Light-front vacuum

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The Light-Front Vacuum

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Background: The vacuum in the light-front representation of quantum field theory is trivial while vacuum in the equivalent canonical representation of the same theory is non-trivial.
Purpose: Understand the relation between the vacuum in light-front and canonical representations of quantum field theory and the role of zero-modes in this relation.
Method: Vacuua are defined as linear functionals on an algebra of field operators. The role of the algebra in the definition of the vacuum is exploited to understand this relation.
Results: The vacuum functional can be extended from the light-front Fock algebra to an algebra of local observables. The extension to the algebra of local observables is responsible for the inequivalence. The extension defines a unitary mapping between the physical representation of the local algebra and a sub-algebra of the light-front Fock algebra.
Conclusion: There is a unitary mapping from the physical representation of the algebra of local observables to a sub-algebra of the light-front Fock algebra with the free light-front Fock vacuum. The dynamics appears in the mapping and the structure of the sub-algebra. This correspondence provides a formulation of locality and Poincaré invariance on the light-front Fock space.

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I. INTRODUCTION

The light-front [1] representation of quantum field theory has a number of properties that make it advantageous for some applications, particularly for applications involving electro-weak probes of the strong interaction and non-perturbative treatments of the strong interaction [2]. The properties of the light-front representation of quantum field theory that lead to these advantages are (1) there is a seven parameter subgroup of the Poincaré group that is free of interactions (2) there is a three-parameter group of Lorentz boosts that is also free of interactions (3) the generator of translations in the x^- direction tangent to the light front is free of interaction and satisfies a spectral condition (4) and the vacuum of the interacting theory is the same as the vacuum of the free field theory.

This is in contrast to the canonical formulation of quantum field theory, where the six-parameter Euclidean group is free of interactions, boosts depend on interactions and do not form a subgroup, the spectrum of the translation generators is the real line, and the vacuum of the interacting theory is not a vector in the free-field Fock space.

These differences have motivated many investigations into the nature of the light-front vacuum [3] [4] [5] [6] [7] [8] [9] [10] [11] [12] [13].

While the light-front formulation of quantum field theory has advantages in these applications, the predictions should be independent of the representation of the theory. It is found that some light-front calculations require additional "zero-mode" contributions in order to maintain the equivalence with covariant perturbation theory [14] [15].

This work has two goals. The first is to understand how the vacuum can be trivial in one representation of field theory and not in another equivalent representation of the same theory. A second goal is to understand the role of zero modes in the light-front vacuum.

It is instructive to consider what happens in the case of a free scalar field theory. The ground state of a harmonic oscillator is uniquely determined by the condition that it is annihilated by the annihilation operator. Free scalar fields behave like infinite collections of harmonic oscillators. The analogy suggests that the vacuum of the free-field theory is defined by the condition that it is annihilated by a collection of annihilation operators, where the annihilation operators are labeled by the three momentum. Free fields can be expressed as integrals over the 3-momentum that are linear in the creation and annihilation operators. A change of variables gives an equivalent expression for the same field as an integral over the light-front components of the momenta. The resulting light-front and canonical creation and annihilation operators are related by a multiplicative factor, which is the square root of the Jacobian of the variable change from the three momenta to the three light-front components of the four momentum. The multiplication of the annihilation operator by a Jacobian should not impact the linear equation that defines the vacuum, except possibly for momenta where the Jacobian becomes singular. This perspective suggests that both representations of the free field theory should have the same vacuum.

If two free scalar fields with different masses are restricted to the light front, the masses do not appear in representations of fields, or in the commutators of the creation and annihilation operators. In addition, the fields restricted to the light front are irreducible in the sense that it is possible to extract both the creation an annihilation operators directly from the field restricted to the light front. It follows from these properties that the two scalar fields with different masses restricted to the light front and their associated vacuum vectors are unitarily equivalent [3].

On the other hand, two free scalar fields with different masses are related by a canonical transformation that expresses the annihilation operator for the mass-1 field theory as a linear combination of a creation and annihilation operators for the mass-2 field theory. It was explicitly shown by Haag [16] that, while this canonical transformation can be represented by a unitary transformation for finite numbers of degrees of freedom, this is no longer true for canonical free fields. For free fields of different mass, the transformation is unitary if there is a momentum cutoff, however when the cutoff is removed the generator of this unitary transformation becomes ill-defined in the sense that it maps every vector in the mass 1 representation of the Hilbert space, including the vacuum, to a vector with infinite norm. In this case the theories are not related by a unitary transformation and the vacuua live in different Hilbert space representations. This is the well-known problem of inequivalent representations of the canonical field algebra for theories of an infinite number of degrees of freedom [17][18][19].

These observations imply that the conventional mass-1 annihilation operator is related to the corresponding light-front annihilation operator by a trivial Jacobian, which is unitarily equivalent to the mass-2 light-front annihilation operator. This operator is in turn related to the conventional mass-2 annihilation operator by another Jacobian. This suggests that the vacua for both theories are the same, or at least unitarily related, however this contradicts the observation that the annihilation operators of the conventional free field theories are related by a canonical transformation that cannot be realized unitarily.

The virtue of free fields is that they are well understood. Others [20][4][12] have used free fields to develop insight into various aspects of this problem. Algebraic methods provide a resolution of the apparent inconsistency discussed above. Algebraic methods were used in the seminal work of Leutwyler, Klauder and Streit [3] and Schlieder and Seiler [4]. This will be discussed in more detail in the subsequent sections.

The problem is that the requirement that the vacuum is the state annihilated by an annihilation operator does not give a complete characterization of the vacuum of a local field theory. In a local field theory the vacuum is also a positive invariant linear functional on the field algebra. The relevant algebra is the algebra of local observables, which is not the same as the canonical equal-time algebra or the algebra generated by fields restricted to a light front. While the desired positive linear functional can be expressed by taking vacuum expectation values of elements of the algebra with the vacuum defined by the annihilation operator, the definition of the vacuum functional also depends on the choice of algebra. Specifically, functionals that agree on a sub-algebra do not have to agree on the parent algebra. The physically relevant algebra for a quantum field theory must be large enough to be Poincaré invariant and to contain local observables. Both the canonical and light-front algebras (which are defined later) are irreducible in the sense that they can be used to build both the Hilbert space and operators on the Hilbert space, but they are not closed under Poincaré transformations and do not contain local observables. While they are not sub-algebras of the local algebra, for the case of free fields the irreducibility allows the linear functional that defines the vacuum on these algebras to be extended to the local algebra. In the case of the canonical equal-time algebra the extension is essentially unique [21][22], while in the case of the light-front algebra there are many extensions that lead to inequivalent representations of the local algebra. These extensions define a unitary map that relates the local algebra and physical vacuum to a sub-algebra of the light-front algebra with the free Fock vacuum.

In the light-front case the extension to the local algebra requires additional attention to what happens when the $+$ component of the momentum is 0. For free fields the extension to the local algebra regularizes apparent singularities that are associated with $p^+ = 0$.

The algebraic methods discussed for free fields can also be applied to interacting fields by first representing them using a Haag expansion [23][24][16] as a series in a complete set of normal products of asymptotic (*in* or *out*) fields. The asymptotic fields are free fields, and each of them can be expressed as an extension of a free field restricted to the light front. This results in an extension of the light-front algebra to the local algebra generated by the interacting field.

The coefficients of the Haag expansion of the interacting Heisenberg field are invariant (covariant) functions [23]. Additional properties of these functions follow because the Heisenberg fields and asymptotic fields are operator-valued tempered distributions. When the asymptotic fields in the Haag expansion are replaced by the extensions of free fields on the light-front to the asymptotic fields, the result is an expansion of the Heisenberg fields in terms of normal products of free fields restricted to a light front. The coefficient functions in this expansion regulate the $p^+ = 0$ behavior of the free light-front fields.

More singular behavior can occur in operators, like Poincaré generators, that involve products of fields at the same point on the light front. These products are ill-defined and a renormalization is necessary for them to make sense as operators. Divergences appear for both large momenta as well as $p^+ = 0$. The resulting finite theory needs to be independent of the orientation of the light front. Invariance under change of orientation of the light front is equivalent to rotational covariance of the theory [25][26] [27][28] [29][30]. Since $p^+ = 0$ for one light front corresponds to some component of p becoming infinite with a different light front, rotational covariance necessarily puts important constraints on the renormalization.

This also suggests that the zero mode issue is more complicated in 3+1 dimensional theories than 1+1 dimensional theories, where rotational covariance plays no role. Specifically, the extension to the local algebra must recover both the rotational covariance and locality of the theory.

In the next section we define our notation and conventions. In section three we discuss inequivalent representations. In section four we discuss the light-front vacuum. In section five we introduce four different field algebras that we use in this paper. Section six discusses the light-front Fock algebra. In section seven we discuss the meaning of equivalent theories. In section eight we discuss extensions of the light-front Fock algebra. In section 9 we discuss dynamical theories. Zero modes are discussed in section ten. The results are summarized in section eleven.

