Hilbert series for constructing Lagrangians: Expanding the phenomenologist's toolbox

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Hilbert Series for Constructing Lagrangians: expanding the phenomenologist's toolbox

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Abstract: This note presents the Hilbert series technique to a wider audience in the context of constructing group-invariant Lagrangians. This technique provides a fast way to calculate the number of operators of a specified mass dimension for a given field content, and is a useful cross check on more well-known group theoretical methods. In addition, at least when restricted to invariants without derivatives, the Hilbert series technique supplies a robust way of counting invariants in scenarios which, due to the large number of fields involved or to high dimensional group representations, are intractable by traditional methods. We work out several practical examples.

1 Introduction

In this post-Higgs-discovery era of fundamental physics, phenomenological models of physics beyond the Standard Model (BSM) are becoming increasingly baroque. The simplest models have been well-studied and more complex models offer numerous adjustable parameters that can be tuned to avoid ever more stringent experimental limits on new physics. One common way to augment the complexity of a model is to simply add particles transforming under higher-dimensional group representations. For some representative examples using sizable group representations in phenomenological settings, see [1–5] and references therein. In addition to complex model building, another line of attack in the search for new physics takes a general bottom-up approach – parameterizing BSM effects through higher dimensional operators formed from SM fields. These operators are suppressed by powers of some high energy scale, but they become important for detecting new physics in high precision measurements.

Given a BSM model containing new particle multiplets in different group representations, one often wants the most general gauge-invariant Lagrangian containing all of the operators allowed by the postulated symmetries. The formation of this Lagrangian allows a full exploration of the model's experimental signatures. In the simplest cases, with small and familiar group representations such as $SU(2)$ doublets, it is straight forward to assemble the Lagrangian almost automatically. With larger group representations, the calculation becomes harder, and one often needs to take into account relations between group invariants (mathematicians refer to these relations as syzygies) to obtain the correct number of independent terms in the Lagrangian. A similar level of calculational difficulty occurs in the task of forming higher dimensional operators from the Standard Model degrees of freedom. Even though the SM group representations are familiar, the high multiplicity of these representations contained in the operators quickly becomes challenging to deal with as the operator dimension increases.

A fundamental reason for the prevalence of such group-theoretical calculations in phenomenological studies is the modern Wilsonian perspective of effective field theory. From this perspective, the gauge and global symmetries of the proposed field content completely determine the Lagrangian, and in principle one must include all possible invariant terms in every operator dimension as part of the resulting QFT.

The mathematical tool known as the Hilbert series is perfectly suited for such computations. The Hilbert series (or Molien or Poincaré function) is a generating function encoding information about the number of independent group invariants that can be formed from some set of multiplets in different representations. As we explain in this paper, the Hilbert series provides an easy cross-check on calculations performed using more familiar group theoretical techniques, allowing one to ensure the correct number of independent terms in the Lagrangian. This technique is especially convenient when dealing with large representations or higher dimensional operators, since the Hilbert series calculation is easily automated using computer algebra programs such as MATHEMATICA.

The Hilbert series approach has been developed and used extensively in more formal theoretical settings. For example, it is often used in calculations involving the operator spectra of supersymmetric gauge theories [6–11], SUSY theories on D-branes [12–14], and moduli spaces of instantons or vortices [15–19]. Other references from a more mathematical point of view include [20, 21]. On the more phenomenological side, Hilbert series methods have been used to calculate the number of independent flavor invariants in the Standard Model and various extensions. The Hilbert series for leptonic flavor invariants in the Standard Model extended by the dimension-5 Weinberg operator was determined for two and three generations in [22]. This paper also obtained the Hilbert series of flavor invariants for the full type-I seesaw model in the case of two generations, and the series for quark sector flavor invariants for both two and three generations. This paper was followed by [23], which completed the calculation of the Hilbert series for the seesaw model with three generations, and also computed the series for quark flavor invariants with four quark generations. The Hilbert series approach has also been used for studying $SU(3)$ subgroups in the context of flavor symmetry model building [24].

While the Hilbert series technique has been used in formal studies and in the more phenomenological setting of calculating flavor invariants, it is not yet well known and appreciated as a tool that can aid in the construction of general gauge-invariant Lagrangians. In this paper, we aim to rectify this and to add the Hilbert series to the toolbox of a wider audience. Specifically, in Section 2 we introduce the basic Hilbert series concepts through a simple example, then present the general framework. Then, in Section 3, we work through a complete example to solidify the concepts. Sections 4 and 5 show some more complicated examples, and we discuss incorporating derivatives and equations of motion in Section 6. In Section 7, we conclude. Some mathematical background material is included in Appendix A.

2 Hilbert series basics: $U(1)$ symmetry

In this section we introduce the key mathematical ingredients of Hilbert series using a simple example. More formal introductory material to the Hilbert series can be found in the literature [7, 9, 10, 13, 21, 22].

For our example, consider a single complex scalar field charged under a $U(1)$ symmetry: $\phi \to e^{i\theta}, \phi^* \to e^{-i\theta}$. The gauge invariant combinations are $(\phi\phi^*)^n$, and there is exactly one possibility for each n. Writing the set of invariants as a series

$$\mathfrak{H} = \sum_{n=1}^{\infty} c_n (\phi\phi^*)^n, \tag{2.1}$$

where c_n is the number of different invariant possibilities for a given dimension, we have

$$\mathfrak{H} = 1 + (\phi\phi^*) + (\phi\phi^*)^2 + (\phi\phi^*)^3 + \dots \tag{2.2}$$

Formally treating $(\phi\phi^*)$ as numbers less than one[1], this geometric series can be summed,

$$\mathfrak{H} = \frac{1}{1-\phi\phi^*}. \tag{2.3}$$

Let us massage this further; the sum above can be replaced by an integral over θ, the variable that parameterizes the $U(1)$ transformation:

$$\mathfrak{H} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\theta}{(1-\phi\, e^{i\,\theta})(1-\phi^*\, e^{-i\,\theta})} \tag{2.4}$$

Substituting $z = e^{i\,\theta}$, the $\mathrm{d}\theta$ integral becomes a contour integral around $|z| = 1$.

$$\mathfrak{H} = \frac{1}{2\pi i}\oint_{|z|=1} \frac{\mathrm{d}z}{z}\frac{1}{(1-\phi\, z)(1-\frac{\phi^*}{z})}. \tag{2.5}$$

