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Cosmic Neutrino Secret Interactions, Enhancement and Total Cross Section

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The scattering of neutrinos assuming a "secret" interaction at low energy is considered. To leading order in energy, the two-body potential is a δ-potential, and it is used as a motivation to study generic short-range elastic interactions between neutrinos. The scattering cross section and Sommerfeld enhancement depend on two phenomenological parameters deriving from the exact form of the potential, akin to "renormalized" coupling constants. Repulsive potentials lead to a decrease in the total cross section, resulting in an enhancement of the neutrino density. For attractive potentials of the right form, substantial Sommerfeld enhancement can appear.

I. INTRODUCTION

In the last twenty years the nature of dark matter, needed to address the missing mass problem, has been extensively investigated [1]. Specifically, there is a large body of evidence from astronomical observations indicating that there is more matter than can be associated with the luminous part of galaxies. This problem remains unresolved, and opens a window to new physics now called astroparticle physics [2]. It is not yet fully understood what kind of particle dominates the dark matter and what are the mechanisms of production/annihilation for them. Neutrinos would be one possibility, but presently we know that only a small portion of neutrinos (1%) of the CNB (cosmic neutrino background) could be considered as dark matter. Specifically, in the CNB the number of neutrinos per cubic centimeter is about 56 per flavor, and this suggests that a small fraction of the dark matter could be attributable to neutrinos. Therefore, cosmic neutrinos would be responsible for only a small part of dark halos that would explain the flatness of galaxy rotation curves [5, 6], *i.e.* the constant value of the velocity of rotation of a galaxy as a function of distance rather than decreasing as expected. Even though dark neutrinos are very few, they are nevertheless important, as their mass bounds are very robust and can be used as phenomenological data for estimating the total cosmological neutrino flux and cross sections [3, 4].

In the present state of our knowledge the cosmic neutrino background (CNB) is a prediction of the standard cosmological model. There are, however, known data which, when consistently interpreted, might lead to new insights beyond the standard model. In this direction, there are several interesting questions; for example, if there are any enhancement mechanisms for dark halo neutrinos, and how they would affect physical observable processes.

In reference [7] (see also [8]) an enhancement mechanism for weakly interacting particles was proposed, inspired by the Sommerfeld enhancement effect, and several examples illustrating how this mechanism works were presented. A thorough treatment of the effective non-relativistic theory of dark matter long-range interactions was also presented in [9], including their renormalization properties. In spite of the intense interest in this field in the last few years, however, to our knowledge there is no discussion on the role played by very short-range potentials (invisible at long distances), nor relevant extensions of the results presented in [7]. This would be especially relevant for low energy scattering of bosons or fermions with a short-range two-body potential. The dynamics of the CNB neutrinos at very low energy, in particular, could be considered by modeling the neutrino-neutrino interaction with a contact potential.

The goal of this paper is to provide a justification for the short range potential as stemming from possible "secret" neutrino interactions, to give a derivation of the scattering amplitude for such short range potentials, and to show how Sommerfeld enhancement can emerge by computing the corresponding enhancement factor.

This work is organized as follows. In section **II** we calculate the two-body potential for neutrinos from secret interaction in inverse powers of the neutrino mass and show that the leading term is a delta potential (contact interaction). In section **III** we solve the associated scattering problem for a general short-range potential and show that physics depends on a set of phenomenological parameters. In section **IV** we consider the Sommerfeld enhancement and derive its strength in various domains of the scale of the annihilation processes. In section **V** we compare and contrast our approach with standard regularizations procedures for the delta-potential and elucidate its connection to renormalization in effective potentials, and briefly comment on bound states. In section **VI** we present our conclusions, pointing out that the neutrino density can be substantially enhanced if the potential is repulsive, and that there is Sommerfeld enhancement if the potential is attractive. Several calculations and justifications of approximations valid for short-range potentials are presented in the appendix.

II. EFFECTIVE TWO-BODY POTENTIAL FOR INTERACTIONS BETWEEN NEUTRINOS

Effective four-fermion interactions between neutrinos with nonstandard coupling (also called "secret" neutrio interactions) have been considered for quite some time [10], as a way to include new classes of matter as the particles mediating this interaction [11]. Clearly the four-fermion interaction is only a first approximation, capturing the low-energy physics of an otherwise more general interaction, valid for distances larger than some characteristic scale for the interaction. (A pure four-fermion interaction would not even be renormalizable.) Its form is taken to be

$$-\frac{\alpha}{2M^2}(\bar{\psi}_a\gamma^\mu\psi_a)(\bar{\psi}_b\gamma_\mu\psi_b), \tag{1}$$

where a sum over the three neutrino species is understood, $\alpha = \pm 1$ corresponds to an attractive/repulsive potential and M is a given mass scale. This contact neutrino interaction leads to the Born scattering amplitude for two distinct neutrinos (see Fig. 1)

$$\begin{aligned} M_{fi} &= -\frac{\alpha}{M^2}\bar{u}_1(p_1')\gamma^\mu u_1(p_1)\bar{u}_2(p_2')\gamma_\mu u_2(p_2), \\ &= \frac{\alpha}{M^2}\bar{w}_\alpha(p_1')\bar{w}_\beta(p_2')\,U_{\alpha\beta,\gamma\delta}\,w_\gamma(p_1)w_\delta(p_2), \end{aligned} \tag{2}$$

where w_α's denote the positive energy spinors in the non-relativistic limit and $U_{\alpha\beta,\gamma\delta}$ is the two-body potential which will be computed in detail below.

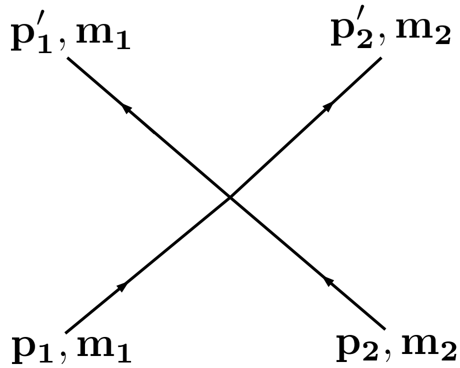

FIG. 1. Contact interaction at the tree level.