II. NOTATION

This section defines the notation and conventions that will be used in the remainder of the paper. The signature of the Minkowski metric is $(-,+,+,+)$. Space-time components of four vectors, x, have Greek indices

$$x^{\mu} := (x^0, \mathbf{x}). \tag{1}$$

A light front is a three-dimensional hyper-plane in Minkowski space satisfying

$$x^{+} := x^0 + \hat{\mathbf{n}} \cdot \mathbf{x} = 0 \tag{2}$$

where $\hat{\mathbf{n}}$ is a fixed unit vector. Points on the light front have either a space-like or light-like separation. Coordinates of points on the light front are

$$\tilde{\mathbf{x}} = (x^{-}, \mathbf{x}_{\perp}) \tag{3}$$

where

$$x^{-} = x^0 - \hat{\mathbf{n}} \cdot \mathbf{x} \qquad \text{and} \qquad \mathbf{x}_{\perp} = \mathbf{x} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}). \tag{4}$$

The light-front components of a four vector x are

$$x = (x^{+}, x^{-}, \mathbf{x}_{\perp}). \tag{5}$$

The Lorentz invariant scalar product of two four vectors in terms of their light-front components is

$$x \cdot y = -\frac{1}{2} x^{+} y^{-} - \frac{1}{2} x^{-} y^{+} + \mathbf{x}_{\perp} \cdot \mathbf{y}_{\perp}. \tag{6}$$

The Fourier transform of a Schwartz test function $f(x) \in S(\mathbb{R}^4)$ is

$$f(p) = \frac{1}{(2\pi)^2} \int e^{-ip\cdot x} f(x) \frac{1}{2} dx^{+} dx^{-} d^2 x_{\perp} = \frac{1}{(2\pi)^2} \int e^{-ip\cdot x} f(x) d^4 x \tag{7}$$

$$f(x) = \frac{1}{(2\pi)^2} \int e^{ix\cdot p} f(p) \frac{1}{2} dp^{+} dp^{-} d^2 p_{\perp} = \frac{1}{(2\pi)^2} \int e^{ix\cdot p} f(p) d^4 p \tag{8}$$

where $d^4 x = \frac{1}{2} dx^{+} dx^{-} d^2 x_{\perp}$. A "mathematically imprecise" notation is used to represent functions $f(x)$ and their Fourier transforms $f(p)$, which are related by (7) and (8). In what follows p, k and q represent momentum variables and x, y and z represent coordinate variables. It is useful to define

$$\tilde{\mathbf{p}} := (p^{+}, \mathbf{p}_{\perp}) \qquad \tilde{\mathbf{x}} \cdot \tilde{\mathbf{p}} := -\frac{1}{2} x^{-} p^{+} + \mathbf{x}_{\perp} \cdot \mathbf{p}_{\perp} \tag{9}$$

and

$$d\tilde{\mathbf{x}} := dx^{-} d\mathbf{x}_{\perp} \qquad d\tilde{\mathbf{p}} := dp^{+} d\mathbf{p}_{\perp}. \tag{10}$$

In this notation the three-dimensional Fourier transform of functions $\tilde{f}(\tilde{\mathbf{x}})$ in the light-front variables are

$$\tilde{f}(\tilde{\mathbf{p}}) = \frac{1}{2^{1/2}(2\pi)^{3/2}} \int e^{-i\tilde{\mathbf{p}}\cdot\tilde{\mathbf{x}}} \tilde{f}(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}} \tag{11}$$

and

$$\tilde{f}(\tilde{\mathbf{x}}) = \frac{1}{2^{1/2}(2\pi)^{3/2}} \int e^{i\tilde{\mathbf{p}}\cdot\tilde{\mathbf{x}}} \tilde{f}(\tilde{\mathbf{p}}) d\tilde{\mathbf{p}}. \tag{12}$$

The ˜ indicates functions supported on the light front and their Fourier transforms.

The light front is invariant under a seven-parameter subgroup, called the light-front kinematic subgroup of the Poincaré group. This subgroup is generated by the three-parameter subgroup of translations tangent to the light front,

$$\tilde{\mathbf{x}} :\rightarrow \tilde{\mathbf{x}}' = \tilde{\mathbf{x}} + \tilde{\mathbf{a}} \tag{13}$$

a three-parameter subgroup of light-front preserving boosts,

$$x^+ \rightarrow x^{+\prime} = q^+ x^+ \qquad \mathbf{x}_\perp \rightarrow \mathbf{x}'_\perp = \mathbf{x}_\perp + \mathbf{q}_\perp x^+ \tag{14}$$

$$x^- \rightarrow x^{-\prime} = \frac{1}{q^+}(x^- + \mathbf{q}_\perp^2 x^+ + 2\mathbf{q}_\perp \cdot \mathbf{x}_\perp) \tag{15}$$

and rotations about the $\hat{\mathbf{n}}$ axis.

Light-front boosts applied to points on the light front only transform x^-:

$$\mathbf{x}_\perp \rightarrow \mathbf{x}'_\perp = \mathbf{x}_\perp \tag{16}$$

$$x^{-\prime} = \frac{1}{q^+}(x^- + 2\mathbf{q}_\perp \cdot \mathbf{x}_\perp), \tag{17}$$

however the conjugate momentum variables $\tilde{\mathbf{p}}$ can take on any value under these transformations provided $p^+ \neq 0$:

$$p^+ \rightarrow p^{+\prime} = q^+ p^+ \qquad \mathbf{p}_\perp \rightarrow \mathbf{p}'_\perp = \mathbf{p}_\perp + \mathbf{q}_\perp p^+. \tag{18}$$

When $p^+ = 0$ the light-front boosts leave $\tilde{\mathbf{p}}$ unchanged.

The light-front inner product is defined by

$$(\tilde{f}, \tilde{g}) = \int \frac{d\tilde{\mathbf{p}}\theta(p^+)}{p^+} \tilde{f}^*(\tilde{\mathbf{p}})\tilde{g}(\tilde{\mathbf{p}}). \tag{19}$$

This inner product is invariant with respect to the kinematic subgroup because (1) the measure is invariant and (2) $\tilde{\mathbf{p}} \rightarrow \tilde{\mathbf{p}}'$ does to involve p^-. This inner product has a logarithmic singularity for functions $\tilde{f}(\tilde{\mathbf{p}})$ that are non-zero at $p^+ = 0$.

III. INEQUIVALENT REPRESENTATIONS

Free fields look like collections of uncoupled harmonic oscillators. For a single oscillator the Hamiltonian in dimensionless variables is

$$H = \frac{1}{2}(x^2 + p^2) = a^\dagger a + \frac{1}{2} \tag{20}$$

where

$$[x, p] = i \qquad a := (x + ip)/\sqrt{2} \qquad [a, a^\dagger] = 1. \tag{21}$$

The equation

$$a|0\rangle = 0. \tag{22}$$

determines the ground state $|0\rangle$ of the oscillator.

The canonical transformation

$$x \rightarrow x' = \alpha x; p \rightarrow p' = \frac{p}{\alpha} \tag{23}$$

preserves $[x,p]=[x',p']=i$, and leads to a linear relation between the original and transformed annihilation operators:

$$a' = \frac{1}{2}(\alpha + \frac{1}{\alpha})a + \frac{1}{2}(\alpha - \frac{1}{\alpha})a^\dagger = \cosh(\eta)a + \sinh(\eta)a^\dagger . \tag{24}$$

This canonical transformation can be implemented by the unitary operator U:

$$a' = UaU^\dagger \qquad U = e^{\frac{\eta}{2}(aa - a^\dagger a^\dagger)} = e^{iG}. \tag{25}$$

The transformed Hamiltonian

$$H' = \frac{1}{2}(x'^2 + p'^2) = a'^\dagger a' + \frac{1}{2} \tag{26}$$

has the same eigenvalues as H. The transformed ground state vector is related to the original ground state vector by

$$|0'\rangle = U|0\rangle. \tag{27}$$

The canonical transformation (23) is the single-oscillator version of the canonical transformation that changes the mass in a free field theory.

The canonically conjugate operators in a free scalar field theory have the form

$$\phi(x) = (2\pi)^{-3/2} \int d\mathbf{p} \frac{1}{\sqrt{2\omega_m(\mathbf{p})}} (a(\mathbf{p})e^{ip\cdot x} + a^\dagger(\mathbf{p})e^{-ip\cdot x}) \tag{28}$$

$$\pi(x) = -i(2\pi)^{-3/2} \int d\mathbf{p} \sqrt{\frac{\omega_m(\mathbf{p})}{2}} (a(\mathbf{p})e^{ip\cdot x} - a^\dagger(\mathbf{p})e^{-ip\cdot x}) \tag{29}$$

where x is restricted to $t=0$ and $p^0 = \omega_m(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$ is the energy.