The second piece of the denominator in Eq. (2.5) can be replaced by

$$\begin{aligned}\frac{1}{(1-\phi\, z)(1-\frac{\phi^*}{z})} &= \exp\Big\{ -\log\,(1-\phi z) - \log\,(1-\frac{\phi^*}{z})\Big\} \\ &= \exp\Big\{ \sum_{r=1}^{\infty}\frac{(\phi z)^r}{r} + \sum_{r=1}^{\infty}\Big(\frac{\phi^*}{z}\Big)^r\frac{1}{r}\Big\}.\end{aligned} \tag{2.6}$$

To get a better idea of what's going on, let us expand the LHS of Eq. (2.6), again treating ϕ and ϕ^* as small, complex numbers rather than quantum fields. To cubic order in both ϕ and ϕ^*

$$\begin{aligned}\frac{1}{(1-\phi\, z)(1-\frac{\phi^*}{z})} &= \big(1+\phi\phi^* + (\phi\phi^*)^2 + (\phi\phi^*)^3 + \cdots\big) + z\,\big(\phi + \phi\,(\phi\phi^*) + \phi\,(\phi\phi^*)^2 + \cdots\big) \\ &\quad + z^2\,\big(\phi^2 + \phi^2\,(\phi\phi^*) + \cdots\big) + z^3\,\phi^3 + \frac{\phi^{*3}}{z^3} + \frac{1}{z^2}\big(\phi^{*2} + \phi^{*2}\,(\phi\phi^*) + \cdots\big) \\ &\quad + \frac{1}{z}\big(\phi^* + \phi^*\,(\phi\phi^*) + \phi^*\,(\phi\phi^*)^2 + \cdots\big)\,.\end{aligned} \tag{2.7}$$

The series of $U(1)$ invariants sits in the term with no z factors, hence it is picked out when we multiply by $1/z$ and perform the contour integral in $\mathrm{d}z$ (Eq. (2.5)). However, inspection of the expansion of Eq. (2.6) shows it contains all possible arrangements of ϕ and ϕ^*; if we wanted to pick out the series of charge $+1$ combinations, we would simply have to multiply by $\frac{1}{z^2}$ before taking the contour integral, the charge -2 can be accessed by multiplying by z, etc.

In picking out a particular charge from Eq. (2.6), we are using two different mathematical

[1] To clarify that the objects we are manipulating, e.g. ϕ, are complex numbers rather than quantum fields, we will refer to ϕ and similar objects as spurions.

facts. The first, already mentioned, is that Eq. (2.6) generates all possible combinations of ϕ and ϕ^*, organized by charge. This exponential form of Eq. (2.6) is an example of a *plethystic exponential* [7, 9, 10, 13, 21], a generating function of all symmetric combinations of its argument (see A.2). The second mathematical construction we employ is integration over the group volume, $d\theta \equiv \frac{dz}{i\,z}$ for $U(1)$. Integrated over $d\theta$, terms containing any non-trivial power of $z \to e^{i\,\theta}$ become integrals $d\theta\, e^{i\,n\,\theta}$ for some integer n and are therefore zero. Terms with no powers of z – the $U(1)$ invariants – remain and are the Hilbert series $\mathfrak{H}$.

While the series of invariants in the $U(1)$ example above could be found without the aid of Eqs. (2.5) and (2.6), the power of this approach lies in its generality. Both the plethystic exponential and the integral over the group parameter θ can be extended to sets of spurions transforming under arbitrary representations of arbitrary compact Lie groups. In addition to generating all possible combinations of spurions such as ϕ and ϕ^*, the plethystic exponential keeps track of relations among invariants, or syzygies (see Ref. [22] for some simple examples of relations among invariants). Our $U(1)$ example is too simplistic to see these syzygies, however we will run into relations among invariants when we consider more complicated setups as in Sections 4 and 5.

To create the plethystic exponential (PE) for a spurion A transforming with representation R of a connected Lie group, we take

$$\mathrm{PE}[A, R] = \exp\Big(\sum_{r=1}^{\infty} \frac{A^r\, \chi_R(z_j^r)}{r}\Big). \tag{2.8}$$

Here $\chi_R(z_j)$ is the character of the representation R expanded as a monomial function of the j complex variables on the Cartan sub-algebra (equivalent to the group rank). For example, consider A to be in the fundamental representation of $SU(2)$, a rank-1 group. The character for the fundamental representation is

$$\Big(z + \frac{1}{z}\Big), \tag{2.9}$$

where z is a complex number with modulus one (often called a fugacity). The argument of the plethystic exponential is then

$$\sum_{r=1}^{\infty} \frac{A^r}{r}\left(z^r + \frac{1}{z^r}\right) = \sum_{r=1}^{\infty}\Big(\frac{(Az)^r}{r} + \frac{A^r}{z^r\, r}\Big) = -\log\left(1 - A/z\right) - \log\left(1 - A\, z\right). \tag{2.10}$$

This can be easily extended to more spurions and representations of different groups – all we need is a list of the characters for different representations of Lie groups. For a brief review of characters and further discussion of the PE, see Appendix A.

Next, the invariants from the PE are picked out by the fact that the characters of compact Lie groups form an orthonormal set of basis functions on the Cartan sub-algebra variables

[25]. As such, any function f of the sub-algebra variables can be expanded in terms of them:

$$f(z_j) = \sum_i c_i\, \chi_i(z_j), \quad \int \mathrm{d}\mu\, \chi_i(z)\, \chi_j^*(z) = \delta_{ij}, \tag{2.11}$$

where the c_i are coefficients. The integration $\mathrm{d}\mu$ in Eq. (2.11) is over the Haar measure – the volume of the group in question projected onto the Cartan sub-algebra variables (maximal torus). These volume elements can be found, for example, in Ref. [10] and are included in Appendix A.3 for convenience. While the PE contains all possible tensor products of spurions, the group integration projects out only the invariant combinations, resulting in the Hilbert series $\mathfrak{H}$.

