20].

Systems dominated by gravitational self-interactions are inherently unstable. In this regard the axion BEC differs from the BECs that occur in superfluid ^{4}He and dilute gases [21]. The axion fluid is subject to the Jeans gravitational instability and this is so whether the axion fluid is a BEC or not [22]. The Jeans instability causes density perturbations to grow at a rate of order the Hubble rate $H(t)$, i.e. on a time scale of order the age of the universe at the moment under consideration. Each mode of the axion fluid is Jeans unstable. However when the thermalization time is shorter than the age of the universe, the rate at which quanta of the axion field jump between modes is faster than the rate at which the Jeans instability develops. So the modes are essentially frozen on the time scale over which the axions thermalize.

Our long term goal is to clarify the dual role of gravity in the evolution of the axion BEC. On the one hand gravity causes Jeans instability of the axion field modes. On the other, gravity causes axions to jump between those field modes. In the present paper, we take two steps towards this goal. In Section II we solve the Schrödinger-Poisson equations for self-gravitating collisionless dark matter. Our solutions describe the homogeneous expanding Friedmann universe and density perturbations therein. They also provide a complete set of states for the axions to occupy. In Section III, we discuss the various relationships between descriptions of cold dark matter in terms of a pressureless fluid, in terms of a wavefunction, of a classical scalar field, and a quantum scalar field, identifying the regimes where the various descriptions coincide and where they differ. In Section IV, we summarize our conclusions.

II. WAVEFUNCTION DESCRIPTION OF LINEAR PERTURBATIONS

Consider a fluid composed of a huge number N of particles that are all in the same quantum-mechanical state. The wavefunction $\psi(\vec{r},t)$ for the state satisfies the Schrödinger equation

$$i\partial_t\psi(\vec{r},t) = \Big(-\frac{1}{2m}\nabla^2 + m\Psi(\vec{r},t)\Big)\psi(\vec{r},t)\,, \tag{2.1}$$

where m is the particle mass and $\Psi(\vec{r},t)$ is the Newtonian gravitational potential. In this section, we set $\hbar = c = 1$. The density of particles in the fluid is

$$n(\vec{r},t) = N\psi^*(\vec{r},t)\psi(\vec{r},t) \quad . \tag{2.2}$$

Let us assume that the only kind of matter present is the fluid of particles. The gravitational potential is then given by the Poisson equation

$$\nabla^2\Psi = 4\pi G m n(\vec{r},t) \quad . \tag{2.3}$$

The fluid density satisfies the continuity equation

$$\partial_t n + \vec{\nabla}\cdot\vec{j} = 0 \quad , \tag{2.4}$$

where

$$\vec{j} = \frac{N}{2mi}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) \quad . \tag{2.5}$$

The fluid velocity $\vec{v}(\vec{r},t)$ is defined by $\vec{j}(\vec{r},t) \equiv n(\vec{r},t)\vec{v}(\vec{r},t)$. If we write $\psi(\vec{r},t) = \sqrt{n(\vec{r},t)}e^{i\beta(\vec{r},t)}$, then

$$\vec{v} = \frac{1}{m}\vec{\nabla}\beta \quad . \tag{2.6}$$

The velocity field satisfies the Euler-like equation

$$\partial_t\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}\Psi - \vec{\nabla}q \quad , \tag{2.7}$$

where

$$q = -\frac{1}{2m^2}\frac{\nabla^2\sqrt{n}}{\sqrt{n}} \quad . \tag{2.8}$$

q is commonly referred to as "quantum pressure". Eqs. (2.4) and (2.7) follow from Eq. (2.1).

We want to use Eqs. (2.1) and (2.3) to describe the evolution of density perturbations in an otherwise homogeneous Friedmann universe. The universe may be open or closed, or in between. However, because our description uses Newtonian gravity, the cosmological constant is set equal to zero.

The wavefuction describing the homogeneous universe is

$$\psi_0(\vec{r},t) = \sqrt{n_0(t)}e^{i\frac{1}{2}mH(t)r^2} \quad , \tag{2.9}$$

where $H(t)$ is the Lemaître-Hubble expansion rate. Indeed Eqs. (2.6) and (2.9) imply

$$\vec{v} = H\vec{r} \quad . \tag{2.10}$$

The imaginary part of the Schrödinger equation is satisfied provided

$$\partial_t n_0 + 3Hn_0 = 0\,, \tag{2.11}$$

and its real part is satisfied provided

$$\Psi_0 = -\frac{1}{2}(\partial_t H + H^2)r^2 \quad . \tag{2.12}$$

The Poisson equation then implies the acceleration equation

$$\partial_t H + H^2 = -\frac{4\pi G}{3} m n_0(t) \quad . \tag{2.13}$$

The continuity and acceleration equations may be combined as usual to yield the Friedmann equation

$$H(t)^2 + \frac{K}{a(t)^2} = \frac{8\pi G}{3} m n_0(t) \quad , \tag{2.14}$$

where $K = -1, 0, +1$ depending on whether the universe is open, critical or closed, and $a(t)$ is the scale factor defined by $H(t) = \frac{\dot{a}}{a}$.