To extract the low-energy dynamics, we work in the non-relativistic regime $p \ll m$, with m the neutrino mass. For a free massive fermion, the Foldy-Wouthyusen transformation that diagonalizes the Hamiltonian in the non-relativistic limit is given by

$$\begin{aligned} U_{FW} &= \exp[\frac{\gamma \cdot \mathrm{p}}{2m}\theta] \\ &= \cos\left(\frac{|\mathbf{p}|\theta}{2m}\right) + \frac{\gamma \cdot \mathbf{p}}{|\mathbf{p}|} \sin\left(\frac{|\mathbf{p}|\theta}{2m}\right), \end{aligned} \tag{3}$$

where

$$\theta = \frac{m}{|\mathbf{p}|} \tan^{-1}\left(\frac{\mathbf{p}}{m}\right) = 1 - \frac{1}{3}\frac{|\mathbf{p}|^{\mathbf{2}}}{m^2} + \frac{1}{5}\left(\frac{|\mathbf{p}|}{m^2}\right)^2 + \cdots, \tag{4}$$

and

$$u(p) = U_{FW} w(p), \quad w(p) = \begin{pmatrix} w_1(p) \\ w_2(p) \end{pmatrix}. \tag{5}$$

We use the normalization

$$w^*_\alpha(p) w_\alpha(p) = 1,$$

Therefore, to order $1/m^2$ we have

$$\begin{aligned} \cos\left(\frac{|\mathbf{p}|\theta}{2m}\right) &= 1 - \frac{|\mathbf{p}|^2}{8m^2} + \cdots, \\ \sin\left(\frac{|\mathbf{p}|\theta}{2m}\right) &= \frac{|\mathbf{p}|}{2m} + \cdots, \end{aligned} \tag{6}$$

and as a consequence

$$\frac{\gamma \cdot \mathbf{p}}{|\mathbf{p}|} \sin\left(\frac{|\mathbf{p}|\theta}{2m}\right) \approx \frac{\gamma \cdot \mathbf{p}}{2m} + \cdots. \tag{7}$$

This leads to

$$U_{FW} \approx \left(1 - \frac{|\mathbf{p}|^{\mathbf{2}}}{8m^2}\right) + \frac{\gamma \cdot \mathbf{p}}{2m} \cdots. \tag{8}$$

The spinor $u(p)$ can now be expanded up to order $1/m^2$ as follows

$$\begin{aligned} u(p) &= U_{FW} w(p) \\ &= \left(1 - \frac{\mathbf{p}^2}{8m^2}\right) w + \frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{2m} w + \mathcal{O}(1/m^3). \end{aligned} \tag{9}$$

Thus by restricting to the positive energy spinor space, we have to order $1/m^2$

$$\begin{aligned} \bar{u}_1(p_1')\gamma^0 u_1(p_1) &= u_1^\dagger(p_1')u_1(p_1) \\ &= w_{1\alpha}^*(p_1') \left[\delta_{\alpha\gamma}\left(1 - \frac{\mathbf{q}^2}{8m_1^2}\right) + i\frac{(\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{p}_1))_{\alpha\gamma}}{4m_1^2}\right] w_{1\gamma}(p_1), \end{aligned} \tag{10}$$

where $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_1' = -(\mathbf{p}_2 - \mathbf{p}_2')$ is the transferred momentum. Similarly for the other spinor we have

$$\begin{aligned} \bar{u}_2(p_2')\gamma^0 u_2(p_2) &= u_2^\dagger(p_2')u_2(p_2) \\ &= w_{2\beta}^*(p_2') \left[\delta_{\beta\delta}\left(1 - \frac{\mathbf{q}^2}{8m_2^2}\right) - i\frac{(\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{p}_2))_{\beta\delta}}{4m_1^2}\right] w_{2\delta}(p_1) \end{aligned} \tag{11}$$

Bilinears involving γ^i can also be calculated and lead to

$$\begin{aligned} \bar{u}_1(p_1')\gamma^i u_1(p_1) &= \frac{1}{2m_1} w_{1\alpha}^*(p_1') \left(\delta_{\alpha\beta}(-2p_1^i + q^i) + i(\boldsymbol{\sigma}\times\mathbf{q})^i_{\alpha\beta}\right) w_{1\beta}(p_1), \\ \bar{u}_2(p_2')\gamma^i u_2(p_2) &= \frac{1}{2m_2} w_{2\alpha}^*(p_2') \left(\delta_{\alpha\beta}(2p_2^i + q^i) + i(\boldsymbol{\sigma}\times\mathbf{q})^i_{\alpha\beta}\right) w_{2\beta}(p_2), \end{aligned} \tag{12}$$

where the indices $1, 2$ correspond to the two particle species and the greek indices $\alpha, \beta, \cdots$ denote spinor components.

Using the identity

$$\mathbf{p}\cdot(\boldsymbol{\sigma}\times\mathbf{q}) = \boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{p}),$$

the scattering amplitude can now be written as

$$M_{fi} = w_{1\alpha}^*(p_1')w_{2\beta}^*(p_2')U_{\alpha\beta,\gamma\delta}(\mathbf{q},\mathbf{p}_1,\mathbf{p}_2)w_{1\gamma}(p_1)w_{2\delta}(p_2), \tag{13}$$

where the two-body potential has the form

$$\begin{aligned} U_{\alpha\beta,\gamma\delta}(\mathbf{q},\mathbf{p}_1,\mathbf{p}_2) = -\frac{\alpha}{M^2}\Bigg[&\delta_{\alpha\gamma}\delta_{\beta\delta}\left(1 - \frac{\mathbf{q}^2}{8}\left(\frac{1}{m_1^2} + \frac{1}{m_2^2}\right)\right) + i\delta_{\beta\delta}\frac{[\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{p}_1)]_{\alpha\gamma}}{4m_1^2} - i\delta_{\alpha\gamma}\frac{[\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{p}_2)]_{\beta\delta}}{4m_2^2} \\ &+ \delta_{\alpha\gamma}\delta_{\beta\delta}\frac{(\mathbf{q}^2 + 2\mathbf{q}\cdot(\mathbf{p}_2 - \mathbf{p}_1) - 4\mathbf{p}_1\cdot\mathbf{p}_2)}{4m_1m_2} - i\delta_{\beta\delta}\frac{[\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{p}_2)]_{\alpha\gamma}}{2m_1m_2} + i\delta_{\alpha\gamma}\frac{[\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{p}_1)]_{\beta\delta}}{2m_1m_2} \\ &+ \frac{(\boldsymbol{\sigma}\cdot\mathbf{q})_{\alpha\gamma}(\boldsymbol{\sigma}\cdot\mathbf{q})_{\beta\delta}}{4m_1m_2} - \frac{\mathbf{q}^2}{4m_1m_2}(\boldsymbol{\sigma})_{\alpha\gamma}\cdot(\boldsymbol{\sigma})_{\beta\gamma}\Bigg]. \end{aligned} \tag{14}$$

