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The Gauss-Bonnet Coupling Constant in Classically Scale-Invariant Gravity

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We discuss the renormalization of higher-derivative gravity, both without and with matter fields, in terms of two primary coupling constants rather than three. A technique for determining the dependence of the Gauss-Bonnet coupling constant on the remaining couplings is explained, and consistency with the local form of the Gauss-Bonnet relation in four dimensions is demonstrated to all orders in perturbation theory. A similar argument is outlined for the Hirzebruch signature and its coupling. We speculate upon the potential implications of instantons on the associated nonperturbative coupling constants.

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I. INTRODUCTION

The general form of the classically scale-invariant theory of the metric takes the form

$$S_1 \equiv \int_{\mathcal{M}} d^4x \sqrt{g} \mathcal{L}_1, \quad (1.1a)$$

$$\mathcal{L}_1 \equiv \frac{1}{2\alpha} C^2 + \frac{1}{3\beta} R^2 + \frac{2}{\gamma} \hat{R}_{\mu\nu}^2, \quad (1.1b)$$

where $C^2 \equiv C_{\kappa\lambda\mu\nu} C^{\kappa\lambda\mu\nu}$ is the square of the Weyl tensor, $\hat{R}_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu} R/4$ is the traceless part of the Ricci tensor, and $R = g^{\mu\nu} R_{\mu\nu}$ is the Ricci scalar. The integral is over the spacetime manifold \mathcal{M} . (See Appendix B for the definition of the Weyl tensor). In general, the action¹Eq. (1.1a) must be augmented by the addition of certain surface or boundary terms in order to have the proper relationship for the composition of the path integral [1]. For the most part, such complications will not concern us since our interest is in perturbation theory, and we shall ignore such terms or simply assume that the manifold has no boundaries. We shall adopt a Euclidean metric throughout this paper, assuming that any spacetime in which we are interested can be realized, with some choice of coordinates, through the replacement of one coordinate, say, x^4 by $-ix^0$, with a corresponding redefinition of the metric $g_{\mu\nu}$.

Classically, the three quadratic invariants in Eq. (1.1b) are in a sense not independent, because of the Gauss-

Bonnet (G-B) relation, whose local form may be written as

$$G \equiv C^2 - 2W, \text{ where } W \equiv \hat{R}_{\mu\nu}^2 - \frac{R^2}{12}, \quad (1.2a)$$

$$G = R^* R^* = R_{\kappa\lambda\mu\nu}^* R^{\kappa\lambda\mu\nu}, R_{\kappa\lambda\mu\nu}^* \equiv \frac{1}{2} \epsilon_{\kappa\lambda\rho\sigma} R^{\rho\sigma}{}_{\mu\nu}. \quad (1.2b)$$

$R_{\kappa\lambda\mu\nu}^*$ is the dual of the Riemann tensor. The first equation, Eq. (1.2a), may be taken as the definition of G in any dimension, whereas the second equation, Eq. (1.2), is valid only in four-dimensions where the totally antisymmetric tensor is well-defined. The fundamental result in four dimensions is that G may be written locally as a divergence $G = \nabla_\mu B^\mu$ of a “vector” B^μ where

$$B^\mu \equiv \epsilon^{\mu\nu\gamma\delta} \epsilon_{\rho\sigma}{}^{\kappa\lambda} \Gamma_{\kappa\nu}^\rho \left[\frac{1}{2} R^\sigma{}_{\lambda\gamma\delta} + \frac{1}{3} \Gamma_{\tau\gamma}^\sigma \Gamma_{\lambda\delta}^\tau \right], \quad (1.3)$$

where $\Gamma_{\mu\nu}^\kappa$ is the Levi-Civita connection associated with the metric. In fact, B^μ does not transform as a vector under general coordinate transformations but transforms like a connection.

Assuming that the four-manifold \mathcal{M} is compact and has no boundaries,

$$\int_{\mathcal{M}} d^4x \sqrt{g} G = 32\pi^2 \chi(\mathcal{M}), \quad (1.4)$$

where the integer χ is the Euler characteristic of the manifold. If we rewrite the original Lagrangian², Eq. (1.1b), as [2–5]

$$\mathcal{L}_2 \equiv \frac{1}{2a} C^2 + \frac{1}{3b} R^2 + c G, \quad (1.5)$$

then since $\sqrt{g} G$ is a total derivative, it makes no contribution to the equations of motion (EoM) and can be

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¹ To be complete, a term $\square R$ should be added as well, but it plays no role in the following. Like G , it appears in the conformal anomaly, but unlike G , it is the covariant divergence of a true vector $\nabla^\mu R$.

² In Eq. (1.5), we have used a different notation than in our earlier work, Ref. [5], for the coefficient of G , where it was called ε .

ignored, thereby reducing the classical theory from three parameters (α, β, γ) to two (a, b) . This is a bit glib, since, in a spacetime that is not Asymptotically Locally Euclidean (ALE), G can give a finite contribution to the classical action, even though it still would contribute nothing to the EoM.

For future reference, a commonly used Lagrangian [6, 7], equivalent to Eq. (1.5), utilizes the combination W in Eq. (1.2a) instead of C^2 , rewriting the Lagrangian density as

$$\mathcal{L}_3 \equiv \frac{1}{a}W + \frac{1}{3b}R^2 + \tilde{c}G. \quad (1.6)$$

As a starting Lagrangian, one may choose G together with any two other linear combinations of C^2 , R^2 , and $\hat{R}_{\mu\nu}^2$, so long as they are linearly independent of G . One could not, for example, choose C^2 , W , and G ³. Any such Lagrangian can be brought to the form of Eq. (1.5). In that sense, the theory is unique.

What about the quantum field theory (QFT)? A distinguishing property is that the theory is renormalizable [6], at least in a topologically trivial background. Since the topology ought not affect the short-distance behavior of correlation functions, it is believed to be renormalizable generally. Insofar as perturbation theory is concerned, the preceding three Lagrangian densities, when expressed in terms of renormalized couplings and operators, require the addition of divergent counterterms in order to obtain finite matrix elements as functions of the renormalized coupling constants. In the process, the operator G cannot be ignored because divergences arise that are not of the form of linear combinations of W or R^2 , but require a third invariant [7].

A fundamental difference between the classical theory and the QFT is that the latter is not scale-invariant after renormalization. In the context of such a scale or conformal anomaly, one might well wonder whether the G-B relation is also anomalous [8]. On the other hand, the G-B relation, especially in its integral form, is a generic result in topology⁴. Like the Bianchi identities, to which it is related, it would be disturbing if the four-dimensional QFT did not recover these topologically-based identities.

Assuming that the G-B relation holds in four dimensions in the QFT, then, under any small variation of the metric $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, the variation of the action is

zero,

$$\delta \int d^4x (\sqrt{g}G) = \int d^4x \partial_\mu \delta(\sqrt{g}B^\mu) = 0. \quad (1.7)$$

By “small variation,” we mean any variation that does not change the spacetime topology. Although it is peculiar to four dimensions, this relation is an algebraic identity and does not depend on any assumptions about the background or require any reference to the EoM. As a result, researchers have tended to ignore G when formulating the Feynman rules for this theory, even though it is essential for renormalizability.

One source of confusion is that the preferred form of gauge-invariant regularization, *viz.*, dimensional-regularization (DREG) requires that we entertain the meaning of the theory outside of four-dimensions. While there are alternative possibilities for a four-dimensional, gauge-invariant regularization, such as the generalized zeta-function method, it is not so clear that they are implementable beyond one-loop. In any case, unlike the dual operators in Eq. (1.2b), it is possible to generalize Eq. (1.2a) to any dimension, so one can expect to recover this linear combination when returning to four dimensions.

