Dark energy from α-attractors

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Dark Energy from α-Attractors

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A class of inflation theories called α-attractors has been investigated recently with interesting properties interpolating between quadratic potentials, the Starobinsky model, and an attractor limit. Here we examine their use for late time cosmic acceleration. We generalize the class and demonstrate how it can interpolate between thawing and freezing dark energy, and reduce the fine tuning of initial conditions, allowing $w \approx -1$ for a prolonged period or as a de Sitter attractor.

I. INTRODUCTION

Scalar fields play a fundamental role in cosmology, with the Higgs field in the standard model of particle physics, the inflaton seeding early universe quantum perturbations that develop into cosmic structure and a primordial gravitational wave background, and possibly a dark energy field that is responsible for late time cosmic acceleration. One of the exciting recent developments in inflation is the discovery of classes of theories that connect different scalar field theories. Here we examine aspects of this in the context of late time dark energy.

The inflationary developments involve conformal ξ-attractors and Kähler curvature α-attractors. These have the property of interpolating between power law scalar field potentials or, e.g., Higgs inflation, Starobinsky inflation, and a fixed point [1–3]. Thus such theories make definite predictions for the primordial scalar perturbation tilt n_s and the tensor (gravitational wave) to scalar power ratio r, both measurable by cosmic microwave background experiments, with all models approaching a particular limit.

We focus here on the minimally coupled α-attractors and investigate the dark energy properties of such models. Several of the forms within this class have high energy physics motivations, from supergravity and other theories. It would be interesting to evaluate whether these models have characteristics useful for late time acceleration, and possibly an improvement over standard quintessence potentials.

In Sec. II we lay out the basic of the models and generalize them to a family of models. Section III addresses the attractor behavior and presents numerical results for the dark energy equation of state evolution. We compare generalized α-models to standard quintessence in Sec. IV and discuss their advantages, summarizing and mentioning future work in Sec. V.

II. α-MODELS

The α-model as written for inflation has a non-canonical kinetic term and a potential, with Lagrangian density [3]

$$\mathcal{L} = \sqrt{-g}\,\left[\frac{1}{2}M_P^2 R - \frac{\alpha}{(1-\varphi^2/6)^2}\,\frac{1}{2}(\partial\varphi)^2 - \alpha f^2\left(\frac{\varphi}{\sqrt{6}}\right)\right] \ , \tag{1}$$

where M_P is the Planck mass, α is a parameter and αf^2 is the potential function. The Starobinsky model corresponds to $\alpha = 1$ and a particular form for f.

Through a field redefinition

$$\phi = \sqrt{6\alpha}\,\tanh^{-1}(\varphi/\sqrt{6}) \ , \tag{2}$$

the kinetic term becomes canonical and the potential function

$$V(\phi) = \alpha f^2\left(\tanh\frac{\phi}{\sqrt{6\alpha}}\right) \ . \tag{3}$$

Note that the function f is not wholly arbitrary since the field redefinition breaks down at $\phi = \infty$.

For example, if we tried to make a standard quintessence model be described by an α-model, this is not generally possible. Consider dark energy with a constant equation of state (pressure to energy density) ratio w. This is given by a potential [4]

$$V \sim \sinh^{-2(1+w)/|w|}(\phi - \phi_\star) \qquad [\text{constant } w] \ . \tag{4}$$

and is accommodated by a function

$$f(x) \sim \left(\frac{x}{\sqrt{1-x^2}}\right)^{-(1+w)/|w|} \ , \tag{5}$$

where $x = \tanh(\phi/\sqrt{6\alpha})$. However this function vanishes at $\phi = \infty$ ($x = 1$), and furthermore blows up at $\phi = 0$.