Looking back at our $U(1)$ example, we can rephrase the results in this more general language. The character for a $U(1)$ representation with charge Q is z^Q, so the PE in Eq. (2.6) is the sum of a representation with charge $Q = +1$ and representation $Q = -1$. The Haar measure for $U(1)$ is $\mathrm{d}z/z$, and the character orthogonality relation is the usual Fourier series orthogonality, as is best seen by setting $z = e^{i\,\theta}$:

$$\int_{\theta=0}^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\, e^{i\,\theta\,(Q-Q')} = \delta(Q - Q'). \tag{2.12}$$

Continuing, Eq. (2.5) can be understood as performing a Fourier expansion of the PE (Eq. (2.6)), multiplying by the trivial representation (i.e. 1) and integrating over the group. Since the characters are orthonormal, only the part of the PE expansion that lies in the trivial representation – the $U(1)$ invariants – is projected out. Had we wanted to project out a different part of the PE, all we have to do is multiply by the conjugate character before doing the group integration. For example, to project out the charge $+1$ combinations, we need to multiply by $1/z = e^{-i\,\theta}$ before integrating.

Having shown how the PE and group integration can generate the Hilbert series of invariants for this simple $U(1)$ example, we now want to apply this method to the Standard Model (and its extensions). Specifically, we will take some subset of the fields Q, u^c, d^c, L, e^c, H, etc. of the SM (plus any extensions) as the spurions, dress them with the characters appropriate to their groups and representations, then form the PE and do the contour integrations. The different SM spurions can transform differently, and some may transform under multiple groups. A separate group integration is carried out for each group under which the fields transform. If the contour integrals can be directly calculated, the resulting Hilbert series will be a rational function of the input spurions, $\mathfrak{H}(Q, u^c, d^c, L, e^c, H, \cdots)$. This function $\mathfrak{H}(Q, u^c, d^c, L, e^c, H, \cdots)$ can then be Taylor expanded as a multivariate power series in the spurions. The coefficient of any particular combination of spurions gives the number of independent group-singlet operators that can be formed from the fields represented by those spurions. For example, a term $2\,Q^\dagger\, Q L^\dagger L$ in the Hilbert series indicates there are two independent singlets that can be formed from one Q field, one L field, and their hermitean conjugates. We emphasize that, while the Hilbert series gives the number of invariants, it does not give the particular index structure, so that must be worked out separately. Before we dive in, a few comments are in order:

- Several of the spurions we want to use are fermionic, meaning they represent fermion fields. To properly count invariants including fermions, we need to extend the PE to handle antisymmetric spurions. This can be done by using the fermionic plethystic exponential (PEF) [21]. For a fermionic spurion A in representation R:

$$PEF[A,R] = \exp\Big(\sum_{r=1}^{\infty} \frac{(-1)^{r+1} A^r \chi_R(z_j^r)}{r}\Big), \tag{2.13}$$

  where, as before, $\chi_R(z_j)$ is the character for representation R as a function of the Cartan sub-algebra variables.

- As we are working with fermions (and, eventually, field-strength tensors) we must include the Lorentz group representations for these spurions. At first sight, this seems problematic since the orthonormality of the group characters (the Peter-Weyl theorem) only holds for compact Lie groups. However, since our purpose is solely to count invariants and does not involve any dynamics, we can work in Euclidean space, where the Lorentz group is compact: $SO(4) \cong SU(2)_R \times SU(2)_L$. For simplicity we will take all fermions to be left-handed objects, transforming as $(0, \frac{1}{2})$ under $SU(2)_L \times SU(2)_R$; hermitian conjugate fermions therefore transform in the right-handed $(\frac{1}{2}, 0)$ representation. When considering field strength tensors, we will work with the objects $X^{\pm}_{\mu\nu} = X_{\mu\nu} \pm \mathrm{i}\,\tilde{X}_{\mu\nu}$, which transform in the $(1,0)$ and $(0,1)$ representations.

- While the PE or PEF generate all possible combinations of spurions, these constructs contain no information regarding the equations of motion – operator relations that go beyond symmetries. How to include the equations of motion, and derivatives in general, lies beyond the scope of this paper but will appear in a companion paper [26]. A few comments will be sketched out in the discussion in Section 6.

3 Using the Hilbert series: a toy example

It is instructive to work through a full example of the Hilbert series technique in order to demonstrate the general procedure. For this purpose, consider the Standard Model left-handed fermion doublet L, which is a weak isodoublet and a color singlet. In this example, we will ignore hypercharge, and we denote the number of generations as N_f. Since L transforms in the fundamental representation of both $SU(2)_L$ and $SU(2)_W$, the argument of the plethystic exponential is

$$N_\mathrm{f}\, L\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right), \tag{3.1}$$

where x is the complex variable for $SU(2)_L$ and y is the variable for $SU(2)_W$. Recall that when carrying out the sum for the PE, we need to include the factor $(-1)^{r+1}$ since L is fermionic. Explicitly, using the Haar measure for the $SU(2)$ groups as given in Eq. (A.7), the Hilbert series for the spurion L is

$$\mathfrak{H}(L) = \frac{1}{(2\pi \mathrm{i})^2} \oint_{|x|=1} \frac{\mathrm{d}x}{x}(1-x^2) \oint_{|y|=1} \frac{\mathrm{d}y}{y}(1-y^2) \exp\left[N_\mathrm{f} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} L^r \left(x^r + \frac{1}{x^r}\right)\left(y^r + \frac{1}{y^r}\right)\right]. \tag{3.2}$$

Note that N_f is not a variable in the PE, but is rather a free parameter. Expanding the PE gives

$$\begin{aligned} \exp\left[N_\mathrm{f} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} L^r \left(x^r + \frac{1}{x^r}\right)\left(y^r + \frac{1}{y^r}\right)\right] = \\ \left(1 + \frac{L}{xy}\right)^{N_\mathrm{f}} \left(1 + \frac{Lx}{y}\right)^{N_\mathrm{f}} \left(1 + \frac{Ly}{x}\right)^{N_\mathrm{f}} (1 + Lxy)^{N_\mathrm{f}} . \end{aligned} \tag{3.3}$$

In general, it can be computationally challenging to do the contour integrals for the Haar integration directly, especially if a large number of fields is under consideration. In this situation, it is better to first expand the integrand in a Taylor series in the spurion L, and then integrate term-by-term up to the desired power [6]. The poles of the integrand will now all be at $x = 0$ and $y = 0$, so the residue calculation is much easier. This method will not give the complete generating function $\mathfrak{H}$, but rather the first few terms in its series expansion. This is sufficient for most purposes. However, if some all-order information about the series is needed, or the asymptotic form of the coefficients as the expansion variables go to infinity is desired,[2] it will be necessary to directly do the integrals and obtain the full functional form of $\mathfrak{H}$.