The canonical transformation that changes the mass $m \to m'$ involves multiplying the integrand of $\phi(x)$ by $\alpha(\mathbf{p}) = \sqrt{\frac{\omega_m(\mathbf{p})}{\omega_{m'}(\mathbf{p})}}$ and the integrand of $\pi(x)$ by $1/\alpha(\mathbf{p})$. This has the same structure as the single oscillator canonical transformation (23) except in this case the parameter α depends on the momentum.

As in the case of the single oscillator, define $\eta(\mathbf{p})$ by

$$\cosh(\eta(\mathbf{p})) = \frac{1}{2}(\alpha(\mathbf{p}) + \frac{1}{\alpha(\mathbf{p})}) \qquad \sinh(\eta(\mathbf{p})) = \frac{1}{2}(\alpha(\mathbf{p}) - \frac{1}{\alpha(\mathbf{p})}). \tag{30}$$

By analogy with (25), the formal generator G of the canonical transform that changes the mass is

$$G \to -i \int \frac{\eta(\mathbf{p})}{2} \left(a(\mathbf{p})a(\mathbf{p}) - a^\dagger(\mathbf{p})a^\dagger(\mathbf{p}) \right) d\mathbf{p}. \tag{31}$$

A simple calculation shows

$$\|G|0\rangle\|^2 = \frac{1}{4} \int \eta(\mathbf{p})^2 d\mathbf{p}\delta(0) = \infty \tag{32}$$

which implies that the domain of G is empty.

The consequence is that the canonical transformation relating free field theories with different masses cannot be realized by a unitary transformation. The second is that the canonical vacuum vectors characterized by

$$a_i(\mathbf{p})|0\rangle_i = 0 \qquad i \in \{1,2\} \tag{33}$$

are not unitarily related. The existence of unitarily inequivalent representations of the canonical commutation for systems of an infinite number of degrees of freedom is well known [17]. The example above was discussed by Haag [16].

The interest in this example is that, in contrast to canonical equal-time fields, for free fields restricted to a light front the vacuum vectors and annihilation operators for free fields of different mass are unitarily related [3].

IV. LIGHT-FRONT VACUUM

In light-front field theory the generator P^+ of translations in the x^- direction is a kinematic operator satisfying the spectral condition, $P^+ \geq 0$. The dynamical operator is P^-, which generates translations normal to the light front, is the light-front Hamiltonian. It can formally be expressed as the sum of a non-interacting term and an interaction

$$P^- = P_0^- + V. \tag{34}$$

Kinematic translational invariance of P^- and P_0^- on the light-front requires that the interaction commutes with P^+:

$$[P^+, V] = 0. \tag{35}$$

It follows that

$$P^+ V|0\rangle = V P^+|0\rangle = 0 \tag{36}$$

for a light-front translationally invariant vacuum.

Because P^+ is kinematic P^+ is the sum of the single particle generators. It satisfies a spectral condition $P^+ = \sum_i P_i^+ \geq 0$. If V can be expressed as a kernel integrated against creation and annihilation operators, then the coefficient of the pure creation terms in the interaction must vanish unless $p_i^+ = 0$ for each creation operator. If this kernel is a continuous function of p^+ the interaction will necessarily leave the vacuum unchanged. This does not rule out singular contributions at $p^+ = 0$ which are associated with zero-modes of the theory.

For the special case of a free field of mass m the light-front representation of the field is obtained by changing the integration variable from $\mathbf{p}$ to the three light-front components of the four momentum. The result is

$$\phi(x) = (2\pi)^{-3/2} \int \frac{dp^+ d\mathbf{p}_\perp \theta(p^+)}{\sqrt{2p^+}} (\frac{\sqrt{\omega_m(\mathbf{p})}}{p^+} a(\mathbf{p}) e^{ip\cdot x} + \sqrt{\frac{\omega_m(\mathbf{p})}{p^+}} a^\dagger(\mathbf{p}) e^{-ip\cdot x}). \tag{37}$$

This leads to the following relation between the light-front and canonical annihilation operators

$$a_{lf}(\tilde{\mathbf{p}}) := a_{lf}(p^+, \mathbf{p}_\perp) = \sqrt{\frac{\omega_m(\mathbf{p})}{p^+}} a(\mathbf{p}) \tag{38}$$

which satisfy

$$[a_{lf}(p^+, \mathbf{p}_\perp), a_{lf}^\dagger(q^+, \mathbf{q}_\perp)] = \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{q}}) = \delta(p^+ - q^+)\delta(\mathbf{p}_\perp - \mathbf{q}_\perp). \tag{39}$$

As in the canonical case, the light front-vacuum is characterized by the condition that it is annihilated by the annihilation operator

$$a_{lf}((p^+, \mathbf{p}_\perp)|0\rangle = \sqrt{\frac{\omega_m(\mathbf{p})}{p^+}} a(\mathbf{p})|0\rangle = 0 \tag{40}$$

If this field is restricted to the light front $x^+ = 0$ the mass-dependent factors disappear both from the field (37) and the commutation relations (39) of a_{lf} and $a_{lf}^\dagger$. In this case vacuum expectation values of a product of fields smeared with functions supported on the light front are all identical, independent of mass.

The equation (40) is the same for all free scalar fields.

V. FIELD ALGEBRAS

An obvious question is how are these different characterizations of the vacuum of a free field of mass m related. If the vacuum is uniquely characterized by the annihilation operator, like it is for a single harmonic oscillator, the arguments of section 3 imply that the vacuua for free fields with different masses are inequivalent. On the other hand, the arguments of the previous section imply that the light-front vacuum is identical to the canonical vacuum and light front vacuum vectors for different masses are unitarily related. The resolution of these apparently conflicting results can be understood by giving up the assumption that the vacuum is uniquely characterized by the annihilation operator.

The critical observation is that each of the vacuua discussed above are implicitly defined on different algebras of operators. The relevant algebra for a local quantum field theory must be invariant with respect to Poincaré transformation and contain observables associated with arbitrarily small spacetime volumes. These two conditions are *not* satisfied by both the light-front and canonical fixed-time field algebras, even though both of these algebras are irreducible.

This suggests that the characterization of the vacuum by an annihilation operator is incomplete. The vacuum functionals generated by both the light-front and canonical annihilation operators require non-trivial extensions in order to define linear functionals on a Poincaré invariant algebra of local observables.

Different algebras are generated by smeared fields of the form

$$\phi(f) = \int \phi(x) f(x) d^4x \tag{41}$$

where we limit our considerations to the case that $\phi(x)$ is a scalar field. The field algebra consist of polynomials in $\phi(f)$ or $e^{i\phi(f)}$. The difference being that the exponential form is bounded so there are no issues with domains, however this distinction is not important in this paper. What is relevant is that for a given scalar field there are different algebras that are distinguished by different choices of the space of test functions.

It is useful to identify the following four free field algebras.

We call the algebra generated by local observables, i.e. fields smeared with Schwartz functions in four space-time variables,

$$\{\phi(f) | f(x) \in S(\mathbb{R}^4)\}, \tag{42}$$

the local algebra.

The algebra generated by fields smeared with Schwartz functions in three coordinates on the light front,

$$\{\phi(f) | f(x) = \delta(x^+) f(\tilde{\text{x}}), \qquad f(\tilde{\text{x}}) \in S(\mathbb{R}^3)\}, \tag{43}$$

is called the light-front algebra. The algebra generated by fields and their time derivatives smeared with Schwartz functions in three spatial coordinates,

$$\{\phi(f) | f(x) = \delta(t) f(\mathbf{x}), -\dot{\delta}(t) f(\mathbf{x}) \qquad f(\mathbf{x}) \in S(\mathbb{R}^3)\}, \tag{44}$$

is called the canonical algebra. Integrating the field over these test functions gives the canonically conjugate $\phi(\cdot)$ and $\pi(\cdot)$ fields restricted to the $t = 0$ hyper-plane. The algebra generated by fields smeared with Schwartz functions in three light-front coordinates, restricted to have zero x^- integral

$$\{\phi(f) | f(x) = \delta(x^+) f(\tilde{\mathbf{x}}) \qquad f(\tilde{\mathbf{x}}) \in S(\mathbb{R}^3); \int f(\tilde{\mathbf{x}}) dx^- = 0\}, \tag{45}$$

is called the Schlieder-Seiler algebra[4]. The restriction on the test functions means the their Fourier transform vanishes at $p^+ = 0$, which makes the light-front inner product (19) finite.

The local algebra is the only one of these four algebras that is preserved under the Poincaré group and contains observables that can be localized in any space-time volume. The light-front and Schlieder-Seiler algebras are preserved under the kinematic subgroup of the light-front and the canonical algebra is preserved under the three-dimensional Euclidean group. The reason for distinguishing the light-front and Schlieder-Seiler algebras is that the light-front Fock vacuum is not defined on the light-front algebra while it is well-behaved on the Schlieder-Seiler algebra.

