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M. Shifman, G. Tallarita, and A. Yung

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’t Hooft-Polyakov Monopoles with Non-Abelian Moduli

M. Shifman[a], G.Tallarita[b,c], and A. Yung[a,d]

[a] *William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA*

[b] *Centro de Estudios Científicos (CECs), Casilla 1469, Valdivia, Chile*

[c] *Departmento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago, Chile*

[d] *National Recearch Center “Kurchatov Institute”, Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia*

Abstract

We extend the Georgi-Glashow model of the t’Hooft-Polyakov monopoles to include additional collective coordinates “orientational isospin moduli.” The low-energy theory of these solitonic solutions can be interpreted as dyons with isospin.

1 Introduction

Magnetic monopole is one of the most venerable constructions in theoretical physics. It dates back to Dirac [1]. Implementation of this construction in field theory is due to 't Hooft [2] and Polyakov [3] (for reviews and references see e.g. [4, 5, 6]). Remarkable effects were shown to be associated with the 't Hooft-Polyakov monopoles, e.g. the Callan-Rubakov effect of the baryon decay catalysis [7, 8, 9]. In the monopole field $SU(2)_{\rm gauge}$ doublet fermions acquire integer spin (e.g. [5]). This phenomenon is called "spin from isospin."

In this paper we will consider an extension of the Georgi-Glashow model [10], in which a *global* $O(3)$ ("isospin") symmetry is present in the Lagrangian.[1] A simple model possessing this property and suitable for our purposes was suggested in [11] (see also [12]). Conceptually it was inspired by Witten's cosmic strings [13]. Our task is to demonstrate that in this model the 't Hooft-Polyakov monopole acquires additional collective coordinates, isospin moduli. The overall set of collective coordinates includes three coordinates of the monopole center, a $U(1)$ phase coordinate which, upon quantization, corresponds to the electric charge and produces dyons out of the monopoles, and two extra collective coordinates associated with the $O(3)$ isospin. Quantization of the isospin collective coordinates is straightforward, paralleling that of the spherical quantum top.

2 The model

Our starting point is the well-known Georgi-Glashow model [10]. The gauge group is $SU(2)$ and the matter sector is described by a triplet real field ϕ^a (belonging to the adjoint representation). The Lagrangian of this model is

$$\mathcal{L}_{\rm GG} = -\frac{1}{4g^2} G^a_{\mu\nu} G^{\mu\nu,a} + \frac{1}{2}(D_\mu \phi^a)(D^\mu \phi^a) - \lambda \left(\phi^a\phi^a - v^2\right)^2 , \tag{1}$$

where

$$D_\mu \phi^a = \partial_\mu \phi^a + \varepsilon^{abc} A^b_\mu \phi^c, \tag{2}$$

is the covariant derivative in the adjoint representation, and

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu \tag{3}$$

[1] This latter isospin results from the global $O(3)$ symmetry, not to be confused with $SU(2)_{\rm gauge}$.

is the non-Abelian field strength. We use the following matrix notation for the fields

$$\phi = \phi^a \frac{\tau_a}{2}, \tag{4}$$

where τ_a denote the standard Pauli matrices. In the Lagrangian, v is a parameter with dimensions of mass, and λ is a dimensionless coupling constant. As is well known, this model supports the t'Hooft-Polyakov magnetic monopoles [2, 3] that are topologically stable as a consequence of the symmetry breaking pattern

$$SU(2)_{\text{gauge}} \to U(1)_{\text{gauge}} \,. \tag{5}$$

The breaking (5) is enforced by the non-vanishing vacuum expectation value of the ϕ field, which can always be chosen aligned in the 3-direction in $SU(2)_{\text{gauge}}$,

$$\phi^a_{vac} = v\delta^{3a} \,. \tag{6}$$

Two components of the gauge field, called $W^{\pm}$, acquire masses

$$m_W = gv \,, \tag{7}$$

whilst the component aligned along the vacuum direction remains massless and plays the role of the $U(1)$ photon. The mapping between the group space and the coordinate space at infinity can be classified by the second homotopy class,

$$\pi_2 \left(SU(2)/U(1)\right) = \mathbb{Z}, \tag{8}$$

which guarantees the topological stability of the monopoles.