For the specific example of Eq. (3.2), we can illustrate this expansion method by leaving N_f unspecified and calculating the first few terms of the Hilbert series for L:

$$\mathfrak{H}(L) = \frac{1}{2}(N_\mathrm{f}^2 - N_\mathrm{f})L^2 + \frac{1}{6}(N_\mathrm{f}^4 + 5N_\mathrm{f}^2)L^4 + \frac{1}{144}(5N_\mathrm{f}^6 - 3N_\mathrm{f}^5 + 17N_\mathrm{f}^4 - 69N_\mathrm{f}^3 + 50N_\mathrm{f}^2)L^6 + \ldots \tag{3.4}$$

If N_f is specified, the Hilbert series can easily be calculated exactly, since the only poles in the integrand of Eq. (3.2) are at $x = 0$ and $y = 0$. Carrying out the contour integrations for the first few values of N_f gives the results displayed in Table 1.

The entries in Table 1 can be understood using standard group theory. In this discussion, $SU(2)_W$ indices will be explicitly displayed, and $SU(2)_L$ (Lorentz group) indices will not be displayed, but will be contracted within parentheses. For $N_\mathrm{f} = 1$, there is no possible L^2 operator, since $\epsilon_{\alpha\beta}(L^\alpha L^\beta) = 0$. At the L^4 level, the indices can be contracted in a single non-zero manner: $\epsilon_{\alpha\beta}\epsilon_{\delta\gamma}(L^\alpha L^\delta)(L^\beta L^\gamma)$. All other methods of index contraction for L^4 are

[2] To obtain asymptotic information about generating functions, see for example the methods described in [27].

N_{f}	**Hilbert series** $\mathfrak{H}(L)$
1	$1+L^4$
2	$1+L^2+6L^4+L^6+L^8$
3	$1+3L^2+21L^4+20L^6+21L^8+3L^{10}+L^{12}$
4	$1+6L^2+56L^4+126L^6+210L^8+126L^{10}+56L^{12}+6L^{14}+L^{16}$

Table 1. The Hilbert series for the first few values of N_{f}, calculated from Eq. (3.2). Note that the resulting polynomials are palindromic, as noted in [22]. Also, for a given N_{f} there is a maximum dimension an operator can have.

either automatically zero, or can be written as this one by using the Schouten identities from the two $SU(2)$ groups. At higher orders, for example L^8, it is always possible to use the freedom given by the two sets of Schouten identities to rewrite the contractions in a form that is manifestly identically zero, which is why the Hilbert series terminates at order L^4.

For $N_{\mathrm{f}}=2$, there is a single operator of order L^2, namely $\epsilon_{\alpha\beta}(L_1^\alpha L_2^\beta)$. The fact that there are two distinct fields keeps this from being identically zero. Similarly, at order L^4, the six possible operators are

$$\begin{aligned}&\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}(L_1^\alpha L_1^\beta)(L_1^\gamma L_1^\delta),\ \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}(L_2^\alpha L_2^\beta)(L_2^\gamma L_2^\delta),\ \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}(L_1^\alpha L_2^\beta)(L_2^\gamma L_2^\delta),\\&\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}(L_1^\alpha L_1^\beta)(L_1^\gamma L_2^\delta),\ \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}(L_1^\alpha L_2^\beta)(L_1^\gamma L_2^\delta),\ \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}(L_1^\alpha L_2^\beta)(L_1^\gamma L_2^\delta).\end{aligned} \tag{3.5}$$

As a final illustration, for $N_{\mathrm{f}}=3$, the three operators of order L^2 are $\epsilon_{\alpha\beta}(L_1^\alpha L_2^\beta)$, $\epsilon_{\alpha\beta}(L_1^\alpha L_3^\beta)$, and $\epsilon_{\alpha\beta}(L_2^\alpha L_3^\beta)$.

The Hilbert series method can actually go further and give us the exact flavor content of the operators as in Eq. (3.5) directly, by doing something called "refining" the series. To do this, instead of putting a generic spurion L into the PE and multiplying by N_{f} as in Eq. (3.2), we put in N_{f} distinct spurions with different labels. For example, if $N_{\mathrm{f}}=3$, use spurions L_1, L_2, and L_3 in the PE. This is also referred to as a "multigraded" Hilbert series. Doing this will in general increase the complexity of the residues that must be computed, so in some cases it may make the problem intractable. However, for the series of Eq. (3.2) it is feasible, at least for low values of N_{f}. Calculating the multigraded series for $N_{\mathrm{f}}=2$ and picking out the L^4 terms replicates the results of Eq. (3.5). Of course, refining the series does not give the exact index contraction structure (since there are multiple equivalent ways to contract the indices), but it does tell us that, for example, there are two independent operators with the flavor content $L_1^2L_2^2$.

Because of the increase in computational power required by refining the Hilbert series, it may sometimes be advantageous to go in the other direction and "unrefine" the series. For example, if the Hilbert series involves both the lepton doublet L and the quark doublet Q, it can be useful to not only lump all of the flavors together, but also to lump together L and

Q by setting both spurions to a common label t in the PE. This can significantly decrease the computational time required by a progam like MATHEMATICA, and may give all of the information that is necessary for a specific application of the Hilbert series method.

The example worked out in this section shows that while the results obtained from the Hilbert series can indeed be replicated using other techniques, the calculations are often tedious and it is easy to make mistakes and miscount the operators. For example, showing that there are 126 independent operators of order L^{10} for $N_\mathrm{f} = 4$ would be a formidable task by standard methods. By refining the series, the Hilbert series method provides an easy automatic way of obtaining the desired operators.

4 Using Hilbert series for Standard Model effective field theory

The Standard Model effective field theory (SMEFT) consists of the Standard Model Lagrangian $\mathcal{L}_{SM}$ plus operators with (mass) dimension greater than four that are invariant under the Standard Model gauge group $SU(3)_C \otimes SU(2)_W \otimes U(1)_Y$ and contain only Standard Model degrees of freedom. At dimension 5, there is only one possible operator, the Weinberg neutrino-mass operator [28]. Continuing on to dimension 6, a classification of the available operators was done in [29], and the reduction to a minimal set of dimension-6 operators was carried out in [30], resulting in a set of 63 operators. More recently, the construction of the minimal set of dimension-7 operators was also completed, giving 20 independent operators [31]. Any BSM physics can be matched onto this effective field theory by integrating out the postulated new heavy particles. In the past few years, much work has been done towards understanding the structure and use of the SMEFT [32–42].