This shows that even though the particles are fermions, at leading order the dominant contribution is a contact potential given by

$$U(\mathbf{x}) = -\frac{\alpha}{M^2}\delta(\mathbf{x}), \tag{15}$$

where $\mathbf{x}$ is the relative coordinate $\mathbf{x}_1 - \mathbf{x}_2$ and the potential can be repulsive or attractive depending of the sign of α. Momentum-dependent higher-order corrections in this potential can also be calculated from (14) and are, generically, spin-dependent.

We can use the above contact interaction as a motivation to consider momentum-independent short-range interactions with a range shorter than the neutrino Compton wavelength. In that regime, the full details of the interaction potential are expected to become irrelevant and be subsumed in some macroscopic phenomenological parameters.

III. THREE-DIMENSIONAL SHORT-RANGE POTENTIALS AND THEIR SCATTERING PROPERTIES

We turn our attention to the scattering properties of short-range potentials of arbitrary form in the regime where their range is much smaller than the de Broglie wavelength of the incident neutrinos. Such potentials would macroscopically look like delta-functions with strength equal to their space integral. Their details, however, will in general matter for scattering, so that more than one parameter may be relevant

We assume that the neutrinos interact with a central two-body interaction potential of the form $V(r)$ where r is the distance between the neutrinos, with the property that $V(r)$ is nonzero only within a very short range $0 < r < a$. We are interested in scattering properties for wavenumbers much smaller that a^{-1}. Clearly, not all the details of the form of $V(r)$ will be relevant. The question is: what are the relevant parameters that fix physics for such wavelengths?

The motion in the above potential is described by the time independent Schrödinger equation for the relative coordinate wavefunction

$$\left(\boldsymbol{\nabla}^2 + k^2\right)\psi(\mathbf{x}) = 2m\,V(r)\psi(\mathbf{x}). \tag{16}$$

where $k^2 = 2mE$ and $m = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass of scattering neutrinos. The integral scattering equation, in this case, is

$$\psi(\mathbf{x}) = \varphi(\mathbf{x}) - 2m\int d^3\mathbf{x}'\; G(\mathbf{x}-\mathbf{x}')\; V(r')\;\psi(\mathbf{x}'), \tag{17}$$

where $\varphi(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}}$ represents the incident plane wave. The retarded (outgoing) Green's function satisfies

$$\left(\boldsymbol{\nabla}^2 + k^2\right)G(\mathbf{x}-\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}') \tag{18}$$

and is explicitly given by

$$G(\mathbf{x}) = \frac{1}{4\pi}\frac{e^{ikr}}{r}, \tag{19}$$

with $r = |\mathbf{x}|$.

In the kinematical regime of interest, only the spherically symmetric s-wave ($\ell = 0$) part of the vavefunction will matter for scattering and Sommerfeld enhancement, since higher-ℓ partial waves vanish at the origin and thus are much smaller over the range of the potential than the s-wave part. (This assertion is fully justified in the appendix.) In fact, restricting to the s-wave sector is exact for the calculation of $\psi(0)$ since the integrand in the scattering equation in that case is spherically symmetric for a central potential $V(r)$ and the higher angular momentum part of $\psi(\mathbf{x})$ drops out. Specifically, putting $\mathbf{x} = 0$ in (17) we obtain

$$\psi(0) = 1 - \frac{m}{2\pi}\int d^3\mathbf{x}\,\frac{e^{ikr}}{r}V(r)\psi(\mathbf{x}) = 1 - 2m\int dr\, re^{ikr}V(r)\psi_s(r). \tag{20}$$

The s-wave part of the wavefunction, $\psi_s(r)$, satisfies the standard radial fixed-energy Schrödinger equation

$$\psi_s(r) = \frac{\phi(r)}{r}\;, \quad \left(\frac{d^2}{dr^2} + k^2\right)\phi(r) = 2mV(r)\phi(r). \tag{21}$$

As usual, $\phi(r)$ vanishes at $r = 0$ and $\psi(0) = \phi'(0)$. We write the full wavefunction in terms of a normalized $u(r) = \phi(r)/\psi(0)$ as

$$\psi(\mathbf{x}) = \psi(0)\frac{u(r)}{r} + \psi_{ns}(x), \tag{22}$$

where $\psi_{ns}(x)$ is the non-s-wave part (nonzero angular momentum) of the wavefunction, satisfying $\psi_{ns}(0) = 0$, and the function $u(r)$ is normalized to satisfy

$$u'' + k^2 u = 2mVu\;, \quad u(0) = 0\;, \quad u'(0) = 1, \tag{23}$$

(primes are r-derivatives). With these initial conditions at $r = 0$ the solution $u(r)$ is unique. The coefficient $\psi(0)$ is fixed by matching the solution for $\psi(\mathbf{x})$ to the appropriate boundary conditions at $r \to \infty$.