We now consider perturbations about this background:

$$\psi(\vec{r}, t) = \psi_0(\vec{r}, t) + \psi_1(\vec{r}, t)\,. \tag{2.15}$$

The perturbation is Fourier transformed in terms of comoving wavevector $\vec{k}$ as follows:

$$\psi_1(\vec{r}, t) = \psi_0(\vec{r}, t) \int d^3k \;\, \psi_1(\vec{k}, t) \mathrm{e}^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}} \quad . \tag{2.16}$$

Likewise the perturbation to the gravitational potential

$$\Psi_1(\vec{r}, t) = \int d^3k \;\, \Psi_1(\vec{k}, t) \mathrm{e}^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}}\,. \tag{2.17}$$

The Schrödinger-Poisson equations expanded to linear order in the perturbations imply

$$i\partial_t \psi_1 = -\frac{1}{2m}\nabla^2\psi_1 + m(\Psi_0\psi_1 + \Psi_1\psi_0)\,, \tag{2.18}$$

and

$$\nabla^2\Psi_1 = 4\pi G m(\psi_0^*\psi_1 + \psi_0\psi_1^*)\,. \tag{2.19}$$

It is useful to introduce the functions

$$\delta(\vec{k}, t) \;\equiv\; \psi_1(\vec{k}, t) + \psi_1^*(-\vec{k}, t)\,, \tag{2.20}$$

$$\eta(\vec{k}, t) \;\equiv\; \psi_1(\vec{k}, t) - \psi_1^*(-\vec{k}, t)\,, \tag{2.21}$$

in terms of which Eqs. (2.18) and (2.19) become

$$i\partial_t\delta(\vec{k},t) \;-\; \frac{k^2}{2ma^2(t)}\eta(\vec{k},t) = 0\,, \tag{2.22}$$

$$i\partial_t\eta(\vec{k},t) \;+\; \left(\frac{8\pi G m^2 n_0(t)}{k^2}a^2(t) - \frac{k^2}{2ma^2(t)}\right)\delta(\vec{k},t) = 0\,. \tag{2.23}$$

These can be combined into one, second order differential equation for $\delta(\vec{k},t)$:

$$\partial_t^2\delta(\vec{k},t) + 2H(t)\partial_t\delta(\vec{k},t) - 4\pi G\rho\delta(\vec{k},t) + \frac{k^4}{4m^2a^4(t)}\delta(\vec{k},t) = 0 \quad , \tag{2.24}$$

where $\rho = mn_0$. The Fourier components of the perturbation to the wavefunction are given by

$$\psi_1(\vec{k},t) = \frac{1}{2}\delta(\vec{k},t) + i\frac{ma(t)^2}{k^2}\partial_t\delta(\vec{k},t) \tag{2.25}$$

in terms of the solutions to Eq. (2.24).

The perturbation to the number density is

$$n_1(\vec{r},t) = |\psi_0(\vec{r},t)|^2 \int d^3k \, \left(\psi_1(\vec{k},t) + \psi_1^*(-\vec{k},t)\right) \mathrm{e}^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}}\,, \tag{2.26}$$

and so the density contrast is

$$\delta(\vec{r},t) = \frac{n_1(\vec{r},t)}{n_0(\vec{r},t)} = \int d^3k \; \delta(\vec{k},t)\mathrm{e}^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}} \quad . \tag{2.27}$$

By expanding

$$\psi(\vec{r},t) = \sqrt{n_0(t) + n_1(\vec{r},t)}\,\mathrm{e}^{i\left(\beta_0(\vec{r},t)+\beta_1(\vec{r},t)\right)} \quad , \tag{2.28}$$

one finds that

$$\beta_1(\vec{r},t) = \frac{1}{2i}\int d^3k \; \eta(\vec{k},t)\mathrm{e}^{i\frac{\vec{k}\cdot\vec{r}}{a(t)}} \quad . \tag{2.29}$$