Historically, the introduction of non-Abelian moduli on topological defects occurred in a rather advanced settings, mostly involving supersymmetric gauge theories (see [14, 15, 16]). In [11] a much simpler setup was suggested resulting in the occurrence of non-Abelian moduli. We will use this setup to study non-Abelian moduli on the world-line of the t'Hooft-Polyakov monopoles. To this end we will end a term $\mathcal{L}_\chi$ to the Lagrangian (1),

$$\mathcal{L} = \mathcal{L}_{\text{GG}} + \mathcal{L}_\chi \tag{9}$$

where

$$\mathcal{L}_\chi = \partial_\mu \chi^i \partial^\mu \chi^i - \gamma \left[(-\mu^2 + \phi^a\phi^a)\chi^i\chi^i + \beta(\chi^i\chi^i)^2 \right] . \tag{10}$$

Here χ^i is an $O(3)$ triplet uncharged under the gauge group. The potential term coupling χ^i to ϕ^a ensures that, if $\mu < v$, the χ^i field will develop in the monopole core and vanish in the vacuum. The mass of the χ field is

$$m^2_\chi = \gamma(v^2 - \mu^2). \tag{11}$$

The global $O(3)$ symmetry of (10) is obvious.

Now, assuming the standard monopole ansatz for the fields ϕ and A

$$\phi^a = vn^a H(r), \quad A_i^a = \epsilon^{aij}\frac{1}{r}n^j F(r), \tag{12}$$

with $n^a = x^a/r$, and adding a self-evident ansatz for χ^i,

$$\chi^i = \sqrt{\frac{\mu^2}{2\beta}}\,\chi(r)\,\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{13}$$

we can reduce the energy functional to

$$\begin{aligned} E &= \frac{4\pi v}{g}\int_0^\infty d\rho\rho^2\Bigg[\frac{(F')^2}{\rho^2}+\frac{(2F-F^2)^2}{2\rho^4}+\frac{H^2(1-F)^2}{\rho^2}+\frac{(H')^2}{2} \\ &\quad +\frac{\lambda}{g^2}\left(H^2-1\right)^2+\frac{\tilde{\mu}^2}{2\beta}\left((\chi')^2+\frac{\gamma}{g^2}\left[(-\tilde{\mu}^2+H^2)\chi^2+\frac{\tilde{\mu}^2}{2}\chi^4\right]\right)\Bigg], \end{aligned} \tag{14}$$

in dimensionless units

$$\rho = gv|\vec{x}|\,. \tag{15}$$

Here $\tilde{\mu}$ is dimensionless,

$$\tilde{\mu} = \mu/v. \tag{16}$$

The energy minimization equations are

$$\left(\rho^2 H'\right)' = H\left[2(F-1)^2+\frac{4\lambda\rho^2}{g^2}\left(H^2-1\right)+\frac{\gamma\tilde{\mu}^2\rho^2}{g^2\beta}\chi^2\right], \tag{17}$$

$$\rho^2 F'' = (-1+F)\left((-2+F)F+\rho^2H^2\right), \tag{18}$$

$$\left(\rho^2\chi'\right)' = \frac{\gamma\rho^2}{g^2}\chi\left(H^2+\tilde{\mu}^2\left(-1+\chi^2\right)\right). \tag{19}$$

2.1 Vacuum structure

In the vacuum, all derivatives of the profile functions must vanish. Clearly equation (18) imposes the condition that $F = 1$ in the vacuum. This leads to four branches of extrema of the energy, the first is

$$H^I_{vac} = \chi^I_{vac} = 0, \tag{20}$$

this is a maximum of the energy functional with $E^I_{max}/4\pi v = \lambda/g^3$. Then, in the vacuum II,

$$H^{II}_{vac} = 1, \quad \chi^{II}_{vac} = 0, \tag{21}$$

for which the vacuum energy density vanishes $E^{II}_{vac} = 0$. The third branch is

$$H^{III}_{vac} = 0, \quad \chi^{III}_{vac} = 1, \tag{22}$$

for which

$$\frac{E^{III}_{vac}}{4\pi v} = \frac{1}{g^3}\left(\lambda - \frac{\tilde{\mu}^4\gamma}{4\beta}\right). \tag{23}$$

Finally, the fourth branch is

$$(H^{IV}_{vac})^2 = \frac{4\beta\lambda - \gamma\tilde{\mu}^2}{4\beta\lambda - \gamma}, \quad (\chi^{IV}_{vac})^2 = \frac{4\beta\lambda(1-\tilde{\mu}^2)}{\tilde{\mu}^2(\gamma - 4\beta\lambda)} \tag{24}$$

for which

$$\frac{E^{IV}_{vac}}{4\pi v} = \frac{\gamma\lambda}{g^3}\frac{(\tilde{\mu}^2-1)^2}{\gamma - 4\beta\lambda}. \tag{25}$$

Hence, when $\gamma < 4\beta\lambda$ this branch corresponds to the actual vacuum. The branches II and IV meet at $\tilde{\mu} = 1$, or when $\mu = v$. When $\gamma > 4\beta\lambda$ and $\lambda > \frac{\tilde{\mu}^4\gamma}{4\beta}$ the vacuum II is the actual vacuum.