Plugging the above form of $\psi_s(r)$ in (20) we obtain

$$\psi(0) = 1 - 2m\psi(0) \int_0^a dr\, e^{ikr} V u, \tag{24}$$

where we used the fact that $V(r)$ vanishes for $r > a$. This fixes $\psi(0)$ as

$$\psi(0) = \frac{1}{1 + 2m \int_0^a dr\, e^{ikr} V u}. \tag{25}$$

Using equation (23) for u and integrating by parts we eventually obtain the exact expression for the wavefunction at the origin

$$\psi(0) = \frac{e^{-ika}}{u'(a) - iku(a)}. \tag{26}$$

Similarly, to find the scattering amplitude we put $|\mathbf{x}| \gg |\mathbf{x}'|$ in the scattering equation (17) and keep only the s-part of the wavefunction $\psi_s(r')$ to obtain

$$\psi(\mathbf{x}) = \varphi(\mathbf{x}) - \frac{e^{ikr}}{4\pi r}\, 2m \int dr' d\theta' d\phi' r'^2 \sin\theta' e^{-ikr'\cos\theta'}\; V(r')\; \psi(0) \frac{u(r')}{r'}, \tag{27}$$

where θ' is the polar angle of $\mathbf{x}'$ measured with respect to $\mathbf{x}$. The factor multiplying the outgoing spherical wave e^{ikr}/r is the scattering amplitude f. Performing the angular integral and using (23) and integration by parts as before we obtain

$$f = -\psi(0) \left(\frac{\sin ka}{k}\, u'(a) - \cos ka\, u(a) \right) = -e^{-ika} \frac{\sin ka\, u'(a) - k \cos ka\, u(a)}{k[u'(a) - iku(a)]}, \tag{28}$$

from which we obtain the scattering cross-section as

$$\sigma = 4\pi |f|^2. \tag{29}$$

The above expressions are exact. We are interested, however, in the limit $ka \ll 1$. In this regime, in general $k^2 \ll 2mV$ and we can treat k^2 as a perturbation parameter. The full solution for $u(r)$ can then be expressed as a series in k^2 in a way analogous to the Lippman-Schwinger expansion. Specifically, we define the unique function u_0 satisfying:

$$u_0'' - 2mVu_0 = 0\ , \quad u_0(0) = 0\ , \quad u_0'(0) = 1, \tag{30}$$

and the sequence of functions u_n, $n = 1, 2, \ldots$

$$u_n'' - 2mVu_n = u_{n-1}\ , \quad u_n(0) = u_n'(0) = 0. \tag{31}$$

Then the full solution for u can be written as

$$u(r) = \sum_{n=0}^{\infty} (ik)^{2n} u_n(r). \tag{32}$$

The above u satisfies equation (23), and also has the correct boundary conditions at $r = 0$ due to the boundary conditions of u_0 and u_n, $n \geq 1$. Formally, the u_n can be written

$$u_n = \left(-\frac{d^2}{dr^2} + 2mV \right)^{-n} u_0. \tag{33}$$

The point of the above expansion is that u_0 represents the zero mode of the operator $-\frac{d^2}{dr^2} + 2mV(r)$. The inverse operator $\left(-\frac{d^2}{dr^2} + 2mV\right)^{-1}$, however, is defined on the set of states with boundary conditions $u(0) = u'(0) = 0$, on which this operator has no zero modes (the unique solution with these boundary conditions is 0), so the inverse exists.

We can now use this expansion in the expressions for $\psi(0)$ and f. We have

$$\frac{1}{\psi(0)} = e^{ika} \sum_n \left[(ik)^{2n} u_n'(a) - (ik)^{2n+1} u_n(a) \right]. \tag{34}$$

The values

$$\lambda_{2n} = u_n'(a)\ , \quad \lambda_{2n+1} = u_n(a), \tag{35}$$

are an (infinite) set of phenomenological parameters that determine the scattering properties of the potential as a function of k. u_0 behaves linearly near $r = 0$ ($u_0(r) \sim r$), while $u_n(r) \sim r^{2n+1}$ and $u_n'(r) \sim r^{2n}$. For a *generic* potential such that $Va^2 \ll 1$

$$\lambda_{2n+1} = u_n(a) \sim a^{2n+1}\ , \quad \lambda_{2n} = u_n'(a) \sim a^{2n}. \tag{36}$$

So the leading term $\lambda_0 = 1 + O(a)$ becomes the only relevant one in the limit $a \to 0$, the rest being negligible. Such potentials have no interesting scattering dynamics, leading to $|\psi(0)| \simeq 1$ and $\sigma \simeq 0$.

Physically interesting potentials are these that lead to at least the first couple of the above parameters to be of the same order of magnitude. Achieving this needs potentials whose magnitude scales as a^{-2} or faster and have nontrivial profiles. (Examples of such potentials and the scaling of u_n are given in the appendix.) Keeping only λ_0 and λ_1 and assuming all higher λ_n are negligible in the limit $ka \ll 1$, we have for the wavefunction

$$\psi(0) = \frac{1}{\lambda_0 - ik\lambda_1}, \tag{37}$$

and the scattering amplitude and cross section

$$f = \frac{\lambda_1}{\lambda_0 - ik\lambda_1}\ , \quad \sigma = \frac{4\pi\lambda_1^2}{\lambda_0^2 + k^2\lambda_1^2}. \tag{38}$$

IV. SOMMERFELD ENHANCEMENT

For short-range annihilation processes, Sommerfeld enhancement is given by $S = |\psi(0)|^2$. In our case, however, the elastic interaction potential is also short-range, so the above formula is not automatically true. We have to distinguish different regimes:

a) If the range of the annihilation process b is much smaller than the range of the potential a, then $\psi(0)$ determines Sommerfeld enhancement:

$$b \ll a: \quad S = |\psi(0)|^2 = \frac{1}{\lambda_0^2 + k^2\lambda_1^2}, \tag{39}$$

with a maximal value for $k = 0$

$$S_{max} = \frac{1}{\lambda_0^2}. \tag{40}$$

b) If the range of the annihilation process is comparable to the range of the potential a, then the values of the wavefunction within a range a are relevant. The exact formula would involve an integral of the profile of the annihilation amplitude over the wavefunction. But as an order of magnitude:

$$b \sim a: \quad S \sim |\psi(a)|^2 = \left|\psi(0)\frac{u(a)}{a}\right|^2 = \frac{\lambda_1^2}{a^2(\lambda_0^2 + k^2\lambda_1^2)} = \frac{\sigma}{4\pi a^2}, \tag{41}$$

with an approximate maximal value for $k = 0$

$$S_{max} \sim \frac{\lambda_1^2}{a^2\lambda_0^2}. \tag{42}$$

c) If the range of the annihilation process is much bigger than the range of the potential a, then the values of the wavefunction outside the range a are relevant. The exact formula would again depend on the exact profile of the annihilation amplitude, but as an order of magnitude:

$$b \gg a: \quad S \sim \left|\frac{\int_0^b \psi(0)\frac{u(a)}{r}4\pi r^2 dr}{\int_0^b 4\pi r^2 dr}\right|^2 \sim |\psi(0)|^2\frac{u(a)^2}{b^2} = \frac{\lambda_1^2}{b^2(\lambda_0^2 + k^2\lambda_1^2)} = \frac{\sigma}{4\pi b^2}. \tag{43}$$

with an approximate maximum value for $k = 0$

$$S_{max} \sim \frac{\lambda_1^2}{b^2\lambda_0^2}. \tag{44}$$

We observe that the maximun enhancement is obtained when the range of the annihilation process is comparable to the range of the interaction.