Hence Eq. (2.22) implies

$$\vec{v}_1(k,t) = \frac{ia(t)\vec{k}}{\vec{k}\cdot\vec{k}}\partial_t\delta(\vec{k},t) \quad , \tag{2.30}$$

which is the same relationship between the velocity perturbation and the density contrast as in the standard description of cold dark matter in terms of a pressureless fluid. Eq. (2.24) is also the standard second order differential equation governing the evolution of the density contrast, except for the last term. It arises due to quantum pressure in Eq. (2.7) and produces a Jeans length [12, 13]

$$\ell_J = (16\pi G\rho m^2)^{-\frac{1}{4}} = 1.01\cdot 10^{14}\mathrm{cm}\left(\frac{10^{-5}\mathrm{eV}}{m}\right)^{\frac{1}{2}}\left(\frac{10^{-29}\mathrm{g/cm}^3}{\rho}\right)^{\frac{1}{4}} \quad . \tag{2.31}$$

For $k > \frac{a(t)}{\ell_J}$, the Fourier components of the density perturbations oscillate in time. For $k << \frac{a(t)}{\ell_J}$, the most general solution of Eq. (2.24) is

$$\delta(\vec{k},t) = A(\vec{k}) \left(\frac{t}{t_0}\right)^{2/3} + B(\vec{k}) \left(\frac{t_0}{t}\right) \quad , \tag{2.32}$$

in the critical universe case $[a(t) \propto t^{\frac{2}{3}}]$. $A(\vec{k})$ and $B(\vec{k})$ are the amplitudes of growing and decaying modes, respectively. On distance scales much larger than the Jeans length, the wavefunction description coincides in all respects with the fluid description.

Let us mention briefly that the wavefunction can also describe rotational modes, provided vortices are introduced. See, for example ref. [20]. In a region where $\vec{\nabla} \times \vec{v} \neq 0$, the vortices have the direction of $\vec{\nabla} \times \vec{v}$ and have density (number of vortices per unit area) $\frac{m}{2\pi}|\vec{\nabla} \times \vec{v}|$. By Kelvin's theorem, the vortices must move with the fluid. Therefore, in an expanding universe, the density of vortices decreases as $a(t)^{-2}$. Hence $\vec{v} \propto a(t)^{-1}$ for rotational modes, which is again the usual result.

III. DISCUSSION

We saw in the previous section that density perturbations in the early universe may be described by a wavefunction which solves the Schrödinger-Poisson equations and that on length scales large compared to the Jeans length, Eq. (2.31), the resulting description coincides with that in terms of a pressureless fluid. It is our purpose in the present section to place this result in a wider physical context.

First let us state that, although it appears that the wavefunction description had not been explicitly given before, it is no surprise that it exists since the Schrödinger equation implies the continuity equation and the Euler-like equation (2.7). These two equations are the basic equations describing a fluid. The only difference is the quantum pressure term in Eq. (2.7) but that term is unimportant on distance scales large compared to the de Broglie wavelength. The Jeans length of Eq. (2.31) can be viewed as the de Broglie wavelength of the minimum energy state in a region of density ρ. Indeed in such a region, the gravitational potential is $\Psi = \frac{2\pi}{3} G\rho r^2$ and hence the energy of a trial wavefunction of width b is of order

$$E(b) \sim \frac{1}{2mb^2} + \frac{2\pi}{3} G\rho m b^2 \quad . \tag{3.1}$$

$E(b)$ reaches its minimum for $b \sim (\frac{4\pi}{3} G\rho m^2)^{-\frac{1}{4}} \sim \ell_J$.

However the mathematical equivalence of the two descriptions hides important physical differences. This is perhaps best illustrated by an example. Consider the wavefunction

$$\psi(\vec{r},t) = A\Big(\mathrm{e}^{i\vec{k}\cdot\vec{r}} + \mathrm{e}^{-i\vec{k}\cdot\vec{r}}\Big)\mathrm{e}^{-i\omega t}\,. \tag{3.2}$$

where A is a constant and $\omega = \frac{\vec{k}\cdot\vec{k}}{2m}$. It solves the Schrödinger equation for a free particle. The fluid with N particles in the state of wavefunction $\psi(\vec{x},t)$ has two flows, both with density $n_1 = n_2 = N|A|^2$, and with velocities $\vec{v}_1 = \frac{\vec{k}}{m}$ and $\vec{v}_2 = -\frac{\vec{k}}{m}$. On the other hand, Eqs. (2.2) and (2.6) map $\psi(\vec{r},t)$ onto a fluid whose density is $n(\vec{r}) = 4N|A|^2\cos^2(\vec{k}\cdot\vec{r})$ and whose velocity $\vec{v} = 0$. The two descriptions are mathematically equivalent in the sense that $n(\vec{r},t)$ and $\vec{v}(\vec{r},t)$ satisfy Eqs. (2.2) and (2.7) because $\psi(\vec{r},t)$ satisfies Eq. (2.1). But the two descriptions are not physically equivalent. They are physically equivalent only if we average over distances large compared to the wavelength $\frac{2\pi}{k}$ and if we ignore the velocity dispersion $\Delta v = \frac{k}{m}$. The spatial averaging is justified in the limit $k \to \infty$. Ignoring the velocity dispersion is justified in the limit, $\frac{k}{m} \to 0$. The two limits are compatible only if $m \to \infty$. This indicates that the physical differences between the wavefunction and fluid description disappear completely only in the limit where the dark matter particle is very heavy.