Below we look for solutions in vacuum II.

3 Numerical results

Here we present the solutions of (17)-(19) obtained by a second order central finite difference numerical procedure with the accuracy $O(10^{-4})$. To launch our numerical procedure we cut-off the radial direction at a large value of ρ, which we will call R. We set $R = 100$. As an example we choose the following values of the parameters:

$$\gamma = 4, \quad \lambda = 0.34, \quad \tilde{\mu} = 0.999, \quad \beta = 2.94, \quad g = 0.6\,. \tag{26}$$

Our boundary conditions are

$$H(0) = F(0) = 0, \quad \chi'(0) = 0, \tag{27}$$

$$H(R) = F(R) = 1, \quad \chi(R) = 0. \tag{28}$$

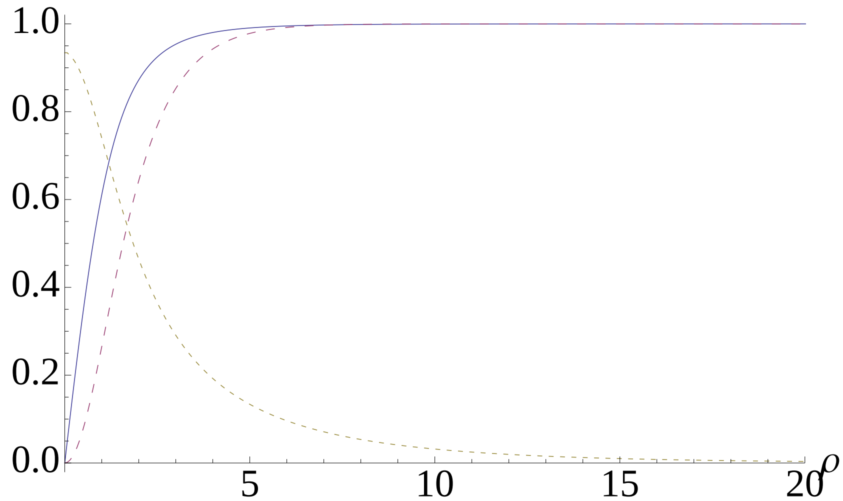

Figure 1: Non-Abelian monopole profiles for $\gamma = 4$, $\lambda = 0.34$, $\tilde{\mu} = 0.999$, $\beta = 2.94$ and $g = 0.6$. The solid curve represents H, medium dashed F and thin dashed χ.

The sought for solution is shown in Figure 1. The energy (monopole mass) M_M of this solution is

$$\frac{M_M}{4\pi v} = 2.14\,, \qquad \text{Monopole with the isospin.} \tag{29}$$

The mass M_M of the standard t'Hooft-Polyakov monopole at $\lambda = 0.34$ and $g = 0.6$ is

$$\frac{M_M}{4\pi v} = 2.33\,. \tag{30}$$

Thus, the energy of the monopole with the nonvanishing χ^i in the core is lower.

4 Stability

In this section we can check that the solution found above with a nonvanishing χ in the core is stable. The energy functional for the χ field is

$$E_\chi = \frac{4\pi v}{g} \int_0^\infty d\rho \rho^2 \left[(\chi')^2 + \frac{\gamma}{g^2} \left((-\tilde{\mu}^2 + H^2)\chi^2 + \frac{\tilde{\mu}^2}{2}\chi^4 \right) \right]. \tag{31}$$

Introducing

$$\psi(\rho) = \rho\chi \tag{32}$$

we can derive a one-dimensional Schrödinger-like equation for the ψ eigenfunctions,

$$-\psi'' + \frac{\gamma}{g^2\rho^2}\left((-\tilde{\mu}^2 + H_1^2) + 3\tilde{\mu}^2\chi_1^2\right)\psi = \epsilon\psi \tag{33}$$

where H_1 and χ_1 are the field profiles shown in Figure 1. Using the same parameters as in (26) we find the lowest-lying mode at $\epsilon \approx 0.00105$. The positivity of ϵ demonstrates the stability of the numerical solution with a non-vanishing χ field in the core.