V. RENORMALIZATION

The concept of renormalization appears quite often in quantum mechanics in the context of effective interactions or singular potentials. In the first approach, the unknown UV properties of a potential with known long-range behavior is parametrized by adding short-range regularization terms and fitting with known physical properties (spectra, phase shifts etc.) (For a nice pedagogical review see [12], and for recent applications related to our considerations see [9].) Since the same physical properties can be obtained for various values of the regularization terms, this leads to the idea of renormalization group.

Renormalization also appears in the context of singular potentials, such as the delta-function potential: they are regularized with a form of cutoff and their coupling constants are renormalized such that they give finite physical results. Again, "bare" constants vary with the cutoff parameter such that they give the same physical observables.

In our approach, the issue of renormalization is, *a priori*, moot: we start with a physical short-range interaction and we work out the physical effects. There is no need to regularize anything, nor do we need to fit any known physical data. Nevertheless, there are parallels with renormalization that are worth pointing out.

We found that the effects of a short-range interaction can be effectively described in terms of a set of phenomenological parameters λ_n. For generic potentials only λ_0 matters; for more interesting potentials λ_1 becomes important, while for potentials with even more nontrivial behavior higher lambdas may become relevant. This is in the spirit of renormalization: physics is parametrized in terms of a set of macroscopic parameters, with the full details of the short-range interaction becoming irrelevant.

In fact, in our parameters λ_0 and λ_1 we see analogs of both coupling constant and wavefunction renormalization: the long-range properties of the potential, such as the scattering cross-section (38) and the Somerfeld enhancement for scales much larger than the scale of the potential (43), depend only the ratio λ_1/λ_0. This ratio can be considered as a renormalized coupling constant, and is the only one relevant for macroscopic physics. The wavefuction at the origin (37), however, depends on both λ_0 and λ_1, thus involving an additional parameter. This can be considered as a wavefunction renormalization at short ranges.

The three-dimensional δ-potential and its scattering properties have also been examined as an example of regularization and renormalization in quantum mechanics [13]. The scattering equation for an exact delta function potential of strength g reads

$$\begin{aligned}\psi(\mathbf{x}) &= \varphi(\mathbf{x}) - 2mg \int d^3\mathbf{x}'\; G(\mathbf{x}-\mathbf{x}')\;\delta(\mathbf{x}')\;\psi(\mathbf{x}'),\\ &= \varphi(\mathbf{x}) - 2mg\; G(\mathbf{x})\;\psi(0).\end{aligned} \tag{45}$$

Putting $\mathbf{x}=0$ in (45) we find

$$\psi(0) = \frac{1}{1+2mg\,G(0)}, \tag{46}$$

$G(0)$ is infinite, and we proceed by regularizing it. It has the momentum integral form

$$G(0) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\mathbf{p}^2-k^2-i\epsilon}. \tag{47}$$

The integral (47) is linearly divergent and can be regularized with a momentum cutoff Λ as in [13, 14], *i.e.*

$$\begin{aligned}G(0) &= \frac{1}{2\pi^2}\int_0^\Lambda \frac{dp\,p^2}{p^2-k^2-i\epsilon}\\ &= \frac{\Lambda}{2\pi^2}+\frac{i\,k}{4\pi}.\end{aligned} \tag{48}$$

Note that we would have obtained essentially the same result if we had, instead, imposed a short-distance regularization a and defined

$$G(0)_{reg} = G(a) = \frac{e^{ika}}{4\pi a} = \frac{1}{4\pi a}+\frac{ik}{4\pi}, \tag{49}$$

where we omitted terms of order a or smaller, which identifies $\Lambda = \frac{\pi}{2a}$. So we obtain

$$\psi(0) = \frac{1}{1+\frac{mg\Lambda}{\pi^2}+\frac{mg}{2\pi}ik}. \tag{50}$$

From (45) for large $|\mathbf{x}|$ we identify the scattering amplitude as

$$f = -\frac{mg}{2\pi}\psi(0) = \frac{1}{-\frac{2\pi}{mg} - \frac{2\Lambda}{\pi} - ik}. \tag{51}$$

We observe that formulae (50) and (51) have the same functional form as our previous formulae (37) and (38) upon identifying

$$\lambda_0 = 1 + \frac{mg\Lambda}{\pi^2} \ , \quad \lambda_1 = -\frac{mg}{2\pi}. \tag{52}$$

Note, however, that there is *no* choice of scaling of the "bare" parameter g with the cutoff Λ that can make both λ_0 and λ_1 finite. The best one can do is demand for the long-range physical properties of the system, such as f, to be finite. Thus we define

$$\frac{2\pi}{mg} + \frac{2\Lambda}{\pi} = \frac{2\pi}{m\tilde{g}}, \tag{53}$$

which defines a renormalized coupling constant $\tilde{g}$. In terms of the phenomenological variables of last section, $\tilde{g}$ is essentially their ratio:

$$\tilde{g} = -\frac{2\pi}{m}\frac{\lambda_1}{\lambda_0} \tag{54}$$

Written in terms of $\tilde{g}$, f and σ are finite:

$$f = -\frac{m\tilde{g}}{2\pi - ikm\tilde{g}} \ , \quad \sigma = \frac{4\pi m^2\tilde{g}^2}{4\pi^2 + k^2m^2\tilde{g}^2} \tag{55}$$

$\psi(0)$, however, would still diverge as $\Lambda \to \infty$ and $\tilde{g}$ remains finite. This calls for wavefunction renormalization and introduces a *new* parameter into the problem. We define the renormalized wavefunction at the origin

$$\tilde{\psi}(0) = Z\psi(0) \ , \quad Z = -\frac{mg}{2\pi\tilde{\lambda}} = \frac{\lambda_1}{\tilde{\lambda}} \tag{56}$$

The above renormalized $\tilde{\psi}(0)$ is finite. $\tilde{\lambda}$ is a new, finite parameter that plays the same role as λ_1 in the previous section. $\tilde{g}$ and $\tilde{\lambda}$ define the physical parameters of the system, $\tilde{g}$ being relevant for macroscopic quantities and $\tilde{\lambda}$ for short-distance ones. The bare parameters g and Z are cutoff-dependent (as manifest in (53) and (56) for fixed $\tilde{g}$ and $\tilde{\lambda}$) and go to zero as Λ becomes large.