A vacuum on any of these algebras is a positive linear functional L. The positivity condition means that

$$L[A^\dagger A] \geq 0 \tag{46}$$

for any element A of the algebra. Each such positive linear functional $L[\cdot]$, can be used to construct a Hilbert-space representation of the algebra with inner product

$$\langle B | A \rangle = L[B^\dagger A]. \tag{47}$$

Formally, the Hilbert space representation is constructed by identifying vectors whose difference has zero norm and completing the space by adding Cauchy sequences. This is the standard GNS construction which is discussed in many texts[31] . There are additional constraints on the linear functional for it to represent a vacuum vector. One of these properties is that when $L[\cdot]$ is invariant with respect to the group that preserves the algebra, then the Hilbert space representation of the group is unitary.

Another property of all of these algebras is that they are irreducible. This means that any bounded linear operators on the Hilbert space can also be formally expressed in terms of operators in the algebra.

Acceptable candidates for vacuum vectors are invariant positive linear functionals on the algebra. While the vacuum expectation value of an element of the algebra with the vector annihilated by the annihilation operator defines a linear functional, the definition of the functional also depends on the algebra. Specifically, since positivity on a sub-algebra does not imply positivity on the parent algebra, and a restricted symmetry on a sub-algebra does not imply the full symmetry on the parent algebra, it follows that the characterization of the vacuum as a positive invariant linear functional depends on the choice of algebra. While this is different than the characterization in terms of annihilation operators, it does not preclude the possibility of using a vacuum characterized by a particular annihilation operator, but the linear functional must be extended to the local algebra.

While the canonical, light-front, and Schlieder-Seiler algebras for free fields are not sub-algebras of the local free-field algebra, these algebras are related to the local algebra by the standard expressions for the field operator, $\phi(x)$.

While local algebras exist for free or interacting Heisenberg fields, because fields are operator-valued distributions the other three algebras may not exist in general because the "test functions" (43,44,45) have a distributional component. However for the case of free fields these restrictions are defined. For the case of free fields, the formal representation of the fields in terms of creation and annihilation operators provides explicit relations among these four algebras.

VI. LIGHT-FRONT FOCK ALGEBRA

In this section we give a more complete description of the light-front and Schlieder-Seiler algebras and review some properties [3] [4] of these algebras.

The light-front Fock algebra is the algebra of free field operators smeared with Schwartz functions in the light-front coordinates $\tilde{\mathbf{x}}$ with $x^+ = 0$. A free scalar field of mass m expressed in terms of light-front coordinates has the form:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\tilde{\mathbf{p}}\theta(p^+)}{\sqrt{2p^+}} (a_{lf}(\tilde{\mathbf{p}})e^{ip\cdot x} + a^\dagger_{lf}(\tilde{\mathbf{p}})e^{-ip\cdot x}) \tag{48}$$

where the mass m only enters in $e^{\pm ip\cdot x}$ for $x^+ \neq 0$:

$$p^- = \frac{\mathbf{p}_\perp^2 + m^2}{p^+}. \tag{49}$$

On the light front, when $x^+ = 0$, all information about the mass is lost. The creation and annihilation operators satisfy

$$[a_{lf}(\tilde{\mathbf{p}}), a^\dagger_{lf}(\tilde{\mathbf{k}})] = \delta(\tilde{\mathbf{p}} - \tilde{\mathbf{k}}), \tag{50}$$

which is also independent of mass.

Equation (49) is where the free field dynamics enters; it extends the field restricted to a light front to a field, $\phi(x)$, on the local algebra that satisfy the Klein-Gordon equation [32]

$$(\Box - m^2)\phi(x) = 0. \tag{51}$$

Each mass defines a distinct and inequivalent extension of the light-front free-field algebra to different local algebras.

The Fourier transform of the field restricted to the light front can be decomposed into terms with positive and negative values of $\mathbf{p}^+$:

$$\tilde{\phi}(\tilde{\mathbf{p}}) := \frac{1}{(2)^{1/2}(2\pi)^{3/2}} \int d\tilde{\mathbf{x}} e^{-i\tilde{\mathbf{p}}\cdot\tilde{\mathbf{x}}} \phi(0, \tilde{\mathbf{x}}) =$$

$$\theta(p^+)\tilde{\phi}(\tilde{\mathbf{p}}) + \theta(-p^+)\tilde{\phi}(\tilde{\mathbf{p}}) = \theta(p^+)\sqrt{\frac{1}{p^+}} a_{lf}(\tilde{\mathbf{p}}) + \theta(-p^+)\sqrt{\frac{-1}{p^+}} a^\dagger_{lf}(-\tilde{\mathbf{p}}). \tag{52}$$

This decomposition only makes sense for $p^+ \neq 0$. This means that this decomposition is only defined on the Schlieder-Seiler algebra, where the test functions vanish at $p^+ = 0$.

This decomposition can be used to separate the creation and annihilation operators:

$$a_{lf}(\tilde{\mathbf{p}}) = \theta(p^+)\sqrt{\frac{p^+}{2}} \frac{1}{(2\pi)^{3/2}} \int d\tilde{\mathbf{x}} e^{-i\tilde{\mathbf{p}}\cdot\tilde{\mathbf{x}}} \phi(0, \tilde{\mathbf{x}}) \tag{53}$$

and

$$a^{\dagger}_{lf}(\tilde{\mathbf{p}}) = \theta(p^+)\sqrt{\frac{p^+}{2}}\frac{1}{(2\pi)^{3/2}}\int d\tilde{\mathbf{x}} e^{i\tilde{\mathbf{x}}\cdot\tilde{\mathbf{p}}}\phi(0,\tilde{\mathbf{x}}). \tag{54}$$

This property, that one can extract both creation and annihilation operators from the field restricted to the light front, is not shared with fields restricted to a space-like hyper-plane. On a space-like hyper-plane, both the field and its time derivative (which requires knowing about the field off of the fixed-time hyper-plane) are needed to independently extract the creation and annihilation operators.

Equation (52) implies that the field restricted to the light front can be decomposed into parts corresponding to the sign of p^+ in the Fourier transform

$$\tilde{\phi}(\tilde{\mathbf{x}}) := \phi(0,\tilde{\mathbf{x}}) = \phi^+(0,\tilde{\mathbf{x}}) + \phi^-(0,\tilde{\mathbf{x}}) \tag{55}$$

where

$$\tilde{\phi}^{\pm}(\tilde{\mathbf{x}}) = \frac{1}{(2)^{1/2}(2\pi)^{3/2}}\int d\tilde{\mathbf{p}} e^{i\tilde{\mathbf{p}}\cdot\tilde{\mathbf{x}}}\tilde{\phi}^{\pm}(\tilde{\mathbf{p}}) \tag{56}$$

and

$$\tilde{\phi}^+(\tilde{\mathbf{p}}) := \frac{\theta(p^+)}{\sqrt{p^+}}a_{lf}(\tilde{\mathbf{p}}) \qquad \tilde{\phi}^-(\tilde{\mathbf{p}}) := \frac{\theta(-p^+)}{\sqrt{-p^+}}a^{\dagger}_{lf}(-\tilde{\mathbf{p}}). \tag{57}$$

Next we argue that the spectral condition on p^+ leads to an algebraic definition of normal ordering. By this we mean that vacuum expectation values are not explicitly needed to define the normal product of light front field operators.

To see this note that translational covariance of the field

$$e^{iP^+a^-}\tilde{\phi}(\tilde{\mathbf{x}})e^{-iP^+a^-} = \tilde{\phi}(\tilde{\mathbf{x}}+a^-). \tag{58}$$

implies

$$[P^+,\tilde{\phi}^{\pm}(\tilde{\mathbf{p}})] = \mp|p^+|\tilde{\phi}^{\pm}(\tilde{\mathbf{p}}). \tag{59}$$

It follows that if $|q^+\rangle$ is an eigenstate of P^+ with eigenvalue q^+ then

$$P^+\tilde{\phi}^{\pm}(\tilde{\mathbf{p}})|q^+\rangle = ([P^+,\tilde{\phi}^{\pm}(\tilde{\mathbf{p}})] + \tilde{\phi}^{\pm}(\tilde{\mathbf{p}})P^+)|q^+\rangle = (q^+ \mp |p^+|)\tilde{\phi}^{\pm}(\tilde{\mathbf{p}})|q^+\rangle. \tag{60}$$

This shows that application of $\tilde{\phi}^+(\tilde{\mathbf{p}})$ to any eigenstate of P^+ results in an eigenstate of P^+ with a lower eigenvalue of P^+. Since kinematic translational invariance in the x^- direction implies that any vacuum is an eigenstate of P^+ with eigenvalue 0, the spectral condition $P^+ \geq 0$ implies that $\tilde{\phi}^+(\tilde{\mathbf{p}})$ must annihilate that vacuum as long as the support of $\tilde{\mathbf{p}}$ does not contain the point $p^+ = 0$. This is always the case for the Schlieder-Seiler algebra. This is a consequence of the algebra - it holds for any Hilbert space representation of the light-front Schlieder-Seiler algebra. It is important to note that the only property of the vacuum that was used was translational invariance, which is a property of any light-front invariant vacuum. The decomposition (55) makes no use of the vacuum.