Since the solution with a non-zero χ in the core has lower energy than the vanishing-χ solution we must also investigate (meta)stability of the vanishing-χ solution. Then, following the argument above, we must solve

$$-\psi'' + \frac{\gamma}{g^2\rho^2}(-\tilde{\mu}^2 + H_1^2)\psi = \epsilon\psi\,. \tag{34}$$

We find that the lowest lying mode is at $\epsilon \approx 0.00103$ thus confirming the classical stability of the $\chi = 0$ solution.

Thus, we observe two classical solutions: one – the standard 't Hooft-Polyakov monopole is a local minimum of the energy functional, while the solution with the nonvanishing χ in the core represents the global minimum.

5 Quantization of the collective coordinates

Quantization of the standard t'Hooft-Polyakov monopole involves four moduli: three translations of the monopole center plus a rotation around the vacuum direction in the $SU(2)_{\rm gauge}$ space. In the adiabatic approximation one finds the Lagrangian

$$\mathcal{L}_{QM} = -M_M + \frac{M_M}{2}(\dot{\vec{x_0}})^2 + \frac{1}{2}\frac{M_M}{m_W^2}\dot{\alpha}^2 \tag{35}$$

where M_M is the mass of the monopole, the vector $\vec{x_0}$ represents the monopole center, whilst α is the collective coordinate related to the remaining $U(1)_{\rm gauge}$ invariance. The full quantum mechanical Hamiltonian for these moduli is

$$\widehat{H}_0 = M_M + \frac{\vec{p}^{\,2}}{2M_M} + \frac{1}{2}\frac{m_W^2}{M_M}\pi_\alpha^2, \tag{36}$$

where

$$\pi_\alpha = \frac{M_M}{m_W^2}\dot{\alpha}, \tag{37}$$

is the canonical momentum conjugated to the angular variable α.

To obtain the orientational moduli we parametrize the χ field as follows

$$\chi^i = \sqrt{\frac{\mu^2}{2\beta}}\chi(\rho)S^i(t) \tag{38}$$

where S^i is a unit vector which depends on time. Substituting (38) in (14) we obtain the low-energy action for the orientational moduli,

$$\mathcal{S}_0 = \frac{I_1}{2}\int dt\ \dot{S}^i\dot{S}^i, \quad S^iS^i = 1, \tag{39}$$

where

$$\frac{I_1 v}{4\pi} = \frac{\tilde{\mu}^2}{g^3\beta}\int_0^\infty d\rho\,\rho^2\chi^2 = 7.81, \tag{40}$$

and the overdot denotes a derivative with respect to time. The above action can be easily quantized as follows.

The action (39) describes the motion of a rigid rotating body, symmetric quantum top, at a fixed spherical radius r_0 (in this case $r_0 = 1$). Its quantization can be carried out in a standard way (see. e.g. [17]), if we parametrize S^i in terms of polar and azimuthal angles,

$$S^1 = \cos\theta\,, \quad S^2 = \sin\theta\,\cos\varphi\,, \quad S^3 = \sin\theta\,\sin\varphi\,. \tag{41}$$

Upon quantization we get the following symmetric top Hamiltonian:

$$\widehat{H} = -\frac{1}{2I_1}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]. \tag{42}$$

Its eigenvalues are labelled by a non-negative integer s and reduce to

$$E_s = \frac{1}{2I_1}s\,(s+1)\ . \tag{43}$$

They have degeneracy $2s+1$. The eigenfunctions are the standard spherical harmonics $Y_s\,(\theta,\varphi)$.

Including the conventional 't Hooft-Polyakov moduli from Eq. (36) we arrive at the monopole mass formula

$$M = M_M + \frac{m_W^2 \, k^2}{2M_M} \; + \frac{1}{2I_1} \, s(s+1), \tag{44}$$

where k and s are integers. The k^2 term here corresponds to dyonic excitations associated with non-zero electric charge, while the last term is associated with non-zero global isospin. The I_1 parameter plays the role of the moment of inertia in the isospace. With our illustrative choice of parameters $I_1^{-1} \sim (1/60) m_W$ while $m_W / M_M \sim 1/40$.

6 Conclusions

In this model we completed the program of constructing the simplest topological defects with non-Abelian moduli. We extended the Georgi-Glashow model of the t'Hooft-Polyakov monopoles to include extra global symmetry $O(3)$ (we referred to it as isospin) and extra collective coordinates "orientational isospin moduli." The adiabatic quantization of these solitonic solutions was carried out. The result can be interpreted as a dyon with isospin.

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