Calculating the full set of operators for a given dimension with the Hilbert series requires including covariant derivatives, field strengths, and the equations of motion. Some thoughts on these ingredients are included in Section 6, and we leave an in-depth discussion to future work [26]. For simplicity, in this section we focus on operators without these complicating factors. This still provides a good example of the practical applications of the technique.

4.1 Dimension-6 baryon number violating operators

The Hilbert series technique proves to be useful for finding independent SMEFT operators, even when restricted to a subset of the field content. For example, consider the set of dimension-6 baryon-number-violating operators. Ref. [30] presented five such operators. However, it has been noted that only four of these five operators are independent if the flavor structure is taken into account [38, 43]. It would be nice to see such operator relations without doing a detailed calculation, and indeed using the Hilbert series can bring the dependence betwen different operator structures to light.

Consider the class of baryon-number violating dimension-6 operators with field content $QQQL$, where Q is the left-handed quark doublet and L is the lepton doublet. Taking $N_\mathrm{f} = 3$, the argument of the PE for these fields is

$$3Q\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)\left(z_1+\frac{z_2}{z_1}+\frac{1}{z_2}\right)u^{1/6}+3L\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)u^{-1/2}, \tag{4.1}$$

where x is the variable for $SU(2)_W$, y is the variable for $SU(2)_L$, u is the variable for $U(1)_Y$, and $\{z_1, z_2\}$ are the variables for $SU(3)_C$. Calculating the multigraded Hilbert series so that we can see the form of the operators gives

$$\mathfrak{H}(L,Q) = 1 + 57\; LQ^3 + 4818\; L^2Q^6 + 162774\; L^3Q^9 + \dots, \tag{4.2}$$

so we expect 57 independent operators of the form $QQQL$ when flavor structure is included.

Now consider the specific operator structure

$$\epsilon_{\alpha\beta\gamma}\epsilon_{ij}\epsilon_{kl}\left(Q_p^{i\alpha}Q_r^{j\beta}\right)\left(Q_s^{k\gamma}L_t^l\right). \tag{4.3}$$

This is symmetric in the flavor indices $\{p, r\}$, so without even considering additional symmetries implied by $SU(2)_L$ Fierz identities, the maximum number of independent flavor permutations is $3\times3\times6 = 54 < 57$ (since a 3×3 symmetric matrix has only 6 independent entries). Therefore the operator structure in Eq. (4.3) does not capture all of the 57 independent operators that we know exist at this order. Perhaps changing the $SU(2)_W$ index contraction structure will enable us to capture all of the operators in a single expression. To this end, consider the following structure with the $SU(2)_W$ contractions "offset" from the $SU(2)_L$ contractions:

$$\epsilon_{\alpha\beta\gamma}\epsilon_{ik}\epsilon_{jl}\left(Q_p^{i\alpha}Q_r^{j\beta}\right)\left(Q_s^{k\gamma}L_t^l\right). \tag{4.4}$$

None of the symmetries are immediately apparent in this form, so naively one might think that we can now just multiply out the flavor possibilities and get $3^4 = 81$ operators with different flavor structures. But this neglects $SU(2)_L$ Fierz identities (see for example [44]), the relevant one of which gives

$$\epsilon_{\alpha\beta\gamma}\epsilon_{ik}\epsilon_{jl}\left(Q_p^{i\alpha}Q_r^{j\beta}\right)\left(Q_s^{k\gamma}L_t^l\right) = -\epsilon_{\alpha\beta\gamma}\epsilon_{ik}\epsilon_{jl}\left[\left(Q_p^{i\alpha}Q_s^{k\gamma}\right)\left(Q_r^{j\beta}L_t^l\right)+\left(Q_p^{i\alpha}L_t^l\right)\left(Q_r^{j\beta}Q_s^{k\gamma}\right)\right]. \tag{4.5}$$

Rearranging the first term on the right-hand side using the $SU(2)_W$ Schouten identity $\epsilon_{ij}\epsilon_{mn} = \epsilon_{im}\epsilon_{jn} - \epsilon_{in}\epsilon_{jm}$, and relabelling the second term on the right at the cost of a sign gives

$$\epsilon_{\alpha\beta\gamma}\epsilon_{ik}\epsilon_{jl}\left(Q_p^{i\alpha}Q_r^{j\beta}\right)\left(Q_s^{k\gamma}L_t^l\right) = \epsilon_{\alpha\beta\gamma}\epsilon_{ik}\epsilon_{jl}\left[2\left(Q_p^{i\alpha}Q_s^{j\beta}\right)\left(Q_r^{k\gamma}L_t^l\right)-\left(Q_s^{i\alpha}Q_r^{j\beta}\right)\left(Q_p^{k\gamma}L_t^l\right)\right]. \tag{4.6}$$

This identity tells us that among the 81 different flavor permutations of the structure in Eq. (4.4), there are $3^3 = 27$ relations among operators differing only in Q flavor permutations. When all of the quark flavors are identical ($p = s = r$), the relation is trivial. Since ($p = s = r$) occurs 3 times for $N_\mathrm{f} = 3$ as in the Standard Model, there are $27 - 3 = 24$ linear relations among the 81 flavor permutations, leaving $81 - 24 = 57$ independent permutations! Since the Hilbert series in Eq. (4.2) revealed that there are only 57 independent operators, we are done; the structure in Eq. (4.4) contains all of the possibilities, and we do not need to write down a different fermion current structure or different $SU(2)_W$ index contractions.

The Hilbert series technique is by no means restricted to invariants of gauge symmetries; it can deal just as easily with global symmetries. As one example, we could have included baryon number for all fields, adding a character and group integration for that global $U(1)$. Had we included baryon number in the above example, we would find no invariants – by construction, since our example concerned baryon number violation. However, had we considered a wider set of spurions, both baryon number violating and respecting terms in the Hilbert series would be generated. In this case, integration over baryon number could be used to project out different subsets, e.g. only baryon number respecting operators, operators violating baryon number by one unit, by two units, etc. Of course, the same technique could be used for finding operators violating lepton number or $B - L$.