In this paper, we wish to make explicit that the renormalized theory can be made consistent with the G-B relation and derive a relation between the renormalization of c and the renormalizations of the other two couplings, say, a, b . One may use DREG, and it is not even necessary to modify the usual mass-independent renormalization procedures such as minimal subtraction (MS). It is necessary to reinterpret the way in which the reduction from three couplings to two has been achieved by previous authors, especially since it plays an important role in our earlier discussion of dimensional transmutation [5]. Since the present paper is the companion promised there, Ref. [5] will henceforth be referred to as [I].

We conclude this introduction with an outline of the remainder of the paper. In the next section II, we discuss aspects of perturbative renormalization for the higher-order theory without matter. This is divided into three parts, reviewing the rather confusing history and status of this puzzle, the details of renormalization using DREG and MS, and finally some insights that may be gleaned by use of the renormalization group equations (RGE). Then, in Section III, we indicate how our results may easily be extended to include matter fields. In Section IV, we discuss briefly another topologically significant parameter, the Hirzebruch signature, that will enter discussions of the axial anomaly, CP-violation, and related issues, such as the $U(1)$ -problem in QCD. That leads us to speculate, in Section V, about the potential role that gravitational instantons, a nonperturbative effect, may have on some of these considerations. Finally, in Section VI, we end with a summary of results and some important remaining questions. Two appendices have been added to clarify some issues in background field quantization (Appendix A) and in the extension of curvature to n -

³ Such a choice would correspond to what has been called conformal or Weyl gravity, with a Lagrangian involving C^2 and G (or W and G). Such models presume that there is a renormalization scheme free of the conformal anomaly. No such a construction has never been displayed. In this paper, we assume that the anomaly exists and only consider models renormalizable in that context.

⁴ For an introduction to differential geometric concepts, see, *e. g.*, Ref. [9, 10]. In its most general form, it does not even require a metric [11].

dimensions (Appendix B).

II. PERTURBATIVE RENORMALIZATION—PURE GRAVITY

A. History and Framework

In their seminal papers on this theory, Fradkin & Tseytlin [7] adopted the form Eq. (1.6). They initially state that the topological term G can be “disregarded” under the usual assumptions, such as the “natural asymptotically flat boundary conditions.” Nevertheless, after obtaining the Feynman rules, which of course requires the addition of gauge-fixing terms and Faddeev-Popov ghosts, they find that there are gauge-invariant divergences not only of the tensor structure of the operators W and R^2 , but also of the form of G . They therefore assign a counterterm to cG , which, because the Feynman rules are independent of c , depend only on the other parameters of the theory. That is, the counterterms assigned to the “coupling constant” c are independent of c . However, when one goes beyond one-loop order, one might think that one must include vertices involving such counterterms for G in addition to those for W and R^2 . Although they feel no need to modify their Feynman rules, this is an unusual prescription, and it is unclear what is precisely going on. In particular, it is not so clear that, when G is expressed as a linear combination of the three renormalized operators as in Eq. (1.2), the resulting renormalized Lagrangian in four dimensions necessarily obeys Eq. (1.7).

Similarly, Avramidi & Barvinsky [2] and Buchbinder *et al.* [12] choose a Lagrangian density of the form Eq. (1.5). Buchbinder *et al.* state (below their eq. (8.3)), that the topological term can only make a finite contribution to the one-loop corrections and, for k -loops, will only contribute to the poles in $1/(n-4)^{k-i}$ with $i \geq 1$. Nevertheless, in their elaboration of the one-loop divergences (see their eqs. (8.102), (8.103)), they encounter a divergent, one-loop contribution to the term cG . In fact, in Ref. [12], it is stated and assumed that the action without the G term is multiplicatively renormalized, which is not true. These paradoxes derive from the conflict between using DREG, on the one hand, and a four-dimensional identity Eq. (1.7) on the other⁵. Beyond one-loop order, it is not obvious that this conflict can always be resolved.

We submit that a consistent formulation exists that regards the beta-function β_c as determined by the beta-functions of the other coupling constants. The point is that, as in the procedure adopted in Ref. [7] at one-loop, the counterterms for c are determined by the counterterms for the other couplings, a, b . This suggests that

we regard $c = c(a, b)$, a function of the other couplings, satisfying the consistency relation

$$\frac{\partial c}{\partial a}\beta_a + \frac{\partial c}{\partial b}\beta_b = \beta_c. \quad (2.1)$$

We shall shortly prove this, viz., c is indeed a function of a, b that obeys Eq. (2.1). This equation will be shown to determine the function c up to its initial value c_0 .

In fact, the β -function β_c described above represents a generalisation to the quantised R^2 -gravity case of the Euler anomaly coefficient, and thus a candidate for an a -function as proposed by Cardy [14], manifesting a 4-dimensional c -theorem. Results for this anomaly coefficient (without quantising gravity) include a non-zero 5-loop contribution involving four quartic scalar couplings [15] and non-zero three loop contributions involving gauge and Yukawa couplings [16]. (For some recent progress on the a -theorem and references, see Refs. [17].)

The relation $c = c(a, b)$ or Eq. (2.1) is reminiscent of the method of coupling constant reduction by Oehme and Zimmermann⁶, employed to seek general relations among renormalized coupling constants that were renormalization group invariant. Their method leads to nontrivial constraints on the parameters of the theory, whereas, in the present case, the relation is a direct consequence of the renormalization properties of the theory. This case is similar in the following sense: Suppose that you had started from Eq. (1.1b) with three coupling constants α, β, γ . Each of the three operators can be defined in n -dimensions. (See Appendix B.) So you can use DREG to regularize and $\overline{\text{MS}}$ to renormalize this theory consistent with gauge invariance. Then, having obtained a finite renormalized theory in four-dimensions, you might ask whether there is some relation among the three renormalized couplings and eventually discover that, for certain linear combinations of couplings, only two linear combinations appear in the beta-functions. So you might eventually arrange them in the form of, say, Eq. (1.5) or Eq. (1.6), hypothesize that $c = c(a, b)$, and discover that the relation Eq. (2.1) can be imposed, in effect, reducing the number of couplings from three to two.

On the other hand, the present situation is dissimilar from coupling constant reduction since, in order to recover the Bianchi identities and maintain the G-B relation, properties that the theory in four-dimensions must have, the relations among the couplings are essential. These relations act like additional symmetries, but ones that only hold in four dimensions. They cannot be anomalous since their validity makes no reference to the EoM or to a conserved current resulting from a symmetry. They are constraints that must follow for a sensible gauge-invariant, renormalized theory in four-dimensions, not a hypothesis to be tested.

⁵ Some of these considerations were taken up in Ref. [13], which also considered the nature of the theory for finite but small $\epsilon = 4 - n$.

⁶ For reviews with references to earlier works, see Ref. [18].

To show that Eq. (2.1) is satisfied, it may be helpful to define $w \equiv a/b$ and to rewrite the Lagrangian density, Eq. (1.5), as⁷

$$\mathcal{L}_4 \equiv \frac{1}{a} \left[\frac{1}{2} C^2 + \frac{w}{3} R^2 \right] + c G. \quad (2.2)$$

We imagine quantizing the theory by the background field method, as briefly explained in Appendix A. We presume that gauge-fixing is done in a manner consistent with background field gauge invariance. In fact, we shall suppress gauge-fixing parameters and ghost terms in the following, because our concern will be with the gauge-invariant beta-functions. Assuming Eq. (1.7), we can ignore $c G$ in formulating our Feynman rules. Then we may identify a with the loop-counting parameter.