The fluid description never allows velocity dispersion since the velocity field $\vec{v}(\vec{r},t)$ has a single value at every point. In contrast, the wavefunction description allows velocity dispersion and multi-streaming. The wavefunction description is richer therefore. It can describe everything that a fluid describes but the reverse is not true. Whether either description is correct depends on the situation at hand.

Consider a dark matter particle with properties typical of a WIMP candidate: $m \sim 100$ GeV, density today $n_0 \sim 10^{-8}/\mathrm{cm}^3$, and primordial velocity dispersion today $\delta v_0 \sim 10^{-12}$. The de Broglie wavelength associated with the primordial velocity dispersion is of order 10^{-3} cm, much smaller than the average interparticle distance of order 5 m. The particles are highly non-degenerate therefore. Provided that their primordial velocity dispersion is in fact irrelevant to whatever phenomenon is under study (free streaming would be an exception since it is a direct result of primordial velocity dispersion), the particles can be described as a pressureless fluid. They can also be described by a wavefunction. The wavefunction description will in almost all respects be equivalent to the pressureless fluid description but, unlike the latter, it allows the inclusion of velocity dispersion and its associated effects. The wavefunction description is also applicable to the non-linear regime, after shell crossing, when the fluid description in terms of a single velocity field $\vec{v}$ breaks down. The wavefunction description is more powerful because it packs more information. The wavefunction varies on a length scale of order 10^{-3} cm in the example given. The fluid description is far coarser.

Next consider a dark matter candidate typical of axions or axion-like particles: spin zero, $m \sim 10^{-5}$ eV, density today $n_0 \sim 10^9/\text{cm}^3$, and primordial velocity dispersion today $\delta v_0 \sim 10^{-17}$. The de Broglie wavelength associated with the primordial velocity dispersion is of order 10^{18} cm. The axion fluid is highly degenerate. The average occupation number of those states that are occupied is huge, of order 10^{61}. This suggests that axion dark matter is well described by a classical scalar field $\varphi(\vec{r}, t)$. The remainder of this section considers whether this is so.

A classical scalar field satisfies the Klein-Gordon equation

$$-c^2 D^\mu \partial_\mu \varphi + \omega_0^2 \varphi + \frac{\lambda}{3!}\varphi^3 = 0 \tag{3.3}$$

where D_μ is the covariant derivative of general relativity. We allow the presence of a self-interaction $\mathcal{L}_{\varphi^4} = -\frac{\lambda}{4!}\varphi^4$ in the action density. First, let us emphasize that the classical field theory, Eq. (3.3), has no notion of axion. The axion is the quantum of the quantized scalar field which we call $\Phi(\vec{r}, t)$. There is no more notion of axion in Eq. (3.3) than there is a notion of photon in Maxwell's equations. Also there is no notion of mass since the mass m is the energy of an axion, divided by c^2. Henceforth, for the sake of clarity, we no longer set $\hbar$ and c equal to one. ω_0 in Eq. (3.3) is not the axion mass but the oscillation frequency of small perturbations in the classical scalar field in the infinite wavelength limit.

In the Newtonian limit of general relativity, the metric is $g_{00} = -c^2 - 2\Psi$, $g_{0i} = 0$, $g_{ij} = \delta_{ij}$. Eq. (3.3) becomes then:

$$(\partial_t^2 - c^2\nabla^2 + \omega_0^2)\varphi + \frac{\lambda}{3!}\varphi^3 - (\frac{2}{c^2}\Psi\partial_t^2 + \vec{\nabla}\Psi\cdot\vec{\nabla} + \frac{1}{c^2}\partial_t\Psi\partial_t)\varphi = 0 \quad . \tag{3.4}$$