4.2 The dimension-7 operator $LLL\bar{e}H$

As another example of applying the Hilbert series to the SMEFT, consider the class of dimension-7 operators with field content $LLL\bar{e}H$ [31]. Here $\bar{e}$ is a left-handed field which can also be written as e^c, and H is the Higgs doublet. In the argument of the plethystic exponential we then have (again with $N_\mathrm{f} = 3$):

$$3L\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right)u^{-1/2} + H\left(x + \frac{1}{x}\right)u^{1/2} + 3\bar{e}\left(y + \frac{1}{y}\right)u\,, \tag{4.7}$$

where x is the variable for $SU(2)_W$, y is the variable for $SU(2)_L$, and u is the variable for $U(1)_Y$. Calculating the unrefined Hilbert series gives

$$\mathfrak{H}(t) = 1 + 3\,t^4 + 57\,t^5 + 171\,t^6 + 6\,t^8 + 144t^9 + 1053\,t^{10} + \ldots \tag{4.8}$$

Order t^5 contains the dimension-7 operators, and calculating the multi-graded Hilbert series shows that this order contains only the dimension-7 operators of the form $L^3\bar{e}H$, so we know that there are 57 operators with various flavor structures.

One such possible structure is

$$\epsilon_{ij}\epsilon_{mn}\left(\bar{e}_p L^i_q\right)\left(L^j_r L^m_s\right)H^n. \tag{4.9}$$

As in the previous section, the symmetries are not manifest, so we need to use a Fierz identity:

$$\epsilon_{ij}\epsilon_{mn}\left(\bar{e}_p L^i_q\right)\left(L^j_r L^m_s\right)H^n = -\epsilon_{ij}\epsilon_{mn}\left(\bar{e}_p L^j_q\right)\left(L^m_s L^i_r\right)H^n - \epsilon_{ij}\epsilon_{mn}\left(\bar{e}_p L^m_s\right)\left(L^i_r L^j_q\right)H^n. \tag{4.10}$$

In the first term on the right-hand side of Eq. (4.10), we can switch the $\{i, j\}$ labels at the cost of a sign (and use the fact that $(z_1 z_2) = (z_2 z_1)$ for anticommuting 2-component spinors). The second term on the right-hand side can be rewritten in a similar way after applying the

$SU(2)_W$ Schouten identity. The final result is

$$\epsilon_{ij}\epsilon_{mn}(\overline{e}_p L^i_q)(L^j_r L^m_s)H^n = \epsilon_{ij}\epsilon_{mn}H^n \left[(\overline{e}_p L^i_q)(L^j_r L^m_s) + (\overline{e}_p L^i_s)(L^j_r L^m_q) - (\overline{e}_p L^i_s)(L^j_q L^m_r)\right]. \tag{4.11}$$

In a similar way as in the previous section, this gives 27 relations among the 81 flavor permutations, this time among operators differing in L flavor permutations. Again only 24 relations are non-trivial, leaving 57 independent operators. So the structure in Eq. (4.9) encapsulates all of the independent operators of the form $LLL\overline{e}H$.

Now suppose that instead of starting with the structure in Eq. (4.9), the first guess had been instead

$$\epsilon_{ij}\epsilon_{mn}(\overline{e}_p L^m_q)(L^i_r L^j_s)H^n. \tag{4.12}$$

Without even considering Fierz identities, we can see that this structure is antisymmetric in $\{r, s\}$, so it contains a maximum of $3 \times 3 \times 3 = 27$ independent flavor permutations, since a 3×3 antisymmetric matrix has only 3 independent entries. Thus we know that another structure is needed in order to get the full 57 independent flavor permutations. In this way the Hilbert series allows a check on the generality of a specific $SU(2)_L$ and $SU(2)_W$ index contraction structure.

5 Hilbert Series for BSM

In the phenomenological study of extensions to the Higgs sector, it is often necessary to write down the most general form of the Higgs potential including various multiplets of $SU(2)_W$ with potentially different hypercharges. Usually only renormalizable terms are included in the potential, but in some cases higher-dimensional terms are also necessary. In either case, this exercise is a perfect candidate for using the Hilbert series to check that the Lagrangian includes a complete set of operators up to a given dimension.

As an explicit example, consider extending the SM Higgs sector by adding a scalar multiplet χ which is a quadruplet under $SU(2)_W$ and has hypercharge $-1/2$, as was done in [1]. The dimension-2 terms for the Higgs sector potential are then trivial to construct, and there are no possible dimension-3 terms, but the task becomes more complicated at dimension 4. Using the character function for the $SU(2)$ quadruplet as given in Appendix A (Table 2), the argument of the PE for the SM Higgs Φ and the new field χ is

$$\chi\left(z^3 + z + \frac{1}{z} + \frac{1}{z^3}\right)u^{-1/2} + \overline{\chi}\left(z^3 + z + \frac{1}{z} + \frac{1}{z^3}\right)u^{1/2} + \Phi\left(z + \frac{1}{z}\right)u^{1/2} + \overline{\Phi}\left(z + \frac{1}{z}\right)u^{-1/2}. \tag{5.1}$$

Generating the first few terms of the unrefined Hilbert series gives

$$\mathfrak{H}(t) = 1 + 2\,t^2 + 11\,t^4 + 31\,t^6 + 94\,t^8 + 222\,t^{10} + \ldots, \tag{5.2}$$

so we see that there are 11 independent operators at the dimension-4 level.[3]

[3] As all operators are scalars, counting dimensions is the same as counting the number of spurions. This

As a side note, the field and group representation content for this example is simple enough that the closed form for the unrefined Hilbert series can be calculated. The result is

$$\mathfrak{H}(t) = \frac{1 + 4t^4 + 9t^6 + 17t^8 + 13t^{10} + 17t^{12} + 9t^{14} + 4t^{16} + t^{20}}{(t^4 + t^2 + 1)^2 (t^2 + 1)^4 (t^2 - 1)^8}. \tag{5.3}$$

Since the complete Hilbert series can be calculated, and since the result is a rational function of t, it is always possible to calculate a closed form for the coefficients of the series (see for example Chapter 7 of [45]) . One way to do this is to use the MATHEMATICA function `SeriesCoefficient`, putting in a general integer n as the argument for the coefficient order. The resulting expression for the coefficients of Eq. (5.3) is complicated and not that useful, since calculating the Taylor series to the necessary order is simple in this case. However, there might be situations where calculating a closed form for the coefficients could be useful for figuring out general information about the form of the series such as the asymptotic form of the coefficients.