It follows that there is an algebraic notion of normal ordering on the free field light-front Schlieder-Seiler algebra. The rule is to decompose every field operator $\tilde{\phi}(\tilde{f}) = \tilde{\phi}^-(\tilde{f}) + \tilde{\phi}^+(\tilde{f})$, and then move all of the $\tilde{\phi}^+(\tilde{f})$ parts of the field to the right of the $\tilde{\phi}^-(\tilde{f})$ parts of the fields. We use the standard double dot :: notation to indicate *algebraic* normal ordering. We refer to it as "algebraic" because a vacuum is not needed to make the decomposition (55).

The light-front vacuum for free fields is uniquely determined by the algebraic normal ordering on the Schlieder-Seiler algebra. To see this note that it follows from the decomposition (57) that

$$U(\tilde{f}) = e^{i\tilde{\phi}(\tilde{f})} = e^{i\tilde{\phi}_+(\tilde{f})+i\tilde{\phi}_-(\tilde{f})} = e^{i\tilde{\phi}_-(\tilde{f})}e^{i\tilde{\phi}_+(\tilde{f})}e^{\frac{1}{2}(f,f)} =: e^{i\tilde{\phi}(\tilde{f})} : e^{\frac{1}{2}(\tilde{f},\tilde{f})} \tag{61}$$

which expresses $U(\tilde{f})$ as an algebraically "normal ordered operator" multiplied by the known coefficient function, $e^{\frac{1}{2}(\tilde{f},\tilde{f})}$. If the vacuum expectation value of the normal product $: e^{i\phi(\tilde{f})} :$ is 1, then the vacuum functional on this algebra by is given by

$$\langle 0|U(\tilde{f})|0\rangle = e^{\frac{1}{2}(\tilde{f},\tilde{f})}. \tag{62}$$

Since $(\tilde{f},\tilde{f})$ is ill-defined for functions that do not vanish at $p^+ = 0$, this vacuum is only defined on the Schlieder-Seiler algebra. Furthermore, on the Schlieder-Seiler algebra the vacuum expectation value of the normal product $: e^{i\tilde{\phi}(\tilde{f})} :$ is 1 as a result of (60) and the fact that the Schlieder-Seiler test functions vanish for $p^+ = 0$.

It follows that for free fields of any mass, the Schlieder-Seiler algebras are unitarily equivalent [3]. This is because the vacuum expectation values of any number of smeared fields are identical.

The formal irreducibility of this algebra follows because it has the structure of a Weyl algebra [33][34], but in this case the algebra has no local observables and the class of test functions is too small to determine any dynamical information. The Weyl structure is contained in the unitary operators

$$U(\tilde{f}) = e^{i\tilde{\phi}(\tilde{f})}. \tag{63}$$

Products of field operators are replaced by products of bounded operators of the form (63). Products of two such operators can be computed using the Campbell-Baker-Hausdorff theorem[35]. The result is

$$U(\tilde{f})U(\tilde{g}) = U(\tilde{f}+\tilde{g})e^{-\frac{1}{2}[\tilde{\phi}(\tilde{f}),\tilde{\phi}(\tilde{g})]} = U(\tilde{f}+\tilde{g})e^{-\frac{1}{2}((\tilde{f},\tilde{g})-(\tilde{g},\tilde{f}))}, \tag{64}$$

which is another operator of the same form multiplied by the scalar coefficient, $e^{-\frac{1}{2}((\tilde{f},\tilde{g})-(\tilde{g},\tilde{f}))}$.

To put (64) in the form of a Weyl algebra first decompose the Fourier transform of a real Schlieder Seiler test function into real and imaginary parts; $\tilde{f} = \tilde{f}_r + i\tilde{f}_i$. Defining

$$U(\tilde{f}) = U(\tilde{f}_r, \tilde{f}_i) \tag{65}$$

equation (64) becomes

$$U(\tilde{f}_r,\tilde{f}_i)U(\tilde{g}_r,\tilde{g}_i) = U(\tilde{f}_r+\tilde{g}_r,\tilde{f}_i+\tilde{g}_i)e^{-\frac{i}{2}((\tilde{f}_r,\tilde{g}_i)-(\tilde{g}_r,\tilde{f}_i))}. \tag{66}$$

This has the same form as the Weyl algebra for the canonical fields if we use

$$U(f,g) := e^{i\phi(f)+i\pi(g)} \tag{67}$$

where the light front-inner product is replaced by the ordinary $L^2(\mathbb{R}^3)$ inner product.

While the light-front inner product (19) is singular for functions that do not vanish at $p^+ = 0$, the difference $(\tilde{f}(-\tilde{\mathbf{p}})\tilde{g}(\tilde{\mathbf{p}}) - \tilde{g}(-\tilde{\mathbf{p}})\tilde{f}(\tilde{\mathbf{p}}))$ vanishes for $p^+ = 0$ for all values of $\mathbf{p}_\perp$. This means that $[\phi(\tilde{f}),\phi(\tilde{g})]$ can be extended to the light-front algebra, however the decomposition (55), and the algebraic normal ordering discussed above is no longer well defined on the full light-front Fock algebra.

In the case of the canonical field algebra, defining a vacuum functional on the Weyl algebra uniquely determines the Hamiltonian [21][22] and hence the dynamics needed to uniquely extend the algebra to the Local algebra. This does not happen in the light-front case.

In the light-front case the representations of the Weyl algebras discussed above are unitarily equivalent [3] for different mass fields.

VII. EQUIVALENCE

In this section the meaning of equivalence of two field theories is discussed. We emphasize that it is important to distinguish the equivalence of the theories and equivalence of the corresponding Weyl representations. The choice of field algebra plays an important role for this characterization.

The relevant algebra for a field theory is the local algebra, which is distinguished from the light-front, canonical and Schlieder-Seiler algebras by being closed under Poincaré transformations. In addition it contains operators that are localized in finite space-time regions, which are needed to formulate locality conditions. The GNS construction using a Poincaré invariant vacuum functional leads to a unitary representation of the Poincaré group on the GNS Hilbert space. Theories that have identical Wightman functions are unitarily equivalent, since the Wightman functions are kernels of the Hilbert space inner product which means that the correspondence between the fields and vacuum in the Wightman functions preserves all inner products.

It is possible for two theories to have Wightman functions that are generally different, but nevertheless are identical on a sub-algebra. An instructive example for the case of two-scalar fields with different masses was given by Schlieder and Seiler [4]. In this example they consider a sub-algebra of the local field algebra.

Let $\phi_1(x)$ and $\phi_2(x)$ be free scalar fields with different masses, m_1 and m_2. If $f(x)$ and $g(x)$ have Fourier transforms $f(p)$ and $g(p)$ that agree on the mass shell for the field of mass m_1, then $\phi_1(f) = \phi_1(g)$. The condition that two functions agree on a given mass shell divides the space of test functions into disjoint equivalence classes of functions.

In general, if two test functions have Fourier transforms that agree on one mass shell, their Fourier transforms are generally unrelated on any other mass shell. However out of the class of all test functions there is a subspace of test functions, $f(p)$ that satisfy

$$\frac{f(\sqrt{m_1^2+\mathbf{p}^2},\mathbf{p})}{(m_1^2+\mathbf{p}^2)^{1/4}} = \frac{f(\sqrt{m_2^2+\mathbf{p}^2},\mathbf{p})}{(m_2^2+\mathbf{p}^2)^{1/4}}. \tag{68}$$

For test functions in this class, calculations imply that

$${}_1\langle 0|\phi_1(f)\phi_1(g)|0\rangle_1 ={}_2 \langle 0|\phi_2(f)\phi_2(g)|0\rangle_2. \tag{69}$$

On this restricted sub-algebra the two-point functions of both fields are identical. In addition, the decomposition of this set of functions into disjoint equivalence classes is identical for both fields. Finally, for free fields, every n-point function is a product of two-point functions. Because the two-point Wightman functions for the different mass fields on this sub-algebra are identical, the correspondence $\phi_1(f) \to \phi_2(f)$ and $|0\rangle_1 \to |0\rangle_2$ is unitary. On the other hand this algebra is not invariant under Poincaré transformations.