We obtain the non-relativistic limit of this equation by setting

$$\varphi(\vec{r}, t) = \sqrt{2}\ \text{Re}[\mathrm{e}^{-i\omega_0 t}\phi(\vec{r}, t)] \tag{3.5}$$

and neglecting Ψ versus c^2, $\partial_t\phi$ versus $\omega_0\phi$, $\partial_t\Psi$ versus $\omega_0\Psi$, and dropping terms proportional to $\mathrm{e}^{2i\omega_0 t}$ and $\mathrm{e}^{-2i\omega_0 t}$ which indeed oscillate so fast as to effectively average to zero. Eq. (3.4) becomes then

$$i\partial_t\phi = -\frac{c^2}{2\omega_0}\nabla^2\phi + \frac{\lambda}{8\omega_0}|\phi|^2\phi + \frac{\omega_0}{c^2}\Psi\phi \quad . \tag{3.6}$$

To obtain the Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r},t)\psi \quad , \tag{3.7}$$

substitute

$$\phi(\vec{r},t) = \sqrt{\frac{\hbar}{\omega_0}}\psi(\vec{r},t) \tag{3.8}$$

in Eq. (3.6) and set $m = \frac{\hbar\omega_0}{c^2}$. The potential energy in Eq. (3.7) is given, in the sense of mean field theory, by

$$V(\vec{r},t) = m\Psi(\vec{r},t) + \frac{\hbar^4\lambda}{8m^2c^4}|\psi(\vec{r},t)|^2 \quad . \tag{3.9}$$

The Newtonian limit of Einstein's equation is the Poisson equation, Eq. (2.3). The non-linear version of Schrödinger's equation obtained by substituting Eq. (3.9) into Eq. (3.7) is commonly called the Gross-Pitaevskii equation.

So the Schrödinger equation describes the dynamics of a classical scalar field in the non-relativistic limit. This result is not new of course. We reproduced it here to prepare the ground for the actual question we want to discuss, namely whether dark matter axions (or axion-like particles) are described by the Schrödinger-Poisson equations. Clearly, if axions are described by a classical scalar field, the answer is yes as we have just seen. But the axion is a quantum field. It may behave like a classical field some of the time or perhaps even all the time, but this is something that has to be proved. It cannot be merely assumed.

Inside a cubic box of volume $V = L^3$ with periodic boundary conditions, the quantum axion field may be expanded (see for example Ref. [18])

$$\Phi(\vec{r},t) = \sum_{\vec{n}} \sqrt{\frac{\hbar}{2\omega_{\vec{n}}V}}[a_{\vec{n}}(t)\mathrm{e}^{\frac{i}{\hbar}\vec{p}_{\vec{n}}\cdot\vec{r}} + a^\dagger_{\vec{n}}(t)\mathrm{e}^{-\frac{i}{\hbar}\vec{p}_{\vec{n}}\cdot\vec{r}}] \quad , \tag{3.10}$$

where $\vec{n} = (n_1, n_2, n_3)$ with n_k $(k = 1, 2, 3)$ integers, $\vec{p}_{\vec{n}} = \frac{2\pi\hbar}{L}\vec{n}$, $\omega = \frac{c}{\hbar}\sqrt{\vec{p}\cdot\vec{p} + c^2m^2}$. The $a_{\vec{n}}$ and $a^\dagger_{\vec{n}}$ are annihilation and creation operators satisfying canonical equal-time commutation relations:

$$[a_{\vec{n}}(t), a^\dagger_{\vec{n}'}(t)] = \delta_{\vec{n},\vec{n}'} \quad , \quad [a_{\vec{n}}(t), a_{\vec{n}'}(t)] = 0 \quad . \tag{3.11}$$

The classical field limit is the limit where $\hbar \to 0$ with $\hbar\mathcal{N}$ held fixed, where $\mathcal{N}$ is the quantum occupation number of the state described by a particular solution of the classical field equations. The Hamiltonian for the quantum field $\Phi(\vec{r},t)$ which satisfies Eqs. (3.6) and (2.3) in the classical field limit is [18]

$$H = \sum_{\vec{n}} \hbar\omega_{\vec{n}}\ a^\dagger_{\vec{n}}a_{\vec{n}} + \sum_{\vec{n}_1,\vec{n}_2,\vec{n}_3,\vec{n}_4} \frac{1}{4}\ \hbar\Lambda^{\vec{n}_3,\vec{n}_4}_{\vec{n}_1,\vec{n}_2}\ a^\dagger_{\vec{n}_1}a^\dagger_{\vec{n}_2}a_{\vec{n}_3}a_{\vec{n}_4} \quad , \tag{3.12}$$

where $\Lambda_{\vec{n}_1,\vec{n}_2}^{\vec{n}_3,\vec{n}_4}$ is the sum of two terms:

$$\Lambda_{\vec{n}_1,\vec{n}_2}^{\vec{n}_3,\vec{n}_4} = \Lambda_{s\ \vec{n}_1,\vec{n}_2}^{\ \vec{n}_3,\vec{n}_4} \quad + \quad \Lambda_{g\ \vec{n}_1,\vec{n}_2}^{\ \vec{n}_3,\vec{n}_4} \quad . \tag{3.13}$$