Returning to the calculation of the (renormalizable) Higgs potential, by using MATHEMATICA or a similar program, we can generate the multi-graded Hilbert series and extract the relevant terms, which gives operators with the following field content:

$$\begin{gathered} 2\,\chi\chi^\dagger\chi\chi^\dagger, \quad \Phi^\dagger\chi\chi^\dagger\chi^\dagger, \quad \Phi^\dagger\Phi^\dagger\chi^\dagger\chi^\dagger, \quad \Phi\chi\chi\chi^\dagger, \quad 2\,\Phi\Phi^\dagger\chi\chi^\dagger, \\ \Phi\Phi^\dagger\Phi^\dagger\chi^\dagger, \quad \Phi\Phi\chi\chi, \quad \Phi\Phi\Phi^\dagger\chi, \quad \Phi\Phi\Phi^\dagger\Phi^\dagger. \end{gathered} \tag{5.4}$$

Counting up the terms we do indeed get eleven. Note that for the field content $\chi\chi^\dagger\chi\chi^\dagger$, the Hilbert series tells us that there are two independent terms, which is indeed the conclusion reached in the paragraph following Eq. (13) of Ref. [1]. The same conclusion holds for the field content $\Phi^\dagger\Phi\chi^\dagger\chi$. The Hilbert series provides these results without the need for any detailed calculations involving group invariants. These calculations can of course be done without much trouble for these dimension-4 operators, but it is nice to have this check. Also note that the Hilbert series result (5.4) gives an operator and its hermitian conjugate that are not included in the result for the most general renormalizable Higgs potential given in Eq. (12) of Ref. [1], namely the operator with field content $\chi\chi\chi^\dagger\Phi$. Working out the correct $SU(2)_W$ index contractions necessary (using the same index notation as Ref. [1]) gives the operator structure

$$\chi_{ijk}\chi_{k'mn}\chi^\dagger_{ijm}\Phi_{n'}\epsilon_{kk'}\epsilon_{nn'} + \text{h.c.} \tag{5.5}$$

Using the Hilbert series has allowed the quick identification of an operator that was missed in the original potential; a clear demonstration of the practical efficacy of this technique.

The 31 independent dimension-6 operators for this example can also be easily generated.

can be generalized by weighting each spurion by its mass dimension, i.e $H \to \epsilon H, Q \to \epsilon^{3/2} Q$, then collecting like terms.

The resulting operators are

$$\begin{gathered}3\,(\chi^\dagger\chi)^3, \quad 2\,\Phi^\dagger(\chi^\dagger)^3\chi^2, \quad 2\,(\Phi^\dagger)^2\chi(\chi^\dagger)^3, \quad (\Phi^\dagger)^3(\chi^\dagger)^3, \quad 2\,\Phi\chi^3(\chi^\dagger)^2, \\ 4\,\Phi\Phi^\dagger\chi^2(\chi^\dagger)^2, \quad 3\,\Phi(\Phi^\dagger)^2\chi(\chi^\dagger)^2, \quad \Phi(\Phi^\dagger)^3(\chi^\dagger)^2, \quad 2\,\Phi^2\chi^3\chi^\dagger, \quad 3\,\Phi^2\Phi^\dagger\chi^2\chi^\dagger, \\ 3\,\Phi^2(\Phi^\dagger)^2\chi\chi^\dagger, \quad \Phi^2(\Phi^\dagger)^3\chi^\dagger, \quad \Phi^3\chi^3, \quad \Phi^3\Phi^\dagger\chi^2, \quad \Phi^3(\Phi^\dagger)^2\chi, \quad (\Phi^\dagger\Phi)^3.\end{gathered} \tag{5.6}$$

6 Thoughts on derivatives

The examples we have worked through so far have been limited to collections of spurions without derivatives. Derivatives are necessary if we wish to apply the Hilbert series to a wider set of problems, such as the full set of SMEFT operators of a given dimension. An immediate complication when including derivatives are the equations of motion (EOM) – relations among operators that are not governed by symmetries or invariances.

At first glance, it seems like the Hilbert series is ill-equipped to handle EOM. However, the Hilbert series is not a Lagrangian and is merely a tool to count invariants formed from whatever spurions are put into the PE/PEF. For the purposes of counting invariants, the role of the EOM is to remove spurions – namely derivatives on fields – by swapping them for different combinations of spurions with no derivatives, which *is* something that can be handled by the Hilbert series methodology. Consider the EOM for the left-handed Standard Model quark doublet Q:

$$i\not{D}Q = y_u^\dagger\, u^{c\dagger}\epsilon H^* + y_d^\dagger\, d^{c\dagger}H, \tag{6.1}$$

which allows one to remove $\not{D}Q$ from the set of spurions. Derivatives of fermions can still appear in invariants, first showing up at dimension 7 [31]. For example, consider the operator

$$\mathcal{O}_{LLud} = \epsilon_{ij}\,(d^c\sigma_\mu u^{c\dagger})(L^i D^\mu L^j). \tag{6.2}$$

The difference between the derivatives in the two equations above lies in the Lorentz group representation. Acting on a left-handed fermion with a derivative, we get

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left(0,\frac{1}{2}\right) = \left(\frac{1}{2},0\right)\oplus\left(\frac{1}{2},1\right). \tag{6.3}$$

The EOM in Eq. (6.1) involves the $(\frac{1}{2},0)$ representation only. Therefore we can incorporate the EOM for a fermion field ψ by including an additional spurion for the $(\frac{1}{2},1)$ part of $D_\mu\psi$, but *omitting* the $(\frac{1}{2},0)$ spurion.

If we follow this logic, as we add further derivatives $D_\mu D_\nu\psi$, etc. we should add a new spurion to the PE/PEF for every new representation of the Lorentz group that is formed. For derivatives of scalars such as the Higgs field H, $D_\mu H$ is new and should be added to the PE. Further derivatives, such as $D_\mu D_\nu H$, will contain a piece with $(0,0)$ Lorentz structure; this piece should not be included as a separate spurion as it is eliminated by the Higgs EOM, $\Box H = m^2 H + \cdots$. Similarly, for derivatives of the field strength tensors $D_\lambda X^\pm_{\mu\nu}$, the $(\frac{1}{2},\frac{1}{2})$ representation should be omitted.