The moral of the above is that a formal renormalization procedure of a singular delta function potential can reproduce the physics of a short-range interaction upon proper identification of renormalized parameters. The true physics of the situation are in the details of the short-range interaction, and its physical content is encoded in the phenomenological parameters λ_n that remain relevant for wavelengths much larger than the interaction scale.

If the short-range potential is sufficiently attractive there can be bound states (see [13–15] for a discussion in the renormalized delta-potential case). These are easily found through our formula (37). For a bound state at $E = -B$ ($B > 0$) we put $ik = -\kappa$, $2mB = \kappa^2$, and look for a resonance in (37) where the denominator vanishes:

$$\lambda_0 + \kappa\lambda_1 = 0 \quad \Rightarrow \quad \kappa = -\frac{\lambda_0}{\lambda_1}. \tag{57}$$

In the presence of a bound state, the total cross section can be expressed as

$$\sigma = \frac{4\pi}{2mB + k^2}. \tag{58}$$

We see that the bound state energy is determined by the same macroscopic parameter λ_1/λ_0 that determines the scattering properties (and defines the renormalized strength in the delta-potential case). This is sensible, as a bound state energy can be detected macroscopically without the need to probe short-distance physics at the scale of the potential.

VI. DISCUSSION AND CONCLUSIONS

In this paper we studied the neutrino scattering problem in the low energy regime by assuming that the interaction between neutrinos is short range. As in this limit we are considering only tree level processes, the only renormalization involved is related to the effective physics at long distances as arising from short-range properties of the potential.

We would like to emphasize two important facts. First, for the scattering cross section, a unique "renormalized" coupling constant is relevant and plays a role analogous to an effective Fermi constant G_X. In principle, bounds for that constant can be found as in [10]. The total cross section has a maximum, as can be seen from

$$\sigma_{tot}^{\max} = 4\pi G_X^2 m_\nu^2, \tag{59}$$

where m_ν represents the reduced neutrino mass. This behavior is independent of the potential being atractive or repulsive.

Using an appropriate rescaling of the total cross section ($\sigma_{tot}/\sigma_{tot}^{max}$) and the energy (k^2/B), the energy dependence of the cross section is as plotted in figure 2.

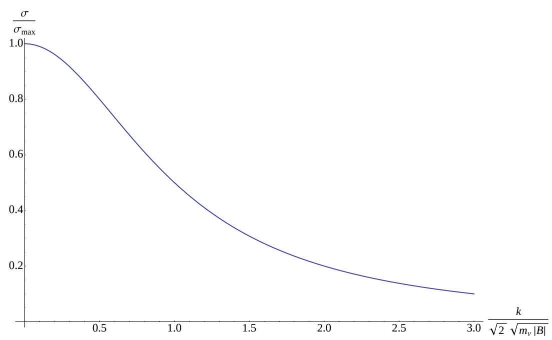

FIG. 2. Qualitative behavior neutrino-neutrino total cross section.

The second important point is that the Sommerfeld enhancement factor depens on additional short-range properties of the potential, and can be quite large. So, if dark matter scattering processes occur via the mechanism proposed in [7], Sommerfeld enhancement could be significant.

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APPENDIX

We first point our that unless the potential scales as a^{-2}, that is, unless $2mVa^2$ is not of negligible magnitude, it will have no effect on scattering. We simply rescale the variable in the equation for $u(r)$ to $s = r/a$, thus expanding the range of the potential to the interval $s \in [0, 1]$, and also define

$$\bar{u}(s) = \frac{1}{a} u(as). \tag{60}$$

In terms of $\bar{u}(s)$

$$\bar{u}'' + a^2(k^2 - 2mV)\bar{u} = 0 \ , \quad \bar{u}'(0) = 1. \tag{61}$$

For small a, the effect of the $k^2 - 2mV$ term is negligible and only the zero-order solution $\bar{u} = s$ survives, the deviation from it being of order a^2; that is,

$$\bar{u} = s + a^2 \int_0^s ds' \int_0^{s'} ds'' \left[2mV(s'') - k^2\right] s'' + O(a^4). \tag{62}$$

So

$$u(a) = a\bar{u}(1) = a + O(a^3) \ , \quad u'(a) = \bar{u}'(1) = 1 + O(a^2), \tag{63}$$

leading to negligible scattering effects.

Even a generic potential scaling like a^{-2} would have limited effect: $\bar{u}_0(s)$ would be a function of order 1 and we would get

$$u_0(a) = a\bar{u}_0(1) \ , \quad u_0'(a) = \bar{u}_0'(1) \ (\neq 1). \tag{64}$$

So, generically, λ_1 would still be neglibible, although λ_0 would now be different than 1 and thus would produce a (momentum-independent) Sommerfeld enhancement factor.

The other interesting possibility would arise if $\bar{u}_0'(1)$ is small (so $s = 1$ would be near a zero of the function $\bar{u}_0'(s)$) and of order a. In that case both λ_0 and λ_1 are small, but their ratio is of order 1, thus leading to nontrivial scattering effects but divergent Sommerfeld enhancement. (This is, essentially, what the standard "renormalization" of the delta-potential [13] achieves.)