The first term

$$\Lambda_{s\ \vec{n}_1,\vec{n}_2}^{\ \vec{n}_3,\vec{n}_4} = +\frac{\lambda\hbar^3}{4m^2c^4V}\ \delta_{\vec{n}_1+\vec{n}_2,\vec{n}_3+\vec{n}_4} \tag{3.14}$$

is due to the $\lambda\Phi^4$ type self-interactions. The second term

$$\Lambda_{g\ \vec{n}_1,\vec{n}_2}^{\ \vec{n}_3,\vec{n}_4} = -\frac{4\pi Gm^2\hbar}{V}\delta_{\vec{n}_1+\vec{n}_2,\vec{n}_3+\vec{n}_4}\ \left(\frac{1}{|\vec{p}_{\vec{n}_1}-\vec{p}_{\vec{n}_3}|^2}+\frac{1}{|\vec{p}_{\vec{n}_1}-\vec{p}_{\vec{n}_4}|^2}\right) \tag{3.15}$$

is due to the gravitational self-interactions. The Heisenberg equations of motion are

$$i\dot{a}_{\vec{n}_1} = -\frac{1}{\hbar}[H,a_{\vec{n}_1}] = \omega_{\vec{n}_1}a_{\vec{n}_1} + \frac{1}{2}\sum_{\vec{n}_2,\vec{n}_3,\vec{n}_4}\Lambda_{\vec{n}_1,\vec{n}_2}^{\vec{n}_3,\vec{n}_4}a_{\vec{n}_2}^\dagger a_{\vec{n}_3}a_{\vec{n}_4} \quad . \tag{3.16}$$

We may likewise expand the classical field

$$\varphi(\vec{r},t) = \sum_{\vec{n}}\sqrt{\frac{\hbar}{2\omega_{\vec{n}}V}}[A_{\vec{n}}(t)\mathrm{e}^{\frac{i}{\hbar}\vec{p}_{\vec{n}}\cdot\vec{r}} + A_{\vec{n}}^*(t)\mathrm{e}^{-\frac{i}{\hbar}\vec{p}_{\vec{n}}\cdot\vec{r}}] \quad . \tag{3.17}$$

The Fourier components $A_{\vec{n}}(t)$ satisfy

$$i\dot{A}_{\vec{n}_1} = \omega_{\vec{n}_1}A_{\vec{n}_1} + \frac{1}{2}\sum_{\vec{n}_2,\vec{n}_3,\vec{n}_4}\Lambda_{\vec{n}_1,\vec{n}_2}^{\vec{n}_3,\vec{n}_4}A_{\vec{n}_2}^*A_{\vec{n}_3}A_{\vec{n}_4} \quad . \tag{3.18}$$

Eqs. (3.16) and (3.18) look similar but, as we will see, their physical implications are different because the $a_{\vec{n}}(t)$ are operators whereas the $A_{\vec{n}}(t)$ are c-numbers.

Let us define the operator

$$\mathcal{N}_{\vec{n}}(t) = a_{\vec{n}}^\dagger(t)\ a_{\vec{n}}(t) \quad , \tag{3.19}$$

i.e. the occupation number at time t of the state labeled $\vec{n}$. It was shown in Ref. [18] that the Hamiltonian of Eq. (3.12) implies the following operator evolution equation

$$\dot{\mathcal{N}}_l = \sum_{k,i,j=1}\frac{1}{2}|\Lambda_{ij}^{kl}|^2\,[\mathcal{N}_i\mathcal{N}_j(\mathcal{N}_l+1)(\mathcal{N}_k+1) - \mathcal{N}_l\mathcal{N}_k(\mathcal{N}_i+1)(\mathcal{N}_j+1)]\,2\pi\delta(\omega_i+\omega_j-\omega_k-\omega_l) \quad . \tag{3.20}$$