For inflation, two functional forms have been used, the T-model

$$f(x) = cx \ , \tag{6}$$

and the Starobinsky form (with the Starobinsky model having $\alpha = 1$)

$$f(x) = c\,\frac{x}{1+x} \ . \tag{7}$$

Note that they are equivalent at small x, and because $x \propto \phi$ at small ϕ then near the origin both models give quadratic potentials. If we were to take $f(x) \propto x^{p/2}$ for $x \ll 1$ then we could match onto any monomial potential ϕ^p (see, e.g., [3]). However here we will instead generalize to interpolating and extrapolating these two models, to study the deviations from quadratic behavior (but see Sec. IV). Our generalized α-model takes

$$f(x) = c\,\frac{x}{(1+x)^n}\ . \tag{8}$$

The quantity c scales the amplitude of the potential, and can be fixed by requiring a certain dark energy density today; that is, c is effectively equivalent to $\Omega_{\phi,0}$, the present dark energy density in units of the critical density. The parameter α scales the field value ϕ.

As $x \to 1$, its maximum value, the potential goes to a constant. Basically it plateaus at a level $V_p = \alpha c^2\, 2^{-2n}$. Thus the generalized α-model potential looks like a quadratic potential at small ϕ, out to a width $\phi \approx \sqrt{\alpha}$, and then flattens to a plateau at large ϕ. Figure 1 shows this potential for values of $n=$ 0-3, and the comparison to a standard quadratic potential at small ϕ. (The potential is symmetric about $x=0$ but we show only the positive half since the dynamics is symmetric.)

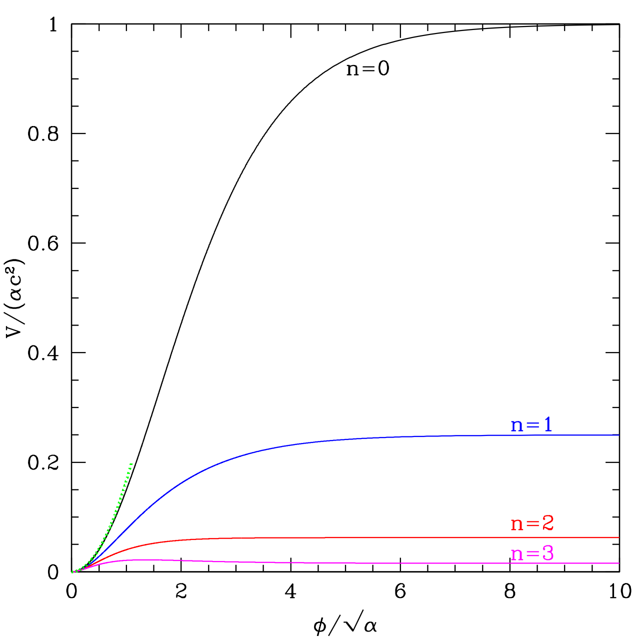

FIG. 1. The potential is plotted as a function of the field ϕ in units of $\sqrt{\alpha}$, for various values n. At small field values $\phi \ll \sqrt{\alpha}$ the α-model resembles the quadratic potential $V \propto \phi^2$ (short, dotted green curve), while at large values it flattens to an uplifted plateau.

III. EVOLUTION AND ATTRACTORS

In the α-attractor family of inflation theories, the word "attractor" refers to their common values for the scalar perturbation power spectrum tilt n_s and tensor to scalar power ratio r in the limit $\alpha \to 0$, while for large α the models resemble monomial potentials ϕ^p [2]. Here we are interested in the dynamics of the scalar field, and its equation of state evolution, in particular whether it gives rise to late time cosmic acceleration.