Nontrivial potentials where both λ_0 and λ_1 are finite and non-negligible require a behavior that is stronger than a^{-2} for a range of values of r, as well as a change of sign in the range $(0, a)$. To illustrate the possibility, we will present a "proof of concept" potential, of no particular physical significance. We start by giving the desired $u_0(r)$, from which the appropriate potential $2mV = u_0''/u_0$ can be obtained. Take

$$u_0(r) = r + \frac{A + aB}{3}\left(\frac{r}{a}\right)^3 - \frac{A}{5}\left(\frac{r}{a}\right)^5. \tag{65}$$

(The choice of odd powers eliminates a non-essential divergence of V at $r = 0$.) It is clear that the above u_0 satisfies $u_0(0) = 0$ and $u_0'(0) = 1$. For $r = a$ we have

$$u_0(a) = \frac{2A}{15} + \frac{B+3}{3}a \ , \quad u_0'(a) = 1 + B. \tag{66}$$

For small a, both $u_0(a)$ and $u_0'(a)$ remain nonzero and finite. The potential corresponding to the above u_0 is

$$V(r) = \frac{1}{2m}\frac{u_0''(r)}{u_0(r)} = \frac{1}{ma^2}\frac{(A + aB)\left(\frac{r}{a}\right) - 2A\left(\frac{r}{a}\right)^3}{r + \frac{A+aB}{3}\left(\frac{r}{a}\right)^3 - \frac{A}{5}\left(\frac{r}{a}\right)^5}. \tag{67}$$

The above potential in nonsingular everywhere, of order a^{-2} for most of the range $0 < r < a$ but becoming of order a^{-3} as r nears 0. It is positive for $r < a/\sqrt{2} + O(a^2)$ and becomes negative for $r > a/\sqrt{2} + O(a^2)$. These features are generic: a purely positive potential cannot produce $u_0(a)$ and $u_0'(a)$ that are both nonzero and finite in the limit of small a.

We can also show that $u_n(a)$ and $u_n'(a)$ ($n \geq 1$) will go to zero as $a \to 0$. To start, we point out that higher u_n can be expressed recursively in terms of integrals

$$u_n(r) = u_0(r)\int_0^r dr' \frac{1}{u_0^2(r')}\int_0^{r'} dr'' u_0(r'')\, u_{n-1}(r''). \tag{68}$$

The argument for the smallness of higher u_n is based on the fact that

$$u_0(r) = f_0(r/a), \tag{69}$$

where $f_0(s)$ is a function that remains finite in the limit $a \to 0$. By changing variables $r = as$ in (68) we see that

$$u_1(r) = a^2 f_1(r/a) \quad \text{where} \quad f_1(s) = f_0(s)\int_0^s ds' \frac{1}{f_0^2(s')}\int_0^{s'} ds'' f_0(s''). \tag{70}$$

Since f_0 remains finite in the limit $a \to 0$, u_1 will scale like a^2 *unless* the integrals above develop a singularity as $a \to 0$. This, however, is not the case: the only singularity could arise at $s' \to 0$, where $f_0 \to as'$ and the denominator diverges, but we can check that the s'-integral does not diverge. Therefore, $u_1(a) \sim a^2$. A similar argument shows that $u_1'(a) \sim a$, and recursively $u_n(a) \sim a^{2n}$, $u_n'(a) \sim a^{2n-1}$, so the only survining parameters are $u_0(a)$ and $u_0'(a)$. Clearly, to get potentials with nonvanishing u_1 or higher, we need to pick u_0 that becomes divergent (of order a^{-1} or higher) somewhere in the range $0 < r < a$. Such behavior is highly unphysical and provides a reason why we do not expect potentials with nonvanishing higher λ_n to arise.

We conclude by showing that for the scattering cross-section higher angular momentum sectors have a negligible contribution in the limit $ka \to 0$. (For the Sommerfeld factor their contribution is exactly zero.) The physical reason is that the radial part of the wavefunction for such sectors satisfies

$$u'' + k^2 u = 2mVu + \frac{\ell(\ell+1)}{r^2}u, \tag{71}$$

the same as for the s-wave sector but with an additional centrifugal potential. The classical inflection point of a particle with energy $k^2/2m$ off the centrifugal potential is at

$$k^2 = \frac{\ell(\ell+1)}{r^2} \implies r_c = \frac{\sqrt{\ell(\ell+1)}}{k}. \tag{72}$$

Since $ka \ll 1$, $r_c \gg a$. So the region where the potential V is nonzero is deep inside the classically forbidden region of the particle. The potential is effectively shielded by the centrifugal barrier, accessible only through tunneling effects. Only for $\ell = 0$ there is no barrier and the wavefunction can access the potential and feel its effects. In what follows we will back this intuition with a calculation.

We will define, again, the unique solution of the radial equation (71) with boundary conditions

$$u \sim r^{\ell+1} \text{ as } r \to 0 \;, \quad \text{or} \quad u^{(\ell+1)}(0) = (\ell+1)! \;, \quad u^{(n)}(0) = 0 \text{ for } n < \ell+1, \tag{73}$$

(exponents in parenthesis indicate derivatives). Then the radial wavefunction in this sector is

$$\psi_\ell(r) = A\frac{u(r)}{r}, \tag{74}$$

with A a scale parameter that is fixed by boundary conditions at $r \to \infty$. For $\ell = 0$ (the s-wave), $A = \psi(0)$, but for higher ℓ, A does not contribute to $\psi(0)$.

To determine the bounday conditions at $r \to \infty$ we write the plane wave decomposition in terms of spherical harmonics and spherical Bessel functions

$$e^{ikx} = \sum_\ell i^\ell (2\ell+1) j_\ell(kr) P_\ell(\theta), \tag{75}$$

with θ measured with respect to the axis $\vec{k}$. So the radial part of this plane wave in the u-parametrization ($u = re^{ikx}$) in the ℓ sector is

$$u_{plane}(r) = i^\ell (2\ell+1) r j_\ell(kr). \tag{76}$$

To isolate the incoming and outgoing part, we write it in terms of Hankel functions

$$h_\ell = j_\ell + i y_\ell. \tag{77}$$

In fact, we define the modified Hankel functions

$$U_\ell(x) = ixh_\ell(x) = x(-y_\ell(x) + ij_\ell(x)), \tag{78}$$

in terms of which the radial ℓ-plane wave function is

$$u_{plane}(r) = \frac{1}{k}\, i^\ell\, \frac{2\ell+1}{2i}\left(U_\ell(kr) - \bar{U}_\ell(kr)\right) \tag{79}$$