Even after removing the derivative spurions, the Hilbert series still lacks information about integration by parts, so invariants involving derivatives need to be checked manually for redundancy. This check will get tedious if we Taylor expand the Hilbert series to arbitrarily high mass dimension operators, but it should be reasonably manageable for the dimension-8 operators. Further investigation along these lines will appear in Ref. [26].

7 Conclusion

The Hilbert series is a mathematical method providing a generating function for group invariants. Since this method is not well known in the phenomenological community, we have introduced the Hilbert series through simple examples and provided several practical illustrations of its use. Through the presented cases, we see that the Hilbert series technique proves very useful for common applications. Any calculation where a BSM or effective field theory Lagrangian contains fields transforming under uncommon group representations or where operators of higher order in mass dimension are needed, the Hilbert series is an easy way to get the right number of invariants. This is important in particular when looking for small deviations from known physics by comparing to precision measurements as redundant or missing terms in Lagrangians could invalidate analyses.

With the inclusion of derivatives and the equations of motion, and applying sufficient skill towards reducing the computational load of calculating the necessary residues, it should be possible to compute a complete Hilbert series for the Standard Model effective field theory. This generating function would contain at each dimension a coefficient specifying the total number of independent operators of that dimension. If the task was extended to calculate the multi-graded Hilbert series, one would then be able to directly see the different forms of operators at each order simply by expanding out the Hilbert series. We intend to work towards this goal in an upcoming study [26].

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A Mathematical background

A.1 Characters

The character of an irreducible group representation is the trace of the matrix giving the representation. Any matrix representing a group element in a given representation will have the same character.

In general, the characters of irreducible Lie group representations are obtained from the Cartan matrices, using the Freudenthal recursion formula to find the correct multiplicities of weights (see [21] and references therein). In practice, this calculation can be trivially done using the Mathematica package LieART by utilizing the `WeightSystem` command [46]. The results for some common representations are shown in Table 2. The characters can also be found using the character generating functions outlined in [21].

Representation	**Character function**
$SU(2)$ fundamental	$z + \frac{1}{z}$
$SU(2)$ quadruplet	$z^3 + z + \frac{1}{z} + \frac{1}{z^3}$
$SU(3)$ fundamental	$z_1 + \frac{z_2}{z_1} + \frac{1}{z_2}$
$SU(3)$ anti-fundamental	$\frac{1}{z_1} + \frac{z_1}{z_2} + z_2$
$SU(3)$ adjoint	$z_1 z_2 + \frac{z_2^2}{z_1} + \frac{z_1^2}{z_2} + 2 + \frac{z_2}{z_1^2} + \frac{z_1}{z_2^2} + \frac{1}{z_1 z_2}$
$SU(2)$ adjoint	$z^2 + 1 + \frac{1}{z^2}$
$U(1)$ with charge Q	z^Q

Table 2. Character functions for several common group representations. Note that setting all of the variables to 1 gives the dimension of the representation.

Many common group calculations can be done using characters. For example, consider taking the tensor product of two $SU(2)$ fundamentals. Using the characters from Table 2, this looks like

$$\left(z + \frac{1}{z}\right)\left(z + \frac{1}{z}\right) = z^2 + 2 + \frac{1}{z^2} = 1 + \left(z^2 + 1 + \frac{1}{z^2}\right), \tag{A.1}$$

which is just the familiar triplet-singlet spin decomposition $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.

A.2 The plethystic exponential

For a multivariable function $f(t_1, \ldots, t_n)$ satisfying the property of going to zero at the origin, the plethystic exponential (PE) [7, 9, 10, 13, 21] is

$$\mathrm{PE}[f(t_1, \ldots, t_n)] \equiv \exp\left(\sum_{r=1}^{\infty} \frac{1}{r} f(t_1^r, \ldots, t_n^r)\right). \tag{A.2}$$

The plethystic exponential generates all symmetric combinations of the variables of the function $f(t_1, \ldots, t_n)$. For example, the plethystic exponential of $f(A, B) = A + B$ is

$$\mathrm{PE}[A + B] = \frac{1}{(1 - A)(1 - B)} = 1 + A + B + A^2 + AB + B^2 + \ldots \tag{A.3}$$

For fermionic variables we are interested in antisymmetric combinations instead of symmetric combinations, since fermions obey Fermi-Dirac statistics. The fermionic plethystic exponential is defined as [21]:

$$\mathrm{PEF}[f(t_1,\ldots,t_n)] \equiv \exp\left(\sum_{r=1}^{\infty}\frac{(-1)^{r+1}}{r}f(t_1^r,\ldots,t_n^r)\right). \tag{A.4}$$

For example,

$$\mathrm{PEF}[A+B] = 1 + A + B + AB, \tag{A.5}$$

where the first three terms are trivial, and the last term is indeed antisymmetric under $A \leftrightarrow B$, since fermions anticommute. In the body of this paper, the fermionic plethystic exponential is not always explicitly differentiated from the ordinary plethystic exponential, but the factor of $(-1)^{r+1}$ is always included with fermionic variables (spurions).

A.3 Group integration with the Haar measure

It is possible to integrate over the manifold of a Lie group by using an invariant measure known as the Haar measure. This group integration projects out invariant quantities from the combinatorial expansion of characters provided by the plethystic exponential. The Haar measures that are used in this paper are as follows (taken from [10]):

$$\int_{U(1)} \mathrm{d}\mu_{U(1)} = \frac{1}{2\pi \mathrm{i}}\oint_{|z|=1|}\frac{\mathrm{d}z}{z}, \tag{A.6}$$

$$\int_{SU(2)} \mathrm{d}\mu_{SU(2)} = \frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\frac{\mathrm{d}z}{z}(1-z^2), \tag{A.7}$$

$$\int_{SU(3)} \mathrm{d}\mu_{SU(3)} = \frac{1}{(2\pi \mathrm{i})^2}\oint_{|z_1|=1}\frac{\mathrm{d}z_1}{z_1}\oint_{|z_2|=1}\frac{\mathrm{d}z_2}{z_2}(1-z_1z_2)\left(1-\frac{z_1^2}{z_2}\right)\left(1-\frac{z_2^2}{z_1}\right). \tag{A.8}$$

Further examples of Haar measures for various Lie groups can be found in [10], where a general formula is also presented.

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