From Fig. 1 we can already expect the answer: if the field rolls into the quadratic region of the potential, then it should act like dark energy from a quadratic potential at late times, while if it remains on the plateau then it should act like a cosmological constant. Looking at this a bit more carefully, we see that the potential for large ϕ is

$$V(\phi \gg \sqrt{\alpha}) \approx \alpha c^2\, 2^{-2n}\left[1 - 2(2-n)\, e^{-2\phi/\sqrt{6\alpha}}\right]\ . \tag{9}$$

This is basically an uplifted exponential potential. The exponential potential was one of the original dark energy models [5], and the uplifted exponential was studied even earlier as one of the first inflation models [6, 7]. Moreover, the quadratic potential is in the thawing class of dark energy, moving away from a cosmological constant state, while the exponential potential is in the freezing class, moving toward a cosmological constant. (See [8, 9] for discussion of these two main classes of dark energy.)

Thus these α-models appear to conjoin in a way these two classes. Furthermore, because the transition depends on the value of α, we see that large α moves the plateau (and hence freezing behavior) to larger and larger ϕ, essentially shrinking the plateau to a point maximum and making the potential look like a hilltop potential [10].

In fact, while these behaviors effectively hold, there is a formal de Sitter fixed point at $w=-1$ only for a particular range of n (though for other n the field has nearly this behavior, for tens of e-folds or more, as we discuss below). Therefore we use the terminology α-model rather than α-attractor, in the dynamical sense. However, in the inflationary sense of a common limit for large α models, and for small α models, there are attractors as we show in Sec. IV.

We numerically solved the scalar field equations of motion and Friedmann equations for the background expansion to verify these behaviors. Indeed, for initial field values $\phi_i \ll \sqrt{\alpha}$ the field gradually thaws and rolls down the potential, growing from equation of state ratio $w=-1$ to less negative values. It eventually (perhaps in the far future) oscillates around the quadratic minimum, giving a time averaged equation of state ratio $\langle w\rangle = 0$. However, it can produce a long (if temporary) period of cosmic acceleration. Figure 2 shows the present equation of state ratio w_0 as a function of ϕ_i for various values of n.

For $\phi \gg \sqrt{\alpha}$, one might suspect that since the plateau is not perfectly flat, but tilted upward, that the field could slide down into the minimum. However, since the field kinetic energy is rapidly damped away in an accelerating universe, this is essentially negligible. Indeed, traditional skating fields [9] where $V=0$ have kinetic energy vanishing as a^{-6}, where a is the cosmic expansion factor; fields on the plateau could be called elevated

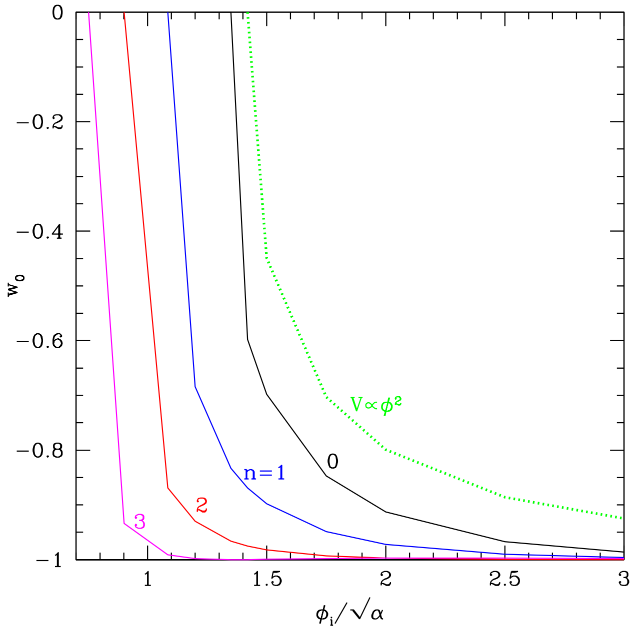

FIG. 2. The present value of the equation of state ratio w_0 is plotted as a function of the initial field value ϕ_i in units of $\sqrt{\alpha}$, for various values of n (solid curves). For contrast, this relation is also shown for a quadratic potential (dotted green curve, where the x-axis is simply ϕ_i). For a given value ϕ_i, the α-model can achieve w_0 closer to -1 (as preferred by observations). Conversely for a given bound on the deviation of w_0 from -1, the α-model can be less fine tuned in the initial field value.

skaters. For example, for $n = 0$, where Fig. 1 shows the plateau begins at $\phi > 5\sqrt{\alpha}$, the equation of state evolves from $w_0 = -0.9995$ today to only $w(a = 7.5) = -0.9988$ for $\phi_i = 5\sqrt{\alpha}$. If $\phi_i = 20\sqrt{\alpha}$, then the field rolls only $\Delta\phi/M_P = 6 \times 10^{-7}$.