Although a slight digression, a word of warning must be added. Using a running coupling to count loops only works so long as the renormalization scale is held fixed. Recall from [I] that, at one-loop order, the scale dependence of a is

$$a(\mu) = \frac{a_0}{1 + a_0 L_t} = a_0 - a_0^2 L_t + a_0^3 L_t^2 + \dots, \quad (2.3)$$

where $L_t \equiv \kappa \beta_2 t$, with $t \equiv \ln(\mu/\mu_0)$; κ and β_2 are constants. Thus, the coupling constant at scale μ involves the coupling constant a_0 at some reference scale μ_0 to arbitrary orders in a_0 . This observation becomes especially important in higher loops or, even at one-loop, for couplings such as w that mix with others. To discuss the renormalization group, one must use a different, fixed parameter such as \hbar to count loops. These seemingly trivial observations will become extremely important below when a function of $w(\mu)$ will be re-expressed in terms of $a(\mu)$, at the same order in the loop expansion. (See Eq. (2.25) below.)

B. Renormalization in Detail

First, we shall review some details of DREG and MS to establish notation and to emphasize certain features of MS. Following 't Hooft [19], we renormalize the couplings a and w as follows:

$$\frac{1}{a_B} = \mu^{-\epsilon} \left[\frac{1}{a} + \frac{A_1(a, w)}{\epsilon} + \frac{A_2(a, w)}{\epsilon^2} + \dots \right], \quad (2.4)$$

where $n \equiv 4 - \epsilon$, μ is the renormalisation scale, and the ellipses represent higher order terms in powers of $1/\epsilon$. The factor $\mu^{-\epsilon}$ in front appears in order to make the

renormalized coupling a dimensionless, independent of the dimension n . Similarly,

$$\frac{1}{b_B} \equiv \frac{w_B}{a_B} = \mu^{-\epsilon} \left[\frac{w}{a} + \frac{B_1(a, w)}{\epsilon} + \frac{B_2(a, w)}{\epsilon^2} + \dots \right], \quad (2.5)$$

or, dividing by Eq. (2.4),

$$w_B = w + \frac{a(B_1 - w A_1)}{\epsilon} + \dots \quad (2.6)$$

From Eq. (2.4), the variation of a with scale $t \equiv \ln(\mu/\mu_0)$ is given by

$$0 = -\epsilon \left[\frac{1}{a} + \frac{A_1(a, w)}{\epsilon} \right] + \frac{da}{dt} \left[-\frac{1}{a^2} + \frac{1}{\epsilon} \frac{\partial A_1}{\partial a} \right] + \frac{dw}{dt} \frac{1}{\epsilon} \frac{\partial A_1}{\partial w} + \dots \quad (2.7)$$

To obtain the beta-functions, we want to isolate the terms of $O(1)$ or higher in ϵ . (In order for the theory to be renormalizable, terms involving negative powers of ϵ must cancel among themselves [19] in the limit $\epsilon \rightarrow 0$.) As expected from its definition, the coupling w_B is dimensionless for all n , so dw/dt will have no terms of order ϵ , and the last term can be neglected. Then we find

$$\frac{da}{dt} = -\epsilon a + \beta_a, \text{ with } \beta_a = -a^2 \frac{\partial (a A_1)}{\partial a}. \quad (2.8)$$

Similarly, from Eq. (2.6),

$$\frac{dw}{dt} = \beta_w = a \frac{\partial}{\partial a} [a(B_1 - w A_1)]. \quad (2.9)$$

The counterterms A_n, B_n may in principle be calculated order-by-order in the loop expansion. In a given order N , the counterterms A_n, B_n vanish for $n \geq N + 1$, so there are only a finite number of counterterms to each order. Further, as 't Hooft showed [19], at a given order, the counterterms A_n, B_n for $n \geq 2$ are completely determined the results of lower-order calculations. (This is why the beta-functions depended only on A_1 and B_1 to each order.) We exploited this fact in [I] to determine the dilaton mass, which first arises at two-loops in this model, from the results at one-loop.

The one-loop divergences have been calculated [2, 7] with the result that

$$A_1 = \beta_2 = \frac{133}{10}, \quad B_1 = \beta_3(w) = \frac{10w^2}{3} - 5w + \frac{5}{12}. \quad (2.10)$$

Thinking for a moment of a as a loop-counting parameter, with the tree approximation of order $1/a$, it comes as no surprise that the one-loop divergences are independent of a . This is why, in [I], we found that at one-loop, $\beta_w = a \bar{\beta}_w(w)$, with $\bar{\beta}_w(w) = \beta_3(w) - w \beta_2$. (As we shall see in Section II C, this scaling relation will not persist in higher orders.)

Presuming that these beta-functions are known, at least to some loop order, we wish to obtain the running couplings $a(t), w(t)$ by solving the coupled system of equations,

$$\frac{da}{dt} = \beta_a(a, w), \quad \frac{dw}{dt} = \beta_w(a, w), \quad (2.11)$$

⁷ As explained in I, there are reasons why it would be more logical to use the ratio b/a rather than a/b , but, as before, we choose to remain faithful to the usual convention.

where we have taken the limit $\epsilon \rightarrow 0$ in Eq. (2.8). It is well known that the general solution of a first-order system of this kind is unique up to the specification of the initial values (a_0, w_0) at some reference scale μ_0 , which we have defined to be $t = 0$. The fixed points of the system are obtained from the simultaneous zeros of the beta-functions $\beta_a(a, w) = 0, \beta_w(a, w) = 0$. As remarked above, at one-loop order, $\beta_a = -\beta_2 a^2$ for a positive constant β_2 . Within the perturbative regime, we may conclude that $\beta_a/a^2 < 0$ to all orders, so that $a(t)$ is monotonically decreasing from its initial value $a_0 > 0$. Thus, in this simple model, the fixed points are determined by the zeros of β_w .

Since the counterterms A_1 and B_1 can in principle be calculated order-by-order in the loop-expansion, the beta-functions $\{\beta_a(a, w), \beta_w(a, w)\}$ can be presumed known to arbitrary order. The running couplings $a(t), w(t)$ are therefore in principle known from the solutions to their defining equations, Eqs. (2.8), (2.9), up to their initial values a_0, w_0 . We now want to discuss the coupling c and its beta-function. As described earlier, having chosen Feynman rules that are independent of the coupling c , its counterterms, $C_n(a, w)$ are also completely fixed in terms of (a, w) . For example, the counterterm $C_1(a, w)/\epsilon$ is determined by what is “left over” from the divergences assigned to $A_1(a, w)/\epsilon$ and $B_1(a, w)/\epsilon$ (as well as any contribution to $\square R$, which, as discussed earlier, we can ignore.) The renormalization of c therefore proceeds more or less like the renormalization of any other coupling constant,

$$c_B = \mu^{-\epsilon} \left[c(\epsilon) + \frac{C_1(a, w)}{\epsilon} + \frac{C_2(a, w)}{\epsilon^2} + \dots \right], \quad (2.12)$$

where we assume that the function $c(\epsilon)$ may be expanded as a power series in ϵ with nonnegative powers, so that the renormalized coupling $c \equiv \lim_{\epsilon \rightarrow 0} c(\epsilon)$ exists. What is different about the renormalization of c is that all the counterterms $C_n(a, w)$ are independent of c . Hence, we have⁸

$$\frac{dc(\epsilon)}{dt} = \epsilon c(\epsilon) + \beta_c, \quad (2.13a)$$

$$\text{where } \beta_c(a, w) = \frac{\partial(aC_1(a, w))}{\partial a} \quad (2.13b)$$

The one-loop calculation [2, 7] gives $C_1 = -\beta_1$ with $\beta_1 = +196/45$, a constant.