In the limit $kr \gg 1$ the functions U_ℓ behave as

$$U_\ell(kr) \to (-i)^\ell e^{ikr}, \tag{80}$$

so U_ℓ is the outgoing part and $\bar{U}_\ell$ is the incoming part. U_ℓ and its conjugate $\bar{U}_\ell$ separately satisfy the free radial equation

$$\frac{d^2 U_\ell(kr)}{dr^2} + k^2 U_\ell(kr) = \frac{\ell(\ell+1)}{r^2} U_\ell(kr). \tag{81}$$

The first few U_ℓ are

$$U_0(x) = e^{ix}, \quad U_1(x) = e^{ix}\left(\frac{1}{x} - i\right), \quad U_2(x) = e^{ix}\left(\frac{3}{x^2} - \frac{3i}{x} - 1\right) \text{ , etc.} \tag{82}$$

Returning to the solution of our problem, the radial wavefunction for $r > a$ (where the potential vanishes) will still be a superposition of U_ℓ and $\bar{U}_\ell$. The incoming part, proportional to $\bar{U}_\ell$, must be the same as in the plane wave (79), while the outgoing part U_ℓ will have a different coefficient, due to the existence of the outgoing scattered wave. The extra scattering part will have the asymptotic form

$$\psi_{sc} \sim f\frac{e^{ikr}}{r} \quad \text{so} \quad u_{sc} \sim f e^{ikr}, \tag{83}$$

as $r \to \infty$. From the asymptotic behavior (80) of U_ℓ we see that the extra scattering part must be of the form

$$u_{sc} = (-i)^\ell f U_\ell(kr) \tag{84}$$

so the full wavefunction for $r > a$ is

$$\begin{aligned} u_{out} &= \left(\frac{1}{k}\, i^\ell\, \frac{2\ell+1}{2i} + (-i)^\ell f\right) U_\ell(kr) - \frac{1}{k}\, i^\ell\, \frac{2\ell+1}{2i} \bar{U}_\ell(kr) \\ &= \frac{1}{k}\, i^\ell\, \frac{2\ell+1}{2i} \left(b\, U_\ell(kr) - \bar{U}_\ell(kr)\right), \end{aligned} \tag{85}$$

where

$$b = 1 + (-1)^\ell \frac{2ik}{2\ell+1}\, f. \tag{86}$$

From unitarity, b must be a pure phase (since incoming and outgoing waves must have equal amplitudes) and it defines the scattering phase shift, while the above equation relates the scattering amplitude to the scattering phase shift (and leads, eventually, to the forward-scattering formula for the cross-section).

Solving the full problem for the radial wavefunction amounts to matching the interior solution (74), defined in terms of the solution u and A, to the exterior solution (85) at $r = a$. Equating the values of the wavefunctions as well as their derivatives on either side of $r = a$ we obtain

$$\begin{aligned} Bu(a) &= b\, U_\ell(ka) - \bar{U}_\ell(ka), \\ Bu'(a) &= kb\, U'_\ell(ka) - k\bar{U}'_\ell(ka), \end{aligned} \tag{87}$$

where we defined

$$B = (-i)^\ell \frac{2ik}{2\ell+1} A, \tag{88}$$

from which we obtain

$$\begin{aligned} B &= k\, \frac{\bar{U}_\ell U'_\ell - \bar{U}'_\ell U_\ell}{U_\ell u' - kU'_\ell u}, \\ b &= \frac{\bar{U}_\ell u' - k\bar{U}'_\ell u}{U_\ell u' - kU'_\ell u}, \end{aligned} \tag{89}$$

where we suppressed the dependence on a or ka to alleviate the form. Since u is real, it is clear that b is a pure phase since $\bar{b} = b^{-1}$. The scattering amplitude f and the coefficient A are determined through their relation with b and B.

As a check, we apply the formulae for $\ell = 0$. In that case, $U_0 = e^{ikr}$ and $A = \psi(0)$. We obtain

$$B = \frac{2ik}{u'(a) - iku(a)} e^{-ika} \;\Rightarrow\; A = \psi(0) = \frac{e^{-ika}}{u'(a) - iku(a)}, \tag{90}$$

which is our earlier result, while

$$b = e^{-2ika}\frac{u'(a) + iku(a)}{u'(a) - iku(a)} \;\Rightarrow\; f = e^{-ika}\frac{1}{k}\frac{k\cos(ka)\, u(a) - \sin(ka)\, u'(a)}{u'(a) - iku(a)}, \tag{91}$$

In the limit $ka \to 0$ this gives

$$f = \frac{u(a)}{u'(a) - iku(a)}, \tag{92}$$

as before.

We can now tackle the issue of whether higher angular momenta contribute to the scattering amplitude for a very small. We will work out the case $\ell = 1$, the others being qualitatively similar.

For $\ell = 1$ we can substitute the explicit form of U_1 from (82) and obtain for b

$$b = e^{-2ika} \frac{(1 + ika - k^2a^2)u(a) + a(1 + ika)u'(a)}{(1 - ika - k^2a^2)u(a) + a(1 - ika)u'(a)}. \tag{93}$$

So the scattering phase shift is $-2ka$ plus twice the phase of the complex number in the numerator. We observe that for $ka \ll 1$ this phase will be negligible (and in fact of order k^3a^3) *no matter what the scaling of* $u(a)$ *and* $u'(a)$ *with* a. So, completely generically, the contribution of the $\ell = 1$ sector to the scattering amplitude is vanishingly small. The only possibility to get an appreciable phase shift and scattering amplitude is for the real and imaginary parts of the numerator to be of comparable magnitude. For this to happen we need

$$au'(a) = -(1 - k^2a^2 + \lambda k^3a^3)u(a), \tag{94}$$

for some finite λ. Thus, not only do we need $au'(a)$ to be close to $-u(a)$, but we need it tuned to such a value to an accuracy of order k^3a^3 (!). This is a tremendous level of fine-tuning. Moreover, it is k-dependent. So even if it were to hold for some k, it would stop holding as soon as k moves away from that value by the tiniest amount.

For higher ℓ, a similar pattern emerges: the dominant terms are such that the scattering amplitude vanishes, and an increasingly accurate fine tuning is required to have an "accidental" scattering resonance that would happen for only one specific value of k, if at all. The final result is that only the $\ell = 0$ sector contributes.

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