While this is very restrictive class of test functions, it is still large enough to be irreducible. To see this note that given any $g(p)$ (not necessarily in this class) there is a function $f_1(p)$ in this class satisfying $\phi_1(f_1) = \phi_1(g)$ (they only have to have Fourier transforms that agree on the mass shell). There is also an $f_2(p)$ in this class satisfying $\phi_2(f_2) = \phi_2(g)$. However there is no relation between $\phi_1(g)$ and $\phi_2(g)$ or $\phi_1(f_1)$ and $\phi_2(f_2)$.

Thus by limiting the space of test functions to a class that is not closed under Poincaré transformations, one gets irreducible unitarily equivalent representations of a sub-algebra of the full four dimensional algebra for scalar fields with different mass. This equivalence is not preserved when the algebra is extended to the local algebra.

While the light-front algebra is not a sub-algebra of the local algebra, it also has a limited set of test functions that cannot distinguish fields of different mass, that are however large enough to be irreducible.

Theories with different masses become inequivalent when the light-front Fock algebra is extended to the local algebra. This will be discussed in the next section.

Another example that makes the role of the underlying algebra clear is comparing the two-point function of the local algebra restricted to the light front to the two-point function constructed from the light-front Fock algebra.

The text book representation for the two-point function of a scalar field of mass m in the local algebra can be found in [36]

$$\langle 0|\phi(x)\phi(y)|0\rangle =$$

$$-i\frac{\epsilon(z^0)\delta(z_0^2-\mathbf{z}^2)}{4\pi} + \frac{im\theta(z_0^2-\mathbf{z}^2)}{8\pi\sqrt{(z_0^2-\mathbf{z}^2)}}(\epsilon(z^0)J_1(m\sqrt{(z_0^2-\mathbf{z}^2)}) - iN_1(m\sqrt{(z_0^2-\mathbf{z}^2)}))$$

$$-\frac{m\theta(\mathbf{z}^2-z_0^2)}{4\pi^2\sqrt{\mathbf{z}^2-z_0^2}}K_1(m\sqrt{\mathbf{z}^2-z_0^2}) \tag{70}$$

where $z^\mu = x^\mu - y^\mu$. Because this is Lorentz invariant, it is a function of z^2, so when $z^+ = 0$, there can be no z^- dependence (except for the sign), and this becomes

$$\langle 0|\phi(x)\phi(y)|0\rangle \to -i\frac{\epsilon(z^-)\delta(\mathbf{z}_\perp^2)}{4\pi} - \frac{m}{4\pi^2\sqrt{\mathbf{z}_\perp^2}}K_1(m\sqrt{\mathbf{z}_\perp^2}). \tag{71}$$

This can be compared to the direct construction of this quantity using the light-front Fock algebra, which knows nothing about the Lorentz symmetry. The result is

$$\frac{1}{(2\pi)^3}\int\frac{d\tilde{\mathbf{p}}\theta(p^+)}{2p^+}e^{i\tilde{\mathbf{p}}\cdot\tilde{\mathbf{z}}}. \tag{72}$$

This is a well-behaved distribution on the Schlieder-Seiler algebra, and it has a non-trivial dependence on z^-. On the other hand the Lorentz invariance of (71) means that it has no dependence on z^-. Furthermore, (71) is not even a distribution on the Schlieder-Seiler functions because the integral behaves like $\frac{1}{\mathbf{z}_\perp^2}$ at the origin [12]. The difference in (71) and (72) is because the order of the light-front limit and integral matters.

VIII. EXTENSION TO THE LOCAL ALGEBRA

The explicit representation of the free scalar field in terms of the light-front creation and annihilation operators, (48), and the light-front creation and annihilation operators in terms of the fields restricted to the light front, (53) and (54), can be combined to get the following expression for the field on the local algebra in terms of the field restricted to the light-front:

$$\phi(y) = \frac{1}{2(2\pi)^3} \int d\tilde{\mathbf{x}} d\tilde{\mathbf{k}} e^{-\frac{i}{2}\frac{\mathbf{k}_\perp^2+m^2}{k^+}y^+ + i\tilde{\mathbf{k}}\cdot(\tilde{\mathbf{y}}-\tilde{\mathbf{x}})}\tilde{\phi}(\tilde{\mathbf{x}}) \tag{73}$$

where in this expression the k^+ integral is over *both positive and negative values.* This provides the desired extension from the light-front or Schlieder-Seiler algebras to the local algebra of the free field.

When $y^+ = 0$ this becomes a delta function in the light-front coordinates and one recovers the field, $\tilde{\phi}(\tilde{\mathbf{x}})$, restricted to the light front. The specification of the mass m in equation (73) puts in the dynamical information in $\phi(y)$.

Equation (73) has the structure

$$\phi(y) = \int d\tilde{\mathbf{x}} F_m(y^+ : \tilde{\mathbf{y}} - \tilde{\mathbf{x}})\tilde{\phi}(\tilde{\mathbf{x}}). \tag{74}$$

If $f(y)$ is a Schwartz function in four-space-time variables and we define

$$\tilde{g}_f(\tilde{\mathbf{x}}) := \int d^4y f(y) F_m(y^+ : \tilde{\mathbf{y}} - \tilde{\mathbf{x}}) \tag{75}$$

then

$$\phi(f) = \int \phi(x) f(x) d^4x = \int \tilde{g}_f(\tilde{\mathbf{x}})\tilde{\phi}(\tilde{\mathbf{x}})d\tilde{\mathbf{x}} = \tilde{\phi}(\tilde{g}_f) \tag{76}$$

where $\tilde{g}_f$ is a function of variables on the light-front hyper-plane. This shows that the local free-field algebra can be expressed in terms of the free-field light-front algebra.

The kernel $F_m(y^+ : \tilde{\mathbf{y}} - \tilde{\mathbf{x}})$ satisfies the Klein-Gordon equation for a given mass. Thus, it restores full Lorentz covariance. The choice of F_m also provides a dynamical distinction between free fields with different masses. It is responsible for the physical inequivalence of free-field theories with different masses. This inequivalence is preserved if we use this to generate the canonical algebra by integrating against test functions in three variables multiplied by delta functions in time and their derivatives.

A free field theory is completely defined by its two-point Wightman function. To compute the two-point Wightman function it is also necessary to have the vacuum functional in addition to the algebra.

If the vacuum is annihilated by the light-front annihilation operator then (74) can be used to calculate the two-point Wightman function in terms of the restriction to the light front

$$\langle 0|\phi(x)\phi(y)|0\rangle =$$

$$\frac{1}{8(2\pi)^9}\int d\tilde{\mathbf{x}}_1 d\tilde{\mathbf{y}}_1 d\tilde{\mathbf{k}} d\tilde{\mathbf{p}} \frac{d\tilde{\mathbf{q}}}{q^+}\theta(q^+) e^{-i\frac{\mathbf{p}_\perp^2+m^2}{2p^+}x^+ + i\tilde{\mathbf{p}}\cdot(\tilde{\mathbf{x}}-\tilde{\mathbf{x}}_1)} e^{-i\frac{\mathbf{k}_\perp^2+m^2}{2k^+}y^+ + i\tilde{\mathbf{k}}\cdot(\tilde{\mathbf{y}}-\tilde{\mathbf{y}}_1)} e^{i\tilde{\mathbf{q}}\cdot(\tilde{\mathbf{x}}_1-\tilde{\mathbf{y}}_1)} =$$

$$\frac{1}{2(2\pi)^3}\int \frac{d\tilde{\mathbf{q}}}{q^+}\theta(q^+) e^{-i\frac{\mathbf{q}_\perp^2+m^2}{2q^+}(x^+-y^+) + i\tilde{\mathbf{q}}(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})} =$$

$$\frac{1}{(2\pi)^3}\int \theta(q^+)\delta(q^2+m^2)e^{iq\cdot(x-y)}d^4q \tag{77}$$

which is the standard representation of the two-point Wightman function.

The interesting property of expression (77) is that while the $1/q^+$ denominator is log divergent for q^+ near zero, $e^{-i\frac{\mathbf{q}_\perp^2+m^2}{q^+}x^+}$ undergoes violent oscillations in a neighborhood of $q^+ = 0$. These oscillations regularize the $1/q^+$ divergences. To see that this happens note that near the origin the integral has the same form as

$$\int_0^a \frac{e^{ic/q}}{q}dq = \int_{c/a}^\infty \frac{e^{iu}}{u}du = \frac{\pi}{2} - (Ci(c/a) + iSi(c/a)) \tag{78}$$

where we have substituted $u = c/q$ and used equations (5.5), (5.5.27) (5.5.26) in [37] to get this result. The finiteness of this expression shows that the oscillations in the exponent regulate the singularity at $q^+ = 0$. It follows that the sharp restriction to the light front turns this "regulator" off. It is not something that can be continuously tuned on.

This shows that smearing with a test function in x^+ leads to an additional p^+ dependence that makes the $1/p^+$ in the light-front inner product (19) harmless.