(One might imagine stranger situations, in which the field starts near the minimum but with initial velocity away from it high enough to reach the plateau, or starting on the plateau and shooting through the minimum to reach the $x < 0$ plateau. However, these are basically moot because the rapid Hubble expansion at early times redshifts away any large kinetic energy, requiring extreme fine tuning for these situations to occur.)

Our generalization shows further interesting properties. Figure 1 illustrates that the plateau begins at smaller ϕ as n increases. Indeed, for $n = 2$ and $\phi_i = 3\sqrt{\alpha}$, the equation of state evolves from $w_0 = -0.9999$ to $w(a = 7.5) = -0.9998$. Notably, for $n > 2$ we have a true attractor since the potential has a maximum on the plateau (at $x = 1/(n-1)$), and slopes down, rather than up, to its asymptotic value as seen from Eq. (9). For example, for $n = 3$ and $\phi_i > 1.35\sqrt{\alpha}$ the asymptotic future state is de Sitter, $w = -1$.

The potential slope $V' = dV/d\phi$ is given by

$$V' = \sqrt{\frac{2\alpha}{3}}\, c^2\, \frac{x(1-x)[1+(1-n)x]}{(1+x)^{2n}} \ . \qquad (10)$$

Figure 3 shows the slope for various values of n. We see that the slope rapidly approaches zero for large ϕ, making the plateau quite flat. Recall from the Klein-Gordon equation of motion for the scalar field that once the slope is negligible then the field quickly freezes in place. For example, for $n = 1$ and $\phi_i = 5\sqrt{\alpha}$, the field rolls less than $\Delta\phi/M_P = 0.06$ in its entire history. Thus even when $n \leq 2$ and there is no formal de Sitter attractor, one can still have $w \approx -1$ for a long time.

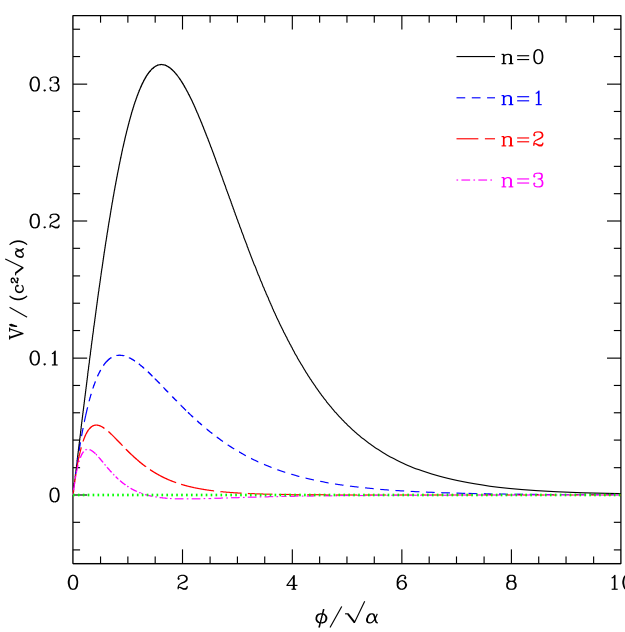

FIG. 3. The first derivative of the potential is plotted as a function of the field ϕ in units of $\sqrt{\alpha}$, for various values n (increasing from top to bottom curves). Note that all derivatives approach zero at large field values (since the potential plateau is asymptotically flat).