To remove unnecessary clutter, we replaced $\vec{n}_j$ by j. The derivation of Eq. (3.20) assumes only that the energy dispersion $\delta\omega$ of the highly occupied states is much larger than the relaxation rate $\Gamma = \frac{1}{\tau}$. τ is the relaxation time, defined as the time scale over which the distribution $\{\mathcal{N}_j\}$ changes completely. When $\delta\omega >> \Gamma$, the system is said to be in the "particle kinetic" regime. The same derivation that yields Eq. (3.20) but applied to the classical counterparts

$$N_{\vec{n}}(t) = A^*_{\vec{n}}(t) A_{\vec{n}}(t) \tag{3.21}$$

yields

$$\dot{N}_l = \sum_{k,i,j=1} \frac{1}{2} |\Lambda^{kl}_{ij}|^2 \left[N_i N_j N_l + N_i N_j N_k - N_l N_k N_i - N_k N_l N_j \right] 2\pi \delta(\omega_i + \omega_j - \omega_k - \omega_l) \quad , \tag{3.22}$$

again in the particle kinetic regime. Eqs. (3.20) and (3.22) are clearly different, and they imply different outcomes.

Consider the process $i + j \to k + l$ where two quanta, initially in states i and j move to states k and l. Assuming $\Lambda^{ij}_{kl} \neq 0$, this process always occurs in the quantum theory when the initial states are occupied. In the classical theory, the corresponding process occurs only if, in addition, at least one of the final states is occupied. In particular the scattering of two waves does not happen in the classical theory of Eq. (3.6) if the only waves present are the two waves in the initial state. The quantum theory behaves differently because the final state oscillators have zero point oscillations. Incidentally, this observation shows that the oft repeated statement that the quantum and classical theories differ only by loop effects is incorrect.

After a sufficiently long time, the classical and quantum systems thermalize and reach an equilibrium distribution. The time scale of thermalization of the classical system is of the same order of magnitude as that of the quantum system [18] but the outcomes of thermalization are different. In the quantum case, Eq. (3.20) implies that the equilibrium distribution $\{\mathcal{N}_j\}$ is such that

$$\mathcal{N}_i \mathcal{N}_j (\mathcal{N}_l + 1)(\mathcal{N}_k + 1) - \mathcal{N}_l \mathcal{N}_k (\mathcal{N}_i + 1)(\mathcal{N}_j + 1) = 0 \tag{3.23}$$

for every quartet of states such that $\omega_i + \omega_j = \omega_k + \omega_l$. Let us call $\epsilon = \hbar\omega$, and rewrite Eq. (3.23) as

$$(1 + \frac{1}{\mathcal{N}_i})(1 + \frac{1}{\mathcal{N}_j}) = (1 + \frac{1}{\mathcal{N}_k})(1 + \frac{1}{\mathcal{N}_l}) \tag{3.24}$$

whenever $\epsilon_i + \epsilon_j = \epsilon_k + \epsilon_l$. Eq. (3.24) is solved by

$$\epsilon_i = C \ln \left[1 + \frac{1}{\mathcal{N}_i} \right] \tag{3.25}$$

where C is a constant. Upon identifying $C = k_B T$, this is seen to be the Bose-Einstein distribution

$$\mathcal{N}_i = \frac{1}{e^{\frac{\epsilon_i}{k_B T}} - 1} \quad . \tag{3.26}$$

On the other hand Eq. (3.22) implies that the equilibrium distribution $\{N_j\}$ for the classical case satisfies

$$(N_i + N_j)N_k N_l = N_i N_j (N_k + N_l) \tag{3.27}$$

whenever $\epsilon_i + \epsilon_j = \epsilon_k + \epsilon_l$. Eq. (3.27), which may be rewritten as

$$\frac{1}{N_i} + \frac{1}{N_j} = \frac{1}{N_k} + \frac{1}{N_l} \quad , \tag{3.28}$$

is solved by

$$\mathcal{N}_i = C\frac{1}{\epsilon_i} \quad . \tag{3.29}$$

Upon identifying $C = k_B T$, we have

$$N_i \epsilon_i = k_B T \quad , \tag{3.30}$$

which is indeed the standard result for classical oscillators at temperature T: each oscillator has energy $k_B T$ on average.

We conclude that axion dark matter is not described by a classical field when it thermalizes. Interactions are seen to have a dual role. They determine the behaviour of the classical field as described by Eq. (3.6), or Eqs. (3.18) and (3.22) which follow directly from Eq. (3.6). Eq. (3.6) has a set of solutions which we may label $\phi_{\vec{\alpha}}(\vec{r}, t)$. The solutions describe the states that the axions may occupy in the quantum theory. In the quantum theory, however, the interactions have the additional role of causing transitions between the various states $\phi_{\vec{\alpha}}(\vec{r}, t)$. Indeed if there were no such transitions the outcomes of thermalization in the quantum and classical theories would be the same. We just saw that they are not.