To see this it is constructive to consider a test function that is a product of a test function in x^+ and a test function in the light-front coordinates $\tilde{\mathbf{x}}$ of the form

$$f(x) = f_1(x^+) f_2(\tilde{\mathbf{x}}). \tag{79}$$

It follows that $\tilde{g}_f(\tilde{\mathbf{x}})$ defined by (75) has a Fourier transform of the form

$$g_f(\tilde{\mathbf{p}}) = f_1(\frac{\mathbf{p}_\perp^2 + m^2}{p^+}) f_2(\tilde{\mathbf{p}}). \tag{80}$$

Thus even if $f_2(0, \mathbf{p}_\perp) \neq 0$, if $f_1(p^-)$ is a Schwartz function, $g_f(\tilde{\mathbf{p}})$ will vanish faster than any power of p^+ as $p^+ \to 0$.

This means that for free-field theories, F_m maps all Schwartz test functions in four variables into Schlieder-Seiler functions on the light-front hyper-plane. This uniquely fixes the light-front vacuum by (61).

To understand the significance of the mapping $f(x) \to g_f(\tilde{\mathbf{x}})$ note that the operators $\tilde{\phi}(\tilde{g}_f)$ for $f(x) \in \mathcal{S}(\mathbb{R}^4)$ generate a sub-algebra of the Schlieder-Seiler algebra.

On this sub-algebra we have the identity

$${}_m\langle 0|\phi(f_1)\cdots\phi(f_n)|0\rangle_m = {}_{lf}\langle 0|\tilde{\phi}(\tilde{g}_{f_1})\cdots\tilde{\phi}(\tilde{g}_{f_n})|0\rangle_{lf} \tag{81}$$

which means that this correspondence preserves all Wightman distributions on the local algebra. This defines a unitary mapping between the physical representation of the local algebra and this representation of this sub-algebra of the Schlieder-Seiler Fock algebra.

This unitary transformation maps the vacuum of the local free field theory to the light front Fock vacuum. If f_1 and f_2 have space-like separated support then this correspondence implies

$$[\tilde{\phi}(\tilde{g}_{f_1}), \tilde{\phi}(\tilde{g}_{f_2})] = 0, \tag{82}$$

in addition $f_i(x) \to f_i'(x) = f_i(\Lambda x + a)$ implies that

$${}_{lf}\langle 0|\tilde{\phi}(\tilde{g}_{f_1})\cdots\tilde{\phi}(\tilde{g}_{f_n})|0\rangle_{lf} = {}_{lf}\langle 0|\tilde{\phi}(\tilde{g}_{f_1'})\cdots\tilde{\phi}(\tilde{g}_{f_n'})|0\rangle_{lf}. \tag{83}$$

These equations show how locality and a unitary representation of the Poincaré group are realized in this light front-Fock representation of this sub-algebra of the Schlieder-Seiler algebra.

Free fields of different mass involve different maps that map to different sub-algebras of the Schlieder-Seiler algebra.

IX. DYNAMICS

In this section we discuss the extension of these results to the case of interacting theories. In particular we show how the local algebra generated by the Heisenberg field operators of an interacting theory can be mapped into a sub-algebra of the light-front Fock algebra.

The asymptotic completeness of the S matrix means that the theory has an irreducible set of asymptotic fields. These are the "in" or "out" fields of the theory. They are local free fields with the masses of physical one-particle states of the theory. In general there may also be local asymptotic fields [38] for composite particles.

In [16][23] it is shown under mild assumptions that any linear operator A on the Hilbert space can be expanded as a series of normal products of asymptotic fields. This expansion is referred to as the Haag expansion [24]. For the Heisenberg field of an interacting theory the Haag expansion has the form:

$$\phi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n L_n(x; x_1, \cdots, x_n) : \phi_{in}(x_1) \cdots \phi_{in}(x_n) : . \tag{84}$$

This expansion is non-trivial - the masses in asymptotic fields are physical particle masses and in general there are asymptotic fields for both composite and elementary particles.

The Poincaré covariance of the Heisenberg and asymptotic fields means that the coefficient functions $L_n(x;x_1,\cdots,x_n)$ are invariant

$$L_n(x;x_1,\cdots,x_n) = L_n(\Lambda x + a;\Lambda x_1 + a,\cdots,\Lambda x_n + a) \tag{85}$$

under Poincaré transformations (Λ, a). For higher spin composite fields the invariance is replaced by an obvious covariance.

Furthermore, if the Heisenberg field $\phi(x)$ is an operator-valued tempered distribution, then $\phi(f)$ should be a Hilbert space operator when $f(x)$ is Schwartz function. For the Hagg expansion of $\phi(f)$ to also be an operator, the smeared coefficient functions

$$L_n(f,x_1,\cdots,x_n) := \int f(x) L_n(x;x_1,\cdots,x_n) d^4x \tag{86}$$

should behave like Schwartz test functions in $4n$ variables, since the asymptotic fields are all operator valued tempered distributions.

Using, (73-74), each of the asymptotic fields can be expressed as the extension of a light-front field, using the kernels $F_m(x^+;\tilde{\mathbf{x}}-\tilde{\mathbf{y}})$. Using these in the Haag expansion gives the following representation of the Heisenberg field in terms of products of algebraically normal-ordered fields restricted to a light front:

$$\phi(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n L_n(x;x_1,\cdots,x_n) F_{m_1}(x_1^+;\tilde{\mathbf{x}}_1-\tilde{\mathbf{y}}_1)\cdots F_{m_n}(x_n^+;\tilde{\mathbf{x}}_n-\tilde{\mathbf{y}}_n)\times$$

$$d\tilde{\mathbf{y}}_1\cdots d\tilde{\mathbf{y}}_n : \tilde{\phi}_{10}(\tilde{\mathbf{y}}_1)\cdots\tilde{\phi}_{n0}(\tilde{\mathbf{y}}_n) : \tag{87}$$

If $\phi(x)$ is smeared with a Schwartz function, $f(x)$, and $L_n(f,x_1,\cdots,x_n)$ is a Schwartz function in $4n$ variables, then the $p^+=0$ behavior of the light-front fields will be suppressed in (87) by the mechanism (78). In this representation the vacuum functional is fixed by (61), as it is in the case of the free field. The non-trivial aspects of the dynamics appear in the extension to the full Heisenberg algebra.

It follows that elements of the local Heisenberg algebra can be expressed as elements of the light-front Fock algebra

$$\phi(f) = \sum \frac{1}{n!} \int \tilde{L}_n(f,\tilde{\mathbf{y}}_1,\cdots,\tilde{\mathbf{y}}_n) d\tilde{\mathbf{y}}_1\cdots d\tilde{\mathbf{y}}_n : \tilde{\phi}_{10}(\tilde{\mathbf{y}}_1)\cdots\tilde{\phi}_{n0}(\tilde{\mathbf{y}}_n) : \tag{88}$$

where

$$\tilde{L}_n(f,\tilde{\mathbf{y}}_1,\cdots,\tilde{\mathbf{y}}_n) = \int \tilde{L}_n(x,\tilde{\mathbf{y}}_1,\cdots,\tilde{\mathbf{y}}_n) d^4x f(x) =$$

$$\int d^4x_1 \cdots d^4x_n L_n(f;x_1,\cdots,x_n) F_{m_1}(x_1^+;\tilde{\mathbf{x}}_1-\tilde{\mathbf{y}}_1)\cdots F_{m_n}(x_n^+;\tilde{\mathbf{x}}_n-\tilde{\mathbf{y}}_n). \tag{89}$$

As in the free field case, this correspondence generates a unitary mapping from the local algebra of the Heisenberg field to a sub-algebra of the Schlieder-Seiler algebra.

Specifically this correspondence has the form

$$\phi(f) \to \tilde{A}[f] \tag{90}$$

where $\tilde{A}[f]$ is the right hand side of (88), which is an element of the light-front Schlieder Seiler algebra. The correspondence

$$\langle 0|\phi(f)_1\cdots\phi(f_n)|0\rangle =_{lf} \langle 0|A[f_1]\cdots A[f_n]|0\rangle_{lf} \tag{91}$$

defines a unitary map from the Hilbert space generated by the local field $\phi(f)$ to the Hilbert space generated by the $A[f]$ on the light-front Fock vacuum.

This correspondence relates the Heisenberg vacuum to the light-front Fock vacuum. It preserves local commutation relations, the sub-algebra is Poincaré invariant, and the Fock vacuum is also Poincaré invariant on this sub-algebra.

In this case all of the dynamical information is contained in the mapping which defines the sub-algebra of the light front Fock algebra. All of the dynamical information is removed from the vacuum.

This discussion does not directly apply to QCD. In the case of QCD the asymptotic fields are composite and color singlets. See ref [39] for a discussion. While these asymptotic fields generate the physical Hilbert space, they do not generate the non-singlet part of the Hilbert space. One possible advantage of the expansion (88) is that the light-front fields that appear in the expansion do not carry information about asymptotic masses, so they might provide a means to extend the light front "Haag expansion" (85) to the non-singlet sectors, where there are no asymptotic fields.