Given its defining equation (now taking the limit $\epsilon \rightarrow 0$ in Eq. (2.13a)),

$$\frac{dc}{dt} = \beta_c(a(t), w(t)), \quad (2.14)$$

⁸ There are potentially negative powers of ϵ in Eq. (2.13b) in addition to these nonnegative ones, but, as 't Hooft [19] showed, these all must cancel among themselves.

with the running couplings $a(t), w(t)$ and the function $\beta_c(a, w)$ presumed known, the formal solution is

$$c(t) - c_0 = \int_0^t dt' \beta_c(a(t'), w(t')). \quad (2.15)$$

Thus, the renormalized coupling $c(t)$ is completely determined up to its initial value c_0 , which was our first claim. Eq. (2.15) does not determine that $c(t) - c_0$ is a function of $(a(t), w(t))$ at the same scale t ; it appears to depend upon their history, *i.e.*, it appears as if $c(t) - c_0$ is actually a functional $F[a(t), w(t)]$. That is an illusion. Since $a(t)$ is monotonically decreasing, its inverse $t = t(a)$ is well-defined, so that, in the integral in Eq. (2.15), we can change variables from t' to a' , writing

$$c(t) - c_0 = \int_{a_0}^{a(t)} da' \frac{\beta_c(a', w(t(a')))}{\beta_a(a', w(t(a')))} \quad (2.16)$$

This shows that $c(t)$ is actually an ordinary function of the value $a(t)$ at the same scale. Further, it shows that the only t dependence of $c(t)$ is implicit through its dependence on $a(t)$, just like other couplings. A similar argument applies to the dependence on w . By definition, $\beta_w(a, w)$ does not vanish except at a fixed point, so that, within a given phase, $\beta_w(a, w)$ will have a definite sign. Therefore, although $w(t)$ may be increasing or decreasing, it too is monotonic and may be inverted $t = t(w)$. As with $a(t)$, we may change variables in Eq. (2.15) from t' to w' to establish that $c(t)$ only depends on the function $w(t')$ through its value $w(t)$ at the same scale. Therefore, we have established our second claim⁹: $c - c_0 = \mathcal{C}(a(t), w(t))$ for some function \mathcal{C} .

To summarize what has been determined thus far, in Eq. (2.15), we have displayed a solution to Eq. (2.14). If we know the functions $a(t), b(t)$, then the solution is unique up to the constant c_0 . Further, we know that the functions $a(t), b(t)$ are uniquely determined by the values of (a_0, b_0) . If we do not know these initial values, then we could regard the solution in Eq. (2.15) as a three-parameter family of solutions $c(t; a_0, w_0, c_0)$.

If this were a real theory of nature rather than a model, we believe that, in principle, (a_0, w_0) would be experimental observables. We have not determined that c_0 is observable, and we will return to this question later. We know in addition that $c(t)$ is in fact a function of (a, w) , *i.e.*, $c = c_0 + \mathcal{C}(a(t), w(t))$. Can we say more about the function $\mathcal{C}(a, w)$? The answer is yes since c obeys the renormalization group equations.

⁹ This conclusion may be generalized to include additional dimensionless coupling constants λ_i associated with the inclusion of matter, but, since not all couplings necessarily run monotonically, the preceding argument must be modified slightly. As the couplings $\{w(t), \lambda_i(t)\}$ evolve, the interval $(0, t)$ may be broken up into a finite number of closed subintervals $[0, t_1], [t_1, t_2], \dots, [t_N, t]$ between which all the couplings run monotonically. Since the couplings are continuous, they must agree at the end-points t_p . Thus, the result may be built up piecewise.

C. Renormalization Group Equations

Knowing that $c(t) - c_0$ is a function of $(a(t), w(t))$, we may write

$$\frac{\partial c}{\partial a} \frac{da}{dt} + \frac{\partial c}{\partial w} \frac{dw}{dt} = \frac{dc}{dt}, \quad (2.17a)$$

$$\text{or } \beta_a(a, w) \frac{\partial c}{\partial a} + \beta_w(a, w) \frac{\partial c}{\partial w} = \beta_c(a, w). \quad (2.17b)$$

These equations are very powerful; each is a form of the RGE for the function $\mathcal{C}(a, w)$. One of its applications is to relate the functions in different orders in perturbation theory. For example, it is clear from Eq. (2.17b) that the one-loop approximation to the three beta-functions constrains the tree-approximation to the function $\mathcal{C}(a, w)$. The two-loop approximation to the beta-functions will constrain the one-loop correction to $\mathcal{C}(a, w)$, etc.

Note that Eq. (2.17b) makes no reference to the scale parameter t and poses the problem of finding c as one of determining the solutions of a first-order, inhomogeneous partial differential equation. Although these equations are not linear in c , the difference between any two solutions satisfies the homogeneous equation, which is linear. The generic approach to the study of such equations employs the method of characteristics¹⁰. In the present context, however, we believe that it is simpler to exploit the loop expansion, especially because nothing much is known beyond one-loop order about theories of this type.

We may take advantage of the fact that Eq. (2.17b) makes no explicit reference to the scale parameter to parameterize the loop expansion in terms of the coupling a , at some fixed scale. In particular, the counterterms may be expanded as

$$\begin{aligned} A_1 &= \sum_{k=1}^{\infty} a_k(w) a^{k-1}, \quad B_1 = \sum_{k=1}^{\infty} b_k(w) a^{k-1}, \\ C_1 &= \sum_{k=1}^{\infty} c_k(w) a^{k-1}, \end{aligned} \quad (2.18)$$

where the a_k, b_k, c_k corresponds to the k^{th} term in the loop-expansion. Then, from Eqs. (2.8), (2.9) and Eq. (2.13b), we have

$$\begin{aligned} -\frac{\beta_a}{a^2} &= \sum_{k=1}^{\infty} k a_k(w) a^{k-1}, \\ \frac{\beta_w}{a} &= \sum_{k=1}^{\infty} k w_k(w) a^{k-1}, \quad \beta_c = \sum_{k=1}^{\infty} k c_k(w) a^{k-1}, \end{aligned} \quad (2.19)$$

where, for brevity, we defined $w_k(w) \equiv b_k(w) - w a_k(w)$. Similarly, we may expand $c(a, w)$

$$c(a, w) = c_0 + \frac{e_0(w)}{a} + \sum_{k=1}^{\infty} e_k(w) a^{k-1}, \quad (2.20)$$

where, in addition to the constant c_0 , a tree-level contribution $e_0(w)/a$ has been included.