Let τ be the time scale over which the distribution $\{\mathcal{N}_{\vec{\alpha}}\}$ of the axions, over the states described by the classical solutions $\phi_{\vec{\alpha}}(\vec{r}, t)$, changes completely. We call τ the *relaxation* or *thermalization* time scale. Note that full thermalization only happens generally on a time scale much longer than τ. (The time scale for "full" thermalization depends on the degree of thermalization required and is therefore less robustly defined than τ.) On time scales short compared to τ, axion dark matter behaves as a classical field because only relatively few transitions take place between the states described by the classical solutions $\phi_{\vec{\alpha}}(\vec{r}, t)$.

On time scales long compared to τ, the axions are not described by a classical field because their distribution over those states changes completely. On time scales long compared to τ dark matter axions form a Bose-Einstein condensate (BEC) since they are highly degenerate and their number is conserved. The time scale for BEC formation is the relaxation time τ [18, 23–25]. Axion BEC means that almost all axions go the lowest energy state available. The question is then: does τ ever become shorter than the age of the universe t at that moment? It was found in Ref.[17, 18] that $\tau \sim t$ during the QCD phase transition at a temperature of order 1 GeV when cold dark matter axions are first produced. The axions thermalize briefly then as a result of their $\lambda\Phi^4$ interactions. Here and elsewhere, by the word 'thermalize', we mean that the axion distribution relaxes and begins to approach a thermal distribution. We do not mean that they thermalize fully. This brief period of thermalization has no known observational consequences. However, when the photon temperature reaches of order 500 eV, cold dark matter axions thermalize anew as a result of their gravitational self-interactions and this does have observational consequences, as was already mentioned in the Introduction.

IV. SUMMARY

We derived solutions of the coupled Schrödinger and Poisson equations. The solutions describe the homogeneous expanding matter-dominated universe and density perturbations therein. The description is identical to that obtained by treating the dark matter as a pressureless fluid, on all scales much larger than the de Broglie wavelength of the wavefunction. In a number of respects, the wavefunction description is simpler and hence superior. It has fewer degrees of freedom since a wavefunction is two real fields whereas a fluid is described by four real fields, the density and the three components of velocity. The mathematics is simpler as well. Even though it has only two real fields, the wavefunction can describe rotational modes. On the other hand, the meaning of the wavefunction is less intuitively obvious.

We considered whether the wavefunction and fluid descriptions are equivalent in general. They are equivalent in the sense that the density and velocity fields, given by Eqs. (2.2) and (2.6), satisfy the fluid equations if the wavefunction satisfies the Schrödinger equation. However, the wavefunction and the fluid describe objects which in general are not physically the same. In particular the wavefunction description allows velocity dispersion and multi-streaming whereas the fluid description does not. The physical distinction between the two descriptions disappears completely only in the limit where the mass of the dark matter particle goes to infinity. Because WIMPs are relatively heavy, the wavefunction and fluid descriptions of WIMP dark matter are equivalent whenever velocity dispersion does not play a role. The wavefunction description has the advantage that it can describe phenomena associated with velocity dispersion, such as free-streaming, and that it can be used not only in the linear regime but also in the non-linear regime, after shell crossing.

We asked whether the Schrödinger-Poisson equations correctly describe axion dark matter. The answer is *yes* on time scales short compared to the relaxation time scale τ, and *no* on time scales long compared to τ. Whenever τ is shorter than the age of the universe t, axion dark matter is not correctly described by the Poisson-Schrödinger equations. Indeed axions move towards a Bose-Einstein distribution on the time scale τ whereas the Schrödinger-Poisson equations would predict that they move towards a Boltzmann distribution. Interactions, such as gravity or $\lambda\Phi^4$ interactions, play a dual role. On the one hand the interaction influences the evolution of the classical field. The solutions of the classical field equation, which is equivalent to the Schrödinger equation in the non-relativistic limit, describe the quantum states that the axions may occupy. But the interaction has the additional role of causing the axions to jump between those states. On time scales large compared to τ, the distribution of quanta over the states described by the classical field changes completely. It was shown in Refs.[17, 18] that, as a result of axion gravitational self-interactions, τ becomes and remains shorter than the age of the universe after the photon temperature has reached approximately 500 eV. Therefore the commonly made assumption that axion dark matter is adequately described by classical field equations at all times is incorrect.

In summary then, we have obtained the solutions of the Schrödinger-Poisson equations that describe the homogeneous expanding Friedmann matter-dominated universe and density perturbations therein. Each solution corresponds to a possible quantum-mechanical state of dark matter axions. However the gravitational and other self-interactions of the axions cause them to jump between those states. The resulting dynamical evolution and observational implications of axion dark matter is the object of future work [26].

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