X. ZERO MODES

In the previous section we demonstrated that it was possible to express a smeared interacting Heisenberg field as an expansion in terms of algebraically normal ordered free fields restricted to a light front, where the vacuum is the free light-front Fock vacuum. In this expansion contributions associated with $p^+ = 0$ are suppressed by the coefficient functions, $\tilde{L}_n(f, \tilde{\mathbf{x}}_1 \cdots \tilde{\mathbf{x}}_n)$ which are Schlieder-Seiler functions on the light front.

Unfortunately the operators that appear in dynamical equations, like the Poincaré generators, involve local products of fields rather than products of smeared fields. A characteristic property of any quantum field theory is that local products of local fields are not defined. This is because the fields are operator-valued distributions and products of distributions are not always defined.

The leading term in the Haag expansion is

$$\phi(x) = Z\phi_{in}(x) + \cdots = Z \int F_m(x^+; \tilde{\mathbf{x}} - \tilde{\mathbf{y}})\phi_0(\tilde{\mathbf{y}}) d\tilde{y} + \cdots \tag{92}$$

where Z is a constant that relates the normalization of $\phi(x)$ to the normalization of the asymptotic field. The presence of $Z\phi_{in}(x)$ in this expansion means that singularities of local products of Heisenberg fields on the light front are determined in part by the singularities of the corresponding local products of asymptotic fields on the light front:

$$\phi(x)^n \to Z^n \phi_{in}(x)^n + \cdots \tag{93}$$

What is relevant is that local products of extensions of the fields do not suppress $p^+ = 0$ singularities.

Recall that previously we showed that for Schlieder-Seiler test functions the vacuum was given by a functional of the form

$$E[f] := \langle 0 | e^{i\tilde{\phi}(\tilde{f})} | 0 \rangle = \langle 0 | : e^{i\phi(\tilde{f})} : | 0 \rangle e^{\frac{1}{2}(\tilde{f}, \tilde{f})}. \tag{94}$$

If we expand this out

$$\langle 0 | e^{i\tilde{\phi}(\tilde{f})} | 0 \rangle = \sum_{m=0}^{\infty} \frac{i^n}{n!} \langle 0 | \tilde{\phi}(\tilde{f})^n | 0 \rangle \tag{95}$$

all of the fields are smeared with Schlieder-Seiler test functions before computing the vacuum expectation value.

Removing the test functions by taking functional derivatives with respect to the Schlieder-Seiler functions gives kernels that represent Schlieder-Seiler distributions. These are not sensitive to what happens at $p_i^+ = 0$ for Schlieder-Seiler test functions. In order to deal with local operator products the vacuum functional needs to be extended to treat points with $p^+ = 0$. The two-point function on the light front in the light-front algebra is up to a constant the ill-defined light-front scalar product.

This has a logarithmic singularity at $p^+ = 0$. This scalar product can be regularized in a number of ways. For example

$$(\tilde{f}, \tilde{f}) \to \int \frac{dp^+ d\mathbf{p}_\perp \theta(p^+)}{p^+} \tilde{f}(-\tilde{p}) \tilde{f}(\tilde{p}) - \frac{dp^+ d\mathbf{p}_\perp \theta(p^+)}{p^+} \tilde{f}(0, -\mathbf{p}_\perp) \tilde{f}(0, \mathbf{p}_\perp) e^{-\beta p^+}. \tag{96}$$

This recovers the expected result on Schlieder-Seiler functions, but it is well-defined on test functions that do not vanish at $p^+ = 0$. One apparent problem with this regularization is that it breaks boost invariance in the $\hat{\mathbf{n}}$ direction. This means that the extension to the local algebra plus renormalization must be designed to recover both rotational invariance and longitudinal boost invariance.

Once the space of test functions is extended to include functions that do not vanish at $p^+ = 0$, it is possible to have

$$\langle 0 | : e^{i\tilde{\phi}(\tilde{f})} : | 0 \rangle \neq 1. \tag{97}$$

While the algebraic normal ordering means that this is 1 for functions with positive p^+ support, one can extend this functional to include additional contributions that are concentrated at $p^+ = 0$. To minimize the effort needed to recover exact Poincaré invariance in the extension to the local algebra, it is advantageous to work with $p^+ = 0$ contributions that have the full kinematic symmetry on the light front. Allowed contributions that are invariant with respect to the seven-parameter kinematic subgroup have the form

$$\langle 0 | : e^{i\tilde{\phi}(\tilde{f})} : | 0 \rangle = \sum \frac{i^n}{n!} \langle 0 | : \tilde{\phi}(\tilde{f})^n : | 0 \rangle = e^{\sum \frac{i^n}{n!} \langle 0 | : \tilde{\phi}(\tilde{f})^n : | 0 \rangle_c}$$

$$= e^{\sum \frac{i^n}{n!} \int w_n^c(\mathbf{p}_{1\perp},\cdots,\mathbf{p}_{n\perp})\tilde{f}(0,\mathbf{p}_{1\perp})\cdots\tilde{f}(0,\mathbf{p}_{n\perp})\delta(\sum \mathbf{p}_{n\perp})d\mathbf{p}_{1\perp}\cdots d\mathbf{p}_{n\perp}} \tag{98}$$

where the subscript c indicates the connected part of the n-point function which is defined in terms of the distributions of the form

$$w_n^c(\mathbf{p}_{1\perp},\cdots,\mathbf{p}_{n\perp}). \tag{99}$$

The construction of the dynamical extension to the local algebra must recover exact Poincaré invariance. Since the regularization of the scalar product (96) that allows test functions that do not vanish at $p^+ = 0$ breaks scale invariance in the longitudinal direction, restoration of scale invariance may require $\mathbf{p}^+ = 0$ contributions with the same six parameter symmetry. A large class of expressions with this property are given in [8].

These do not exhaust the possible zero-mode contributions; distributions of the form

$$w_n^c(\tilde{f},\cdots\tilde{f}) = \int w_n(\mathbf{p}_{1\perp},\cdots\mathbf{p}_{n\perp},\xi_1,\cdots\xi_n)\delta(\sum_i p_i^+)\delta(\sum \xi_j - 1)\prod \theta(\xi_k)\prod_{i=1}^n f(\tilde{\mathbf{p}}_i)d\tilde{\mathbf{p}}_i \tag{100}$$

where the ξ_k are light-front momentum fractions, can also be considered. What is relevant is that the extension to the local algebra must recover the full Poincaré symmetry.

It is clear that the need for zero modes is related to renormalization of operator products. This is non trivial because both ultraviolet and infrared singularities must be treated together in a manner that preserves the rotational and $\hat{n}$-boost covariance of the theory and preserves the positivity of the Hilbert space inner product. The presence of zero modes defines an extension of the light-front Fock vacuum. These extensions may be needed in perturbative expansions of expressions that involve local products of fields. While they may play a role in constructing the coefficient functions in the light-front Haag expansion, they do not directly contribute to the final representation of the smeared Heisenberg operators.

XI. SUMMARY

The goal in this paper was to understand (1) why the light-front vacuum of an interacting theory is the same as the Fock vacuum while the vacuua differ in the conventional formulation of field theory and (2) the role of zero modes in the light-front vacuum.

The first issue involves determining the meaning of the vacuum. For free fields the characterization of the vacuum by an annihilation operator is incomplete. The physically relevant characterization of the vacuum is as a positive linear functional of an algebra of field operators.

When the vacuum is characterized as a linear functional on an algebra of operators, the choice of algebra matters. In order to realize Poincaré symmetry manifestly, or localize field observables to finite regions of space time, the space of test functions of the algebra should include functions with support in finite space-time volumes and should be invariant under space-time translations and Lorentz transformations.

From a physics point of view the relevant algebra is generated by fields smeared with test functions of four space-time variables. This algebra is Poincaré invariant.

In this work we showed that the local algebra of both free and interacting fields could be mapped into sub-algebras of the Schlieder-Seiler algebra. We also showed that the vacuum is trivial and uniquely defined on the Schlieder-Seiler algebra. This mapping moves the dynamics in the light-front vacuum into the mapping. The mapping defines a unitary transformation from the physical representation of the local algebra generated by a scalar field to a sub-algebra of the Schlieder-Seiler algebra with the Fock vacuum. This unitary correspondence leads to a formulation of locality and Poincaré invariance on a subspace of the light front Fock space. Sub-algebras of the Schlieder-Seiler algebra associated with different local fields are not unitarily related.

The Schlieder-Seiler algebra has no place for zero modes, but extension to include zero modes may be required to treat local operator products that arise in perturbation theory. A large class of zero-mode contributions are possible, but they are restricted by positivity and Poincaré invariance.

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