To determine $e_0(w)$ explicitly, we must insert the one-loop contributions to the beta-functions into Eq. (2.17b) to obtain

$$a_1(w) e_0(w) + w_1(w) e'_0(w) = c_1(w), \quad (2.21a)$$

$$\beta_2 e_0(w) + \bar{\beta}_w(w) e'_0(w) = -\beta_1, \quad (2.21b)$$

where, in Eq. (2.21b), we inserted the one-loop values for a_1, c_1 and $w_1(w) = \bar{\beta}_w(w)$ from from Eq. (2.10) and from immediately below Eq. (2.13b). The actual values are not so important as the fact that β_1, β_2 are constants. Then we observe that a solution of Eq. (2.21b) is simply $e_0(w) = -\beta_1/\beta_2$, a constant, regardless of the form¹¹. Therefore, the tree approximation to c is

$$c(a, w) = c_0 - \frac{\beta_1}{\beta_2 a}. \quad (2.22)$$

This is a rather remarkable result in some ways. As advertised, the one-loop beta-functions in Eq. (2.17b) determine the tree approximation for c . On the other hand, unlike ordinary coupling constants, the only arbitrariness in c is the constant c_0 , so rather than a consistency check, the RGE actually **determines** the tree approximation. Even though β_1, β_2 are quantum corrections of $O(\hbar)$, their ratio is $O(1)$.

It is convenient but not crucial that the one-loop corrections β_1, β_2 be independent of w ; however, if β_2 were dependent on w , there may be a danger that their ratio would either be singular or vanish for certain values of w . Nevertheless, there remains a paradox: although Eq. (2.22) corresponds to one solution, there appear to be others, since, to any solution $e_0(w)$ of Eq. (2.21b) may be added a solution $e_h(w)$ of the homogeneous equation

$$\beta_2 e_h(w) + \bar{\beta}_w(w) e'_h(w) = 0. \quad (2.23)$$

Thus, we could replace the solution Eq. (2.22) by

$$c(a, w) = c_0 + \frac{e_h(w)}{a} - \frac{\beta_1}{\beta_2 a}. \quad (2.24)$$

On the other hand, we argued earlier that the solution Eq. (2.15) was unique up to the constant c_0 . How can

¹⁰ See, e.g., Ref. [20]. For an application in an analogous context, see Ref. [21]. If the initial values (a_0, w_0, c_0) are regarded as unknown, this method can provide insight into the manifold of all solutions.

¹¹ In other words, we would not need to know the one-loop renormalization $b_3(w)$ of the R^2 term. Although we will not demonstrate it here, this persists in higher orders in the sense that, in order to determine the $O(N)$ -loop contribution to c , we only need know the renormalization of R^2 to $O(N-1)$.

both statements be true? The answer is that, like c_0 , $e_h(w(t))/a(t)$ is renormalization group invariant, *i.e.*, to one-loop order, it is independent of t . This is easily seen:

$$\begin{aligned} \frac{d}{dt} \left(\frac{e_h(w(t))}{a(t)} \right) &= -\frac{\beta_a}{a^2} e_h(w) + e'_h(w) \frac{\beta_w(a, w)}{a} \\ &= \beta_2 e_h(w) + e'_h(w) \bar{\beta}_w, \end{aligned} \quad (2.25)$$

which is identical to Eq. (2.23) and therefore zero. (Recall our earlier warning surrounding Eq. (2.3).) Thus, the ambiguity simply corresponds to the freedom to choose a different value of c_0 . (It is easy enough to verify this explicitly by writing down the general solution of Eq. (2.23), using β_2 and $\beta_3(w)$ as defined in Eq. (2.10)). Our general arguments above assure us that this result remains true to arbitrary order in perturbation theory; we can choose any solution for the $e_k(w)$ and the ambiguity can eventually be absorbed into the freedom to choose c_0 arbitrarily.

Therefore, for R^2 -gravity without matter, we have shown that, formulating the theory in terms of Feynman rules depending on only two coupling constants is self-consistent, provided the coupling constant associated with the Gauss-Bonnet term G is correspondingly renormalized.

We have only discussed the dimensionless coupling constants because, in a mass-independent renormalization scheme, the addition of UV irrelevant operators, such as an Einstein-Hilbert term or a cosmological constant, does not change the counterterms for the dimensionless couplings. Thus, they may be added without consequences for this proof. The preceding proof in no way required classical scale invariance.

This result may be rewritten in a number of other ways. Most commonly, C^2 is exchanged for W as in Eq. (1.6). We have that $\tilde{c} = c + 1/2a$, so that $\beta_{\tilde{c}} = \beta_c - \beta_a/(2a^2)$. Therefore, to one-loop order, $\beta_{\tilde{c}} = \kappa(-\beta_1 + \beta_2/2)$, and the tree approximation to \tilde{c} will be

$$\tilde{c} = c_0 + \left(\frac{1}{2} - \frac{\beta_1}{\beta_2} \right) \frac{1}{a}. \quad (2.26)$$

III. EXTENSION TO MATTER

The results of the preceding section may be extended to the incorporation of matter with only slight modifications. The fundamental consistency relation must be extended to

$$\beta_c = \frac{\partial c}{\partial a} \beta_a + \frac{\partial c}{\partial b} \beta_b + \sum_i \frac{\partial c}{\partial \lambda_i} \beta_{\lambda_i}, \quad (3.1)$$

where $\{\lambda_i\}$ represents all the additional dimensionless coupling constants in the theory. For example, the addition of a scalar field in the form

$$S_m = \int d^4x \sqrt{g} \left[\frac{1}{2} (\nabla \phi)^2 + \frac{\lambda}{4} \phi^4 - \frac{\xi \phi^2}{2} R \right]. \quad (3.2)$$

(Again, one could add mass terms or cubic couplings without changing the results for the dimensionless couplings.) The divergences for a and c are modified by the inclusion of matter, but, at one-loop, they simply change the values of the constants β_2 and β_1 , respectively [2, 7]. The nonminimal coupling ξ adds to the divergences proportional to R^2 , modifying the beta-function for w . Similarly, the divergences for λ as well as for the wavefunction renormalization of ϕ receive gravitational contributions. Their structure is well-understood [3]. The form of these renormalizations can be brought into the same form as before as follows: it turns out to be natural to rescale the field $\phi = \tilde{\phi}/\sqrt{a}$ and coupling $\lambda \equiv ay$, so that the matter action Eq. (3.2) takes the form

$$S_m[\phi, g_{\mu\nu}] = \int d^4x \frac{\sqrt{g}}{a} \left[\frac{1}{2} (\nabla \tilde{\phi})^2 + \frac{y}{4} \tilde{\phi}^4 - \frac{\xi \tilde{\phi}^2}{2} R \right]. \quad (3.3)$$

Thus, $1/a$ factors out, so that a remains a loop-counting parameter, and the beta-functions for w , y , and ξ may be written in the form

$$\frac{\partial w}{\partial u} = \bar{\beta}_w(w, \xi), \quad \frac{\partial \xi}{\partial u} = \bar{\beta}_\xi(w, \xi, y), \quad \frac{\partial y}{\partial u} = \bar{\beta}_y(w, \xi, y), \quad (3.4)$$

where $du = -da/(\beta_2 a)$. (See [I] for further details.) In this case, the fixed point behavior is far more complicated. We found that there are six fixed points, only one of which is a UV fixed point for all three couplings. Its basin of attraction is limited and does not include all values of the couplings, which is to say that these parameters do not always approach finite fixed points. It therefore depends on the initial conditions whether, as $a \rightarrow 0$, all other couplings are AF or finite. Thus, in the loop expansion of the renormalized couplings, the coefficients depend on the three parameters w, ξ, y in general.

Finally, one may add scalars, fermions and non-Abelian gauge fields. Each species makes a contributions to the constants β_1 and β_2 , but these gravitational couplings remain independent of other coupling constants. This is not true for β_w , which can depend on the non-minimal couplings ξ_i of the scalar fields as well as other dimensionless coupling constants. On the other hand, there are more interrelated matter couplings that complicate the determination of fixed points.