IV. RELATION TO QUINTESSENCE MODELS

At small ϕ, the generalized α-model reduces to the quadratic potential,

$$V(\phi \ll \sqrt{\alpha}) \approx \frac{c^2}{6\alpha}\,\phi^2 \ . \qquad (11)$$

That is a thawing dark energy model, and to keep the present equation of state ratio w_0 near enough to -1 to be compatible with observational constraints, the initial position of the field requires some fine tuning. In particular, starting the field too close to the minimum means that it will never achieve a present dark energy density in units of the critical density $\Omega_{\phi,0} = 0.7$. For example, in order to reach this requires $\phi_i > 1.42\, M_P$, while to also keep $w_0 < -0.9$ requires $\phi_i > 2.65\, M_P$.

Now we consider this situation for the α-model. From Fig. 2 we see the fine tuning can be ameliorated considerably. For $n = 0$ ($n = 3$) the minimum field value to achieve $\Omega_{\phi,0} = 0.7$ and $w_0 < -0.9$ is $\phi_i > 1.95\, M_P$ ($0.9\, M_P$) for $\alpha = 1$. The conditions to reach w_0 closer to -1 are even more softened, since once the α-model field is on the potential plateau then w can easily be extremely close to -1. Again, the plateau effectively stretches out hilltop potentials, such as from pseudo-Nambu goldstone boson or axion fields, or Higgs symmetry breaking, vastly increasing the initial conditions delivering $w \approx -1$.

We can now further generalize the α-model by taking

$$V(x) = \alpha c^2\, \frac{x^p}{(1+x)^{2n}} \ , \tag{12}$$

(where before we restricted to $p = 2$). This now acts at small ϕ like any chosen monomial potential ϕ^p, while retaining the plateau at larger field values. True attractors to $w = -1$ can now be achieved when the field is at values $x > p/(2n - p)$, requiring $n > p$. For example, rather than matching the quadratic potential near the minimum we can use the axion monodromy potential $\phi^{2/3}$, i.e. $p = 2/3$. With $n = 1$, say, this has a late time de Sitter attractor for $x > 0.5$ or $\phi_i > 1.35\,\sqrt{\alpha}$.

Let us explore the attractor behavior in the inflationary sense of an α-attractor, where there is a common behavior for observables for some limit of α. When $\alpha \gg 1$, all models regardless of n will look like the $V \propto \phi^p$ case, e.g. a quadratic potential for our baseline of $p = 2$. Conversely, when $\alpha \to 0$, all models will see a plateau potential. In particular, when $n > p$ and the field is beyond the potential maximum at $x = p/(2n - p)$, then all models are attracted to the de Sitter state.

Figure 4 shows the dynamics in the $w - w'$ phase space, where $w' = dw/d\ln a$, for three values of p, fixing $n = 3$ and taking $\phi_i = 1.5\,\sqrt{\alpha}$. For $\alpha = 1$, the dynamics are well separated, though with common general characteristics. Note that they all start off as thawing dark energy, evolving along the canonical thawing behavior of $w' = 3(1+w)$, and even at present (denoted by x's) they lie in the thawing region, at roughly $w' = 1+w$. However in the future they turn around and convert to freezing behavior, heading along an attractor toward the de Sitter point $w = -1$, $w' = 0$. For small α, these characteristics persist but curves for all p stay much closer to the ultimate cosmological constant behavior.

Returning to the baseline ($p = 2$) case, if we expand the α-model potential for small $\phi/\sqrt{\alpha}$, we see

$$V(\phi \ll \sqrt{\alpha}) \approx \frac{c^2}{6\alpha}\,\phi^2 \left(1 - \frac{2n}{\sqrt{6\alpha}}\,\phi + \mathcal{O}(\phi^2/\alpha)\right) \ . \tag{13}$$

So there is an attractor behavior for large α where for any n the potential acts like a simple quadratic potential. Figure 5 illustrates this behavior. (An analogous result will hold for any p.)