We shall not discuss these in detail here, but some examples have been worked out previously. (For a summary of models, see Chapter 9 of Ref. [3].) At one-loop, the gauge couplings receive no contributions from the gravitational couplings, a vestige of their classical conformal symmetry. Generally, the Yukawa couplings vanish more rapidly than the bosonic couplings, but the top quark coupling is so large in the SM that it is often necessary to include it, at least up to the scale of the Planck mass, to obtain realistic predictions. For present purposes, the important point is that none of these complications will alter the conclusions of this paper concerning the treatment of the couplings of the topologically-significant operators discussed herein.

IV. THE HIRZEBRUCH SIGNATURE

The G-B relation is not the only topologically motivated relation in theories such as these. Another is the Hirzebruch signature whose topological density R^*R is the gravitational contribution to the axial anomaly. Our excuse for neglecting it until now is that, unlike G , it is only needed for renormalization in models that include fermions whose couplings imply CP -violation, as in the Standard Model. Analogous to Eq. (1.3), the local form of the relation

$$R^*R = \nabla_\mu H^\mu; \quad H^\mu \equiv \epsilon^{\mu\nu\gamma\delta} \Gamma_{\nu\kappa}^\rho \left[\frac{1}{2} R^\kappa_{\rho\gamma\delta} + \frac{1}{3} \Gamma_{\gamma\lambda}^\kappa \Gamma_{\rho\delta}^\lambda \right]. \quad (4.1)$$

Like B^μ , H^μ transforms like a connection. The corresponding integral for for a compact manifold without boundaries is

$$48\pi^2\tau = \int_{\mathcal{M}} d^4x \sqrt{g} R^*R = \int_{\mathcal{M}} d^4x \sqrt{g} C^*C, \quad (4.2)$$

where the integer τ is referred to as the Hirzebruch signature or Hirzebruch index. (If \mathcal{M} has boundaries, then there will be additional terms representing their contributions.) Since $(C \pm C^*)^2 \geq 0$, $C^2 \geq |C^*C|$, with equality only for $C = \pm C^*$ (self-dual or anti-self-dual.) Thus,

$$\int_{\mathcal{M}} d^4x C^2 \geq 48\pi^2 |\tau|. \quad (4.3)$$

Consequently, only a compact spacetime that is not conformally flat can have nonzero signature. Since

$$C^*C = \left(\frac{C + C^*}{2} \right)^2 - \left(\frac{C - C^*}{2} \right)^2, \quad (4.4)$$

it may come as no surprise that τ can be related to the number of self-dual (b_2^+) or anti-self-dual (b_2^-) harmonic two-forms. In fact, $\tau = b_2^+ - b_2^-$.

The upshot of this is that another term may be added to the Lagrangian Eq. (1.1a) of the form $i\vartheta C^*C$. Obviously, ϑ is analogous to the θ -parameter of QCD, and a nonzero value of ϑ implies the model is P- and CP-violating. As with G , since R^*R is a total derivative, ϑ will not contribute to the Feynman rules. On the other hand, we expect that, if renormalized, it will obey an equation like Eq. (3.1), since fermions contribute to the beta-functions for a , b , as well as to those for other couplings, in particular, the beta-functions for Yukawa couplings.

V. INSTANTONS

The inclusion of these topologically significant terms in the action suggests that they could become even more relevant nonperturbatively, although this is not the focus of this paper. There has been a great deal of discussion about instantons in the context of Einstein-Hilbert

theory¹² and in string theory in higher dimensions, especially their role in anomalies [22]. For higher-order gravity of the type considered herein, there has been speculation about instantons and their potential effects assuming that the theory has a sensible conformal limit [23, 24]. Although our work specifically assumes that the QFT is not scale-invariant, let alone conformal-invariant, the potential physical implications of instantons may well be similar to those that were discussed for the conformal theory. Thus, an instanton that has a nonzero Euler characteristic χ would presumably be topology-changing, representing a tunneling amplitude from an initial state that represents one genus (*e.g.*, a sphere) to a final state that represents another (*e.g.*, a torus). Similarly, if an instanton carries a nonzero Hirzebruch signature τ , transitions between states of different “winding numbers” should occur. Since $\tau \neq 0$ will affect the chiral anomaly, it would be interesting to investigate what changes, if any, such instantons would imply for the usual picture of nonperturbative effects in QCD.

We have not yet investigated the role of instantons in these theories, but, as pointed out in Ref. [23], some of the instantons presented there for the conformal theory ought to survive in a scale-invariant theory, although these authors appear to have in mind a theory without anomalies. Motivated by the considerations in this paper and in [I], we suggest that classically scale-invariant theories may well provide a hospitable setting for treating such instantons semi-classically, even though their QFT’s are anomalous. The point is that, at sufficiently high scales, their background fields will be approximately scale-invariant. By this, we mean that, if all relevant couplings are AF, then the degree of scale-breaking becomes small asymptotically. Even if one supposes that the background has constant curvature, the actual magnitude of the curvature will still be undetermined. It remains to be seen whether topological characteristics can be discussed within such a framework.

Earlier work assumes that instantons are ALE, but in order to consider spacetimes such as de Sitter space, anti-de Sitter space, and others where curvature is essential and persistent, one must apparently give up this requirement. Exactly what alternative constraints are mandated for such theories has yet to be determined.

For cosmological applications, one probably should be discussing only initial states with the time evolution determined by an “in-in” or Schwinger-Keldysh formalism¹³. The Hartle-Hawking no-boundary hypothesis [26] is one such possibility, with a transition at the birth of the universe from Euclidean to Lorentzian signature. It has been argued that such a framework strongly favors inflationary cosmologies [27]. Just how starting from R^2

¹² For a review of early work, see, *e.g.*, [9]. Ref. [10] reviews some of the subsequent developments.

¹³ For some recent perspectives, see, *e.g.*, Refs. [25].

gravity and including instantons might affect such deliberations, if at all, remains unclear.

VI. CONCLUSIONS AND OPEN QUESTIONS

We have demonstrated that, quite generally, renormalizable gravity allows reduction from three to two primary operators and their associated couplings, as required by the local Gauss-Bonnet relation in four dimensions¹⁴. This has been tacitly assumed by previous authors, but there can be confusion concerning the precise role of topological terms such as G in the renormalization of the theory, since it must be included among the renormalized operators. It holds quite generally for the extension of pure gravity to include matter consisting of an arbitrary collection of scalars, vectors, and fermions. A similar discussion undoubtedly applies to the Hirzebruch signature density C^*C , which is also a covariant divergence of a “current” H^μ . When fermions are added in such a way that CP is violated, ϑ is expected to be renormalized, but in a manner similar to $c(a,b)$. The idea then is that the only arbitrariness in couplings such as $c(a,b)$ or $\vartheta(a,b)$ would be in the constants c_0 and or ϑ_0 .

One open question is whether the parameter c_0 , the only free parameter in the Gauss-Bonnet coupling, is in principle observable. We have our doubts that it can be observed in a purely perturbative framework, but if instantons come to play a role in the determination of acceptable states of the theory, then c_0 may well affect the outcome. Similar remarks should apply to ϑ_0 as well.

All these speculations presume that there are extensions of our earlier work [I] to classically scale-invariant models in which there is dimensional transmutation with an induced Planck mass in the same phase in which the coupling constants are asymptotically free. We suspect that such models exist, as other authors have usually assumed about models that explicitly break scale invariance classically¹⁵.

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Appendix A: Background Field Quantization

In this section, we elaborate what we mean by the background field method of quantizing Eq. (2.2). This is completely standard, except for the way in which the term cG enters the theory. We shall follow the notation and conventions of Appendix B of Ref. [5], employing DeWitt’s condensed notation [30], using a single index to denote all indices, including spacetime x^μ or other continuous parameters. Repeated indices are (usually) summed or integrated over.

For a classical action $S[\phi_i]$, the effective action may be formally defined by $\Gamma[\phi_i] \equiv S[\phi_i] + \Delta\Gamma[\phi_i]$, where

$$e^{-\Delta\Gamma[\phi_i]} = \int_{\mathcal{B}} \mathcal{D}h_i e^{-\Delta S[\phi_i, h_i] + h_k \frac{\delta \Delta\Gamma[\phi_i]}{\delta \phi_k}}, \quad (\text{A1a})$$

$$\text{with } \Delta S[\phi_i, h_i] \equiv S[\phi_i + h_i] - S[\phi_i] - h_j \frac{\delta S[\phi_i]}{\delta \phi_j}. \quad (\text{A1b})$$

\mathcal{B} denotes the background manifold associated with ϕ_i . Eq. (A1a) is a complicated integro-differential equation, whose meaning we have summarized previously in [I]. Here, we want to focus on Eq. (A1b), with ϕ_i replaced by the background metric, $g_{\mu\nu}$, and h_i , by the metric fluctuations¹⁶, $h_{\mu\nu}$. The point is that, according to Eqs. (1.7), (A1b), the operator G enters only into the classical action $S[g_{\mu\nu}]$ and not into $\Delta S[g_{\mu\nu}, h_{\mu\nu}]$, Eq. (A1b), and therefore does not contribute to the integral Eq. (A1a) that determines the QFT in the classical background. The term cG contributes neither to the propagator nor to the vertices.

Next, since one is dealing with a gauge theory, one must add gauge-fixing terms to ΔS , together with their associated Faddeev-Popov ghosts, although for the most general background field, this may not be necessary¹⁷. Of course, one then finds that the Feynman rules lead to divergent integrals, so that the theory must be regularized and renormalized. The canonical procedure is to express the classical action in terms of finite renormalized fields and couplings plus divergent but local counterterms chosen to cancel these divergences order-by-order in perturbation theory. In the case of interest, even though G contributes nothing to the Feynman rules arising from ΔS , there are divergences arising that contribute to the

¹⁴ Concerns about the viability of DREG were expressed in Ref. [28]. One consequence of our results is that these concerns have finally been laid to rest. See also Ref. [13].

¹⁵ A recent application of this type [29] attempts to include the effects of the Gauss-Bonnet coupling.

¹⁶ An expression for ΔS to second order in $h_{\mu\nu}$ may be found in Ref. [4], eqns. (4.53-4.55), spanning more than two full pages. To go beyond one-loop requires adding vertices arising in higher order, a formidable task!

¹⁷ The quadratic terms in $h_{\mu\nu}$ may be invertible without gauge-fixing, at least off-shell, which may be sufficient for determining beta-functions. For further discussion, see the Appendix of Ref. [31].

renormalization of the coupling c . Such phenomena are familiar already from QFT in curved spacetime even without quantizing the gravitational field. (For example, see Ref. [32].) If one is to use DREG, this procedure requires extending the operators in the classical action to n -dimensions. This can easily be done for C^2 , R^2 , as well as for G in the form of Eq. (1.2a) (but not in the form of Eq. (1.2b).)

By this reasoning, we believe that there is no obstruction to renormalization (as there are with anomalies), and the renormalization program can proceed as usual. Fortunately, we are not alone in our belief, inasmuch as this has also been implicitly assumed by all previous authors.

Had one defined the QFT by extending the operators to n -dimensions at the outset, *e.g.*, in the form of Eqs. (1.1b), (1.5), or Eq. (1.6), one could not use Eq. (1.7) to develop the Feynman rules¹⁸. ΔS would include terms from cG in the QFT contributing to the propagator and to vertices of order ϵ or higher. In that case, it must be shown that the renormalized operators in four-dimensions actually respect Eq. (1.7), the Bianchi identities, and other special properties peculiar to the four-dimensional theory. It would be nice to have a proof of this, but we have not found such an argument in the literature. Nevertheless, by our previous argument above, it seems that the coupling constant c can be renormalized without including it in the Feynman rules for the QFT.

Appendix B: Curvature in n -dimensions.

The Riemann curvature $R^\kappa_{\mu\lambda\nu}$ can be defined in n -dimensions, from which one can obtain the Ricci tensor $R_{\mu\nu} \equiv R^\lambda_{\mu\lambda\nu}$ and scalar $R \equiv R^\mu_{\mu}$. The Weyl tensor $C_{\kappa\lambda\mu\nu}$ can then be defined by the linear relation¹⁹

$$C_{\kappa\lambda\mu\nu} \equiv R_{\kappa\lambda\mu\nu} - \frac{1}{n-2} \left(g_{\kappa[\mu} \hat{R}_{\nu]\lambda} - g_{\lambda[\mu} \hat{R}_{\nu]\kappa} \right) + \frac{R}{n(n-1)} (g_{\kappa[\mu} g_{\nu]\lambda}), \quad (\text{B1})$$

where $\hat{R}_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu}R/n$. Exchanging the positions of the Riemann and Weyl tensors, we may regard this as the decomposition of the Riemann tensor into its irreducible components under $SO(n)$, symbolically as $R = C \oplus \hat{R} \oplus R$. This decomposition is orthogonal in the sense that

$$R^2_{\kappa\lambda\mu\nu} = C^2_{\kappa\lambda\mu\nu} + \frac{4\hat{R}^2_{\mu\nu}}{n-2} + \frac{2R^2}{n(n-1)}. \quad (\text{B2})$$

In four-dimensions, this becomes²

$$R^2_{\kappa\lambda\mu\nu} = C^2_{\kappa\lambda\mu\nu} + 2\hat{R}^2_{\mu\nu} + \frac{R^2}{6}. \quad (\text{B3})$$

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- [1] G. W. Gibbons and S. W. Hawking, "Action Integrals and Partition Functions in Quantum Gravity," *Phys. Rev. D* **15** (1977) 2752.
 - [2] I. G. Avramidi and A. O. Barvinsky, "Asymptotic Freedom In Higher Derivative Quantum Gravity," *Phys. Lett. B* **159** (1985) 269.
 - [3] I. L. Buchbinder, S. D. Odintsov and I. L. Shapiro, "Effective action in quantum gravity," Bristol, UK: IOP (1992).
 - [4] A detailed review can be found in I. G. Avramidi, "Heat kernel and quantum gravity," *Lect. Notes Phys. M* **64** (2000) 1. Springer, Berlin (2000). DOI: 10.1007/3-540-46523-5
 - [5] M. B. Einhorn and D. R. T. Jones, "Naturalness and Dimensional Transmutation in Classically Scale-Invariant Gravity," *JHEP*, to appear. [arXiv:1410.8513 [hep-th]]
 - [6] K. S. Stelle, "Renormalization of Higher Derivative Quantum Gravity," *Phys. Rev. D* **16** (1977) 953.
 - [7] E. S. Fradkin and A. A. Tseytlin, "Renormalizable Asymptotically Free Quantum Theory of Gravity," *Phys. Lett. B* **104** (1981) 377.