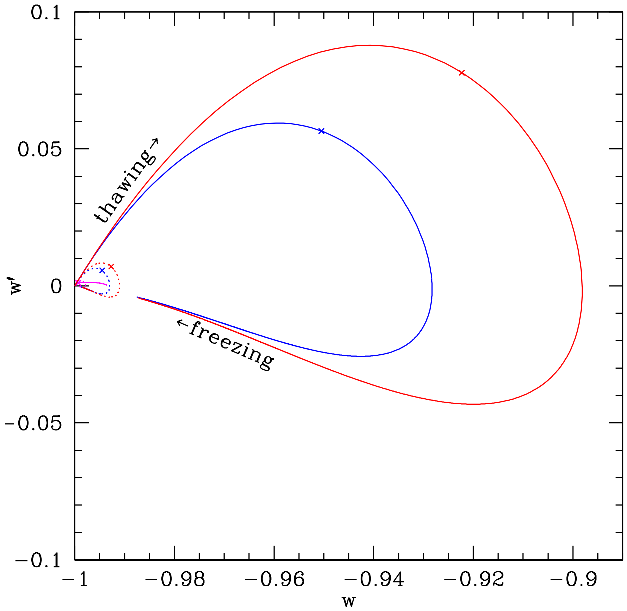

FIG. 4. The dynamics in the $w - w'$ phase space transitions between thawing and freezing dark energy, with an attractor behavior both toward a cosmological constant and toward common dynamics as $\alpha \to 0$. Models with $p = 2/3$, 1, and 2 (red, blue, magenta, or from outer to inner) are shown, for $\alpha = 1$ (solid curves) and $\alpha = 0.1$ (dotted curves). The present is denoted by x's and the curves are stopped at $a = 220$ for clarity. Note the $p = 2$, $\alpha = 0.1$ case has not quite turned around since it is very close to the potential maximum.

As an interesting additional point, Eq. (13) shows that away from the minimum the potential becomes anharmonic. That is, the potential slope becomes shallower, just as for hilltop potentials. We know these have negative mass squared terms near the maximum (also see [11–13] for other anharmonic instabilities), and we find this as well for the α-model. Specifically,

$$\begin{aligned} m^2 = V'' = \frac{c^2}{3}\,\frac{1-x}{(1+x)^{2n}}\, &[1 + x(1-4n) \\ &-x^2(3-n-2n^2) - x^3(3-5n+2n^2)] \ . \end{aligned} \tag{14}$$

Note that the mass vanishes for $\phi \gg 1$, as do all the derivatives of the potential; the plateau becomes flat. Figure 6 exhibits the mass squared as a function of ϕ for various n.

For hilltop potentials the spinodal, or tachyonic, instability when the mass squared is negative is not a major concern [14–16], but for α-models it does exist for a wider range of field values. This raises the possibility of interesting phenomenology, such as clustering dark energy. Here we consider only the background, zero momentum modes, but will investigate this further in future work.