- E. S. Fradkin and A. A. Tseytlin, "Renormalizable asymptotically free quantum theory of gravity," *Nucl. Phys. B* **201** (1982) 469.
- [8] S. A. Brunini and M. Gomes, "The Gauss-Bonnet identity in fourth order gravity," *Mod. Phys. Lett. A* **8** (1993) 1977.
- [9] T. Eguchi, P. B. Gilkey and A. J. Hanson, "Gravitation, Gauge Theories and Differential Geometry," *Phys. Rept.* **66** (1980) 213.
- L. Alvarez-Gaume and P. H. Ginsparg, "The Structure of Gauge and Gravitational Anomalies," *Annals Phys.* **161** (1985) 423 [Erratum-ibid. **171** (1986) 233].
- [10] E. Alvarez, "Quantum Gravity: A Pedagogical Introduction To Some Recent Results," *Rev. Mod. Phys.* **61** (1989) 561.
- L. Alvarez-Gaume, "Supersymmetry and the Atiyah-Singer Index Theorem," *Commun. Math. Phys.* **90** (1983) 161.
- [11] S. S. Chern, "A Simple Intrinsic Proof of The Gauss-Bonnet Formula for Closed Riemannian Manifolds," *Ann. Math.* **46** (1944) 747.
- [12] I. L. Buchbinder and S. D. Odintsov, "Asymptotics Of Effective Action In Higher Derivative Quantum Gravity," *Czech. J. Phys. B* **37** (1987) 682.
- [13] G. de Berredo-Peixoto and I. L. Shapiro, "Higher derivative quantum gravity with Gauss-Bonnet term," *Phys. Rev. D* **71** (2005) 064005 [hep-th/0412249].
- [14] J. L. Cardy, "Is There a c Theorem in Four-Dimensions?," *Phys. Lett. B* **215** (1988) 749.
- [15] S. J. Hathrell, "Trace Anomalies and $\lambda\phi^4$ Theory in

¹⁸ In flat background, it has been shown by Zwiebach [33] that, even in higher dimensions, G suprisingly remains a total derivative. See also Ref. [34].

¹⁹ Our convention concerning bracketed indices is to antisymmetrize those contained, so, *e.g.*, $g_{\kappa[\mu} \hat{R}_{\nu]\lambda} \equiv g_{\kappa\mu} \hat{R}_{\nu\lambda} - g_{\kappa\nu} \hat{R}_{\mu\lambda}$. A common alternative takes the brackets to mean half the difference.

- Curved Space,” *Annals Phys.* **139** (1982) 136.
- [16] D. Z. Freedman and H. Osborn, “Constructing a c function for SUSY gauge theories,” *Phys. Lett. B* **432** (1998) 353 [hep-th/9804101].
- [17] I. Jack and H. Osborn, “Constraints on RG Flow for Four Dimensional Quantum Field Theories,” *Nucl. Phys. B* **883** (2014) 425 [arXiv:1312.0428 [hep-th]].
I. Jack and C. Poole, “The a -function for gauge theories,” arXiv:1411.1301 [hep-th].
- [18] W. Zimmermann, “Reduction in the Number of Coupling Parameters,” *Commun. Math. Phys.* **97** (1985) 211.
R. Oehme, “Reduction and Reparametrization of Quantum Field Theories,” *Prog. Theor. Phys. Suppl.* **86** (1986) 215.
R. Oehme, “Reduction of coupling parameters in quantum field theories,” in “Concise Encyclopedia of Supersymmetry”, Kluwer Academic Publishers, Dordrecht, (Editors: J. Bagger, S. Duplij and W. Siegel) 2001 [hep-th/0012162].
R. Oehme, “Reduction of coupling parameters and duality,” *Lect. Notes Phys.* **558** (2000) 136 [hep-th/9903092].
W. Zimmermann, “Scheme independence of the reduction principle and asymptotic freedom in several couplings,” *Commun. Math. Phys.* **219** (2001) 221.
- [19] G. 't Hooft, “Dimensional regularization and the renormalization group,” *Nucl. Phys. B* **61** (1973) 455.
- [20] V. I. Arnold, “Ordinary Differential Equations,” The MIT Press, Boston (1978) [978-0262510189].
- [21] M. B. Einhorn and D. R. T. Jones, “Scale Fixing by Dimensional Transmutation: Supersymmetric Unified Models and the Renormalization Group,” *Nucl. Phys. B* **211** (1983) 29.
- [22] L. Alvarez-Gaume and E. Witten, “Gravitational Anomalies,” *Nucl. Phys. B* **234** (1984) 269.
- [23] A. Strominger, G. T. Horowitz and M. J. Perry, “Instantons in Conformal Gravity,” *Nucl. Phys. B* **238** (1984) 653.
- [24] M. J. Perry and E. Teo, “Topological conformal gravity in four-dimensions,” *Nucl. Phys. B* **401** (1993) 206 [hep-th/9211063].
- [25] P. Adshead, R. Easter and E. A. Lim, “The ‘in-in’ Formalism and Cosmological Perturbations,” *Phys. Rev. D* **80** (2009) 083521 [arXiv:0904.4207 [hep-th]].
A. Higuchi, D. Marolf and I. A. Morrison, “On the Equivalence between Euclidean and In-In Formalisms in de Sitter QFT,” *Phys. Rev. D* **83** (2011) 084029 [arXiv:1012.3415 [gr-qc]].
A. Kaya, “Stationary phase approximation and instanton-like states for cosmological in-in path integrals,” *Phys. Rev. D* **86** (2012) 123511 [arXiv:1209.4694 [hep-th]].
A. Kaya, “In-in formalism, pseudo-instantons and rethinking quantum cosmology,” *Int. J. Mod. Phys. D* **23** (2014) 1450026 [arXiv:1306.6541 [gr-qc]].
- [26] J. B. Hartle and S. W. Hawking, “Wave Function of the Universe,” *Phys. Rev. D* **28** (1983) 2960.
- [27] J. B. Hartle, S. W. Hawking and T. Hertog, “No-Boundary Measure of the Universe,” *Phys. Rev. Lett.* **100** (2008) 201301 [arXiv:0711.4630 [hep-th]].
J. B. Hartle, S. W. Hawking and T. Hertog, “The Classical Universes of the No-Boundary Quantum State,” *Phys. Rev. D* **77** (2008) 123537 [arXiv:0803.1663 [hep-th]].
- [28] D. M. Capper and D. Kimber, “An Ambiguity in One Loop Quantum Gravity,” *J. Phys. A* **13** (1980) 3671.
- [29] R. Myrzakulov, S. Odintsov and L. Sebastiani, “Inflationary universe from higher-derivative quantum gravity,” arXiv:1412.1073 [gr-qc].
- [30] B. S. DeWitt, “Dynamical theory of groups and fields,” *Conf. Proc. C* **630701** (1964) 585 [Les Houches Lect. Notes **13** (1964) 585].
- [31] E. S. Fradkin and A. A. Tseytlin, “One Loop Effective Potential in Gauged $O(4)$ Supergravity,” *Nucl. Phys. B* **234**, 472 (1984).
- [32] I. Jack and H. Osborn, *Nucl. Phys. B* **234** (1984) 331.
- [33] B. Zwiebach, “Curvature Squared Terms and String Theories,” *Phys. Lett. B* **156** (1985) 315.
- [34] B. Zumino, “Gravity Theories in More Than Four-Dimensions,” *Phys. Rept.* **137** (1986) 109.