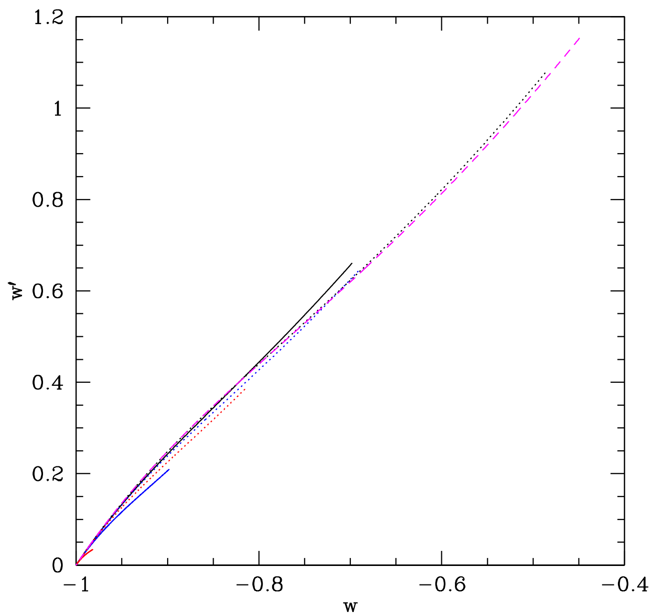

FIG. 5. The dynamics in the $w - w'$ phase space increasingly resembles a quadratic potential (dashed, long magenta curve) as $\alpha \to \infty$. Models with $n = 0$, 1, and 2 (black, blue, red, or from longest to shortest curves for each line type) are shown, for $\alpha = 1$ (solid curves) and $\alpha = 10$ (dotted curves). For $\alpha = 10$, the curves, and the endpoints at $a = 1$, move noticeably closer to the quadratic result.

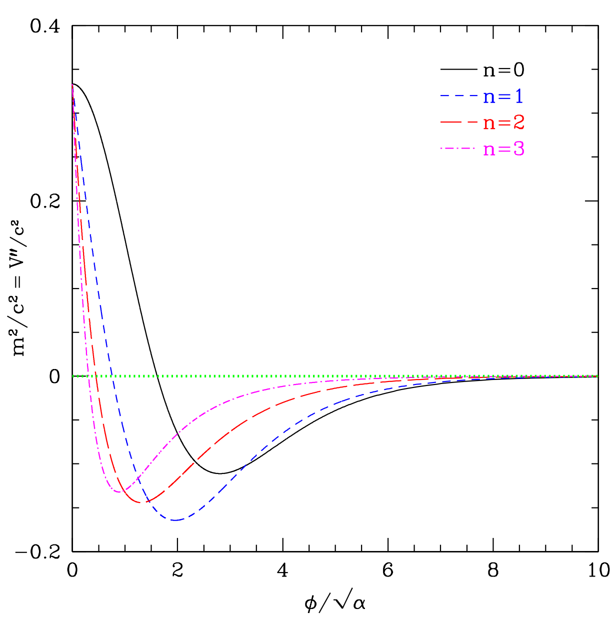

FIG. 6. The second derivative of the potential, giving the effective mass squared, is plotted as a function of the field ϕ in units of $\sqrt{\alpha}$, for various values n (increasing from top to bottom curves at small ϕ). Note that all derivatives approach zero at large field values (since the potential plateau is asymptotically flat).

V. CONCLUSIONS

We investigated the properties of α-attractors, as recently highlighted for inflation, as dark energy for late time acceleration instead. Such models have several interesting properties, combining aspects of the thawing and freezing classes of dark energy together. By generalizing the usual α-attractors we derive models that have a true de Sitter attractor, as well as ones that have a metastable acceleration with equation of state ratio lingering near $w \approx -1$.

The dynamics near the potential minimum, where the α-model looks like a quadratic potential, is found to be less fine tuned that a standard quadratic potential, especially for values of w consistent with observations. Further from the minimum, the model resembles a stretched hilltop model, with elements of an exponential potential as well.

A family of theories can be defined as one varies the value of α. As $\alpha \to 0$, the field sees predominantly a plateau, with a slight slope up or down, depending on the value of the generalized parameter n. This determines whether there is a true de Sitter attractor or not, but either way $w \approx -1$ for many e-folds of expansion. For $\alpha \gg 1$, the potential increasingly resembles a quadratic potential, or any monomial ϕ^p in a further generalization.

We note the α-model has anharmonic properties, and will have negative effective mass squared in some regions. Either of these can induce clustering in the scalar field, possibly leading to interesting effects. While here we concentrate on the background field, future work will explore the possibility of observable signatures of these fluctuation properties.

ACKNOWLEDGMENTS

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